EECS 16A Designing Information Devices and Systems I
Spring 2023 Homework 2

This homework is due February 3rd, 2023, at 23:59.
Self-grades are due February 6th, 2023, at 23:59.

Submission Format

Your homework submission should consist of a single PDF file that contains all of your answers (any hand-written answers should be scanned) as well as your IPython notebook saved as a PDF.

If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

1. **Reading Assignment**

   For this homework, please read Note 2A and Note 2B. Notes 2A and 2B provide an overview of vectors, matrices, and operations among them.

   Please answer the following questions:

   (a) Can you name a few applications where we might use vectors? What do the components represent in each application?

   (b) Are there special matrices where the order of their multiplication does *not* matter? If so come up with an example and explain why.

   Solution:

   (a) See Note 2A 2.2.1.

   (b) Identity with any matrix, 0 matrix with any matrix, diagonal matrices with each other.

2. **Image Masks**

   Learning Objective: *Learn to setup imaging problems with matrices.*

   For these word problems, you only need to setup the problem in augmented matrix or matrix-vector notation. Of course, you may solve the system for practice (e.g., with Gaussian elimination), but no additional credit is awarded.

   Solution: Full credit is awarded for setting up the augmented matrix or matrix-vector notation correctly.

   After your first EECS16A lecture, you decide to try to build a single-pixel camera. You want to take a 2x2 image, i.e. 4 tiles, and based on the first lecture, you choose to take 4 measurements. Recall that each measurement is the sum of the illuminated tiles. For each measurement, you will use a different mask.

   (a) Initially, you want to illuminate only one tile for each measurement. That is, you will first illuminate x_1, then you will illuminate x_2, etc. The outputs of your 4 measurements are y_1, y_2, y_3, and y_4 respectively. The 4 measurements you take are shown in Figure 1. Explicitly setup the matrix problem for this in the $A\vec{x} = \vec{b}$ form.

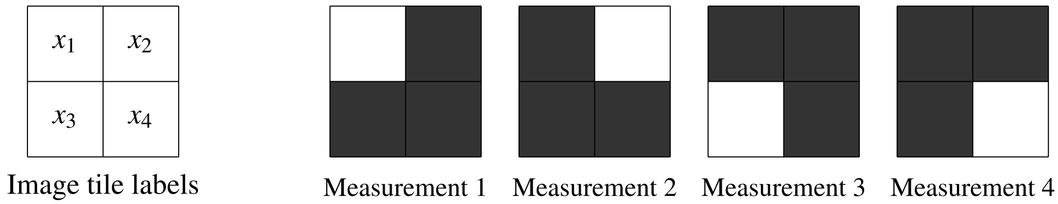


Figure 1: Four image masks.

Solution: The augmented matrix setup looks like this:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & y_1 \\ 0 & 1 & 0 & 0 & y_2 \\ 0 & 0 & 1 & 0 & y_3 \\ 0 & 0 & 0 & 1 & y_4 \end{array}\right]$$

The matrix-vector setup looks like this:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

(b) While setting up your code to create the masks, you forget to turn off the illuminated tiles from the previous measurement. As a result, measurement one contains x_1, measurement two contains $x_1 + x_2$, etc. The outputs of your 4 measurements are z_1, z_2, z_3, and z_4 respectively. The 4 measurements you take are shown in Figure 2. Explicitly setup the matrix problem for this in $A\vec{x} = \vec{b}$ form.

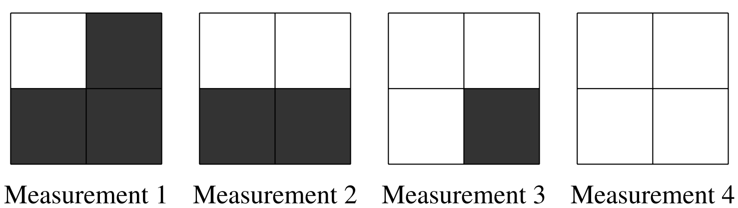


Figure 2: Four image masks.

