EECS 16A Spring 2023 Designing Information Devices and Systems I Discussion 1B

1. Gaussian Elimination

Use Gaussian elimination to solve the following systems. Does a solution exist? Is it unique?

(a)

$$\left[\begin{array}{ccc|c} 2 & 0 & 4 & 6 \\ 0 & 1 & 2 & -3 \\ 1 & 2 & 0 & 3 \end{array}\right]$$

Answer:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}$$

(b)

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 1 & 2 & 8 & 0 \\ 1 & 3 & 5 & 3 \end{array}\right]$$

Answer:

No solution. Performing Gaussian Elimination on the augmented matrix will lead to a row of zeros with a non-zero constant term.

(c)

$$\left[\begin{array}{ccc|c} 2 & 2 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 2 & 0 & 1 & 1 \end{array}\right]$$

Answer:
There are an infinite number of solutions. One solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

(d) True or False: A system of equations with more equations than unknowns will always have either infinite solutions or no solutions.

Answer: False, a counterexample of this is when we have N equations and K unknowns ($N > K$) and $N-K$ of the equations are linear combinations of the first K. This means that those $N-K$ equations are just multiples of the K equations and do not present any new or unique information. This, in turn, means that there are actually K unique equations and K unique unknowns; therefore a unique solution will exist. This can be observed when Gaussian elimination is performed and the last $N-K$ rows are all 0, meaning they provide redundant information.

For example, here is a system of four equations and two unknowns,

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 4 & 2 \\ 2 & 5 & 3 \end{array}\right] \xRightarrow{-2R_1+R_3 \to R_3} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 5 & 3 \end{array}\right] \xRightarrow{-2R_1-R_2+R_3 \to R_3} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{1}$$

Solving, we get a single exact solution, $x = -1$ and $y = 1$.

(e) **(Practice)**

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

Answer:

There are an infinite number of solutions. One solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ \frac{1}{2} \end{bmatrix}$$

(f) (**Practice**)

$$\begin{bmatrix} 2x & + & 4y & + & 2z & = & 8 \\ x & + & y & + & z & = & 6 \\ x & - & y & - & z & = & 4 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

2. Finding The Bright Cave

Nara the one-handed druid and Kody the one-handed ranger find themselves in dire straits. Before them is a cliff with four cave entrances arranged in a square: two upper caves and two lower caves. Each entrance emits a certain amount of light, and the two wish to find exactly the amount of light coming from each cave. Here's the catch: after contracting a particularly potent strain of ghoul fever, our intrepid heroes are only able to see the total intensity of light before them (so their eyes operate like a single-pixel camera). Kody and Nara are capable adventurers, but they don't know any linear algebra – and they need your help.

Kody proposes an imaging strategy where he uses his hand to completely block the light from two caves at a time. He is able to take measurements using the following four masks (black means the light is blocked from that cave):

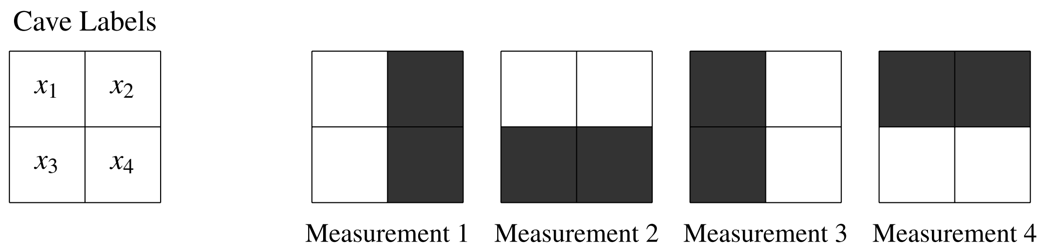


Figure 1: Four image masks.

(a) Let x_1, x_2, x_3, and x_4 represent the magnitude of light emanating from the four cave entrances shown in the image above. Write an equation for each masking process in Figure 1 which results in the four measurements of total light: m_1, m_2, m_3, and m_4. Then, create an augmented matrix that represents this system.

