EECS 16A Foundations of Signals, Systems, & Information Processing
Fall 2025 Homework 7

This homework is due Friday, October 17, 2025, at 23:59.

Submission Format

Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned). as well as your IPython notebook saved as a PDF.

If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

Some problems are adapted from the textbook "Introduction to Applied Linear Algebra - Vectors, Matrices, and Least Squares" (henceforth referred to as VMLS) by Stephen Boyd and Lieven Vandenberghe. A link to the pdf version of the textbook is provided here: `https://web.stanford.edu/~boyd/vmls/vmls.pdf`.

On occasion, a problem set contains one more problems designated as optional. Your solutions to such problems are not to be evaluated, and will not count toward your grade. Nevertheless, you are still responsible for learning the subject matter within their scope. In other words, we may test you on their scope.

1. Mechanical Projections

(a) Find the projection of $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ onto $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. What is the squared error between the projection and $\mathbf{b}$, i.e. $\|\varepsilon\|^2 = \|\text{proj}_{\mathbf{a}}(\mathbf{b}) - \mathbf{b}\|^2$?

(b) Find the projection of $\mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix}$ onto the column space of $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. What is the squared error between the projection and $\mathbf{b}$, i.e. $\|\varepsilon\|^2 = \|\mathbf{P}_{\mathbf{A}}(\mathbf{b}) - \mathbf{b}\|^2$?

2. Least squares and QR factorization (Adapted from VMLS 12.8)

Suppose $\mathbf{A}$ is an $m \times n$ matrix with linearly independent columns and QR factorization $\mathbf{A} = \mathbf{QR}$, and $\mathbf{b} \in \mathbb{R}^m$. The vector $\mathbf{A}\hat{\mathbf{x}}$ is the linear combination of the columns of $\mathbf{A}$ that is closest to the vector $\mathbf{b}$, i.e., it is the projection of $\mathbf{b}$ onto the column space of $\mathbf{A}$.

(a) Show that $\mathbf{A}\hat{\mathbf{x}} = \mathbf{Q}\mathbf{Q}^\top\mathbf{b}$, i.e. that $\mathbf{Q}\mathbf{Q}^\top\mathbf{b}$ is the projection of $\mathbf{b}$ onto the column space of $\mathbf{A}$.
Hint: What is the relationship between the column space of $\mathbf{Q}$ *and the column space of* $\mathbf{A}$*?*

(b) Show that $\|\mathbf{A}\hat{\mathbf{x}} - \mathbf{b}\|^2 = \|\mathbf{b}\|^2 - \|\mathbf{Q}^\top\mathbf{b}\|^2$. (This is the square of the distance between $\mathbf{b}$ and the closest linear combination of the columns of $\mathbf{A}$.)

3. Linear Regression

An object moves along a straight line at an unknown but constant velocity $\dot{y}(t) = \beta$, where $\dot{y}(t) = dy(t)/dt$ denotes the derivative of y with respect to time t. We want to determine its velocity and initial position, which characterize the straight line

$$\forall t \in \mathbb{R} \qquad y(t) = \alpha + \beta\, t,$$

where α is the unknown position of the object at time $t = 0$—that is, $y(0) = \alpha$.

To do this, we measure the position of the object at M time instants $t_1, t_2, \ldots, t_M$. We denote our measurements by $y_1, \ldots, y_M$. Unfortunately, our measurements are prone to error, caused by imperfections of our measurement instruments. What we have is the following set of M equations in two unknowns:

$$\begin{aligned} \alpha + t_1\,\beta &= y_1 \\ &\vdots \\ \alpha + t_m\,\beta &= y_m \\ &\vdots \\ \alpha + t_M\,\beta &= y_M. \end{aligned}$$

We can write our measurement equations in matrix-vector form $\mathbf{A}\mathbf{x} = \mathbf{y}$ as follows:

$$\underbrace{\begin{bmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \\ \vdots & \vdots \\ 1 & t_M \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_m \\ \vdots \\ y_M \end{bmatrix}}_{\mathbf{y}} \tag{1}$$

We know the least-squares solution $\widehat{\mathbf{x}}$ to Equation (1) is given by

$$\widehat{\mathbf{x}} = \left(\mathbf{A}^\top\mathbf{A}\right)^{-1}\mathbf{A}^\top\mathbf{y}.$$