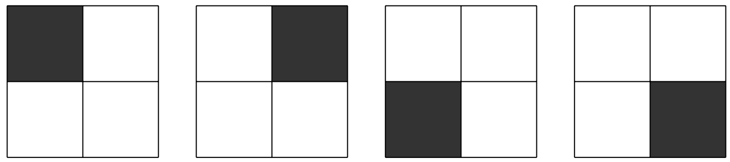
Solution: The augmented matrix setup looks like this:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & z_1 \\ 1 & 1 & 0 & 0 & z_2 \\ 1 & 1 & 1 & 0 & z_3 \\ 1 & 1 & 1 & 1 & z_4 \end{array}\right]$$

The matrix-vector setup looks like this:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

(c) Your friend is also building their own single pixel camera. However, they make a different mistake in their code and during each measurement, instead of lighting up one tile, you light up the other 3 tiles instead. That is, instead of measuring x_1, they measure $x_2 + x_3 + x_4$. The output of the 4 measurements are w_1, w_2, w_3, and w_4. The 4 measurements from their setup are shown in Figure 3. Explicitly setup the matrix problem for this in $A\vec{x} = \vec{b}$ form.



Measurement 1 Measurement 2 Measurement 3 Measurement 4

Figure 3: Four image masks.

Solution: The augmented matrix setup looks like this:

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & w_1 \\ 1 & 0 & 1 & 1 & w_2 \\ 1 & 1 & 0 & 1 & w_3 \\ 1 & 1 & 1 & 0 & w_4 \end{array}\right]$$

The matrix-vector setup looks like this:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

3. Gaussian Elimination

Learning Goal: *Understand the relationship between Gaussian elimination and the graphical representation of linear equations, and explore different types of solutions to a system of equations. You will also practice determining the parametric solutions when there are infinitely many solutions.*

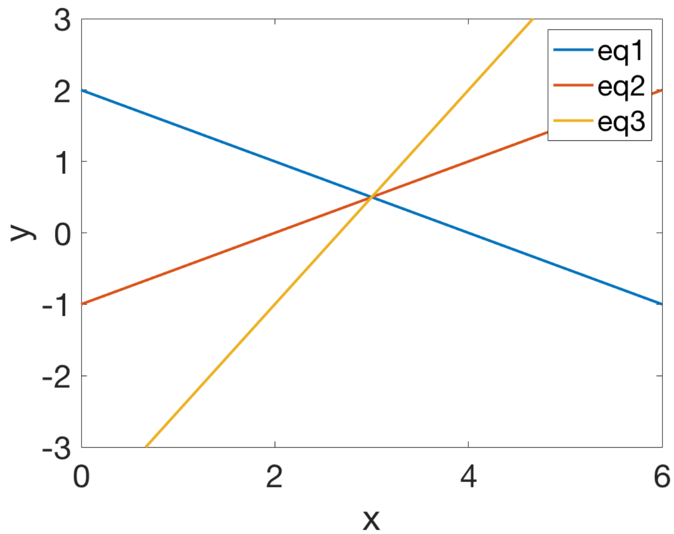
(a) In this problem we will investigate the relationship between Gaussian elimination and the geometric interpretation of linear equations. You are welcome to draw plots by hand or using software. Please be sure to label your equations with a legend on the plot.

i. Plot the following set of linear equations in the x-y plane. If the lines intersect, write down the point or points of intersection.

$$x + 2y = 4 \tag{1}$$
$$2x - 4y = 4 \tag{2}$$
$$3x - 2y = 8 \tag{3}$$

Solution:

The three lines intersect at the point (3,0.5).



ii. Write the above set of linear equations in augmented matrix form and do the first step of Gaussian elimination to eliminate the x variable from equation 2. Now, the second row of the augmented matrix has changed. Plot the corresponding new equation created in this step on the same graph as above. What do you notice about the new line you draw?

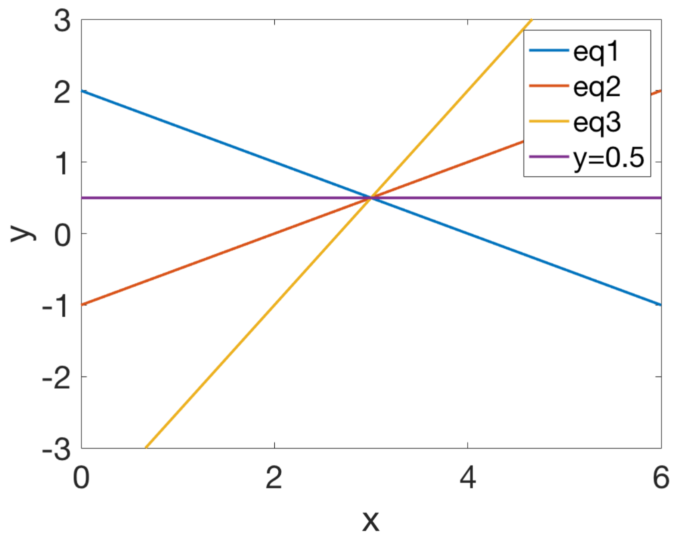
Solution: We start with the following augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & -4 & 4 \\ 3 & -2 & 8 \end{array}\right]$$

