
EECS 16A
Fall 2022

Designing Information Devices and Systems I

Homework 12

This homework is due November 29, 2022, at 14:00.

Self-grades are due December 2, 2022, at 23:59.

Submission Format

Your homework submission should consist of **one** file.

- `hw12.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

Submit the file to the appropriate assignment on Gradescope.

1. Reading Assignment

For this homework, please read Note 21 (Inner Products and GPS) and Note 22 (Trilateration and Correlation). You are always encouraged to read beyond this as well.

2. Inner Product Properties

***Learning Goal:** The objective of this problem is to exercise useful identities for inner products.*

Our definition of the inner product in \mathbb{R}^n is:

$$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n = \vec{x}^T \vec{y}, \quad \text{for any } \vec{x}, \vec{y} \in \mathbb{R}^n$$

Prove the following identities in \mathbb{R}^n :

(a) $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

Solution: This is seen by direct expansion:

Let $x_i, y_i \in \mathbb{R}$, then

$$\begin{aligned} \left\langle \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \right\rangle &= x_1 \cdot y_1 + x_2 \cdot y_2 + \dots + x_n \cdot y_n \\ &= y_1 \cdot x_1 + y_2 \cdot x_2 + \dots + y_n \cdot x_n \\ &= \left\langle \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right\rangle \end{aligned}$$

So the inner product is commutative.

(b) $\langle \vec{x}, \vec{x} \rangle = \|\vec{x}\|^2$

Solution:

$$\begin{aligned} \left\langle \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right\rangle &= x_1 \cdot x_1 + x_2 \cdot x_2 + \cdots + x_n \cdot x_n \\ &= x_1^2 + x_2^2 + \cdots + x_n^2 \\ &= (\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2})^2 \\ &= (\|\vec{x}\|)^2 \end{aligned}$$

The inner product of a vector with itself is its norm squared.

(c) $\langle -\vec{x}, \vec{y} \rangle = -\langle \vec{x}, \vec{y} \rangle$.

Solution:

$$\begin{aligned} \langle -\vec{x}, \vec{y} \rangle &= \left\langle \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \right\rangle \\ &= -x_1 \cdot y_1 - x_2 \cdot y_2 - \cdots - x_n \cdot y_n \\ &= -(x_1 \cdot y_1 + x_2 \cdot y_2 + \cdots + x_n \cdot y_n) \\ &= -\langle \vec{x}, \vec{y} \rangle \end{aligned}$$

Flipping the sign of one of the vectors in the inner product flips the sign of the inner product, but does not change the magnitude.

(d) $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

Solution:

$$\begin{aligned} \langle \vec{x}, \vec{y} + \vec{z} \rangle &= \vec{x}^T (\vec{y} + \vec{z}) \\ &= \vec{x}^T \vec{y} + \vec{x}^T \vec{z} \\ &= \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle \end{aligned}$$

The inner product is distributive.

(e) $\langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle = \langle \vec{x}, \vec{x} \rangle + 2\langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{y} \rangle$

Solution:

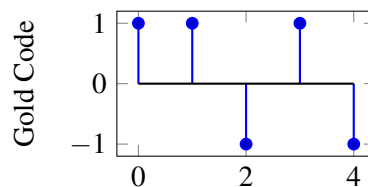
$$\begin{aligned}
 \langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle &= \langle \vec{x}, \vec{x} + \vec{y} \rangle + \langle \vec{y}, \vec{x} + \vec{y} \rangle \\
 &= \langle \vec{x}, \vec{x} \rangle + \langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle \\
 &= \langle \vec{x}, \vec{x} \rangle + 2 \langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{y} \rangle
 \end{aligned}$$

3. Golden Positioning System

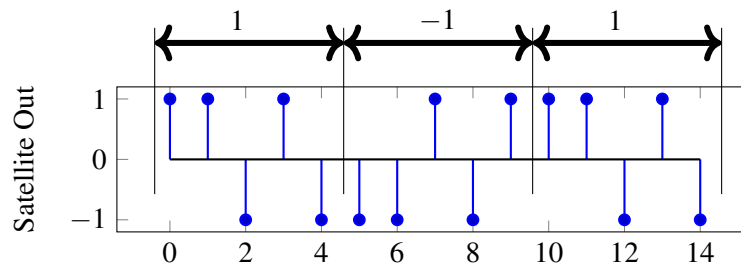
Learning Goal: This problem is meant to help practice computing cross-correlation. It also covers the concept of trilateration.