(a) Determine reasonably simple forms for each of $\mathbf{A}^\top\mathbf{A}$, $\left(\mathbf{A}^\top\mathbf{A}\right)^{-1}$, and $\mathbf{A}^\top\mathbf{y}$. Each of your expressions should be in terms of an appropriate subset of M, t_m, and y_m, where $m = 1, \ldots, M$.
Note: Do NOT carry out the multiplication

$$\left(\mathbf{A}^\top\mathbf{A}\right)^{-1}\mathbf{A}^\top\mathbf{y}.$$

You should be relieved that we did not ask you to determine

$$\widehat{\mathbf{x}} = \left(\mathbf{A}^\top\mathbf{A}\right)^{-1}\mathbf{A}^\top\mathbf{y},$$

because the expression for it would not be pretty! In what follows, we'll apply the Gram-Schmidt method to orthogonalize (but not orthonormalize) the columns of $\mathbf{A} = [\mathbf{a}_1\ \mathbf{a}_2]$. In particular, let the columns of $\mathbf{A}$ be given by

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \mathbf{1} \qquad \text{and} \qquad \mathbf{a}_2 = \begin{bmatrix} t_1 \\ \vdots \\ t_M \end{bmatrix},$$

where $\mathbf{1}$ denotes the all-ones vector of size M—that is, the vector each of whose M entries is 1. Take as your first orthogonal vector $\mathbf{z}_1 = \mathbf{a}_1$; note that $\mathbf{z}_1$ does not have unit length, and we won't bother normalizing its length.

Now we determine the second vector $\mathbf{z}_2$ in our orthogonal set.

(b) Show that the second orthogonal vector $\mathbf{z}_2$ is given by

$$\mathbf{z}_2 = \big(\mathbf{I} - \mathbf{P}_1\big)\mathbf{a}_2, \tag{2}$$

where $\mathbf{I}$ is the identity matrix of appropriate size, and $\mathbf{P}_1$ is the matrix given by

$$\mathbf{P}_1 = \frac{\mathbf{a}_1\,\mathbf{a}_1^\top}{\langle \mathbf{a}_1, \mathbf{a}_1 \rangle}.$$

Be sure to prove that $\mathbf{z}_2$, as given by Equation (2), is, in fact, orthogonal to the first vector, which we chose to be $\mathbf{z}_1 = \mathbf{a}_1$. Also, explain what the matrix $\mathbf{P}_1$ does to any vector that it acts on. In particular, explain what $\mathbf{P}_1\mathbf{a}_2$ is.

(c) Determine a reasonably simple form for $\mathbf{z}_2$. Your expressions for the entries in $\mathbf{z}_2$ should be in terms of M, t_m, and $\bar{t}$, where

$$\bar{t} = \frac{1}{M}\sum_{m=1}^{M} t_m.$$

Explain what the orthogonalization of $\mathbf{a}_1$ and $\mathbf{a}_2$ to the forms $\mathbf{z}_1 = \mathbf{a}_1$ and $\mathbf{z}_2 = \big(\mathbf{I} - \mathbf{P}_1\big)\mathbf{a}_2$ means in terms of relocating the origin of the time axis along which we've made our measurements.

(d) Suppose we measure the position of the moving object at time instants $t_1 = 0$, $t_2 = 1$, and $t_3 = 2$ seconds, and that we've registered the following position values:

$$y(0) = 1, \qquad y(1) = 3, \qquad y(2) = 5.$$

Determine α and β, which define the straight line that best fits the data in a least-squares sense.