We then eliminate x from the second equation by subtracting $2\times$Row 1 from Row 2:

$$\text{Row 2: subtract } 2\times\text{Row 1} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -8 & -4 \\ 3 & -2 & 8 \end{array}\right]$$

So equation 2 becomes $-8y = -4$, which is equivalent to $y = 0.5$. You will notice that the line $y = 0.5$ intersects with the three lines you drew previously.



iii. Complete all of the steps of Gaussian elimination including back substitution. Now plot the new equations represented by the rows of the augmented matrix in the last step (after completing back substitution) on the same graph as above. What do you notice about the new line you draw?

Solution:

We continue from the previous part, where we had the following augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -8 & -4 \\ 3 & -2 & 8 \end{array}\right]$$

and take the following steps to complete Gaussian elimination:

$$\text{Row 3: subtract } 3\times\text{Row 1} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -8 & -4 \\ 0 & -8 & -4 \end{array}\right]$$

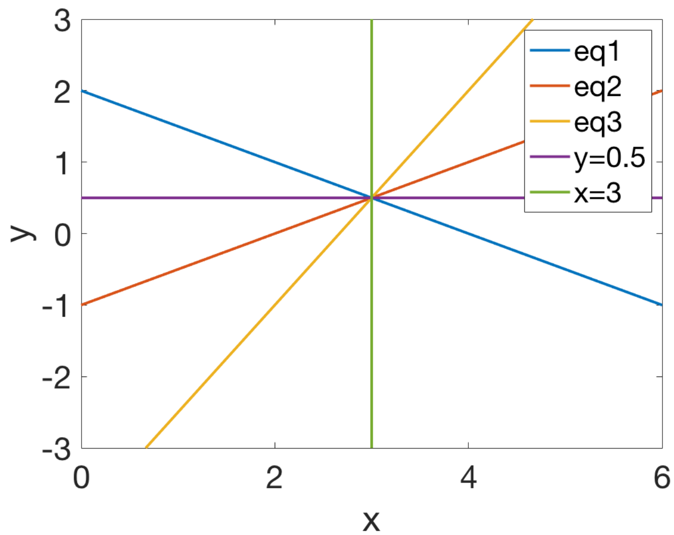
$$\text{Row 2: divide by } -8 \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 0.5 \\ 0 & -8 & -4 \end{array}\right]$$

$$\text{Row 3: subtract } -8\times\text{Row 2} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{array}\right]$$

$$\text{Row 1: subtract } 2\times\text{Row 2} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, we end up with the solution $x = 3$ and $y = 0.5$.

Plotting the new equation $x = 3$ on the same graph as before, we see that all five lines intersect at the same point (3,0.5).



(b) Write the following set of linear equations in augmented matrix form and use Gaussian elimination to determine if there are no solutions, infinite solutions, or a unique solution. If any solutions exist, determine what they are. You may do this problem by hand or use a computer. We encourage you to try it by hand to ensure you understand Gaussian elimination. Remember that it is possible to end up with fractions during Gaussian elimination.

$$\begin{aligned} x+2y+5z &= 3 \\ x+12y+6z &= 1 \\ 2y+z &= 4 \\ 3x+16y+16z &= 7 \end{aligned}$$

Solution:

Writing the system in augmented matrix form we get the following:

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 1 & 12 & 6 & 1 \\ 0 & 2 & 1 & 4 \\ 3 & 16 & 16 & 7 \end{array}\right]$$

We eliminate the x variables from the second and fourth equations:

$$\begin{array}{c} \text{Row 2: subtract Row 1} \\ \text{Row 4: subtract } 3\times\text{Row 1} \end{array} \implies \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 10 & 1 & -2 \\ 0 & 2 & 1 & 4 \\ 0 & 10 & 1 & -2 \end{array}\right]$$

