EECS 16A Designing Information Devices and Systems I
Spring 2023 Discussion 1A

1. Solving Systems of Equations

While we'd love every system of linear equations to have a unique solution, in reality we can either have (a) one unique solution, (b) an infinite number of solutions, or (c) no solution at all. We are going to walk through some examples to see what sorts of equations create these types of solutions.

(a) Let's consider the system (where each $a, b \in \mathbb{R}$ can be any real number):

$$\begin{aligned} ax + y &= 3 \\ -x + 2y &= b \end{aligned}$$

For each of the selected values for a and b, sketch or plot out the lines $y(x)$ each of these equations form.

Can you conclude which values result in a unique solution? Infinite solutions? No solutions?

- i. $a = 1$, $b = 0$
- ii. $a = 0$, $b = 2$
- iii. $a = -1/2$, $b = 6$
- iv. $a = -1/2$, $b = 4$

Answer: First, we re-write our system into line equations by isolating y on one side:

$$y = 3 - ax \qquad\qquad y = \frac{b}{2} + \frac{1}{2}x$$

By directly inserting the values for a, b we can get a collection of lines to plot

i. $y = 3 - x$ ii. $y = 3 + 0x$ iii. $y = 3 + x/2$ iv. $y = 3 + x/2$

. $y = 0 + x/2$ $y = 1 + x/2$ $y = 3 + x/2$ $y = 2 + x/2$

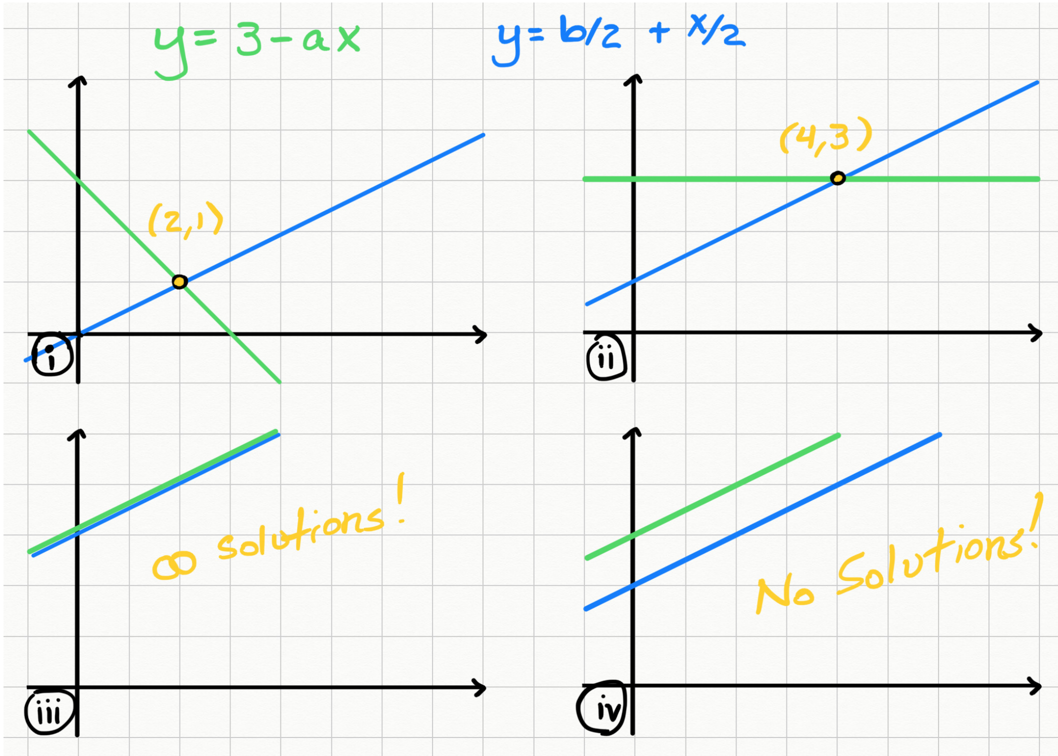
Essentially, each line represents the values of (x,y) that satisfy that equation, so we need values that satisfy both equations (i.e. the intersection).

