EECS 16A Designing Information Devices and Systems I
Spring 2023 Homework 5

This homework is due Friday, February 24, 2023 at 23:59.
Self-grades are due Friday, March 3, 2023 at 23:59.

Submission Format
Your homework submission should consist of **one** file.

We strongly recommended that you submit your self-grades PRIOR to taking Midterm 1 on March 1, 2023, since looking at the solutions earlier will help you to study for the midterm.

1. **Reading Assignment**

For this homework, please read Notes 8 and 9. These notes will give you an overview of matrix subspaces and eigenvalues/eigenvectors. Note that Note 10 covers change of basis and diagonalizaiton, which is not in-scope for this couse; however you are welcome to read it if interested, as these topics will be emphasized in EECS 16B. You are always welcome and encouraged to read beyond this as well.

How do we compute eigenvalues and, subsequently, corresponding eigenvectors? What is the eigenvalue corresponding to the steady state of a system?

2. **Mechanical Determinants**

For each of the following matrices, compute their determinant and state whether they are invertible.

(a) $\begin{bmatrix} 6 & 9 \\ 4 & 6 \end{bmatrix}$.

Solution:

We can use the form of a 2×2 determinant from lecture:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Therefore,

$$\det\left(\begin{bmatrix} 6 & 9 \\ 4 & 6 \end{bmatrix}\right) = 6 \cdot 6 - 9 \cdot 4 = 0$$

Since the determinant is 0, the matrix is non-invertible. Note that the columns of the matrix are linearly dependent.

(b) $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

Solution:

$$\det\left(\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\right) = 2 \cdot 3 - 1 \cdot 0 = 6$$

Since the determinant is not 0, the matrix is invertible.

(c) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Solution:

$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right) = 2 \cdot 3 - 0 = 6$$

Since the determinant is not 0, the matrix is invertible.

(d) $\begin{bmatrix} -4 & 2 & 1 \\ 5 & 1 & -3 \\ 7 & 3 & 1 \end{bmatrix}$.

Solution: To find the determinant of a 3 by 3 matrix, we can use the formula:

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a \cdot \det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b \cdot \det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c \cdot \det\left(\begin{bmatrix} d & d \\ g & h \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} -4 & 2 & 1 \\ 5 & 1 & -3 \\ 7 & 3 & 1 \end{bmatrix}\right) = -4 \cdot \det\left(\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}\right) - 2 \cdot \det\left(\begin{bmatrix} 5 & -3 \\ 7 & 1 \end{bmatrix}\right) + 1 \cdot \det\left(\begin{bmatrix} 5 & 1 \\ 7 & 3 \end{bmatrix}\right)$$

$$\begin{aligned} &= -4 \cdot [(1 \cdot 1) - (-3 \cdot 3)] - 2 \cdot [(5 \cdot 1) - (-3 \cdot 7)] + 1 \cdot [(5 \cdot 3) - (1 \cdot 7)] \\ &= -4 \cdot [10] - 2 \cdot [26] + 1 \cdot [8] \\ &= -84 \end{aligned}$$

Since the determinant is not 0, the matrix is invertible.

(e) $\begin{bmatrix} -4 & 0 & 0 \\ 5 & 1 & -3 \\ 7 & 3 & 1 \end{bmatrix}$.

Solution:

$$\det\left(\begin{bmatrix} -4 & 0 & 0 \\ 5 & 1 & -3 \\ 7 & 3 & 1 \end{bmatrix}\right) = -4 \cdot \det\left(\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}\right) - 0 \cdot \det\left(\begin{bmatrix} 5 & -3 \\ 7 & 1 \end{bmatrix}\right) + 0 \cdot \det\left(\begin{bmatrix} 5 & 1 \\ 7 & 3 \end{bmatrix}\right)$$

$$\begin{aligned} &= -4 \cdot [(1 \cdot 1) - (-3 \cdot 3)] - 0 + 0 \\ &= -40 \end{aligned}$$

Since the determinant is not 0, the matrix is invertible.

3. Introduction to Eigenvalues and Eigenvectors

Learning Goal: *Practice calculating eigenvalues and eigenvectors. The importance of eigenvalues and eigenvectors will become clear in the following problems.*

For each of the following matrices, find their eigenvalues and the corresponding eigenvectors. For simple matrices, you may do this by inspection if you prefer.