We then divide Row 2 by 10 to get a 1 in the pivot position:

$$\text{Row 2: divide by 10} \quad \Longrightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 1 & 0.1 & -0.2 \\ 0 & 2 & 1 & 4 \\ 0 & 10 & 1 & -2 \end{array}\right]$$

Next, we eliminate the y variables from the third and fourth equations:

$$\begin{array}{c}\text{Row 3: subtract } 2\times\text{Row 2} \\ \text{Row 4: subtract } 10\times\text{Row 2}\end{array} \quad \Longrightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 1 & 0.1 & -0.2 \\ 0 & 0 & 0.8 & 4.4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We divide Row 3 by 0.8 to get a 1 in the pivot position:

$$\text{Row 3: divide by 0.8} \quad \Longrightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 1 & 0.1 & -0.2 \\ 0 & 0 & 1 & 5.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We then proceed with back-substitution:

$$\begin{array}{c}\text{Row 2: subtract } 0.1\times\text{Row 3} \\ \text{Row 1: subtract } 5\times\text{Row 3}\end{array} \quad \Longrightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & 0 & -24.5 \\ 0 & 1 & 0 & -0.75 \\ 0 & 0 & 1 & 5.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{Row 1: subtract } 2\times\text{Row 2} \quad \Longrightarrow \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -23 \\ 0 & 1 & 0 & -0.75 \\ 0 & 0 & 1 & 5.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This final matrix is in reduced row echelon form. The first three rows of the matrix have non-zero elements in pivot position, for a system with three unknowns, and the fourth row is a row of zeros, so we can conclude there is a unique solution: $x = -23$, $y = -0.75$, and $z = 5.5$.

(c) Consider the following system:

$$\begin{aligned} 4x + 4y + 4z + w + v &= 1 \\ x + y + 2z + 4w + v &= 2 \\ 5x + 5y + 5z + w + v &= 0 \end{aligned}$$

If you were to write the above equations in augmented matrix form and use Gaussian elimination to solve the system, you would get the following (for extra practice, you can try and do this yourself):

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 3 & 16 \\ 0 & 0 & 1 & 0 & -3 & -17 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

How many variables are free variables? Which ones? Find the general form of the solutions in terms of real constants (e.g., $s \in \mathbb{R}$).

Solution:

We first note that the given augmented matrix is in reduced row echelon form, which makes sense as it is the final output of the Gaussian elimination algorithm. We observe that the second and fifth columns do not have 1s in pivot position so there are two free variables corresponding to y and v.

Let $y = s$ and let $v = t$, where $s \in \mathbb{R}$ and $t \in \mathbb{R}$.

Using back substitution, we can solve for x, y, z, w, and v in terms of s and t:

$$\begin{aligned}
&\text{Row 1:} \quad x + y + 3v = 16 \quad \implies x = 16 - 3t - s \\
&\text{Row 2:} \quad z - 3v = -17 \quad \implies z = -17 + 3t \\
&\text{Row 3:} \quad w + v = 5 \quad \implies w = 5 - t
\end{aligned}$$

The solutions to the system of equations are therefore:

$$\begin{aligned}
x &= 16 - 3t - s \\
y &= s \\
z &= -17 + 3t \\
w &= 5 - t \\
v &= t
\end{aligned}$$

The solutions can also be represented by a set as:

$$S = \left\{ \vec{u} \;\middle|\; \vec{u} = \begin{bmatrix} 16 \\ 0 \\ -17 \\ 5 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3 \\ 0 \\ 3 \\ -1 \\ 1 \end{bmatrix} t \quad , \quad s \in \mathbb{R}, \quad t \in \mathbb{R} \right\}.$$

4. Vector-Vector, Matrix-Vector, and Matrix-Matrix Multiplication

Learning Objective: *Practice evaluating vector-vector, matrix-vector, and matrix-matrix multiplication.*

(a) For the following multiplications, state the dimensions of the result. If the product is not defined and thus has no solution, state this and justify your reasoning. For this problem $\vec{x} \in \mathbb{R}^N, \vec{y} \in \mathbb{R}^N, \vec{z} \in \mathbb{R}^M$, with $N \neq M$.

i. $\vec{x}^T \cdot \vec{z}$

Solution: This is invalid. $\vec{x}^T$ is an $1 \times N$ vector meaning that it has 1 rows and N column but $\vec{z}$ is an $M \times 1$ vector meaning that is has M rows and 1 column. Since $\vec{x}^T$ does not have the same number of columns as $\vec{z}$ has rows there is no solution.

