EECS 16A Designing Information Devices and Systems I
Spring 2023 Homework 3

This homework is due February 10th, 2023, at 23:59.
Self-grades are due February 17th, 2023, at 23:59.

Submission Format
Your homework submission should consist of a single PDF file that contains all of your answers (any hand-written answers should be scanned).

1. Reading Assignment

For this homework, please read Notes 3 and 4. Note 3 provides an overview of linear dependence and span and Note 4 gives an introduction to thinking about and writing proofs.

Please answer the following question:

(a) Why are there two definitions of linear dependence? What value does each definition provide?

(b) List in your own words the steps used to construct a proof.

Solution:

(a) See Note 3.1 for the two definitions. Definition (I) is more useful for mathematically proving linear dependence while Definition (II) provides a more intuitive understanding of linear dependence and formalizes the notion of redundancy.

(b) From Note 4, the steps used to construct a proof are

i. Read the entire proof statement carefully.
ii. Identify what we know from the proof statement.
iii. Identify what we want to prove.
iv. Observe the beginning (i.e., knowns) and end (i.e., final result) of the proof and try to identify connecting similarities.
v. Manipulate both sides of the claim and connect them together. Make sure to carefully justify each step.
vi. Repeatedly try different approaches until you find an idea that works.
vii. Complete the proof.

2. Study Group Survey

Please help us understand how your study groups are going! **Fill out the following survey** (even if you are not in a study group) to help us create better matchings in the future. In case you have not been able to connect with a study group, or would like to try a new study group, there will be an opportunity for you to request a new study group as well in this form.

`https://forms.gle/8q6YzjSNhkcQ66ua7`

To get full credit for this question you must both

1. Fill out the survey (it will record your email)
2. Indicate in your homework submission that you filled out the survey.

3. Linear Dependence

Learning Objectives: *Evaluate the linear dependency of a set of vectors.*

State if the following sets of vectors are linearly independent or dependent. If the set is linearly dependent, provide a linear combination of the vectors that sum to the zero vector.

(a) $\left\{ \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$

Solution:

The vectors are linearly independent. A set of two vectors can only be linearly dependent if one of the vectors is a scaled version of the other. $\begin{bmatrix} -5 \\ 2 \end{bmatrix} \neq \alpha \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ for $\alpha \in \mathbb{R}$.

(b) $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Solution:

This set of vectors are linearly dependent. To both show this and find a linear combination that yields the vector $\vec{0}$, solve the equation $A\vec{x} = \vec{0}$ for a valid nonzero solution $\vec{x} \in \mathbb{R}^3$ and where the given vectors form the columns of the matrix A.

$$A\vec{x} = \vec{0} \quad \rightarrow \quad \begin{bmatrix} -1 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 3 & -1 \\ -1 & -2 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Gaussian elimination, we can solve this linear system of equations by finding the RREF of the following augmented matrix:

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ -1 & -2 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow -R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ -1 & -2 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow[R_4 \leftarrow R_4 + R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & -3 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 \leftarrow R_2/3]{R_5 \leftarrow R_5 - R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 3 & -1 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_4 \leftarrow R_4 + 3R_2]{R_3 \leftarrow R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The free variable can be chosen to be the variable of third column $\left[1, -\frac{1}{3}, 0, 0, 0\right]^T$. We can assign the value of the free variable to be $x_3 = 3\alpha$ (chosen arbitrarily to make subsequent calculations easier) for some arbitrary scalar α. If we solve for the second column variable from the second row after substitution, we obtain the variable of the second column to be $x_2 = \alpha$.

$$x_2 - \frac{1}{3}x_3 = x_2 - \frac{1}{3}(3\alpha) = 0 \quad \rightarrow \quad x_2 = \alpha$$

We can do similarly with the first variable to obtain a value of $x_1 = -2\alpha$ from the first row.

$$x_1 - x_2 + x_3 = x_1 - (\alpha) + (3\alpha) = 0 \quad \rightarrow \quad x_1 = -2\alpha$$

Finally, we can express a linear combination of the original vectors as depending on a nonzero α and showing linear dependence.

$$x_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2\alpha \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix} + 3\alpha \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If you have a set of coefficients that match a specific value of α, give yourself full credit.