The key take-away here is that for i. and ii. we have different slopes, but for iii. and iv. the slopes are both $1/2$, which means that they either never intersect (they're parallel) OR they're the same line.

Thus, for i. and ii. there must be one point of intersection somewhere and so there is one solution! We can get this by equating the two equations, $3 - x = x/2$, solving for $x = 2$, and substituting in for $y = 1$. A similar method works on ii. $3 = 1 + x/2 \to x = 4 \to y = 1 + 4/2 = 3$.

However, iii. produces just one line, meaning there are infinite solutions along the line $y = 3 + x/2$. Problem iv. has no solution, since the lines are parallel (the lines have the same slope with different intercepts) and never intersect. Thus, we have *inconsistent* equations.

See the sketched plots below:



(b) Now, assume we are using the tomography imaging technique described in lecture to image a 2x2 grid, as shown below.

2x2 Tomography Example

x_1	x_2
x_3	x_4

i. We record the following measurements:

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 + x_3 &= 2 \\ x_2 + x_4 &= 2 \\ x_3 + x_4 &= 3 \end{aligned}$$

Solve this system using any method. Is this a valid set of measurements? Why or why not?

Answer: No, this is not valid. For instance, $(x_1 + x_2) + (x_3 + x_4) = 2 + 3 = 5$, however, $(x_1 + x_3) + (x_2 + x_4) = 2 + 2 = 4$, which is a contradiction.

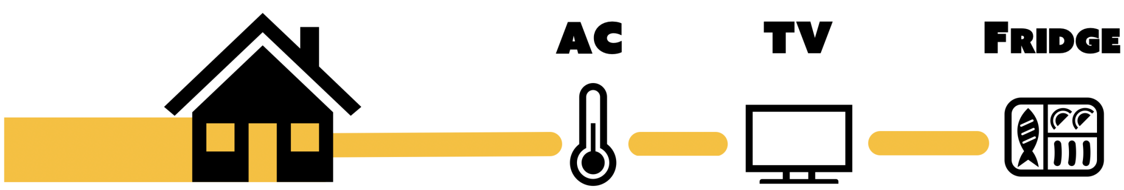
ii. We are led to believe our measurements might be faulty, so we record the following new measurements:

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 + x_3 &= 3 \\ x_2 + x_4 &= 2 \\ x_3 + x_4 &= 3 \end{aligned}$$

Now, solve this system using any method. Is this a valid set of measurements? Why or why not?

Answer: These are valid measurements. Doing the check from the previous part, we get $(x_1 + x_2) + (x_3 + x_4) = 2 + 3 = (x_1 + x_3) + (x_2 + x_4)$. By plugging in a random initial value for x_1, we can in fact find infinite solutions — for instance, if we set $x_1 = 1$, we get the solution $(x_1, x_2, x_3, x_4) = (1, 1, 2, 1)$, and if we set $x_1 = 0.5$, we get the solution $(x_1, x_2, x_3, x_4) = (0.5, 1.5, 2.5, 0.5)$. This system wasn't that different from the one in part i. – instead of 2, the value of $x_1 + x_3$ was 3. This goes to show that even a little noise or interference in our measurements can cause our real-life data to be mathematically inconsistent. Some level of noise is inevitable in most real-life scenarios, so we will learn techniques to combat that later in this course.
Now, while this given system is technically mathematically consistent, there are infinite solutions, so there is no way to tell exactly which solution is true and in fact, this will always be the case with our current measurement technique (measuring the sum of the rows and columns).

2. Energy Disaggregation



Suppose you live in a home with just these **three appliances: an air conditioning unit (AC), a television (TV), and a refrigerator (R)**. Now, say you want to find the amount of electricity these appliances use individually, but the only measurement you can take is of the total power your home draws using your meter outside (this is often mounted on the side of the house and shows a running total of your electricity usage).

To do this, you will turn some appliances on and off and then read different total measurements. You can turn off the TV at any time, but you **can't unplug the fridge** since the food would spoil. We keep the air conditioner off throughout the morning, but then it must stay on during the afternoon. However, the breaker trips (meaning the electricity suddenly shuts off) if all three are running, so **the TV and AC cannot run at the same time**.

