EECS 16A Designing Information Devices and Systems I
Spring 2023 Discussion 2B

1. Span Basics

(a) What is $\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$?

Answer:

$\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ contains any vector $\vec{u}$ that can be written as

$$\vec{u} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Any vector whose last component is zero can be written in this form and any vector whose last component is nonzero cannot. Hence, the required span is the set of all vectors that can be written in the form $\begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$.

(b) Is $\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ in $\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$?

Answer:

From the definition of span, we know that if we can express $\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, it is in the span of those two vectors. Assume such a linear combination exists, then we can set up a vector equation

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}.$$

We can solve this system of equations for the coefficients α_1 and α_2 by expressing it as an augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right]$$

Solving with Gaussian Elimination, we get

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{5}{3} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 \end{array}\right]$$

From this result, $\alpha_1 = \frac{5}{3}$ and $\alpha_2 = \frac{5}{3}$, so we can conclude the statement is true.

(c) What is a possible choice for $\vec{v}$ that would make $\text{span}\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \vec{v}\right\} = \mathbb{R}^3$?

Answer: From part (a), any vector whose last component is zero can be written as a linear combination of the two vectors $\begin{bmatrix}1\\2\\0\end{bmatrix}$ and $\begin{bmatrix}2\\1\\0\end{bmatrix}$. Hence if we include, for example, $\begin{bmatrix}0\\0\\1\end{bmatrix}$ into the set, then we should be able to reach any vector in $\mathbb{R}^3$. Any vector $\vec{v}$ whose last component is non-zero is a valid choice to achieve the desired span of $\mathbb{R}^3$.

(d) For what values of b_1, b_2, b_3 is the following system of linear equations consistent? *Note: "Consistent" means there is at least one solution.*

$$\begin{bmatrix}1 & 2\\2 & 1\\0 & 0\end{bmatrix}\vec{x} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

Answer: For the system of linear equations to be consistent, there must exist some $\vec{x}$ such that the equality above holds. Performing matrix vector multiplication, we can rewrite the above expression as

$$x_1\begin{bmatrix}1\\2\\0\end{bmatrix} + x_2\begin{bmatrix}2\\1\\0\end{bmatrix} = \vec{b}$$

The question now becomes: which vectors $\vec{b}$ can be written in the above form, i.e as a linear combination of the columns of A? This is exactly the definition of span, and the answer must be the same as that from part (a).

2. Visualizing Matrices as Operations

This problem is going to help you visualize matrices as operations. For example, when we multiply a vector by a "rotation matrix," we will see it "rotate" in the true sense here. Similarly, when we multiply a vector by a "reflection matrix," we will see it be "reflected." The way we will see this is by applying the operation to all the vertices of a polygon and seeing how the polygon changes.

Your TA will now show you how a unit square can be rotated, scaled, or reflected using matrices! Note that in this exercise we are applying a matrix transformation on each of the vertices of the unit square separately.

(a) First, we will look at reflections. The transformation matrix that reflects a vector about the y-axis is:

$$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$$

since any vector of the form $\begin{bmatrix}x_0\\y_0\end{bmatrix}$ is transformed to $\begin{bmatrix}-x_0\\y_0\end{bmatrix}$.

What are the matrices that reflect a vector about the (i) x-axis and (ii) line $x = y$?

Answer: The matrix that reflects about the x-axis:

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \rightarrow \begin{bmatrix} x_0 \\ -y_0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ -y_0 \end{bmatrix} \tag{1}$$

and the matrix that reflects about $x = y$:

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \rightarrow \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} y_0 \\ x_0 \end{bmatrix} \tag{2}$$

(b) We are given matrices $\mathbf{T}_1$ and $\mathbf{T}_2$, and we are told that they will rotate the unit square by 15° and 30° respectively. Suggest some methods to rotate the unit square by 45° using only $\mathbf{T}_1$ and $\mathbf{T}_2$. How would you rotate the square by 60°? Your TA will show you the result in the iPython notebook.

Answer:

Apply $\mathbf{T}_1$ and $\mathbf{T}_2$ in succession to rotate the unit square by 45°. To rotate the square by 60°, you can either apply $\mathbf{T}_2$ twice, or if you prefer variety, apply $\mathbf{T}_1$ twice and $\mathbf{T}_2$ once.

(c) Find a single matrix $\mathbf{T}_3$ to rotate the unit square by 60°. Your TA will show you the result in the iPython notebook.

Answer: This matrix will look like the rotation matrix that rotates a vector by 60°. This matrix can be composed by multiplying $\mathbf{T}_1$ by $\mathbf{T}_1$ by $\mathbf{T}_2$ (or equivalently, $\mathbf{T}_2$ by $\mathbf{T}_2$).

(d) $\mathbf{T}_1$, $\mathbf{T}_2$, and the matrix you used in part (b) are called "rotation matrices." They rotate any vector by an angle θ. Show that a rotation matrix has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where θ is the angle of rotation. To do this, consider rotating the unit vector $\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ by θ degrees using the matrix $\mathbf{R}$.

(**Definition:** A vector, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix}$, is a unit vector if $\sqrt{v_1^2 + v_2^2 + \dots} = 1$.)

(*Hint: Use your trigonometric identities:* $\cos(a)\cos(b) - \sin(a)\sin(b) = \cos(a+b)$, $\cos(a)\sin(b) + \sin(a)\cos(b) = \sin(a+b)$.)