(c) $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

Solution:

This set of vectors are linearly dependent. To both show this and find a linear combination that yields the vector $\vec{0}$, solve the equation $A\vec{x} = \vec{0}$ for a valid nonzero solution $\vec{x} \in \mathbb{R}^3$ and where the given vectors form the columns of the matrix A.

$$A\vec{x} = \vec{0} \quad \rightarrow \quad \begin{bmatrix} 2 & 0 & 2 & 0 \\ 2 & 1 & 4 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Gaussian elimination, we can solve this linear system of equations by finding the RREF of the following augmented matrix:

$$\left[\begin{array}{cccc|c} 2 & 0 & 2 & 0 & 0 \\ 2 & 1 & 4 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1/2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 4 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & -3 & 2 & 0 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3/-3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 \end{array}\right] \xrightarrow[R_1 \leftarrow R_1 - R_3]{R_2 \leftarrow R_2 - 2R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 \end{array}\right]$$

The free variable can be chosen to be the variable of fourth column $\left[\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right]^T$. We can assign the value of the free variable to be $x_4 = 3\alpha$ (chosen arbitrarily to make subsequent calculations easier) for some arbitrary scalar α. Once again, using back substitution into the other rows of the matrix, we can find

$$\begin{aligned} & & x_4 &= 3\alpha \\ x_3 - \frac{2}{3}x_4 = x_3 - \frac{2}{3}(3\alpha) = 0 & \quad\rightarrow & x_3 &= 2\alpha \\ x_2 + \frac{1}{3}x_4 = x_2 + \frac{1}{3}(3\alpha) = 0 & \quad\rightarrow & x_2 &= -\alpha \\ x_1 + \frac{2}{3}x_4 = x_1 + \frac{2}{3}(3\alpha) = 0 & \quad\rightarrow & x_1 &= -2\alpha \end{aligned}$$

Finally, we can express a linear combination of the original vectors as depending on a nonzero α and showing linear dependence.

$$x_1 \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2\alpha \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \alpha \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 2\alpha \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} + 3\alpha \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If you have a set of coefficients that match a specific value of α, give yourself full credit.

Alternatively, since the subset of the first 3 vectors is linearly independent, they span $\mathbb{R}^3$. We know this because the 3 vectors have no linear combination that yields 0 other than the trivial solution where all coefficients are also 0.

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1/2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 1 & -4 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow (R_3 - 2R_1 - R2)/3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The solution to this system has one unique solution, where all variables must be 0. Thus, there is no linear combination of the first three column vectors that can be equal to 0 (other than the trivial solution), so those vectors are linearly indepedent. Since the fourth vector is in $\mathbb{R}^4$, we are guaranteed some linear combination of the first three vectors that yields the fourth.

(d) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Solution:

Since this set contains $\vec{0}$, it is linearly dependent as we can take the linear combination

$$0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} + \alpha \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $\alpha \neq 0$ to get $\vec{0}$.

4. **Easing into Proofs**

Learning Objectives: *This is an opportunity to practice your proof development skills.*

Proof: Show that if the system of linear equations, $\mathbf{A}\vec{x} = \vec{b}$, has infinitely many solutions, then columns of $\mathbf{A}$ are linearly dependent.

To approach this proof, let us use a simplified version of the methodology delineated in Note 4. Although the final proof would read sequentially as in the *Proof Steps* indicated in Table 1, each part of this question will tackle each proof step but in a non-sequential order.

Table 1: Fundamental steps to a proof

Proof Steps	**Description**	**Question Part**
1	Identify what is known	(a)
2	Manipulate what is known	(c)
3	Connecting it up	(d)
4	Identify what is to be proved	(b)

(a) **Proof Step 1: Write what is known**
Think about the information we already know from the problem statement. Every detail could be important and some details could be implicit.

We know that system of equations, $\mathbf{A}\vec{x} = \vec{b}$, has infinitely many solutions which can be difficult to work with, but perhaps we can simplify to a case that we can work with. It turns out that if a linear system has at least **two** distinct solutions, then it must also have an infinite number of solutions.

We can also construct arbitrary vectors $\vec{u}$ and $\vec{v}$ which, in this case, are each a solution to the system of equations $\mathbf{A}\vec{x} = \vec{b}$ but not the same vector. Express the previous sentence in a mathematical form (just writing the equations will suffice; no need to take do further mathematical manipulation).