ii. $\vec{x} \cdot \vec{x}^T$

Solution:

$$N \times N$$

$\vec{x}$ has N row and 1 columns and $\vec{x}^T$ has 1 row and N columns. Since the number of columns of $\vec{x}$ is the same as the number of rows of $\vec{x}^T$, there is a solution. The solution would have the dimensions of the number of rows of $\vec{x}$ times the number of columns of $\vec{x}^T$.

iii. $\vec{x} \cdot \vec{y}^T$

Solution:

$$N \times N$$

$\vec{x}$ has N row and 1 columns and $\vec{y}^T$ has 1 row and N columns. Since the number of columns of $\vec{x}$ is the same as the number of rows of $\vec{y}^T$, there is a solution. The solution would have the dimensions of the number of rows of $\vec{x}$ times the number of columns of $\vec{y}^T$.

iv. $\vec{x} \cdot \vec{z}^T$

Solution:

$$N \times M$$

$\vec{x}$ has N row and 1 columns and $\vec{z}^T$ has 1 row and M columns. Since the number of columns of $\vec{x}$ is the same as the number of rows of $\vec{z}^T$, there is a solution. The solution would have the dimensions of the number of rows of $\vec{x}$ times the number of columns of $\vec{z}^T$.

For questions (b) through (d), complete the matrix-vector multiplication. If the product is not defined and thus has no solution, state this and justify your reasoning:

(b)

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution: No solution. The number of columns in the matrix does not match the number of elements (aka the number of rows) in the column vector.

(c)

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 * \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 * \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 1 * \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix}1 & 0\\1 & 0\\0 & 0\\0 & 1\\1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix} = 2*\begin{bmatrix}1\\1\\0\\0\\1\\1\end{bmatrix} - \begin{bmatrix}0\\0\\0\\1\\1\\0\end{bmatrix} = \begin{bmatrix}2\\2\\0\\-1\\1\\2\end{bmatrix}$$

(e) Compute $\mathbf{AB}$ by hand, where $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{bmatrix}1 & 0\\2 & 1\\0 & 1\end{bmatrix}, \text{ and } \qquad \mathbf{B} = \begin{bmatrix}1 & 2 & -1 & 0\\-3 & 0 & 2 & -1\end{bmatrix}$$

What are the dimensions of $\mathbf{AB}$? Compute $\mathbf{BA}$ too if the operation is valid. If it is invalid, explain why. Make sure you show the work for your calculations.

Solution:

$$\begin{aligned}\mathbf{AB} &= \begin{bmatrix}1 & 0\\2 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & -1 & 0\\-3 & 0 & 2 & -1\end{bmatrix}\\
&= \begin{bmatrix}1\times 1+0\times -3 & 1\times 2+0\times 0 & 1\times -1+0\times 2 & 1\times 0+0\times -1\\2\times 1+1\times -3 & 2\times 2+1\times 0 & 2\times -1+1\times 2 & 2\times 0+1\times -1\\0\times 1+1\times -3 & 0\times 2+1\times 0 & 0\times -1+1\times 2 & 0\times 0+1\times -1\end{bmatrix}\\
&= \begin{bmatrix}1 & 2 & -1 & 0\\-1 & 4 & 0 & -1\\-3 & 0 & 2 & -1\end{bmatrix}\end{aligned}$$

$\mathbf{BA}$ does not exist since the number of columns in $\mathbf{B}$ is not equal to the number of rows in $\mathbf{A}$.

(f) Compute $\mathbf{AB}$ by hand, where $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{bmatrix}3 & 21 & 9\\-1 & 14 & 4\\7 & -8 & 2\end{bmatrix}, \text{ and } \qquad \mathbf{B} = \begin{bmatrix}-2 & 4\\-1 & 2\\3 & -6\end{bmatrix}$$

Compute $\mathbf{BA}$ too if the operation is valid. If it is invalid, explain why. Make sure you show the work for your calculations.