4. Proof: Least Squares

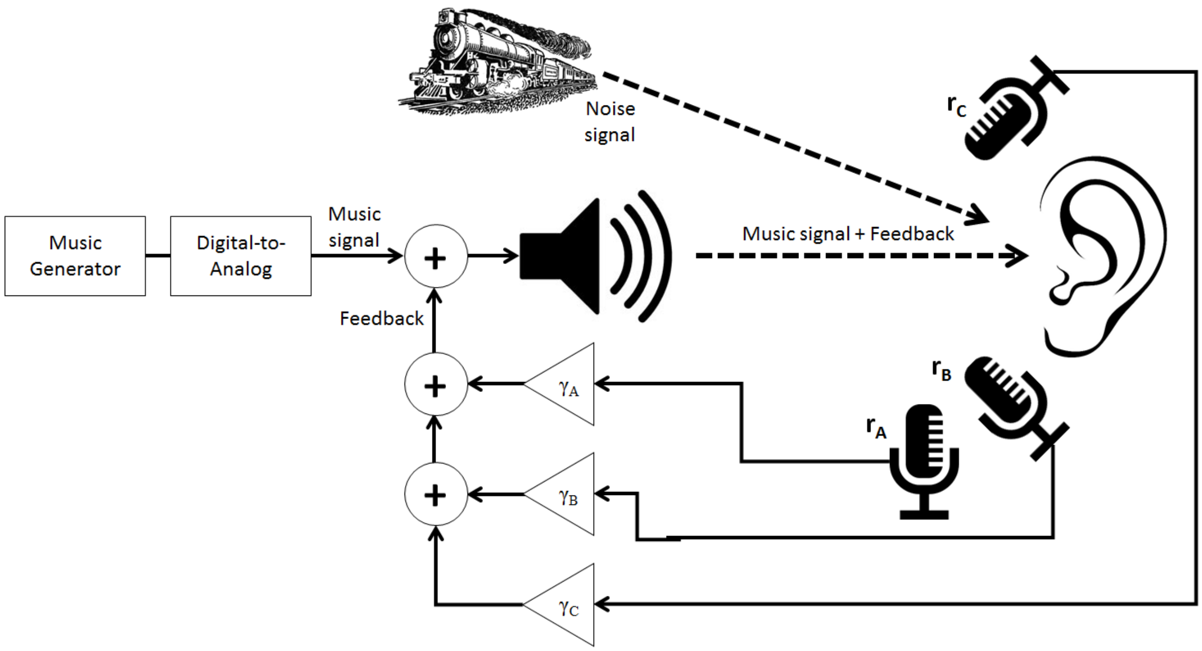
Let $\hat{\mathbf{x}}$ be the solution to a linear least squares problem.

$$\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{b} - \mathbf{A}\mathbf{x}\|^2$$

Show that the minimizing least squares error vector $\varepsilon = \mathbf{b} - \mathbf{A}\hat{\mathbf{x}}$ is orthogonal to the columns of $\mathbf{A}$ by direct manipulation (i.e., plug the formula for the linear least squares solution $\hat{\mathbf{x}}$ into the error vector and then check if $\mathbf{A}^T\varepsilon = \mathbf{0}$.)

5. Noise Canceling Headphones

In this problem, we will explore a common design for noise cancellation using noise-canceling headphones as an example application. We will work with the model shown in the figure below.



A music signal is generated at a speaker and transmitted to the listener's ear. If there is noise in the environment (*e.g.* other people's voices, a train going by), this noise signal will be superimposed (i.e. added) on the music signal and the listener will hear both. In order to cancel the noise, we will try **to record the noise and subtract it directly from the transmitted signal** with the hope that we can achieve perfect cancellation of everything but the music. Since our system is imperfect, we'll have to solve a **least squares problem**.

The gain blocks marked by γ **(Greek "gamma") represent scalar multiplication**, and we will assume that they can take on any real number, positive or negative.

Assume we have an **additive noise signal** noted by $\mathbf{n}$, which we want to cancel. This represents the extra noise coming from the train.

$$\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{bmatrix}.$$

Let us assume the music signal is represented by

$$\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{bmatrix}.$$

In absence of any noise cancellation, the user will hear:

$$\mathbf{s} = \mathbf{m} + \mathbf{n}.$$

However, we want to cancel this noise. For this we use three microphones to record this noise, Mic A, Mic B, and Mic C. However, they cannot perfectly record the noise $\mathbf{n}$ and have erroneous recordings. Let $\mathbf{r}_A$, $\mathbf{r}_B$, and $\mathbf{r}_C$ represent the **noise that each microphone picks up**:

$$\mathbf{r}_A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}, \mathbf{r}_B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}, \mathbf{r}_C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}.$$

We can arrange the recordings into a matrix $\mathbf{R}$ and the microphone gains, γ, into a vector γ to get:

$$\mathbf{R} = \begin{bmatrix} \mathbf{r}_A & \mathbf{r}_B & \mathbf{r}_C \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \\ a_5 & b_5 & c_5 \end{bmatrix}, \gamma = \begin{bmatrix} \gamma_A \\ \gamma_B \\ \gamma_C \end{bmatrix}.$$

We want to choose a γ such that we minimize the impact of the noise.