(a) Can you design a way to calculate how much power each appliance uses? What type of measurements will you need to make, and how many?

Let x_R be the power consumed by the refrigerator, x_{TV} by the TV, and x_{AC} by the AC, and let m_i represent the power measured in measurement i. To find out the values of three variables, we somehow need three equations/measurements.

Answer:
We must collect 3 different measurements, and each one needs to give us new information. Since we can only toggle (turn on and off) two of the appliances (AC & TV) and we cannot have all 3 appliances running, there are exactly 3 unique measurements we can take.

i. Measure in the morning (AM) with the TV off $\rightarrow$ $m_1 = x_R$
ii. Measure in the morning (AM) with the TV on $\rightarrow$ $m_2 = x_{\text{TV}} + x_R$
iii. Measure in the afternoon (PM) with the TV off $\rightarrow$ $m_3 = x_{\text{AC}} + x_R$

We have three equations and three unknowns and can now try to solve the problem!

(b) Write out a system of equations that describes your measurements. Can you solve this system so that each appliance's power is written in terms of measurements m_i?
For example: if you measure the power m_1 in the afternoon with the AC and refrigerator on but the TV is off, then the equation might look like $x_{\text{AC}} + x_{\text{R}} = m_1$.

Answer: As written above, the unique measurements are

$$\begin{aligned} x_{\text{R}} &= m_1 \\ x_{\text{TV}} + x_{\text{R}} &= m_2 \\ x_{\text{AC}} + x_{\text{R}} &= m_3 \end{aligned}$$

We can solve this by substitution.

$$\begin{aligned} x_{\text{R}} &= m_1 & & \\ x_{\text{TV}} + m_1 &= m_2 & \rightarrow \quad & x_{\text{TV}} = m_2 - m_1 \\ x_{\text{AC}} + m_1 &= m_3 & \rightarrow \quad & x_{\text{AC}} = m_3 - m_1 \end{aligned}$$

(c) Let us say the breaker is fixed, so now we can safely run the AC and TV at the same time. Is there another way (or ways) you could create a new system of equations to solve? If so, see if you can solve your new system!

Answer:
Without the breaker constraint, there are now 3 new possible systems of equations we could solve to determine the power drawn from each appliance.

Two of these systems can be solved via the substitution method like shown above, but for the system omitting the measurement with both AC and TV off we must add/subtract equations to find our result.

$$\begin{aligned} x_{\text{TV}} + x_R &= m_2 \\ x_{\text{AC}} + x_R &= m_3 \\ x_{\text{AC}} + x_{\text{TV}} + x_R &= m_4 \end{aligned}$$

We can identify x_{AC} by subtracting the m_2 equation from the m_4 equation:

$$m_4 - m_2 \quad = \quad x_{\text{AC}} + x_{\text{TV}} + x_R \; - x_{\text{TV}} - x_R \quad = \quad x_{\text{AC}}$$

From this point we can utilize the result via substitution into equations m_3 and m_2.

$$\begin{aligned} (m_4 - m_2) + x_R = m_3 \quad &\rightarrow \quad x_R = m_2 + m_3 - m_4 \\ x_{\text{TV}} + (m_2 + m_3 - m_4) = m_2 \quad &\rightarrow \quad x_{\text{TV}} = m_4 - m_3 \end{aligned}$$

(d) Lastly, suppose as a busy Berkeley student, you only get a chance to take two measurements. Can you determine how much power each of the three appliances draw? If not, what combinations of power consumption can you find out?

Answer:

As you may expect, you cannot figure out the values of three unknowns from two equations. For three unknowns, we would need a minimum of three equations. But, you can still determine some limited information on the amount of electric power that the appliances draw.

- measurements 1 & 2: You can measure x_R and x_{TV}.
- measurements 1 & 3: You can measure x_R and x_{AC}.
- measurements 1 & 4: You can measure x_R and the combined $(x_{\text{AC}} + x_{\text{TV}})$.
- measurements 2 & 3: You can measure the combined $(x_{\text{TV}} + x_R)$ and $(x_{\text{AC}} + x_R)$. Tricky one!
- measurements 2 & 4: You can measure x_{AC} and the combined $(x_{\text{TV}} + x_R)$.
- measurements 3 & 4: You can measure x_{TV} and the combined $(x_{\text{AC}} + x_R)$.