Solution:

$$\begin{aligned}\mathbf{AB} &= \begin{bmatrix}3 & 21 & 9\\-1 & 14 & 4\\7 & -8 & 2\end{bmatrix}\begin{bmatrix}-2 & 4\\-1 & 2\\3 & -6\end{bmatrix}\\
&= \begin{bmatrix}3\times -2+21\times -1+9\times 3 & 3\times 4+21\times 2+9\times -6\\-1\times -2+14\times -1+4\times 3 & -1\times 4+14\times 2+4\times -6\\7\times -2+-8\times -1+2\times 3 & 7\times 4+-8\times 2+2\times -6\end{bmatrix}\\
&= \begin{bmatrix}0 & 0\\0 & 0\\0 & 0\end{bmatrix}\end{aligned}$$

$\mathbf{BA}$ does not exist since the number of columns in $\mathbf{B}$ is not equal to the number of rows in $\mathbf{A}$.

5. Image Stitching

Learning Objective: *This problem is similar to one that students might experience in an upper division computer vision course. Our goal is to give students a flavor of the power of tools from fundamental linear algebra and their wide range of applications.*

Often, when people take pictures of a large object, they are constrained by the field of vision of the camera. This means that they have two options to capture the entire object:

- Stand as far away as they need to include the entire object in the camera's field of view (clearly, we do not want to do this as it reduces the amount of detail in the image)
- (This is more exciting) Take several pictures of different parts of the object and stitch them together like a jigsaw puzzle.

We are going to explore the second option in this problem. Daniel, who is a professional photographer, wants to construct an image by using "image stitching". Unfortunately, Daniel took some of the pictures from different angles as well as from different positions and distances from the object. While processing these pictures, Daniel lost information about the positions and orientations from which the pictures were taken. Luckily, you and your friend Marcela, with your wealth of newly acquired knowledge about vectors and matrices, can help him!

You and Marcela are designing an iPhone app that stitches photographs together into one larger image. Marcela has already written an algorithm that finds common points in overlapping images. **It's your job to figure out how to stitch the images together using Marcela's common points to reconstruct the larger image.**

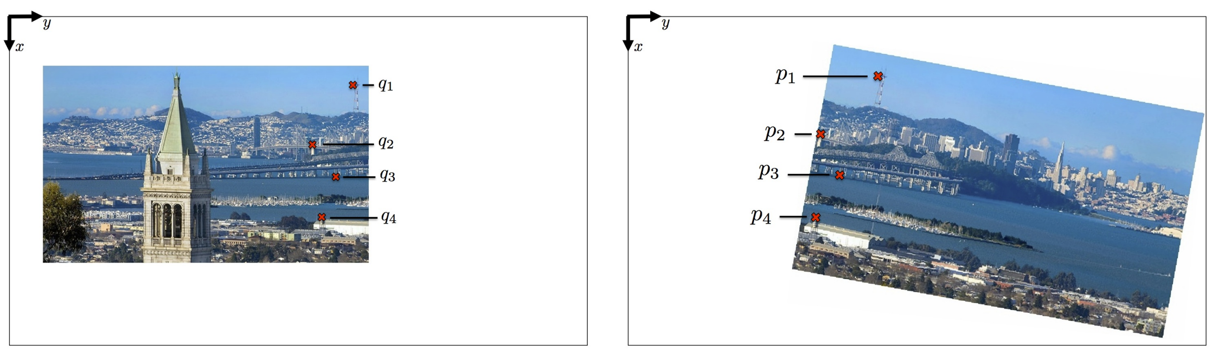


Figure 4: Two images to be stitched together with pairs of matching points labeled.

We will use vectors to represent the common points which are related by a affine transformation. Your idea is to find this affine transformation. For this you will use a single matrix, $\mathbf{R}$, and a vector, $\vec{t}$, that transforms every common point in one image to their corresponding point in the other image. Once you find $\mathbf{R}$ and $\vec{t}$ you will be able to transform one image so that it lines up with the other image.

Suppose $\vec{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$ is a point in one image , which is transformed to $\vec{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$, the corresponding point in the other image (i.e., they represent the same object in the scene). For example, Fig. 4 shows how the points $\vec{p}_1$, $\vec{p}_2$ ... in the right image are transformed to points $\vec{q}_1$, $\vec{q}_2$ ... on the left image. You write down the following relationship between $\vec{p}$ and $\vec{q}$.