The listener's ear will hear a total signal of:

$$\mathbf{s} = \mathbf{m} + \mathbf{R}\gamma + \mathbf{n},$$

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \\ a_5 & b_5 & c_5 \end{bmatrix} \begin{bmatrix} \gamma_A \\ \gamma_B \\ \gamma_C \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{bmatrix}.$$

This problem focuses on how to choose γ so as to have the signal $\mathbf{s}$ be as close to $\mathbf{m}$ as possible.

(a) Ideally, we would want to have a signal at the ear that matches the original music signal $\mathbf{m}$ perfectly. In reality, this is not possible, so we will aim to minimize the effect of the noise. What quantity would we need to minimize to make sure this happens? Write your answer in terms of the matrix $\mathbf{R}$, the vector of mic gains γ, and the noise vector $\mathbf{n}$. Hint: Your answer should be the norm of something.

(b) We can solve the noise minimization problem by the least squares method. In effect, if we have a problem, $\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|$, then the $\mathbf{x}$ that solves this problem is,

$$\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}. \tag{3}$$

Implement this least squares solution in the IPython Notebook helper function `doLeastSquares`. Remember that $\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} + \mathbf{b}\| = \min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - (-\mathbf{b})\|$.

(c) In order to verify the least squares function, we want to compute an example by hand. For this part only, we will be using a system with 2 microphones instead of 3 and a signal of length 3. For the given $\mathbf{n}$ and the recordings, $\mathbf{r}_A$, $\mathbf{r}_B$, below, find the γ's that minimize the effect of noise. You must show your work but you may check your answer with the helper function.

$$\mathbf{n} = \begin{bmatrix} 0.20 \\ 0.10 \\ 0.50 \end{bmatrix}, \ \mathbf{r}_A = \begin{bmatrix} 0.50 \\ 0.00 \\ 0.50 \end{bmatrix}, \ \mathbf{r}_B = \begin{bmatrix} 0.00 \\ 0.10 \\ 0.00 \end{bmatrix}.$$

(d) For the given $\mathbf{n}$ and the recordings, $\mathbf{r}_A$, $\mathbf{r}_B$, $\mathbf{r}_C$, below, find the γ's that minimize the effect of noise. Use the helper function `doLeastSquares` you made to solve this.

$$\mathbf{n} = \begin{bmatrix} 0.100 \\ 0.370 \\ -0.450 \\ 0.068 \\ 0.036 \end{bmatrix}, \ \mathbf{r}_A = \begin{bmatrix} 0.000 \\ 0.110 \\ -0.310 \\ -0.012 \\ -0.018 \end{bmatrix}, \ \mathbf{r}_B = \begin{bmatrix} 0.000 \\ 0.220 \\ -0.200 \\ 0.080 \\ 0.056 \end{bmatrix}, \ \mathbf{r}_C = \begin{bmatrix} 0.000 \\ 0.370 \\ -0.440 \\ 0.065 \\ 0.038 \end{bmatrix}.$$

The next few questions can be answered in the IPython notebook by running the associated cells.

(e) Follow the instructions in the IPython notebook to load a **music signal** and some **noise signals**. Listen to the music signal `music_y` and the two noise signals `noise1_y` and `noise2_y`. Which ones are full of static and which ones are not?

Also listen to the **noisy music signal** `noisyMusic`, generated by adding the music signal `music_y` and the first noise signal `noise1_y`. What do you hear?

(f) For this part we assume the noise signal $\mathbf{n}$ is given by `noise1_y` only. $\mathbf{R}$, i.e. the matrix of recorded noise from 3 microphones, is simulated with the `recordAmbientNoise` function. Calculate a vector γ that minimizes the effect of noise using `doLeastSquares`.

(g) Use you answers from the previous part to find the **noise-cancellation signal $\mathbf{R}\gamma$**. Add this noise-cancellation signal ($\mathbf{R}\gamma$), the music signal $\mathbf{m}$ (`music_y`), and the noise signal $\mathbf{n}$ (`noise1_y`), in order to generate a **noise-cancelled signal**. Play the noisy signal from part (e) and the noise-cancelled signal. Can you hear a difference?

