EECS 16A Designing Information Devices and Systems I
Spring 2023 Discussion 4A

1. Exploring Column Spaces and Null Spaces

- The **column space** is the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that when multiplied with the matrix result in the zero vector.

For the following matrices, answer the following questions:

i. What is the column space of **A**? What is its dimension? (The **dimension** of a vector space is defined as the minimum number of vectors needed to span the space.)

ii. What is the null space of **A**? What is its dimension?

iii. Do the columns of **A** span $\mathbb{R}^2$?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Answer: Column space: span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Null space: span $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

The column space does not span $\mathbb{R}^2$.

(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

Answer:

Column space: $\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Null space: $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Columns do not span $\mathbb{R}^2$.

(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

Answer:

Column space: $\mathbb{R}^2$

Null space: span$\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$. This is known as the trivial nullspace, because it only contains the zero vector, which is always a solution to $\mathbf{A}\vec{x} = \vec{0}$.

Columns do span $\mathbb{R}^2$.

(d) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$

Answer:

Column space: span$\left\{ \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} \right\}$

Null space: span$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

Does not span $\mathbb{R}^2$.

(e) $\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

Answer:

i. The column space of the columns is $\mathbb{R}^2$.

ii. The following "algorithm" can be used to solve for the null space of a matrix: the definition of the null space is the set of vectors that satisfy $\mathbf{A}\vec{x} = \vec{0}$, so we can solve this equation by performing Gaussian elimination on $\mathbf{A}$ and writing the solution in terms of free variables.

We start by performing Gaussian elimination on matrix $\mathbf{A}$ to get the matrix into upper-triangular form. Normally the augmented matrix is used, but because the augmented vector is just $\vec{0}$, we know multiplying/adding 0's will still be 0, so we won't write it explicitly.

$$\begin{aligned}
\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix} &\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{7}{2} \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \text{reduced row echelon form}
\end{aligned}$$

$$\begin{aligned}
x_1 + \frac{1}{2}x_3 - \frac{7}{2}x_4 &= 0 \\
x_2 + \frac{5}{2}x_3 + \frac{1}{2}x_4 &= 0
\end{aligned}$$

$$x_3 \text{ is free and } x_4 \text{ is free (columns 3 and 4 don't have pivots)}$$

Now let $x_3 = s$ and $x_4 = t$. Then we have:

$$\begin{aligned}
x_1 + \frac{1}{2}s - \frac{7}{2}t &= 0 \\
x_2 + \frac{5}{2}s + \frac{1}{2}t &= 0
\end{aligned}$$

Now writing all the unknowns (x_1, x_2, x_3, x_4) in terms of the dummy variables:

$$\begin{aligned}
x_1 &= -\frac{1}{2}s + \frac{7}{2}t \\
x_2 &= -\frac{5}{2}s - \frac{1}{2}t \\
x_3 &= s \\
x_4 &= t
\end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s + \frac{7}{2}t \\ -\frac{5}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s \\ -\frac{5}{2}s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{7}{2}t \\ -\frac{1}{2}t \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

So every vector in the null space of $\mathbf{A}$ can be written as follows:

$$\text{Null space}(\mathbf{A}) = s\begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Therefore the null space of $\mathbf{A}$ is

$$\text{span}\left\{\begin{bmatrix}\begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}\end{bmatrix}\right\}$$

$\mathbf{A}$ has a 2-dimensional null space.

iii. The columns of $\mathbf{A}$ do span $\mathbb{R}^2$.

(f) What do you notice about the relationship between the dimension of the column space, the dimension of the null space, and their sum in all of these matrices?

Answer:

The dimension of the column space and null space add up to the total number of columns of the original matrix! The dimension of the column space indicates the number of linearly independent columns in the matrix, while the dimension of the null space indicates the number of linearly dependent columns in the matrix. Formally, this is called the rank-nullity theorem: for a matrix with m columns, $\dim(Col(A)) + \dim(N(A)) = m$

2. Row Space

Consider:

$$\mathbf{V} = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 0 & 4 \\ 6 & 4 & 10 \\ -2 & 4 & 2 \end{bmatrix}$$

Row reducing this matrix yields:

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Show that the row spaces of **U** and **V** are the same. Argue that in general, Gaussian elimination preserves the row space.

Answer:

To show that the spans of two sets of vectors are the same, show that each vector in one set can be written as a linear combination of the vectors in the other. Let's call the rows of **U** $\vec{u}_i^T$ and the rows of **V** $\vec{v}_i^T$. Because of the structure of **U**, it is easier to write the $\vec{v}_i$ in terms of the $\vec{u}_i$:

$$\begin{aligned} \vec{v}_1 &= 2\vec{u}_1 + 4\vec{u}_2 \\ \vec{v}_2 &= 4\vec{u}_1 \\ \vec{v}_3 &= 6\vec{u}_1 + 4\vec{u}_2 \\ \vec{v}_4 &= -2\vec{u}_1 + 4\vec{u}_2 \end{aligned}$$

We have now shown that all the $\vec{v}_i$ lie in the span of the $\vec{u}_i$. By rearranging the equations to be in terms of the $\vec{u}_i$, we can then show that all the $\vec{u}_i$ lie in the span of the $\vec{v}_i$

In general: the valid Gaussian elimination operations are scaling rows, swapping rows, and adding rows to each other. Since the span of a set of vectors consists of all linear combinations of the vectors, it is not affected by these operations.

(b) Show that the null spaces of **U** and **V** are the same. Argue that in general, Gaussian elimination preserves the null space.

Answer:

To show the null spaces are the same, compute them and check. The null space for both is span $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$.

In general: if there is a vector $\vec{x}$ in the null space of $\mathbf{V}$, then $\mathbf{V}\vec{x} = \vec{0}$. One interpretation is that for every row $\vec{v}_i^T$, $\vec{v}_i^T \vec{x} = 0$. During Gaussian elimination, we scale/swap/add rows to each other, and all of them satisfy $\vec{v}_i^T \vec{x} = 0$. Therefore, all the rows will always continue to satisfy this relation during GE.

(c) We define the row rank of a matrix as the dimension of the row space. We say that a matrix has full row rank if its row rank is equal to the number of rows in the matrix. What is the row rank of **U** and **V**, and do they have full row rank?

Answer:

From part (a), **U** and **V** have the same row space, so they have the same row rank. Looking at **U**, the first two rows are linearly independent and the last two are zeros, so the row space has dimension 2. This is not full row rank, and in fact a matrix with more rows than columns can never have full row rank.

Also note that since the rows of a matrix $\mathbf{A}$ are the columns of its transpose $\mathbf{A}^T$, the row rank of $\mathbf{A}$ is equal to the column rank of $\mathbf{A}^T$.

Full row rank is relevant to the singular value decomposition, which will be covered in EECS 16B.