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \underbrace{\begin{bmatrix} r_{xx} & r_{xy} \\ r_{yx} & r_{yy} \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\vec{t}} \tag{4}$$

This problem focuses on finding the unknowns (i.e. the components of $\mathbf{R}$ and $\vec{t}$), so that you will be able to stitch the image together. *Note that this is the opposite from our usual setting in which we would solve for $\vec{p}$ given all other variables.*

(a) To understand how the matrix $\mathbf{R}$ and vector $\vec{t}$ transforms any vector representing a point on a image, Consider this example equation similar to Equation (4),

$$\vec{v} = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \vec{u} + \vec{w} = \vec{v}_1 + \vec{w}. \tag{5}$$

Use $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for this part.
We want to find out what geometric transformation(s) can be applied on $\vec{u}$ to give $\vec{v}$.

Step 1: Find out how $\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$ is transforming $\vec{u}$. Evaluate $\vec{v}_1 = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \vec{u}$.
What **geometric transformation(s)** might be applied to $\vec{u}$ to get $\vec{v}_1$? Choose the options that answers the question and explain your choice.

(i) Rotation
(ii) Scaling
(iii) Shifting/Translation

Drawing the vectors $\vec{u}$, and $\vec{v}_1$ in two dimensions on a single plot might help you to visualize the transformations.

Step 2: Find out $\vec{v} = \vec{v}_1 + \vec{w}$. Find out how **addition of $\vec{w}$ is geometrically transforming** $\vec{v}_1$. Choose the option that answers the question and explain your choice.

(i) Rotation
(ii) Scaling
(iii) Shifting/Translation

Drawing the vectors $\vec{v}$, $\vec{w}$, and $\vec{v}_1$ in two dimensions on a single plot might help you to visualize the transformations.

Solution: Plugging in the given vectors and performing the matrix vector multiplication,

$$\vec{v}_1 = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}. \tag{6}$$

It is observable that $\vec{v}_1$ is a scaled, rotated version of $\vec{u}$.
We get $\vec{v} = \vec{v}_1 + \vec{w} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. We can also see that $\vec{v}$ is a shifted version of $\vec{v}_1$.
Hence $\vec{u}$ is scaled, rotated and shifted to get $\vec{v}$.
Note: We can only use matrix transformations to scale and/rotate a vector. We cannot translate a vector through matrix transformations; instead we must use vector addition for this.

(b) Now back to the main problem. First, multiply Equation (4) out into **two equations**.

(i) What are the known values and what are the unknown values in each equation (recall what we are trying to solve for in Equation (4))?

(ii) How many unknown values are there?

(iii) How many independent equations do you need to solve for all the unknowns?

(iv) How many pairs of common points $\vec{p}$ and $\vec{q}$ will you need in order to write down a system of equations that you can use to solve for the unknowns? ***Hint:*** *Remember that each pair of $\vec{p}$ and $\vec{q}$ is related by two equations, one for each coordinate.*

Solution:

We can rewrite the above matrix equation as the following two scalar equations:

$$q_x = p_x r_{xx} + p_y r_{xy} + t_x$$
$$q_y = p_x r_{yx} + p_y r_{yy} + t_y$$

Here, the known values are each pair of points' elements: $q_x,\ q_y,\ p_x,\ p_y$, and the scaling factor of the $\vec{t}$ vector (1). The unknowns are elements of $\mathbf{R}$ and $\vec{t}$: $r_{xx},\ r_{xy},\ r_{yx},\ r_{yy},\ t_x$, and t_y. There are 6 unknowns, so we need a total of 6 equations to solve for them. For every pair of points we add, we get two more equations. Thus, we need 3 pairs of common points to get 6 equations.

(c) Use what you learned in the above two subparts to explicitly write out **just enough** equations of these transformations as you need to solve the system. Assume that the four pairs of points from Fig. 4 are labeled as:

$$\vec{q}_1 = \begin{bmatrix} q_{1x} \\ q_{1y} \end{bmatrix}, \vec{p}_1 = \begin{bmatrix} p_{1x} \\ p_{1y} \end{bmatrix} \qquad \vec{q}_2 = \begin{bmatrix} q_{2x} \\ q_{2y} \end{bmatrix}, \vec{p}_2 = \begin{bmatrix} p_{2x} \\ p_{2y} \end{bmatrix} \qquad \vec{q}_3 = \begin{bmatrix} q_{3x} \\ q_{3y} \end{bmatrix}, \vec{p}_3 = \begin{bmatrix} p_{3x} \\ p_{3y} \end{bmatrix} \qquad \vec{q}_4 = \begin{bmatrix} q_{4x} \\ q_{4y} \end{bmatrix}, \vec{p}_4 = \begin{bmatrix} p_{4x} \\ p_{4y} \end{bmatrix}.$$