(h) **[Optional]** Try adding the other noise signal `noise2_y` to the music signal to generate a new noisy signal. Don't re-calculate new values for γ i.e. don't solve the least squares problem again. Recalculate $\mathbf{R}$ and find the new noise-cancellation signal $\mathbf{R}\gamma$ using γ from part (g). Add this noise-cancellation signal to your noisy signal. Comment on the quality of the resulting noise-cancelled signal. Is it perfect or are there artifacts?

Note: We want to find out how the γ we calculated work for any type of noise.

6. Image Analysis

Applications in medical imaging often require an analysis of images based on the image's pixels. For instance, we might want to count the number of cells in a given biological sample. One way to do this is to take a picture of the cells and use the pixels to determine their locations and how many there are. Automatic detection of shape is useful in image classification as well (e.g. consider a robot trying to find out autonomously where a mug is in its field of vision).

Let us focus back on the medical imaging scenario. You are interested in finding the exact position and shape of a cell in an image. You will do this by finding the **equation of the circle or ellipse** that bounds the cell relative to a given coordinate system in the image. Your collaborator uses edge detection techniques to find **a bunch of points that are approximately along the edge of the cell**. We assume that the origin of the coordinate system is in the center of the image with standard axes (x, y) and your collaborator gives you the following points that approximately bound the cell:

$(0.3, -0.69)$, $(0.5, 0.87)$, $(0.9, -0.86)$, $(1, 0.88)$, $(1.2, -0.82)$, $(1.5, 0.64)$, $(1.8, 0)$.

Recall that an equation of the form

$$a_1x^2 + b_1xy + c_1y^2 + d_1x + e_1y = 1$$

can be used to represent an ellipse (if $b_1^2 - 4a_1c_1 < 0$), and an equation of the form

$$a_1(x^2 + y^2) + d_1x + e_1y = 1$$

is a circle if $d_1^2 + e_1^2 + 4a_1 > 0$. Notice that the circle has fewer parameters.

You don't need to consider these constraints in your least squares setup, but you are encouraged to check whether your least squares solutions satisfy these constraints.

(a) How can you find the equation of a circle that surrounds the cell by fitting the data points? First, provide a setup and formulate a minimization problem to do this, i.e. a least squares problem minimizing the squared error $\|\mathbf{A}\mathbf{x}_c - \mathbf{b}\|^2$, where you attempt to find the **unknown coefficients** a_1**,** d_1**, and** e_1 from your data points. Here your unknown vector $\mathbf{x}_c = \begin{bmatrix} a_1 \\ d_1 \\ e_1 \end{bmatrix}$. *Hint: The quantities $(x^2 + y^2)$, x, and y can be thought of as known values calculated from your data points.*

You do not need to simplify the numerical values for the matrix elements; just writing out the matrix with numerical expressions will suffice.

(b) How can you find the equation of an ellipse (instead of a circle) that surrounds the cell? Provide a setup and formulate a minimization problem similar to that in part (a) with the unknown vector defined as $\mathbf{x}_e$. Note that $\mathbf{x}_e$ will be a different vector than $\mathbf{x}_c$.

You do not need to simplify the numerical values for the matrix elements; just writing out the numerical expressions will suffice.

(c) Now let's try this ourselves with the data given in the problem. In the IPython notebook, write a short program that uses least-squares to fit a circle to the given points. A helper function `plot_circle` is provided. What is $\frac{\|\mathbf{e}\|}{N}$, where $\mathbf{e} = \mathbf{A}\mathbf{x}_c - \mathbf{b}$ and N is the number of data points? Plot your points and the best fit circle in IPython.

(d) In the IPython notebook, write a short program that uses least-squares to fit an ellipse to the given points. A helper function `plot_ellipse` is provided. What is $\frac{\|\mathbf{e}\|}{N}$, where $\mathbf{e} = \mathbf{A}\mathbf{x}_e - \mathbf{b}$ and N is the number of data points? Plot your points and the best fit ellipse in IPython. How does this error compare to the one in the previous subpart?

7. Homework Process and Study Group

Whom did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