Answer:

Note that the only unknowns in this system are x_1, x_2, x_3, and x_4.

$$\begin{aligned} m_1 &= x_1 + x_3 \\ m_2 &= x_1 + x_2 \\ m_3 &= x_2 + x_4 \\ m_4 &= x_3 + x_4 \end{aligned}$$

Represented as an augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 1 & 1 & 0 & 0 & m_2 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right] \tag{2}$$

(b) Does Kody's set of masks give us a unique solution for all four caves' light intensities? Why or why not?

Answer:

There are two ways to arrive at the answer. We will show both.

i. We can perform Gaussian elimination on the matrix. Now, since we don't know Kody's measurements (the vector $\vec{m}$), we will not augment the matrix.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 1 & 1 & 0 & 0 & m_2 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right] \tag{3}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_1 - m_2 + m_3 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_1 - m_2 + m_3 \\ 0 & 0 & 0 & 0 & m_4 - m_1 + m_2 - m_3 \end{array}\right] \tag{4}$$

The matrix above has a row of zeroes, which implies that there will either be infinite solutions or no solutions (if the m values sum up to be a nonzero value). Therefore, Kody's set of masks cannot give us a unique solution for all four caves' light intensities.

ii. The second way we can show that we will not get a unique solution is to notice the equations. If we find that we could get one equation from the other equations, then we know that the solution is not unique. Notice that the sum of the first and the third row is the same as the sum of the second and fourth row.

$$\begin{aligned} m_1 + m_3 &= m_2 + m_4 \\ m_4 &= m_1 + m_3 - m_2 \\ (x_3 + x_4) &= (x_1 + x_3) + (x_2 + x_4) - (x_1 + x_2) \\ x_3 + x_4 &= x_3 + x_4 \end{aligned}$$

(c) Nara, in her infinite wisdom, places her one hand diagonally across the entrances, covering two of the cave entrances. However, her hand is not wide enough, letting in 50% of the light from the caves covered and 100% of the light from the caves not covered. The following diagram shows the percentage of light let through from each cave:

50%	100%
100%	50%

Does this additional measurement give them enough information to solve the problem? Why or why not?

Answer:

The answer is yes; the additional measurement does give them enough information to solve the problem. Since Nara's measurement provides additional information, we are now able to solve for all four light intensities uniquely.

This can be shown using algebra with the addition of the following measurement:

$$m_5 = \frac{1}{2}x_1 + x_2 + x_3 + \frac{1}{2}x_4$$

Note that we can isolate x_3 by combining measurements 2, 3, and 5:

$$x_3 = m_5 - \frac{1}{2}m_2 - \frac{1}{2}m_3$$

We can use further substitution to determine x_1, x_2, and x_4:

$$\begin{aligned}
x_1 &= m_1 - x_3 = m_1 - (m_5 - \frac{1}{2}m_2 - \frac{1}{2}m_3) = m_1 - m_5 + \frac{1}{2}m_2 + \frac{1}{2}m_3 \\
x_2 &= m_2 - x_1 = m_2 - (m_1 - m_5 + \frac{1}{2}m_2 + \frac{1}{2}m_3) = -m_1 + m_5 + \frac{1}{2}m_2 - \frac{1}{2}m_3 \\
x_4 &= m_4 - x_3 = m_4 - (m_5 - \frac{1}{2}m_2 - \frac{1}{2}m_3) = m_4 - m_5 + \frac{1}{2}m_2 + \frac{1}{2}m_3
\end{aligned}$$

We can also use Gaussian elimination to arrive at this result.