In this problem we will explore how real GPS systems work, and touch on a few aspects of implementing GPS receivers.

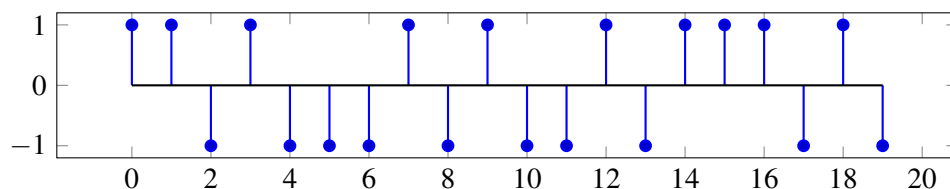
A Gold code is a sequence of 1's and -1 's that has a high autocorrelation at a shift of 0, and small autocorrelations otherwise. Every GPS satellite has a unique Gold code assigned to it, and users are aware of the Gold code used by each satellite. The plot below shows a Gold code of length 5.



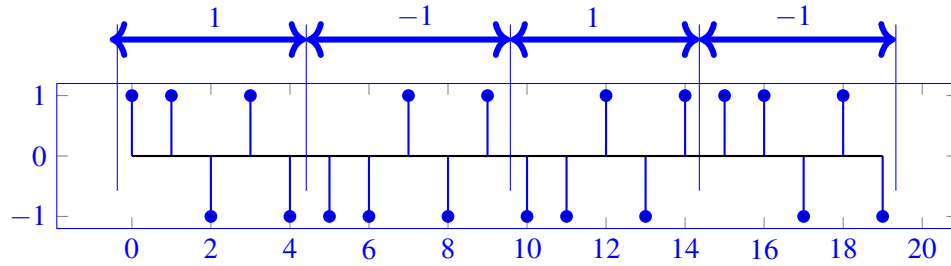
Each GPS satellite has a message that it transmits by modulating the Gold code. When the satellite is transmitting a 1, it sends just the Gold code sequence. When the satellite is transmitting a -1 , it sends -1 times the Gold code. For example, if a satellite were transmitting the message $[1, -1, 1]$, it would transmit the following:



- (a) Suppose you receive the following from a GPS satellite that has the same Gold code as above. **What message is the satellite transmitting?**



Solution: The data sent by the satellite is [1 -1 -1 1].



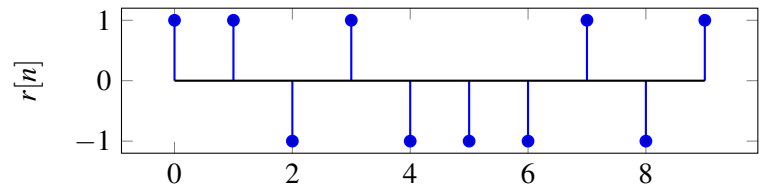
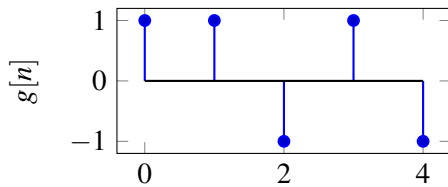
- (b) In order to find the message being sent by the satellite, the receiver will find the linear cross-correlation of the received signal with a replica of the satellite Gold code.

We need to find the **linear cross-correlation** of the signals shown below given by

$$\text{corr}_{\vec{r}}(\vec{g})[k] = \sum_{i=-\infty}^{\infty} r[i]g[i-k]$$

where $r[n]$ is the received signal and $g[n]$ is the Gold code sequence. *Note that neither of these signals is periodic in this part.*

Plot the values of $\text{corr}_{\vec{r}}(\vec{g})[k]$ for $-1 \leq k \leq 7$. What is the significance of the peaks in the linear cross-correlation?