3. Linear or Nonlinear

Determine whether the following functions (f: $\mathbb{R}^2 \rightarrow \mathbb{R}$) are linear or nonlinear.

(a)

$$f(x) = \sin\left(\frac{\pi}{2}\right)x + \cos\left(\frac{\pi}{2}\right)$$

Solution/Answer: First, we can simplify our expression above. Notice that $\sin\left(\frac{\pi}{2}\right) = 1$ and $\cos\left(\frac{\pi}{2}\right) = 0$. With this information we can rewrite the function above as

$$f(x) = x.$$

To check for linearity, check for superposition (additivity) and homogeneity (multiplicative scaling). In other words we must check that:

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \quad \forall \alpha, \beta, x, y \in \mathbb{R}$$

Linear:

$$\begin{aligned} f(\alpha x + \beta y) &= \alpha x + \beta y \\ &= \alpha f(x) + \beta f(y) \end{aligned}$$

Alternatively, you can state that this function is linear because it is of the form:

$$f(x) = ax$$

where a is a constant.

(b)

$$f(x_1, x_2) = 3x_1 + 4x_2$$

Solution/Answer: To check for linearity, check for superposition (additivity) and homogeneity (multiplicative scaling).
In other words we must check that:

$$f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) = \alpha f(x_1, x_2) + \beta f(y_1, y_2) \quad \forall \alpha, \beta, x_1, x_2, y_1, y_2 \in \mathbb{R}$$

Linear:

$$\begin{aligned} f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) &= 3(\alpha x_1 + \beta y_1) + 4(\alpha x_2 + \beta y_2) \\ &= \alpha(3x_1 + 4x_2) + \beta(3y_1 + 4y_2) \\ &= \alpha f(x_1, x_2) + \beta f(y_1, y_2) \end{aligned}$$

Alternatively, you can state that this function is linear because it is of the form:

$$f(x_1, x_2) = a_1 x_1 + a_2 x_2$$

where a_1 and a_2 are constants.

(c)

$$f(x_1, x_2) = e^{x_2} + x_1^2$$

Solution/Answer: To check for linearity, check for additivity and homogeneity (multiplicative scaling).
In other words we must check that:

$$f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) = \alpha f(x_1, x_2) + \beta f(y_1, y_2) \quad \forall \alpha, \beta, x_1, x_2, y_1, y_2 \in \mathbb{R}$$

Nonlinear:

$$\begin{aligned} f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) &= e^{\alpha x_2 + \beta y_2} + (\alpha x_1 + \beta y_1)^2 \\ &\neq \alpha e^{x_2} + \alpha x_1^2 + \beta e^{y_2} + \beta y_1^2 \\ &= \alpha f(x_1, x_2) + \beta f(y_1, y_2) \end{aligned}$$

Alternatively, you can state that this function is nonlinear because it is NOT of the form:

$$f(x_1, x_2) = a_1 x_1 + a_2 x_2$$

where a_1 and a_2 are constants.

(d)

$$f(x_1, x_2) = x_2 - x_1 + 3$$

Solution/Answer: To check for linearity, check for additivity and homogeneity (multiplicative scaling).
In other words we must check that:

$$f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) = \alpha f(x_1, x_2) + \beta f(y_1, y_2) \quad \forall \alpha, \beta, x_1, x_2, y_1, y_2 \in \mathbb{R}$$

Nonlinear (in fact, this function is affine (see notes for more details)):
You may simply state that this function doesn't satisfy homogeneity when scaled by 0.
Alternatively, you can show:

$$\begin{aligned} f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) &= (\alpha x_2 + \beta y_2) - (\alpha x_1 + \beta y_1) + 3 \\ &\neq \alpha(x_2 - x_1 + 3) + \beta(y_2 - y_1 + 3) \\ &= \alpha f(x_1, x_2) + \beta f(y_1, y_2) \end{aligned}$$

Alternatively, you can state that this function is nonlinear because it is NOT of the form:

$$f(x_1, x_2) = a_1 x_1 + a_2 x_2$$

where a_1 and a_2 are constants.