(a) $\mathbf{A} = \begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix}$

Solution:

Here, it is hard to guess the answers.

$$\det\left(\begin{bmatrix} 22-\lambda & 6 \\ 6 & 13-\lambda \end{bmatrix}\right) = 0$$

$$\begin{aligned} (22-\lambda)(13-\lambda) - 36 &= 0 \\ 250 - 35\lambda + \lambda^2 &= 0 \\ (\lambda - 10)(\lambda - 25) &= 0 \\ \implies \lambda &= 10, 25 \end{aligned}$$

$\lambda = 10$:

$$\mathbf{A}\vec{x} = 10\vec{x} \implies (\mathbf{A} - 10\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 12 & 6 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 2x + y = 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -2x \end{bmatrix}$$

where x is a free variable.

Any vector that lies in $\text{span}\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\}$ is an eigenvector with corresponding eigenvalue $\lambda = 10$.

$\lambda = 25$:

$$\mathbf{A}\vec{x} = 25\vec{x} \implies (\mathbf{A} - 25\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix} - \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} -3 & 6 \\ 6 & -12 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 2y = x \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ y \end{bmatrix}$$

where y is a free variable.

Any vector that lies in $\text{span}\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda = 25$.

(b) $\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

Solution:

Self-grading note: For this subproblem and the following subproblems which involve computing eigenvectors, give yourself full credit if the eigenvector(s) you calculated is/are a scaled (i.e, multiplied by a real valued α) version of the eigenvector(s) given in the solutions.

There are two ways to do this.

First, we can do it by inspection. We can see that this matrix multiplies everything in the first coordinate by 5 and everything in the second by 2. Consequently, when given $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, it will return 2 times the input. And when given $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, it will return 5 times the input vector.

Alternatively, we can use determinants.

$$\det(\begin{bmatrix} 5-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}) = 0$$

$$(5-\lambda)(2-\lambda) - 0 = 0$$

This is already factored for you! We see that, by definition, diagonal matrices have their eigenvalues on the diagonal.

$\lambda = 5$:

$$\mathbf{A}\vec{x} = 5\vec{x} \implies (\mathbf{A} - 5\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies y = 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

where x is a free variable.

Any vector in $\text{span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$ is an eigenvector of the matrix with corresponding eigenvalue $\lambda = 5$.

$\lambda = 2$:

$$\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x = 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$$

where y is a free variable.

Any vector in $\text{span}\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ is an eigenvector of the matrix with corresponding eigenvalue $\lambda = 2$.

(c) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Solution:

We can explicitly calculate:

$$\det(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}) = 0$$

$$(1-\lambda)(4-\lambda) - 4 = 0$$

$$\lambda^2 - 5\lambda = 0 \implies \lambda(\lambda - 5) = 0$$

$$\lambda = 0, 5$$

$\lambda = 0$:

$$\mathbf{A}\vec{x} = 0\vec{x} \implies \mathbf{A}\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x = -2y \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2y \\ y \end{bmatrix}$$

where y is a free variable.

Any vector that lies in $\text{span}\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda = 0$.

$\lambda = 5$:

$$\mathbf{A}\vec{x} = 5\vec{x} \implies (\mathbf{A} - 5\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} - \begin{bmatrix}5 & 0\\0 & 5\end{bmatrix}\right)\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies \begin{bmatrix}-4 & 2\\2 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$
$$\implies \begin{bmatrix}2 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies y = 2x \implies \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\2x\end{bmatrix}$$

where x is a free variable.

Any vector that lies in span$\{\begin{bmatrix}1\\2\end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda = 5$.

Alternatively, this can also be seen by inspection. The matrix is not invertible since the first two rows are linearly dependent. Therefore, there must be a 0 eigenvalue. This has the eigenvector $\begin{bmatrix}-2\\1\end{bmatrix}$, which belongs in the **nullspace of the matrix**.

The other eigenvector can be seen by noticing that the second row is twice the first. So $\begin{bmatrix}1\\2\end{bmatrix}$ is a good guess to try and indeed it works with $\lambda = 5$.

(d) Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a general square matrix. Show that the set of eigenvectors corresponding to a particular eigenvalue of $\mathbf{A}$ is a subspace of $\mathbb{R}^n$. In other words, show that

$$\{\vec{x} \in \mathbb{R}^n : \mathbf{A}\vec{x} = \lambda\vec{x}, \lambda \in \mathbb{R}\}$$

is a subspace. You have to show that all three properties of a subspace (as mentioned in Note 8) hold.

