EECS 16A Designing Information Devices and Systems I
Spring 2023 Discussion 3A

1. Proofs

Definition: A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is **linearly dependent** if there exist constants c_1, c_2, $\ldots$ c_n such that

$$\sum_{i=1}^{i=n} c_i \vec{v}_i = \vec{0}$$

and at least one c_i is non-zero.

This condition intuitively states that it is possible to express any one vector in the set in terms of the others.

(a) Suppose for some non-zero vector $\vec{x}$, $\mathbf{A}\vec{x} = \vec{0}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

Answer:

Begin by defining column vectors $\vec{a}_1 \ldots \vec{a}_n$.

$$\mathbf{A} = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

Thus, we can represent the multiplication $\mathbf{A}\vec{x}$ as

$$\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix} = \sum x_i \vec{a}_i = \vec{0}$$

Note that the equation above is the definition of linear dependence. That is, there exist coefficients, at least one which is non-zero, such that the sum of the vectors weighted by the coefficients is zero. These coefficients are the elements of the non-zero vector $\vec{x}$.

(b) For a matrix $\mathbf{A}$, suppose there exist two unique vectors $\vec{x}_1$ and $\vec{x}_2$ that both satisfy $\mathbf{A}\vec{x} = \vec{b}$, that is, $\mathbf{A}\vec{x}_1 = \vec{b}$ and $\mathbf{A}\vec{x}_2 = \vec{b}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

Answer:

Let us consider the difference of the two equations:

$$\mathbf{A}\vec{x}_1 - \mathbf{A}\vec{x}_2 = \mathbf{A}(\vec{x}_1 - \vec{x}_2) = \vec{b} - \vec{b} = \vec{0}$$

Once again, we've reached the definition of linear dependence since $\vec{x}_1 - \vec{x}_2 \neq \vec{0}$. We can apply the results from part (a), setting $\vec{x} = \vec{x}_1 - \vec{x}_2$.

(c) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix for which there exists a non-zero $\vec{y} \in \mathbb{R}^n$ such that $\mathbf{A}\vec{y} = \vec{0}$. Let $\vec{b} \in \mathbb{R}^m$ be some non zero vector. Show that if there is one solution to the system of equations $\mathbf{A}\vec{x} = \vec{b}$, then there are infinitely many solutions.

Answer: The key insight is to use the linearity of Matrix-vector multiplication.

By assumption, let $\vec{x}_1 \in \mathbb{R}^n$ be a solution to $\mathbf{A}\vec{x} = \vec{b}$. Then, for any $c \in \mathbb{R}$

$$\mathbf{A}(\vec{x}_1 + c\vec{y}) = \mathbf{A}\vec{x}_1 + \mathbf{A}(c\vec{y}) = \mathbf{A}\vec{x}_1 + c\mathbf{A}\vec{y} = \mathbf{A}\vec{x}_1 + \vec{0} = \mathbf{A}\vec{x}_1 = \vec{b}$$

where the first two equalities follow by linearity and the last two equalities follow from the assumptions that $\mathbf{A}\vec{y} = \vec{0}$ and that $\vec{x}_1$ is a solution to the system.

Hence, $\mathbf{A}(\vec{x}_1 + c\vec{y}) = \vec{b}$, implying that $(\vec{x}_1 + c\vec{y})$ is also a solution to $\mathbf{A}\vec{x} = \vec{b}$ for **any** constant c. Therefore, there are infinitely many solutions.

2. Exploring Dimension and Linear Independence

In this problem, we are going to talk about the connections between several concepts we have learned about in linear algebra – linear independence and dimension of a vector space/subspace.

Let's consider the vector space $\mathbb{R}^k$ (the k-dimensional real-world) and a set of n vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in $\mathbb{R}^k$.

(a) For the first part of the problem, let $k > n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ span the full $\mathbb{R}^k$ space? If so, prove it. If not, what conditions does it violate/what is missing?

Answer:

No, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ cannot span the space $\mathbb{R}^k$. The dimension of $\mathbb{R}^k$ is k, so you would need k linearly independent vectors to describe the vector space. Since $n < k$, this is not possible.

(b) Let $k = n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ span the full $\mathbb{R}^k$ space? Why/why not? What conditions would we need?

Answer:

Fact: matrix V is invertible $\iff$ V is square and has linearly independent columns

Note you have not proven this, but you will see the proof in EECS16B. Yes, this is possible. The only condition we need is that $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is linearly independent. If the vectors are linearly independent, since there are k of them, we can put them into a square matrix V:

$$V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

This matrix is square because the number of entries in the column vectors (k) is equal to the number of column vectors (n).

Using the fact from above, we know that if the square matrix V has n linearly independent columns, it will be invertible. If the matrix V is invertible, the matrix vector equation $V\vec{x} = \vec{b}$ will always have a unique solution for all vectors $\vec{b}$. Thus all possible $\vec{b} \in \mathbb{R}^k$ are in the span of the columns of matrix V: $(\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\})$.

To summarize, we can conclude then that if $n = k$ and $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is linearly independent, $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ can span the full $\mathbb{R}^k$ space.

(c) Finally, let $k < n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ span the full $\mathbb{R}^k$ space?

Hint: Think about whether the vectors can be linearly independent.

Answer:

If k out of the n vectors are linearly independent, then the set of vectors will be able to span the full $\mathbb{R}^k$ space. However, if we have less than k linearly independent vectors in the set, then there is too much linear dependence for this set of vectors to be able to span the full $\mathbb{R}^k$ space.

The two regimes—one where $n > k$ and one where $n < k$—give rise to two different classes of interesting problems. You might learn more about them in upper division courses!