Solution: The cross-correlation of the two signals is shown below. Note that we show the full cross-correlation for completeness, you were only required to show the values between [-1, 7]. As an example computation:

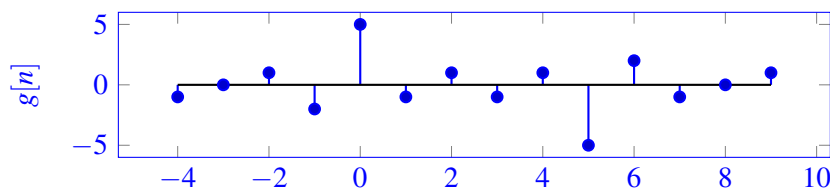
$$\text{corr}_{\vec{r}}(\vec{g})[5] = \sum_{i=-\infty}^{\infty} r[i]g[i-(5)]$$

Notice that since these signals are *not* zero-padded, $r[n]$ is 0 for all $n < 0$ and $n > 9$. This allows us to simplify our sum to:

$$\text{corr}_{\vec{r}}(\vec{g})[5] = \sum_{i=0}^9 r[i]g[i-5].$$

Similarly, notice that $g[n]$ is 0 for all $n < 0$ and $n > 4$; in the sum, this means it is 0 for all $i < 5$ and $i > 9$. This further simplifies the sum as follows:

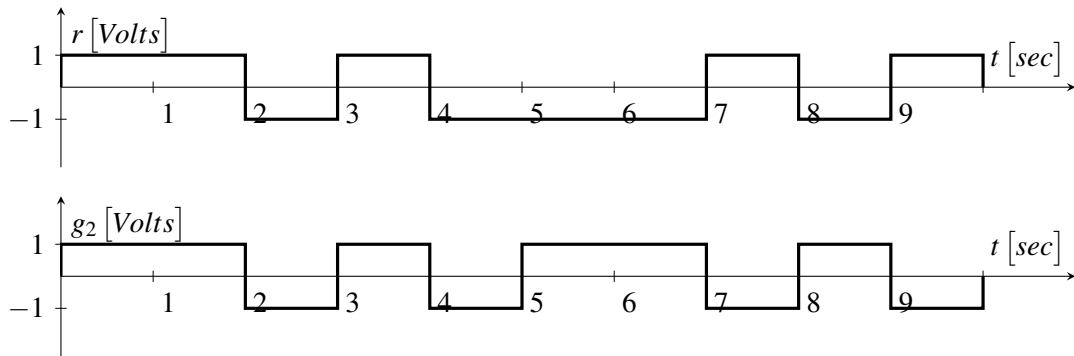
$$\text{corr}_{\vec{r}}(\vec{g})[5] = \sum_{i=5}^9 r[i]g[i-5] = r[5]g[0] + r[6]g[1] + r[7]g[2] + r[8]g[3] + r[9]g[4] = -5.$$



In the above plots, the relative strength of the peaks signify how well the gold code matches with the received signal, with a magnitude of 5 representing a perfect match. The sign of the peak can be used to determine whether the transmitted data was +1 or -1.

- (c) Real GPS receivers have specialized hardware to perform cross-correlation using circuits. However, since these transmissions are continuous signals instead of discrete values, we will model the received

signal $r(t)$ and the Gold code signal $g_2(t)$ as square waves, as shown in the plot below. Notice that $g_2(t)$ shows two periods of the Gold code.