Solution:

Recall the definition of a matrix subspace from Note 8. A subspace $\mathbb{U}$ consists of a subset of the vector space $\mathbb{V}$ if it contains the zero vector, is closed under scalar multiplication, and is closed under vector addition.

i. Zero vector: The zero vector is contained in this set since $\mathbf{A}\vec{0} = \vec{0} = \lambda\vec{0}$.

ii. Scalar multiplication: Let $\vec{v}_1$ be a member of the set. Let $\vec{u} = \alpha\vec{v}_1$. Note that $\vec{u} \in \mathbb{R}^n$, thus a possible value of $\vec{x}$. Now, $\mathbf{A}\vec{u} = \mathbf{A}\alpha\vec{v}_1 = \alpha\mathbf{A}\vec{v}_1 = \alpha\lambda\vec{v}_1 = \lambda\vec{u}$. Hence, $\vec{u}$ is a member of the set as well and the set is closed under scalar multiplication.

iii. Vector addition: Let $\vec{v}_1$ and $\vec{v}_2$ be members of the set. Observe below that the set is closed under vector addition as well.

$$\mathbf{A}(\vec{v}_1 + \vec{v}_2) = \mathbf{A}\vec{v}_1 + \mathbf{A}\vec{v}_2 = \lambda\vec{v}_1 + \lambda\vec{v}_2 = \lambda(\vec{v}_1 + \vec{v}_2)$$

Note that $\vec{v}_1 + \vec{v}_2$ is also a vector in $\mathbb{R}^n$, which corresponds to how $\vec{x}$ is defined in this setup.

Hence, the set defined in the question satisfies the properties of a subspace and is consequently a subspace of $\mathbb{R}^n$.

4. Properties of Pump Systems - II

Learning Objectives: *This problem builds on the pump examples we have been doing, but is meant to help you learn to do proofs in a step by step fashion. Can you generalize intuition from a simple example?*

We consider a system of reservoirs connected to each other through pumps. An example system is shown below in Figure 1, represented as a graph. Each node in the graph is marked with a letter and represents a reservoir. Each edge in the graph represents a pump which moves a fraction of the water from one reservoir to the next at every time step. The fraction of water moved is written on top of the edge.

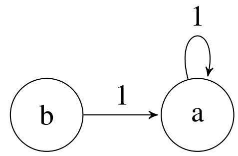


Figure 1: Pump system

We want to prove the following theorem. We will do this step by step.

Theorem: Consider a system consisting of k reservoirs such that the entries of each column in the system's state transition matrix sum to one. If s is the total amount of water in the system at timestep n, then total amount of water at timestep $n+1$ will also be s.

(a) Rewrite the theorem statement for a graph with only two reservoirs.

Solution: Consider a system consisting of 2 reservoirs such that the entries of each column in the system's state transition matrix sum to one. If s is the total amount of water in the system at timestep n, then total amount of water at timestep $n+1$ will also be s.

(b) Since the problem does not specify the transition matrix, let us consider the transition matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and the state vector $\vec{x}[n] = \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix}$. Write out what is "known" or what is given to you in the theorem statement in mathematical form.

Note: In general, it is helpful to write as much out mathematically as you can in proofs. It can also be helpful to draw the transition graph.

Solution: Each column of the transition matrix sums to one:

$$a_{11} + a_{21} = 1, \qquad a_{12} + a_{22} = 1$$

The total amount of water in the system is s at timestep n:

$$x_1[n] + x_2[n] = s$$

We know that the state vector at the next timestep is equal to the transition matrix applied to the state vector at the current timestep:

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n]$$

(c) Now write out the theorem we want to prove mathematically.

Solution: We want to prove that the total amount of water at timestep $n+1$ will also be s:

$$x_1[n+1] + x_2[n+1] = s$$

(d) Prove the statement for the case of two reservoirs. In other words, combine parts b and c to prove the theorem.