Answer:

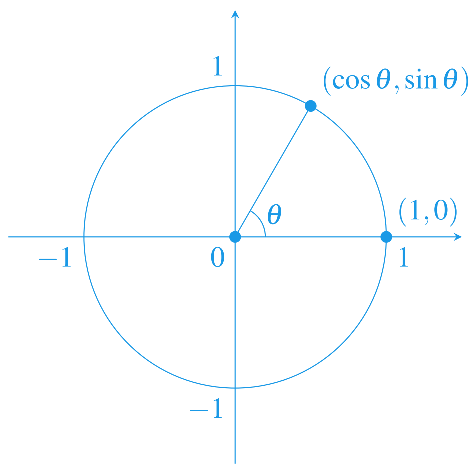
The reason the matrix is called a rotation matrix is because it transforms the unit vector $\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ to give $\begin{bmatrix} \cos(\alpha+\theta) \\ \sin(\alpha+\theta) \end{bmatrix}$.

Proof:

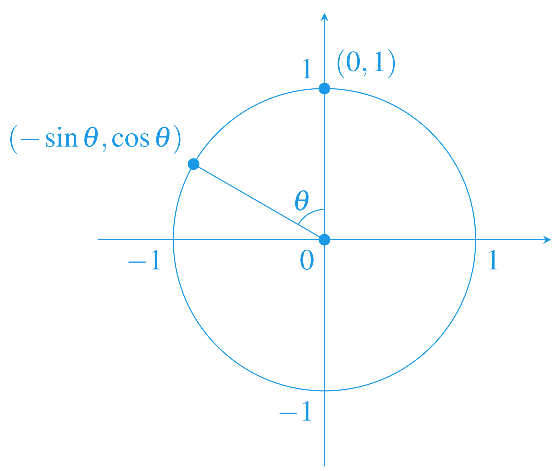
$$\begin{aligned}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix} &= \cos\alpha\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} + \sin\alpha\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}\\ &= \begin{bmatrix}\cos\alpha\cos\theta - \sin\alpha\sin\theta\\ \cos\alpha\sin\theta + \sin\alpha\cos\theta\end{bmatrix}\\ &= \begin{bmatrix}\cos(\alpha+\theta)\\ \sin(\alpha+\theta)\end{bmatrix}\end{aligned}$$

Alternative solution:

Let's try to derive this matrix using trigonometry. Suppose we want to rotate the vector $\begin{bmatrix}1\\0\end{bmatrix}$ by θ.



We can use basic trigonometric relationships to see that $\begin{bmatrix}1\\0\end{bmatrix}$ rotated by θ becomes $\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$. Similarly, rotating the vector $\begin{bmatrix}0\\1\end{bmatrix}$ by θ becomes $\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$:



We can also scale these pre-rotated vectors to any length we want, $\begin{bmatrix}x\\0\end{bmatrix}$ and $\begin{bmatrix}0\\y\end{bmatrix}$, and we can observe graphically that they rotate to $\begin{bmatrix}x\cos\theta\\ x\sin\theta\end{bmatrix}$ and $\begin{bmatrix}-y\sin\theta\\ y\cos\theta\end{bmatrix}$, respectively. Rotating a vector solely in the

x-direction produces a vector with both x and y components, and, likewise, rotating a vector solely in the y-direction produces a vector with both x and y components.

Finally, if we want to rotate an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix}$, we can combine what we derived above. Let x' and y' be the x and y components after rotation. x' has contributions from both x and y: $x' = x\cos\theta - y\sin\theta$. Similarly, y' has contributions from both components as well: $y' = x\sin\theta + y\cos\theta$. Expressing this in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, we've derived the 2-dimensional rotation matrix.

(e) Now, we want to get back the original unit square from the rotated square in part (c). What matrix should we use to do this? (**Note:** Don't use inverses! Answer this question using your intuition; we will visit inverses very soon in lecture!)

Answer:

Use a rotation matrix that rotates by $-60°$.

$$\begin{bmatrix} \cos(-60°) & -\sin(-60°) \\ \sin(-60°) & \cos(-60°) \end{bmatrix}$$

(f) Use part (e) to obtain the rotation matrix that reverses the operation of a matrix that rotates a vector by θ. Multiply the reverse rotation matrix with the rotation matrix and vice-versa. What do you get?

Answer:

The reverse matrix is as follows:

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

We can see that for any $\vec{v} \in \mathbb{R}^2$ that the product of the rotation matrix with $\vec{v}$ followed by the product of the reverse results in the original $\vec{v}$.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \left(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \vec{v} \right) = \vec{v}$$

(g) A natural question to ask is the following: does the *order* in which you apply transformations matter? Let's see what happens to a vector when we rotate it by 60° and then reflect it along the y-axis (matrix given in part (a)). Next, let's see what happens when we first reflect the vector along the y-axis and then rotate it by 60°. You will need to multiply the corresponding rotation and reflection matrices in the correct order. Are the results the same?

Answer: The results are not the same. If you rotate some vector $\vec{v}$ and then reflect along the y-axis you get:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(60°) & -\sin(60°) \\ \sin(60°) & \cos(60°) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\cos(60°) & \sin(60°) \\ \sin(60°) & \cos(60°) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

If you reflect along the y-axis and then rotate you get:

$$\begin{bmatrix} \cos(60°) & -\sin(60°) \\ \sin(60°) & \cos(60°) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\cos(60°) & -\sin(60°) \\ -\sin(60°) & \cos(60°) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

(h) Now, let's perform the operations in part (g) on the unit square in our iPython notebook. Are the results the same?

Answer: The results are not the same as shown in the iPython notebook.

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