Solution: $\vec{u}$ and $\vec{v}$ must satisfy:

$$\mathbf{A}\vec{u} = \vec{b}, \quad \mathbf{A}\vec{v} = \vec{b}. \tag{1}$$

$$\vec{u} \neq \vec{v}. \tag{2}$$

(b) **Proof Step 4: Identify what is to be proved**
We have to show that the columns of $\mathbf{A}$ are linearly dependent. The matrix $\mathbf{A}$ can always be deconstructed and the columns explicitly denoted as vectors $\vec{c}_1$, $\vec{c}_2$, ..., and $\vec{c}_n$, i.e. $\mathbf{A} = \begin{bmatrix} | & | & \cdots & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & \cdots & | \end{bmatrix}$.

Using the Definition 3.1 of linear dependence from Note 3.1, write a mathematical equation that conveys linear dependence of $\vec{c}_1$, $\vec{c}_2$, ..., and $\vec{c}_n$.

Solution: According to the definition of linear dependence:

$$\alpha_1\vec{c}_1 + \alpha_2\vec{c}_2 + \ldots + \alpha_n\vec{c}_n = \vec{0}. \tag{3}$$

where at least one α_i is not equal to zero.

(c) **Proof Step 2: Manipulating what is known**
Now let us try to start from the givens in part (a) and make mathematically logical steps to reach the the final result in part (b). Since your answer to (b) is expressed in terms of the column vectors of $\mathbf{A}$, try to express the mathematical equations from part (a) in terms of the column vectors too. For example, we can write

$$\mathbf{A}\vec{x} = \vec{b} \implies \begin{bmatrix} | & | & \cdots & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \vec{b} \implies x_1\vec{c}_1 + x_2\vec{c}_2 + \ldots + x_n\vec{c}_n = \vec{b}$$

Now use your answer to part (a) to repeat the above formulation for distinct solutions $\vec{u}$ and $\vec{v}$.

Solution:

$$\mathbf{A}\vec{u} = \vec{b} \implies \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \end{bmatrix} = \vec{b} \implies u_1\vec{c}_1 + u_2\vec{c}_2 + \ldots + u_n\vec{c}_n = \vec{b}$$

$$\mathbf{A}\vec{v} = \vec{b} \implies \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{bmatrix} = \vec{b} \implies v_1\vec{c}_1 + v_2\vec{c}_2 + \ldots + v_n\vec{c}_n = \vec{b}$$

(d) **Proof Step 3: Connecting it up**
Now think about how you can mathematically manipulate your answer from part (c) to match the pattern of your desired final proof statement in part (b).

Solution:
Subtracting the second equation from the first equation in part (iii), we have

$$u_1\vec{c}_1 + u_2\vec{c}_2 + \ldots + u_n - v_1\vec{c}_1 - v_2\vec{c}_2 - \ldots - v_n\vec{c}_n = \vec{b} - \vec{b} \tag{4}$$

$$\implies (u_1 - v_1)\vec{c}_1 + (u_2 - v_2)\vec{c}_2 + \ldots + (u_n - v_n)\vec{c}_n = \vec{0} \tag{5}$$

Let $\alpha_1 = u_1 - v_1$, ..., and $\alpha_n = u_n - v_n$, i.e. $\vec{\alpha} = \vec{u} - \vec{v}$. Here, at least one α_i is not equal to zero since $\vec{u} \neq \vec{v}$. In other words, we find that because $\vec{u} \neq \vec{v}$, there exists a $(u_i - v_i)\vec{c}_i = \vec{0}$. Thus, $\mathbf{A}$ must have linearly dependent columns. Hence the final mathematical expression from part (b) is satisfied, i.e. the proof is complete!

5. Span Proofs

Learning Objectives: *This is an opportunity to practice your proof development skills.*

(a) Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$$

In other words, we can replace one vector with the sum of itself and another vector and not change their span.

In order to show this, you have to prove the two following statements:

- If a vector $\vec{q}$ belongs in $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, then it must also belong in $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$.
- If a vector $\vec{r}$ belongs in $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$, then it must also belong in $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$.

In summary, you have to prove the problem statement from both directions.

Solution:
Suppose $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars a_i:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = a_1(\vec{v}_1 + \vec{v}_2) + (-a_1 + a_2)\vec{v}_2 + \cdots + a_n\vec{v}_n$$

We can change the scalar values to adjust for the combined vectors. Thus, we have shown that $\vec{q} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$.
Now, we must show the other direction. Suppose we have some arbitrary $\vec{r} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars b_i:

$$\vec{r} = b_1(\vec{v}_1 + \vec{v}_2) + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n = b_1\vec{v}_1 + (b_1 + b_2)\vec{v}_2 + \cdots + b_n\vec{v}_n.$$

Thus, we have shown that $\vec{r} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Combining this with the earlier result, the spans are the same.