Solution: Consider the product $\mathbf{A}\vec{x}[n] = \vec{x}[n+1]$:

$$\mathbf{A}\vec{x}[n] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} = \begin{bmatrix} a_{11}x_1[n] + a_{12}x_2[n] \\ a_{21}x_1[n] + a_{22}x_2[n] \end{bmatrix}$$

Let's consider the sum of the elements in $\vec{x}[n+1]$:

$$\sum_{i=1}^{2} x_i[n+1] = (a_{11}x_1[n] + a_{12}x_2[n]) + (a_{21}x_1[n] + a_{22}x_2[n])$$

Regrouping terms:

$$(a_{11} + a_{21})x_1[n] + (a_{12} + a_{22})x_2[n] = 1 \cdot x_1[n] + 1 \cdot x_2[n] = x_1[n] + x_2[n] = s$$

(e) Now use what you learned to generalize to the case of k reservoirs. *Hint:* Think about $\mathbf{A}$ in terms of its columns, since you have information about the columns.

Solution:

Let $\vec{x}[n] = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}$ be the amount of water in each reservoir at timestep n. We know:

$$x_1[n] + x_2[n] + \cdots + x_k[n] = s$$

Let $\vec{a}_j$ be the j-th column of the state transition matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_k \end{bmatrix}$$

We know that every column of $\mathbf{A}$ sums to one, so we know for all j,

$$a_{1j} + a_{2j} + \cdots + a_{kj} = 1$$

Now, consider the product $\mathbf{A}\vec{x}[n]$:

$$\mathbf{A}\vec{x}[n] = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_k \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ \vdots \\ x_k[n] \end{bmatrix} = x_1[n]\vec{a}_1 + x_2[n]\vec{a}_2 + \cdots + x_k[n]\vec{a}_k = \vec{x}[n+1]$$

Let's consider the sum of the elements in $\vec{x}[n+1]$:

$$\begin{aligned} x_1[n+1] + x_2[n+1] + \cdots + x_k[n+1] = &(a_{11}x_1[n] + a_{12}x_2[n] + \cdots + a_{1k}x_k[n]) \\ &+ (a_{21}x_1[n] + a_{22}x_2[n] + \cdots + a_{2k}x_k[n]) \\ &+ \ldots \\ &+ (a_{k1}x_1[n] + a_{k2}x_2[n] + \cdots + a_{kk}x_k[n]) \end{aligned}$$

Factoring out each element of $x[n]$ gives:

$$\begin{aligned} &x_1[n](a_{11} + a_{21} + \cdots + a_{k1}) + x_2[n](a_{12} + a_{22} + \cdots + a_{k2}) + \cdots + x_k[n](a_{1k} + a_{2k} + \cdots + a_{kk}) \\ &= 1 \cdot x_1[n] + 1 \cdot x_2[n] + \cdots + 1 \cdot x_k[n] = x_1[n] + x_2[n] + \cdots + x_k[n] = s \end{aligned}$$

5. Page Rank

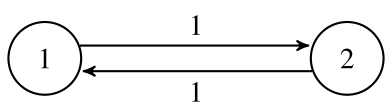
Learning Goal: *This problem highlights the use of transition matrices in modeling dynamical linear systems. Predictions about the steady state of a system can be made using the eigenvalues and eigenvectors of this matrix.*

In homework and discussion, we have discussed the behavior of water flowing in reservoirs and the people flowing in social networks. We now consider the setting of web traffic where the dynamical system can be described with a directed graph, also known as state transition diagram.

As we have seen in lecture and discussion, the "transition matrix", $\mathbf{T}$, can be constructed using the state transition diagram as follows: entries t_{ji} represent the *proportion* of the people who are at website i that click the link for website j.

The **steady-state frequency** (i.e. fraction of visitors in steady-state) for a graph of websites is related to the eigenspace associated with eigenvalue 1 for the "transition matrix" of the graph. Once computed, an eigenvector with eigenvalue 1 will have values which correspond to the steady-state frequency for the fraction of people for each webpage. When the elements of this eigenvector are made to **sum to one** (to conserve population), the i^{th} element of the eigenvector will correspond to the fraction of people on the i^{th} website.

(a) For graph A shown below, what are the steady-state frequencies (i.e. fraction of visitors in steady-state) for the two webpages? Graph A has weights in place to help you construct the transition matrix. Remember to ensure that your steady state-frequencies sum to 1 to maintain conservation.



Graph A

Solution:

To determine the steady-state frequencies for the two pages, we need to find the appropriate eigenvector of the transition matrix. In this case, we are trying to determine the proportion of people who would be on a given page at steady state.