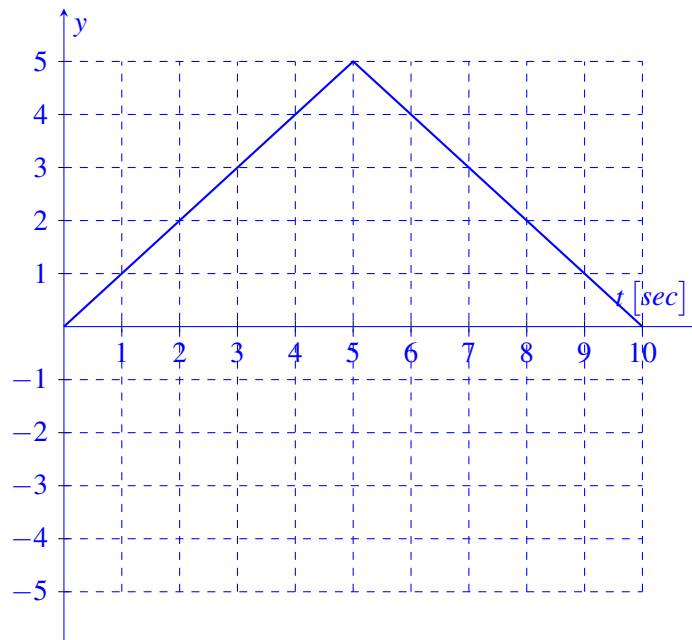
An essential hardware block to implementing a GPS correlator is *Multiply and Integrate*. The Multiply and Integrate block takes in two inputs, then integrates the product of the two inputs over time. For example, the output of the *Multiply and Integrate* block given the above two inputs would be:

$$y(t) = \int_0^t r(\tau)g_2(\tau)d\tau$$

where $y(t)$ is the circuit output at time t . **Draw $y(t)$ as a function of time, for $t = 0$ to $t = 10$ sec.**

Solution:

Notice that for the first 5 seconds, $r(t)$ and $g_2(t)$ are equal, then for the next 5 seconds $r(t) = -g_2(t)$. We can see that because the magnitude of r and g are both 1, when the match is perfect, the magnitude of the correlation is also exactly 1, with the sign again denoting \pm .



- (d) Receivers also need to use the received data to calculate the position of the satellite. Each receiver will receive data from k satellites. Each satellite transmits the time, S_i , at which it started sending the message, where i is the index of the satellite, and $1 \leq i \leq k$. The receiver knows the time, T_i , at which

each message arrives. You may assume the receiver and transmitter clocks are synchronized perfectly. Let c represent the speed of the signal.

Find an expression for d_i , the distance between the receiver and the i^{th} satellite, in terms of S_i , T_i , and other relevant parameters.

Solution:

$$d_i = c \times (T_i - S_i)$$

- (e) Each satellite's position in 3D space is (u_i, v_i, w_i) , where $1 \leq i \leq k$. The receiver position is given by (x, y, z) . We need a linear system of equations the receiver can use to solve for its position, $\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Due to limitations of the hardware, the receiver can only handle **linear** systems of equations. **How many satellites must the receiver get data from to solve for the receivers position?**

Solution: Here we need to solve for 3 variables, but the equations are non-linear. We will need one extra equation to make the equations linear, so 4 satellites total. Feel free to take a look at the first section of Note 22 for more info on linearization.

4. Inner Products

For each of the following functions, show whether it defines an inner product on the given vector space. If not, give a counterexample.

- (a) For \mathbb{R}^2 :

$$\langle \vec{p}, \vec{q} \rangle = \vec{p}^T \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \vec{q}$$

Solution: Yes, the function defines an inner product on \mathbb{R}^2 . To show this, we will show that all of the three axioms apply. Let $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ and $\vec{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$.

- i. Symmetry:

$$\begin{aligned} \langle \vec{p}, \vec{q} \rangle &= \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \\ &= \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} 3q_1 + q_2 \\ q_1 + 2q_2 \end{bmatrix} \\ &= 3p_1q_1 + p_1q_2 + p_2q_1 + 2p_2q_2 \\ &= \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} 3p_1 + p_2 \\ p_1 + 2p_2 \end{bmatrix} \\ &= \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \\ &= \langle \vec{q}, \vec{p} \rangle \end{aligned}$$

- ii. Linearity: Let $\vec{p}_1, \vec{p}_2, \vec{q} \in \mathbb{R}^2$.