(b) Consider the span of the set $(\vec{v}_1, ..., \vec{v}_n, \vec{u})$. Suppose $\vec{u}$ is in the span of $\{\vec{v}_1, ..., \vec{v}_n\}$. Then, show that any vector $\vec{r}$ in $span\{\vec{v}_1, ..., \vec{v}_n, \vec{u}\}$ is in $span\{\vec{v}_1, ..., \vec{v}_n\}$.

Solution: From the first sentence of the question, by definition of span, we know that any vector $\vec{r}$ in $span\{\vec{v}_1, ..., \vec{v}_n, \vec{u}\}$ can be written $\vec{r} = k\vec{u} + a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_n\vec{v}_n$. Using the summation symbol, we can also write $\vec{r} = k\vec{u} + \sum_{i=1}^{n} a_i\vec{v}_i$.
From the second sentence of the question, since $\vec{u}$ is in the span of $\{\vec{v}_1, ..., \vec{v}_n\}$, we can write $\vec{u} = b_1\vec{v}_1 + b_2\vec{v}_2 + ...b_n\vec{v}_n$ or $\sum_{i=1}^{n} b_i\vec{v}_i$. Now that we have an expression for $\vec{u}$, let's substitute it into the previous expression.

$$\vec{r} = k\vec{u} + \sum_{i=1}^{n} a_i\vec{v}_i$$

$$\vec{r} = k(\sum_{i=1}^{n} b_i\vec{v}_i) + \sum_{i=1}^{n} a_i\vec{v}_i$$

Finally, gathering up coefficients, we get:

$$\vec{r} = \sum_{i=1}^{n} (k * b_i + a_i)\vec{v}_i,$$

so this arbitrary vector $\vec{r}$ is also in $span\{\vec{v}_1,...,\vec{v}_n)\}$.
Intuitively, $\vec{u}$ is redundant, so we can safely remove it without reducing our span.

6. Linear Dependence in a Square Matrix

Learning Objective: *This is an opportunity to practice applying proof techniques. This question is specifically focused on linear dependence of rows and columns in a square matrix.*

Let A be a square $n \times n$ matrix, (i.e. both the columns and rows are vectors in $\mathbb{R}^n$). Suppose we are told that the columns of A are linearly dependent. Prove, then, that the rows of A must also be linearly dependent.

You can use the following conclusion in your proof:
If Gaussian elimination is applied to a matrix A, and the resulting matrix (in reduced row echelon form) has at least one row of all zeros, this means that the rows of A are linearly dependent.

(**Hint**: Can you use the linear dependence of the columns to say something about the number of solutions to $A\vec{x} = \vec{0}$? How does the number of solutions relate to the result of Gaussian elimination?)

Solution:

Let $\vec{a}_1, \vec{a}_2, \ldots \vec{a}_n$ be the columns of A. By the definition of linear dependence, there exist scalars, $c_1, c_2, \ldots c_n$, not all zero, such that

$$c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n = \vec{0} \tag{6}$$

We define $\vec{c}$ to be a vector containing the c_i's as follows: $\vec{c} = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix}^T$, where $\vec{c} \neq \vec{0}$ by the definition of linear dependence. We can write Eq. 6 in matrix vector form:

$$A\vec{c} = \vec{0} \tag{7}$$

Let's use the first hint: How many solutions are there to the equation $A\vec{x} = \vec{0}$? We know from Eq. 7 that $\vec{c}$ is a solution, but we can also show that $\alpha\vec{c}$ is a solution for any α:

$$A(\alpha\vec{c}) = \alpha\vec{0} = \vec{0} \tag{8}$$

Since $\vec{c}$ is not zero, every multiple of $\vec{c}$ is a different solution. Therefore there are infinite solutions to the equation $A\vec{x} = \vec{0}$.

What can we say about the result of Gaussian elimination if there are infinite solutions? We know that if there are infinite solutions, there must be a free variable after Gaussian elimination. In other words, there must be a column in the row reduced matrix with no leading entry. Therefore, there must be fewer leading entries than the number of columns. Since the matrix A is square, it has the same number of rows as columns, so there must be fewer leading entries than the number of rows. That means there is at least one row with no leading entry, which is equivalent to saying there must be one row that's all zeros in the row reduced matrix.