The transition matrix of graph A:

$$\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1}$$

To determine the eigenvalues of this matrix:

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 - 1 = 0 \tag{2}$$

$\lambda = 1, -1$. The steady state vector is the eigenvector that corresponds to $\lambda = 1$. To find the eigenvector,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{3}$$

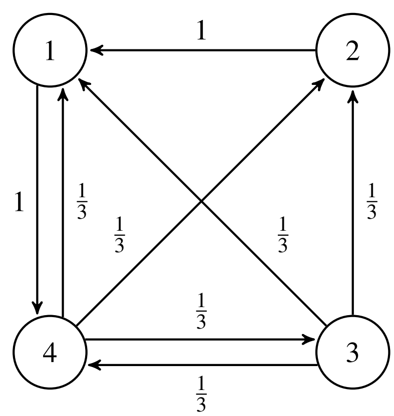
The sum of the values of the vector should equal 1 since the number of people is conserved, so our conditions are:

$$v_1 + v_2 = 1$$
$$v_1 = v_2$$

The steady-state frequency eigenvector is $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ and each webpage has a steady-state frequency of 0.5.

(b) For graph B, what are the steady-state frequencies for the webpages? You may use IPython and the Numpy command `numpy.linalg.eig` for this. Graph B is shown below, with weights in place to help you construct the transition matrix.
Hint: `numpy.linalg.eig` returns eigenvectors and eigenvalues. The eigenvectors are arranged in a matrix in *column-major* order. In other words, given eigenvectors
$\vec{v}_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}$
NumPy will return:

$$\begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} \tag{4}$$



Graph B

Solution:
To determine the steady-state frequencies, we need to create the transition matrix $\mathbf{T}$ first.

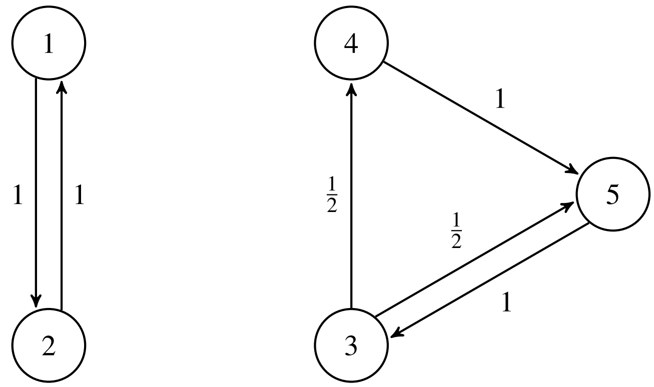
$$\mathbf{T} = \begin{bmatrix} 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} & 0 \end{bmatrix}$$

One possible eigenvector associated with eigenvalue 1 is $\begin{bmatrix} -0.61 & -0.31 & -0.23 & -0.69 \end{bmatrix}^T$ (found using IPython). Scaling it by

$$\frac{1}{(-0.61 + (-0.31) + (-0.23) + (-0.69))}$$

so the elements sum to 1, we get $\begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{8} & \frac{3}{8} \end{bmatrix}^T$
These are the steady-state frequencies for the pages.

(c) Graph C with weights in place is shown below. Find the eigenspace that corresponds to the steady-state for graph C. How many independent systems (disjoint sets of webpages) are there in graph C versus in graph B? What is the dimension of the eigenspace corresponding to the steady-state for graph C? You may use IPython to compute the eigenvalues and eigenvectors again.



Graph C

Solution:

The transition matrix for graph C is

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 \end{bmatrix}$$

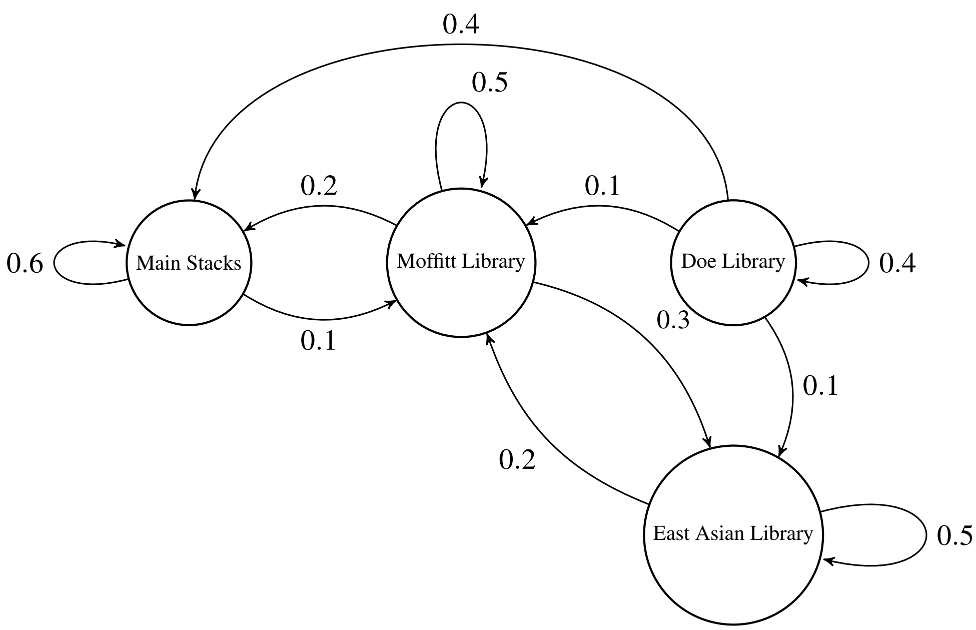
Using IPython, we find that the eigenspace associated with $\lambda = 1$ is spanned by the vectors $\begin{bmatrix} 0 & 0 & 0.4 & 0.2 & 0.4 \end{bmatrix}^T$ and $\begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \end{bmatrix}^T$. While any linear combination of these vectors is an eigenvector, these two particular vectors have a nice interpretation.

The first eigenvector describes the steady-state frequencies for the last three webpages, and the second vector describes the steady-state frequencies for the first two webpages. This makes sense since there are essentially "two internets", or two disjoint sets of webpages. Surfers cannot transition between the two, so you cannot assign steady-state frequencies to webpage 1 and webpage 2 relative to the rest. This is why the eigenspace corresponding to the steady-state has dimension 2.

Assuming that each set of steady-state frequencies needs to add to 1, the first assigns steady-state frequencies of 0.4, 0.2, 0.4 to webpage 3, webpage 4, and webpage 5, respectively. The second assigns steady-state frequencies of 0.5 to both webpage 1 and webpage 2.

6. Favorite Study Spots in Berkeley

Berkeley students are some of the most studious in the nation! Thus, it is not uncommon to find them studying in various spots on campus. Due to class schedules, students often move to different libraries based on their proximity to different classes. The flow of students across the four most popular libraries is as follows:



Let the number of students at the libraries be represented in the following way:

$$A \begin{bmatrix} x_{ML}[t] \\ x_{DL}[t] \\ x_{MS}[t] \\ x_{EAL}[t] \end{bmatrix} = \begin{bmatrix} x_{ML}[t+1] \\ x_{DL}[t+1] \\ x_{MS}[t+1] \\ x_{EAL}[t+1] \end{bmatrix}$$

(a) Write the transition matrix A corresponding to the diagram above. **Solution:**

$$\text{A} = \begin{bmatrix} 0.5 & 0.1 & 0.1 & 0.2 \\ 0 & 0.4 & 0 & 0 \\ 0.2 & 0.4 & 0.6 & 0 \\ 0.3 & 0.1 & 0 & 0.5 \end{bmatrix}$$

(b) Determine if the transition matrix is conservative or not. Explain why or why not either conceptually or mathematically.

Solution:
The definition of a conservative transition matrix implies that its columns must add up to 1. This is not the case for the third and fourth columns of the transition matrix; thus, we can conclude that the transition matrix is not conservative.

(c) For a research project, your friend wants to predict the number of students studying in these libraries in the future. Ignoring your answer from the previous part, use the following transition matrix A and the current number of students given by $\vec{x}[t]$:

$$A = \begin{bmatrix} 0.5 & 0.4 & 0 & 0.3 \\ 0.3 & 0.4 & 0.5 & 0 \\ 0 & 0 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.3 & 0.5 \end{bmatrix} \quad \vec{x}[t] = \begin{bmatrix} 140 \\ 400 \\ 210 \\ 90 \end{bmatrix}$$

Help your friend predict the number of students in each libraries in the next time step.