$$\begin{aligned}
\langle \alpha \vec{p}_1 + \beta \vec{p}_2, \vec{q} \rangle &= (\alpha \vec{p}_1 + \beta \vec{p}_2)^T \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \vec{q} \\
&= \alpha \vec{p}_1^T \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \vec{q} + \beta \vec{p}_2^T \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \vec{q} \\
&= \alpha \langle \vec{p}_1, \vec{q} \rangle + \beta \langle \vec{p}_2, \vec{q} \rangle
\end{aligned}$$

iii. Positive-definiteness:

$$\langle \vec{p}, \vec{p} \rangle = 3p_1p_1 + p_1p_2 + p_2p_1 + 2p_2p_2 = 3p_1^2 + 2p_2^2 + 2p_1p_2 = 2p_1^2 + p_2^2 + (p_1 + p_2)^2$$

Since all of the components are non-negative, $\langle \vec{p}, \vec{p} \rangle \geq 0$.

Furthermore, the inner product will be 0 if and only if $p_1 = p_2 = 0 \implies \vec{p} = \vec{0}$.

(b) For \mathbb{R}^2 :

$$\langle \vec{p}, \vec{q} \rangle = \vec{p}^T \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} \vec{q}$$

Solution:

No, the function does not define an inner product on \mathbb{R}^2 . To show this, we will give a counterexample for the positive-definiteness axiom. Let $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$.

$$\begin{aligned}
\langle \vec{p}, \vec{p} \rangle &= \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \\
&= 3p_1p_1 + p_1p_2 + p_2p_1 - 2p_2p_2 \\
&= 3p_1^2 - 2p_2^2 + 2p_1p_2
\end{aligned}$$

Let $p_1 = 0$ and $p_2 = 1$. Then $3p_1^2 - 2p_2^2 + 2p_1p_2 = -2 < 0$.

(c) For \mathbb{R}^2 :

$$\langle \vec{p}, \vec{q} \rangle = \vec{p}^T \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \vec{q}$$

Solution:

No, the function does not define an inner product on \mathbb{R}^2 . To show this, we will give a counterexample for the symmetry axiom. Let $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ and $\vec{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$.

$$\begin{aligned}
\langle \vec{p}, \vec{q} \rangle &= \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \\
&= 3p_1q_1 + p_1q_2 + 2p_2q_2
\end{aligned}$$

$$\begin{aligned}
\langle \vec{q}, \vec{p} \rangle &= \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \\
&= 3p_1q_1 + p_2q_1 + 2p_2q_2
\end{aligned}$$

Since there exists p_1, q_1, p_2, q_2 such that $p_1q_2 \neq p_2q_1$, we have $\langle \vec{p}, \vec{q} \rangle \neq \langle \vec{q}, \vec{p} \rangle$.

(d) For $\mathbb{R}^{2 \times 2}$, the space of all 2×2 real matrices, the *Frobenius* inner product is defined as:

$$\langle A, B \rangle_F = \text{Tr}(A^T B)$$

Where A and B are 2×2 real matrices, and Tr represents the *Trace* of a matrix, or the sum of its diagonal entries. Prove that the Frobenius inner product is valid over $\mathbb{R}^{2 \times 2}$. **Solution:**

To simplify analysis, let's define the entries of A and B as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad B^T = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}$$

We can now proceed to prove the three axioms.

i. Symmetry: Let's first compute the result of $A^T B$:

$$A^T B = \begin{bmatrix} a_{11}b_{11} + a_{21}b_{21} & a_{11}b_{12} + a_{21}b_{22} \\ a_{12}b_{11} + a_{22}b_{21} & a_{12}b_{12} + a_{22}b_{22} \end{bmatrix}$$