For example consider performing elimination on the following square matrix with infinite solutions:

$$\left[\begin{array}{ccc|c} 2 & 2 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 2 & 0 & 1 & 1 \end{array}\right]$$

Subtracting row 1 from row 3 ($R_3 - R_1 \to R_3$):

$$\left[\begin{array}{ccc|c} 2 & 2 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 0 & -2 & -2 & -6 \end{array}\right]$$

Dividing row 1 by 2 ($R_1/2 \to R_1$):

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 1 & 1 & 3 \\ 0 & -2 & -2 & -6 \end{array}\right]$$

Adding row 2 multiplied by 2 to row 3 ($R_3 + 2 * R_2 \to R_3$):

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

where we see that the last row is missing the leading entry and becomes all 0s.

Finally, we were given that if there is a row of all zeros in the row reduced matrix, then the rows of A must be linearly dependent. Thus, if the columns of an $n \times n$ matrix A are linearly dependent, then the rows are linearly dependent as well.

7. Filtering Out The Troll

Learning Objectives: *The goal of this problem is to explore the problem of sound reconstruction by solving a system of linear equations.*

You were attending the 16A lecture the day before the first exam, and decided to record it using two directional microphones (one microphone receives sound from the 'x' direction and the other from the 'y' direction). However, someone in the audience was trolling around loudly, adding interference to the recording! The troll's interference dominates both of your microphones' recordings, so you cannot hear the recorded speech. Fortunately, since your recording device contained two microphones, you can combine the two individual microphone recordings to remove the troll's interference.

The diagram shown in Figure 1 shows the locations of the speaker, the troll, and you and your two microphones (at the origin).

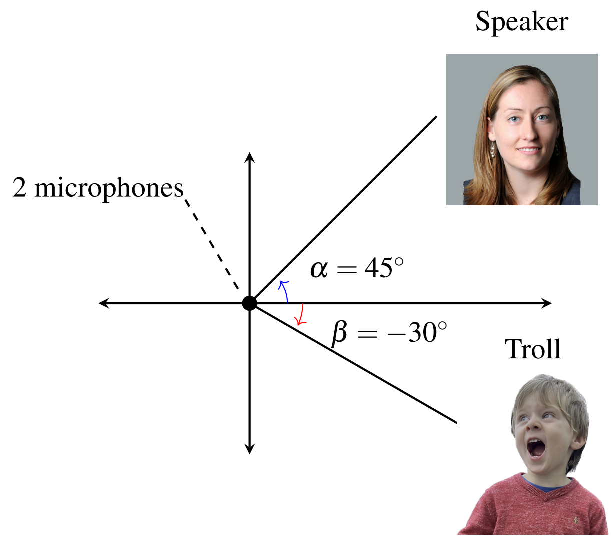

Figure 1: Locations of the speaker and the troll.

Since the microphones are directional, the strength of the recorded signal depends on the angle from which the sound arrives. Suppose that the sound arrives from an angle θ relative to the x-axis (in our case, these angles are 45°and −30°, labeled as α and β, respectively). The first microphone scales the signal by $\cos(\theta)$, while the second microphone scales the signal by $\sin(\theta)$. Each microphone records the weighted sum (or linear combination) of all received signals.

The speech signal can be represented as a vector, $\vec{s}$, and the troll's interference as vector $\vec{r}$, with each entry representing an audio sample at a given time. The recordings of the two microphones are given by $\vec{m}_1$ and $\vec{m}_2$:

$$\vec{m}_1 = \cos(\alpha)\cdot\vec{s} + \cos(\beta)\cdot\vec{r} \tag{9}$$

$$\vec{m}_2 = \sin(\alpha)\cdot\vec{s} + \sin(\beta)\cdot\vec{r} \tag{10}$$

where α and β are the angles at which the professor and the troll respectively are located with respect to the x-axis, and variables $\vec{s}$ and $\vec{r}$ are the audio signals produced by the professor and the troll respectively.

(a) Plug in $\alpha = 45° = \frac{\pi}{4}$ and $\beta = -30° = -\frac{\pi}{6}$ to Equations 9 and 10 to write the recordings of the two microphones $\vec{m}_1$ and $\vec{m}_2$ as a linear combination (i.e. a weighted sum) of $\vec{s}$ and $\vec{r}$.