At this point, you can either add this equation to what we had previously to make a 5×4 system of equations, or you can remove one of Kody's masks to make a 4×4 system of equations. Here, we write it as a 5×4 matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 1 & 1 & 0 & 0 & m_2 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \\ 0.5 & 1 & 1 & 0.5 & m_5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2-m_1 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \\ 0 & 1 & 0.5 & 0.5 & m_5-\frac{m_1}{2} \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2-m_1 \\ 0 & 0 & 1 & 1 & m_3-m_2+m_1 \\ 0 & 0 & 1 & 1 & m_4 \\ 0 & 0 & 1.5 & 0.5 & m_5+\frac{m_1}{2}-m_2 \end{array}\right] \tag{5}$$

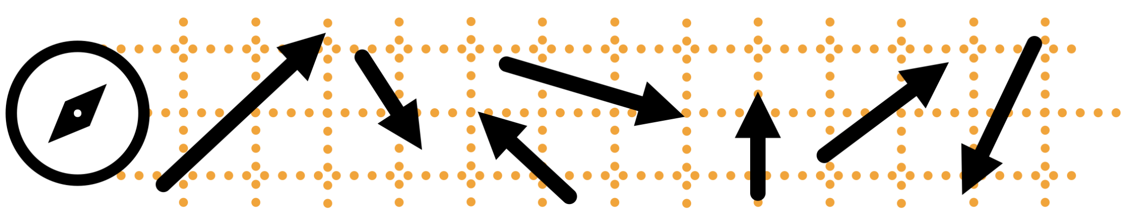
$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2-m_1 \\ 0 & 0 & 1 & 1 & m_3-m_2+m_1 \\ 0 & 0 & 0 & 0 & m_4-m_3+m_2-m_1 \\ 0 & 0 & 0 & -1 & m_5-\frac{3m_3}{2}+\frac{m_2}{2}-m_1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2-m_1 \\ 0 & 0 & 1 & 1 & m_3-m_2+m_1 \\ 0 & 0 & 0 & -1 & m_5-\frac{3m_3}{2}+\frac{m_2}{2}-m_1 \\ 0 & 0 & 0 & 0 & m_4-m_3+m_2-m_1 \end{array}\right] \tag{6}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2-m_1 \\ 0 & 0 & 1 & 1 & m_3-m_2+m_1 \\ 0 & 0 & 0 & 1 & -m_5+\frac{3m_3}{2}-\frac{m_2}{2}+m_1 \\ 0 & 0 & 0 & 0 & m_4-m_3+m_2-m_1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2-m_1 \\ 0 & 0 & 1 & 0 & m_5-\frac{m_3}{2}-\frac{m_2}{2} \\ 0 & 0 & 0 & 1 & -m_5+\frac{3m_3}{2}-\frac{m_2}{2}+m_1 \\ 0 & 0 & 0 & 0 & m_4-m_3+m_2-m_1 \end{array}\right] \tag{7}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -m_5+\frac{m_3}{2}+\frac{m_2}{2}+m_1 \\ 0 & 1 & 0 & 0 & m_5-\frac{m_3}{2}+\frac{m_2}{2}-m_1 \\ 0 & 0 & 1 & 0 & m_5-\frac{m_3}{2}-\frac{m_2}{2} \\ 0 & 0 & 0 & 1 & -m_5+\frac{3m_3}{2}-\frac{m_2}{2}+m_1 \\ 0 & 0 & 0 & 0 & m_4-m_3+m_2-m_1 \end{array}\right] \tag{8}$$

Notice here that, despite of the row of zeros, we still have a system of four unkowns and four unique equations. Therefore, we can uniquely determine all four light intensities given Nara's added measurement. If the last row had a non-zero value in the augmented column, however, that would indicate a measurement error or the introduction of some noise to our measurement, which would cause our system to not have a unique solution.
Also notice here that the measurements do not determine how we perform our Gaussian elimination.