The Trace (sum of diagonal elements) is:

$$a_{11}b_{11} + a_{21}b_{21} + a_{12}b_{12} + a_{22}b_{22}$$

More compactly, it is:

$$\sum_{i=1}^2 \sum_{j=1}^2 a_{ij}b_{ij}$$

Proceeding similarly, we get that the product of $B^T A$ is:

$$B^T A = \begin{bmatrix} a_{11}b_{11} + a_{21}b_{21} & b_{11}a_{12} + b_{21}a_{22} \\ b_{12}a_{11} + b_{22}a_{21} & a_{12}b_{12} + a_{22}b_{22} \end{bmatrix}$$

We can see that the diagonal elements, and hence the Traces are equal, proving symmetry–

$$\langle A, B \rangle_F = \langle B, A \rangle_F$$

ii. Linearity: Let's define $A, B, C \in \mathbb{R}^{2 \times 2}$ and $\alpha, \beta \in \mathbb{R}$. Defining the elements in the same way as in the Symmetry section, let's look at

$$\begin{aligned} \langle \alpha A + \beta B, C \rangle_F &= \text{Tr}((\alpha A + \beta B)^T C) \\ \text{Tr}((\alpha A + \beta B)^T C) &= \text{Tr}(C^T (\alpha A + \beta B)) \\ &= \text{Tr}(C^T \alpha A + C^T \beta B) \\ &= \text{Tr}(\alpha C^T A) + \text{Tr}(\beta C^T B) \\ &= \alpha \text{Tr}(C^T A) + \beta \text{Tr}(C^T B) \\ &= \alpha \langle A, C \rangle_F + \beta \langle B, C \rangle_F \end{aligned}$$

Proving linearity.

- iii. **Positive Definiteness:** If we define the elements of a matrix, $A \in \mathbb{R}^{2 \times 2}$, in the same fashion as above, we find that $A^T A$ takes on the following form:

$$A^T A = \begin{bmatrix} a_{11}^2 & a_{21}a_{12} \\ a_{12}a_{21} & a_{22}^2 \end{bmatrix}$$

With the Trace:

$$\text{Tr}(A^T A) = a_{11}^2 + a_{22}^2$$

Since $a_{11}, a_{22} \in \mathbb{R}$: $\langle A, A \rangle_F \geq 0$, and 0 iff the diagonal of A is 0.

5. Cauchy-Schwarz Inequality

Learning Goal: The objective of this problem is to understand and prove the Cauchy-Schwarz inequality for real-valued vectors.

The Cauchy-Schwarz inequality states that for two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$:

$$|\langle \vec{v}, \vec{w} \rangle| = |\vec{v}^T \vec{w}| \leq \|\vec{v}\| \cdot \|\vec{w}\|$$

In this problem we will prove the Cauchy-Schwarz inequality for vectors in \mathbb{R}^2 .

Take two vectors: $\vec{v} = r \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$ and $\vec{w} = t \begin{bmatrix} \cos \phi \\ \sin \phi \end{bmatrix}$, where $r > 0, t > 0, \theta$, and ϕ are scalars. Make sure you understand why any vector in \mathbb{R}^2 can be expressed this way and why it is acceptable to restrict $r, t > 0$.

- (a) In terms of some or all of the variables r, t, θ , and ϕ , what are $\|\vec{v}\|$ and $\|\vec{w}\|$? *Hint: Recall the trig identity: $\cos^2 x + \sin^2 x = 1$*

Solution: We use the trig identity $\cos^2 x + \sin^2 x = 1$ to show:

$$\begin{aligned} \|\vec{v}\| &= \sqrt{v_1^2 + v_2^2} \\ &= \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta} \\ &= r \end{aligned}$$

Similarly, $\|\vec{w}\| = t$.

