
EECS 16A Designing Information Devices and Systems I

Spring 2023

Discussion 12B

1. Inner Product Properties

For this question, we will verify our definition of the Euclidean inner product in Cartesian coordinates

$$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n, \quad \text{for any } \vec{x}, \vec{y} \in \mathbb{R}^n$$

indeed satisfies the key properties required for all inner products for the 2-dimensional case. Suppose $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^2$ for the following parts (there is a slightly different definition for complex vectors):

(a) Show symmetry: $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$.

Answer: This is seen by direct expansion:

Let $x_i, y_i \in \mathbb{R}$, then

$$\begin{aligned} \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle &= x_1 \cdot y_1 + x_2 \cdot y_2 \\ &= y_1 \cdot x_1 + y_2 \cdot x_2 \\ &= \left\langle \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \end{aligned}$$

(b) Show linearity: $\langle \vec{x}, c\vec{y} + d\vec{z} \rangle = c\langle \vec{x}, \vec{y} \rangle + d\langle \vec{x}, \vec{z} \rangle$, where $c, d \in \mathbb{R}$ are real numbers.

Answer: This is accomplished through a direct expansion:

$$\begin{aligned} \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, c \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + d \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} cy_1 + dz_1 \\ cy_2 + dz_2 \end{bmatrix} \right\rangle \\ &= x_1(cy_1 + dz_1) + x_2(cy_2 + dz_2) \\ &= c(x_1 y_1 + x_2 y_2) + d(x_1 z_1 + x_2 z_2) \\ &= c \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle + d \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle \\ &= c\langle \vec{x}, \vec{y} \rangle + d\langle \vec{x}, \vec{z} \rangle \end{aligned}$$

(c) Show non-negativity: $\langle \vec{x}, \vec{x} \rangle \geq 0$, with equality if and only if $\vec{x} = \vec{0}$.

Answer: This part requires just a bit more thought beyond a direct expansion of $\langle \vec{x}, \vec{x} \rangle$, but we first recognize that this inner product is the definition of the norm (or length) of \vec{x} . So it is at least intuitive that a length of some vector (squared) cannot be negative:

$$\begin{aligned}\langle \vec{x}, \vec{x} \rangle &= \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \\ &= x_1^2 + x_2^2\end{aligned}$$

From this result we notice if either x_1 or x_2 are nonzero (even negative) values, then the inner product HAS to be positive. The only case in which the inner product $\langle \vec{x}, \vec{x} \rangle$ is identically zero is when both $x_1 = 0$ AND $x_2 = 0$, which verifies the final part of the property: $\langle \vec{x}, \vec{x} \rangle = 0$ ONLY IF $\vec{x} = \vec{0}$.

As a bonus, suppose we re-label our vector components $x_1 = a$ and $x_2 = b$.

Then we see $\langle \vec{x}, \vec{x} \rangle = c^2 = a^2 + b^2$, which is the Pythagorean theorem!

This verifies that $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = c$ can be geometrically understood as the length of vector \vec{x} .

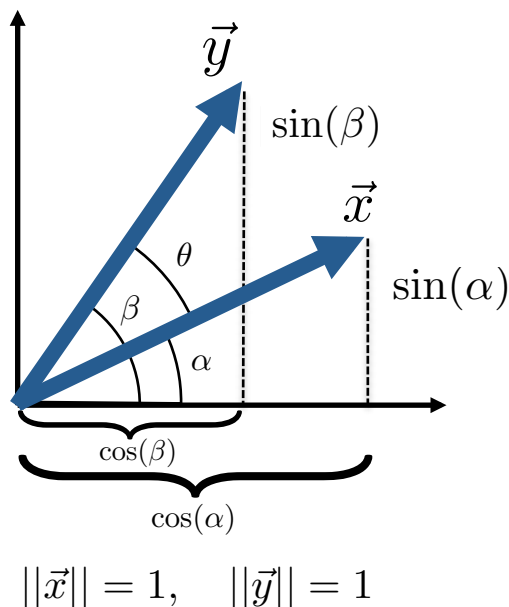
2. Geometric Interpretation of the Inner Product

In this problem, we explore the geometric interpretation of the Euclidean inner product, restricting ourselves to vectors in \mathbb{R}^2 .

Remember that the formula for the inner product of two vectors can be expressed in terms of their magnitudes and the angle between them as follows:

$$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \|\vec{y}\| \cdot \cos \theta$$

The figure below may be helpful in illustrating this property:



For each subpart, give an example of any two (nonzero) vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$ that satisfy the stated condition and compute their inner product.

- (a) Give an example of a pair of parallel vectors (vectors that point in the same direction and have an angle of 0 degrees between them).

Answer: Parallel vectors point in the same direction (have an angle of 0° between them).

This means we must have $\vec{y} = \alpha \vec{x}$ for some $\alpha > 0$.

Having only this condition leaves a lot of freedom.

Let us choose $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{y} = 2 \vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

$$\langle \vec{x}, \vec{y} \rangle = 1 \cdot 2 + 1 \cdot 2 = 4$$

- (b) Give an example of a pair of anti-parallel vectors (vectors that point in opposite directions).

Answer: Anti-parallel vectors point in opposite directions (have an angle of 180° between them).

This means we must have $\vec{y} = \alpha \vec{x}$ again, but now for some negative $\alpha < 0$.

Having only this condition still leaves a lot of freedom.

Let us choose $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and then set $\vec{y} = -2 \vec{x} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$.

$$\langle \vec{x}, \vec{y} \rangle = 1 \cdot -2 + 1 \cdot -2 = -4$$

- (c) Give an example of a pair of perpendicular vectors (vectors that have an angle of 90 degrees between them).

Answer: Perpendicular vectors point in 90° directions with respect to each-other.

Most importantly, the Euclidean inner product $\langle \vec{x}, \vec{y} \rangle = 0$ whenever \vec{x}, \vec{y} are orthogonal, or perpendicular.

For our example we will fix $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and then leave $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ general.

$$\langle \vec{x}, \vec{y} \rangle = 1 \cdot y_1 + 0 \cdot y_2 = y_1 \equiv 0.$$

Thus we must set $y_1 = 0$, but y_2 can assume any nonzero value!