Solution:

$$\begin{aligned}
\vec{m}_1 &= \cos\left(\frac{\pi}{4}\right)\cdot\vec{s} + \cos\left(-\frac{\pi}{6}\right)\cdot\vec{r} \\
&= \frac{1}{\sqrt{2}}\cdot\vec{s} + \frac{\sqrt{3}}{2}\cdot\vec{r} \\
\vec{m}_2 &= \sin\left(\frac{\pi}{4}\right)\cdot\vec{s} + \sin\left(-\frac{\pi}{6}\right)\cdot\vec{r} \\
&= \frac{1}{\sqrt{2}}\cdot\vec{s} - \frac{1}{2}\cdot\vec{r}
\end{aligned}$$

(b) Solve the system from part (a) using any convenient method you prefer to recover the important speech $\vec{s}$ as a weighted combination of $\vec{m}_1$ and $\vec{m}_2$. In other words, write $\vec{s} = c \cdot \vec{m}_1 + k \cdot \vec{m}_2$ (where c and k are scalars). What are the values of c and k?

Solution:

Solving the system of linear equations yields

$$\vec{s} = \frac{\sqrt{2}}{1+\sqrt{3}} \cdot \left(\vec{m}_1 + \sqrt{3}\vec{m}_2\right).$$

Therefore, the values are $c = \frac{\sqrt{2}}{1+\sqrt{3}}$ and $k = \frac{\sqrt{6}}{1+\sqrt{3}}$.

It is fine if you solved this either using IPython or by hand using any valid technique. The easiest approach is to subtract either of the two equations from the other and immediately see that $\vec{r} = \frac{2}{\sqrt{3}+1}(\vec{m}_1 - \vec{m}_2)$. Substituting b back into the second equation and multiplying through by $\sqrt{2}$ gives that $\vec{s} = \sqrt{2}(\vec{m}_2 + \frac{1}{\sqrt{3}+1}(\vec{m}_1 - \vec{m}_2))$, which simplifies to the expression given above.

Notice that subtracting one equation from the other is natural given the symmetry of the microphone patterns and the fact that the patterns intersect at the 45 degree line where the important speech is happening, and the fact that $\sin(45°) = \cos(45°)$. So we know that the result of subtracting one microphone recording from the other results in only the troll's contribution. Once we have the troll contribution, we can remove it and obtain the professor's sole content.

(c) Partial IPython code can be found in `prob2.ipynb`, which you can access through the Datahub link associated with this assignment on the course website. Complete the code to get the signal of the important speech. Write out what the speaker says. (Optional: Where is the speech taken from?)

Note: You may have noticed that the recordings of the two microphones sound remarkably similar. This means that you could recover the real speech from two "trolled" recordings that sound almost identical! Leave out the fact that the recordings are actually different, and have some fun with your friends who aren't lucky enough to be taking EECS16A.

Solution:

The solution code can be found in `sol3.ipynb`. The speaker (Professor Waller) is giving a lecture on the nebulous *turboencabulator* (you can find the full lecture link here: `https://youtu.be/Ac7G7xOG2Ag`), and the audio is recorded while her son was being particularly boisterous singing a song.

The idea of using multiple microphones to isolate speech is interesting and is increasingly used in practice. Furthermore, similar techniques are used in wireless communication both by cellular systems like LTE and increasingly by WiFi hotspots. (This is why they often have multiple antennas).

8. Prelab Questions These questions pertain to the Pre-Lab reading for the Imaging 2 lab. You can find the reading under the Imaging 2 Lab section on the 'Schedule' page of the website. We do not expect in-depth answers for the questions. Please limit your answers to a maximum of 2 sentences.

(a) Briefly explain what the H matrix, $\vec{i}$ vector, and $\vec{s}$ vector each signify.

(b) How will we get the vector $\vec{i}$ from $\vec{s} = H\vec{i}$, the equation representing our imaging system?

Solution:

(a) The matrix H is also known as the mask matrix. It allows us to selectively choose what pixels we want to read (scan) at a given time.

The vector $\vec{i}$ represents the image that we are trying to reconstruct.

The vector $\vec{s}$ represents the scan of the image.

(b) $\vec{i} = H^{-1}\vec{s}$. Multiplying by H^{-1} on both sides of the equation gives us the vector $\vec{i}$.

9. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.