3. Vectors



A vector is an ordered list of numbers. For instance, a point on a plane (x,y) is a vector! We label vectors using an arrow overhead $\vec{v}$, and since vectors can live in ANY dimension of space we'll need to leave our notation general $\vec{v} = (v_1, v_2, \ldots)$. Below are a few more examples (the left-most form is the general definition):

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \qquad \vec{a} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \in \mathbb{R}^3 \qquad \vec{b} = \begin{bmatrix} 2.4 \\ 5.3 \end{bmatrix} \in \mathbb{R}^2$$

Just to unpack this a bit more, $\vec{b} \in \mathbb{R}^3$ in English means "vector $\vec{b}$ lives in 3-dimensional space."

- The $\in$ symbol literally means "in"
- The $\mathbb{R}$ stands for "real numbers" (FUN FACT: $\mathbb{Z}$ means "integers" like $-2, 4, 0, \ldots$)
- The exponent $\mathbb{R}^{n} \leftarrow$ indicates the dimension of space, or the number of elements in the vector.

One last thing: it is standard to write vectors in column-form, like seen with $\vec{a}, \vec{b}, \vec{x}$ above. We call these *column vectors*, in contrast to horizontally written vectors, which we call *row vectors*. Row vectors are denoted with a transpose symbol; for instance, $\vec{x}^T$ denotes a row vector, which is simply $\vec{x}$ but expressed horizontally. This will become important later on when we discuss the importance of dimension matching.

Okay, let's dig into a few examples:

(a) Which of the following vectors lives in $\mathbb{R}^2$ space?

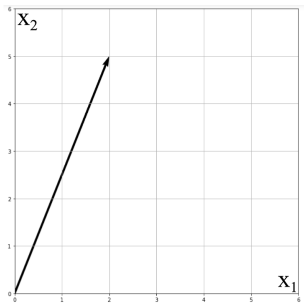
$$i. \begin{bmatrix} 3 \\ 6 \end{bmatrix} \qquad ii. \begin{bmatrix} 5 \\ 0 \\ 3 \\ 5 \end{bmatrix} \qquad iii. \begin{bmatrix} -4.76 \\ 1.32 \\ 0.01 \end{bmatrix} \qquad iv. \begin{bmatrix} -20 \\ 100 \end{bmatrix}$$

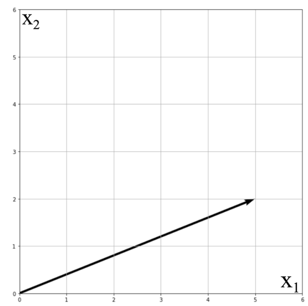
Answer:
i. Yes ii. No iii. No iv. Yes
Remember $\mathbb{R}^2$ means 2D space, which hosts vectors with 2 terms.
We count and see only i. and iv. have 2 terms.

(b) Graphically show the vectors (either in a sketch with axes, or a plot on a computer):

$$i. \begin{bmatrix} 2 \\ 5 \end{bmatrix} \qquad\qquad ii. \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Answer: Although these vectors look similar, remember that the ordering matters!

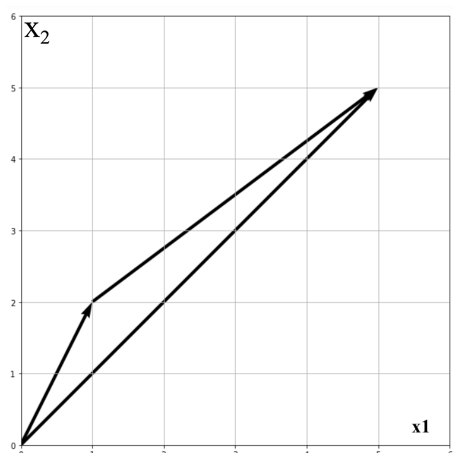




(c) Compute the sum $\vec{a}+\vec{b}=\vec{c}$ from the vectors below, and then graphically sketch or plot these vectors. (show them in a way that forms a triangle; also, is there only one possible triangle?)

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad\qquad \vec{b} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Answer:



Computation is done element-wise:

$$\vec{c} = \vec{a}+\vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$