Solution: From the previous part, we know that we will need six equations, as we have six unknowns. Recalling that each point provides us with two equations, we arbitrarily select the first three points:

$$\vec{q}_1 = \begin{bmatrix} q_{1x} \\ q_{1y} \end{bmatrix}, \quad \vec{p}_1 = \begin{bmatrix} p_{1x} \\ p_{1y} \end{bmatrix} \qquad \vec{q}_2 = \begin{bmatrix} q_{2x} \\ q_{2y} \end{bmatrix}, \quad \vec{p}_2 = \begin{bmatrix} p_{2x} \\ p_{2y} \end{bmatrix} \qquad \vec{q}_3 = \begin{bmatrix} q_{3x} \\ q_{3y} \end{bmatrix}, \quad \vec{p}_3 = \begin{bmatrix} p_{3x} \\ p_{3y} \end{bmatrix}$$

which yield the following system:

$$r_{xx}p_{1x} + r_{xy}p_{1y} + t_x = q_{1x} \tag{7}$$
$$r_{yx}p_{1x} + r_{yy}p_{1y} + t_y = q_{1y} \tag{8}$$
$$r_{xx}p_{2x} + r_{xy}p_{2y} + t_x = q_{2x} \tag{9}$$
$$r_{yx}p_{2x} + r_{yy}p_{2y} + t_y = q_{2y} \tag{10}$$
$$r_{xx}p_{3x} + r_{xy}p_{3y} + t_x = q_{3x} \tag{11}$$
$$r_{yx}p_{3x} + r_{yy}p_{3y} + t_y = q_{3y} \tag{12}$$

(d) Remember that we are ultimately solving for the components of the $\mathbf{R}$ matrix and the vector $\vec{t}$. This is different from our usual setting and so we need to reformulate the problem into something we are more used to (i.e., $A\vec{x} = \vec{b}$ where x is the unknown). In order to do this, let's view the components of $\mathbf{R}$ and $\vec{t}$ as the unknowns in the equations from from part c). We then have a system of linear equations which we should be able to write in the familiar matrix-vector form. Specifically, we can store the unknowns in a vector $\vec{\alpha} = \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{yx} \\ r_{yy} \\ t_x \\ t_y \end{bmatrix}$ and specify 6×6 matrix **A** and vector $\vec{b}$ such that $A\vec{\alpha} = \vec{b}$. Please write out the entries of **A** and $\vec{b}$ to match your equations from part c). To get you started, we provide the first row of **A** and first entry of $\vec{b}$ which corresponds to one possible equation from part c):

$$\begin{bmatrix} p_{1x} & p_{1y} & 0 & 0 & 1 & 0 \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{yx} \\ r_{yy} \\ t_x \\ t_y \end{bmatrix} = \begin{bmatrix} q_{1x} \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Your job is the fill in the remaining entries according to the other equations.

Solution: We write the system of linear equations from the previous part in matrix form.

$$\begin{bmatrix} p_{1x} & p_{1y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{1x} & p_{1y} & 0 & 1 \\ p_{2x} & p_{2y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{2x} & p_{2y} & 0 & 1 \\ p_{3x} & p_{3y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{3x} & p_{3y} & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{yx} \\ r_{yy} \\ t_x \\ t_y \end{bmatrix} = \begin{bmatrix} q_{1x} \\ q_{1y} \\ q_{2x} \\ q_{2y} \\ q_{3x} \\ q_{3y} \end{bmatrix}$$

(e) In the IPython notebook `prob4.ipynb`, you will have a chance to test out your solution. Plug in the values that you are given for p_x, p_y, q_x, and q_y for each pair of points into your system of equations to solve for the matrix, **R**, and vector, $\vec{t}$. The notebook will solve the system of equations, apply your transformation to the second image, and show you if your stitching algorithm works. **You are NOT responsible for understanding the image stitching code or Marcela's algorithm.** What are the values for **R** and $\vec{t}$ which correctly stitch the images together?

Solution:

The parameters for the transformation from the coordinates of the first image to those of the second image are $\mathbf{R} = \begin{bmatrix} 1.1954 & .1046 \\ -.1046 & 1.1954 \end{bmatrix}$ and $\vec{t} = \begin{bmatrix} -150 \\ -250 \end{bmatrix}$.

6. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.