- (b) In terms of some or all of the variables r, t, θ , and ϕ , what is $\langle \vec{v}, \vec{w} \rangle$? *Hint: The trig identity $\cos(a)\cos(b) + \sin(a)\sin(b) = \cos(a-b)$ may be useful.*

Solution: We use the trig identity $\cos(x)\cos(y) + \sin(x)\sin(y) = \cos(x-y)$ to show:

$$\begin{aligned} \langle \vec{v}, \vec{w} \rangle &= (r \cos \theta)(t \cos \phi) + (r \sin \theta)(t \sin \phi) \\ &= r \cdot t (\cos \theta \cos \phi + \sin \theta \sin \phi) \\ &= r \cdot t \cos(\theta - \phi) \end{aligned}$$

- (c) Show that the Cauchy-Schwarz inequality holds for any two vectors in \mathbb{R}^2 . *Hint: consider your results from part (b). Also recall $-1 \leq \cos x \leq 1$ and use both inequalities.*

Solution: We use the fact that $\cos x \leq 1$ to show:

$$\begin{aligned}\langle \vec{v}, \vec{w} \rangle &= r \cdot t \cos(\theta - \phi) \\ &= \|\vec{v}\| \|\vec{w}\| \cos(\theta - \phi) \\ &\leq \|\vec{v}\| \|\vec{w}\|\end{aligned}$$

We use the fact that $\cos x \geq -1$ to show:

$$\begin{aligned}\langle \vec{v}, \vec{w} \rangle &= r \cdot t \cos(\theta - \phi) \\ &= \|\vec{v}\| \|\vec{w}\| \cos(\theta - \phi) \\ &\geq -\|\vec{v}\| \|\vec{w}\|\end{aligned}$$

Therefore:

$$-\|\vec{v}\| \|\vec{w}\| \leq \langle \vec{v}, \vec{w} \rangle \leq \|\vec{v}\| \|\vec{w}\|,$$

which gives us that

$$|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \|\vec{w}\|.$$

- (d) Note that the inequality states that the inner product of two vectors must be less than *or equal to* the product of their magnitudes. What conditions must the vectors satisfy for the equality to hold? In other words, when is $\langle \vec{v}, \vec{w} \rangle = \|\vec{v}\| \cdot \|\vec{w}\|$?

Solution:

$$\begin{aligned}\langle \vec{v}, \vec{w} \rangle &= \|\vec{v}\| \|\vec{w}\| \\ \|\vec{v}\| \|\vec{w}\| \cos(\theta - \phi) &= \|\vec{v}\| \|\vec{w}\| \\ \cos(\theta - \phi) &= 1 \\ \theta - \phi &= 0\end{aligned}$$

We see that the equality holds when the angle between the two vectors is zero. Note that when the angle is zero, the vectors would be linearly dependent.

6. Orthonormal Matrices

Definition: A matrix $U \in \mathbb{R}^{n \times n}$ is called an orthonormal matrix if $U^{-1} = U^T$ and each column of U is a unit vector.

Orthonormal matrices represent linear transformations that preserve angles between vectors and the lengths of vectors. Rotations and reflections, useful in computer graphics, are examples of transformations that can be represented by orthonormal matrices.

- (a) Let U be an orthonormal matrix. For two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, show that $\langle \vec{x}, \vec{y} \rangle = \langle U\vec{x}, U\vec{y} \rangle$, assuming we are working with the Euclidean inner product.

Solution:

$$\langle U\vec{x}, U\vec{y} \rangle = (U\vec{x})^T (U\vec{y}) = \vec{x}^T U^T U \vec{y} = \vec{x}^T U^{-1} U \vec{y} = \vec{x}^T \vec{y} = \langle \vec{x}, \vec{y} \rangle$$

- (b) Show that $\|U\vec{x}\| = \|\vec{x}\|$, where $\|\cdot\|$ is the Euclidean norm.

Solution:

$$\|U\vec{x}\| = \sqrt{\langle U\vec{x}, U\vec{x} \rangle} = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \|\vec{x}\|$$

The second equality follows from the identity proved in part (a).

- (c) How does multiplying \vec{x} by U affect the length of the vector? That is, how do the lengths of $U\vec{x}$ and \vec{x} compare? **Solution:**

Recall that the L_2 , or Euclidean norm of a vector *is* the length. As we proved in part b), U does not affect the norm of \vec{x} . In other words, the length of \vec{x} is the same before and after applying U ! This allows us to transform \vec{x} in ways that may make analysis easier while preserving its length! You will have the opportunity to explore this further in EECS 16B.

7. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.