Solution:

$$x[t+1] = Ax[t] = \begin{bmatrix} 257 \\ 307 \\ 60 \\ 216 \end{bmatrix}$$

(d) You want to expand upon your friend's research. Thus, you tracked the number of students at 2 lesser known libraries (Hangrove Library and Mathematics/Statistics Library) and calculated their corresponding state-transition matrix to form the following model:

$$\begin{bmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{bmatrix} \begin{bmatrix} x_H[t] \\ x_{MS}[t] \end{bmatrix} = \begin{bmatrix} x_H[t+1] \\ x_{MS}[t+1] \end{bmatrix}$$

Given that there are in total 1500 students, determine the number of students in these two libraries after infinite time steps ($\vec{x}[\infty]$). If the answer can not be determined, give a brief explanation why.

Solution: We see that $0.7+0.3=1$ and $0.6+0.4=1$; thus, we know that A is conservative, meaning $\vec{x}[\infty]$ exists. To find the steady-state, we need to calculate the eigenvector of the transition matrix that corresponds to an eigenvalue of 1.

$$(A-\lambda I)\vec{x} = \vec{0}$$
$$\begin{bmatrix} -0.3 & 0.6 \\ 0.3 & -0.6 \end{bmatrix} \vec{x} = \vec{0}$$
$$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Make sure to scale the eigenvector so that the sum of the elements correspond to the total number of students:

$$\vec{x}[\infty] = \begin{bmatrix} 1000 \\ 500 \end{bmatrix}$$

7. Is There A Steady State?

So far, we've seen that for a conservative state transition matrix $\mathbf{A}$, we can find the eigenvector, $\vec{v}$, corresponding to the eigenvalue $\lambda = 1$. This vector is the steady state since $\mathbf{A}\vec{v} = \vec{v}$. However, we've so far taken for granted that the state transition matrix even has the eigenvalue $\lambda = 1$. Let's try to prove this fact.

(a) Show that if λ is an eigenvalue of a matrix $\mathbf{A}$, then it is also an eigenvalue of the matrix $\mathbf{A}^T$.
Hint: The determinants of $\mathbf{A}$ and $\mathbf{A}^T$ are the same. This is because the volumes which these matrices represent are the same.
Solution:

Recall that we find the eigenvalues of a matrix $\mathbf{A}$ by setting the determinant of $\mathbf{A}-\lambda\mathbf{I}$ to 0.

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \det\left((\mathbf{A}-\lambda\mathbf{I})^T\right) = \det\left(\mathbf{A}^T-\lambda\mathbf{I}\right) = 0$$

Since the two determinants are equal, the characteristic polynomials of the two matrices must also be equal. Therefore, they must have the same eigenvalues.

(b) Let a square matrix $\mathbf{A}$ have, for each row, entries that sum to one. Show that $\vec{1} = \begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$ is an eigenvector of $\mathbf{A}$. What is the corresponding eigenvalue?

Solution:

If the rows of $\mathbf{A}$ sum to one, then $\mathbf{A}\vec{1} = \vec{1}$. Therefore, the corresponding eigenvalue is $\lambda = 1$.

We can see this by simple matrix multiplication. For instance, if we look at a 3x3 matrix $\mathbf{A}$ where the first row has elements a_1, a_2, and a_3, such that $a_1 + a_2 + a_3 = 1$. We can perform the matrix multiplication, $\mathbf{A}\cdot\vec{1}$ and analyze the first element in the vector product, which is a product of the first row of $\mathbf{A}$ and $\vec{1}$.

$$\begin{bmatrix}a_1 & a_2 & a_3\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = (1)a_1 + (1)a_2 + (1)a_3$$

$$= a_1 + a_2 + a_3 = 1$$

We see that since the first row of $\mathbf{A}$ will sum to 1, the first element in the matrix multiplication will be 1. Since each row sums to 1, every element in the matrix multiplication will be 1. Thus, the matrix multiplication $\mathbf{A}\vec{1} = \vec{1}$. By inspection, we know that the eigenvalue must be 1.

(c) Let's put it together now. From the previous two parts, show that any conservative state transition matrix will have the eigenvalue $\lambda = 1$. Recall that conservative state transition matrices have, for each column, entries that sum to 1.

Solution:

If we transpose a conservative state transition matrix $\mathbf{A}$, then the rows of $\mathbf{A}^T$ (or the columns of $\mathbf{A}$) sum to one by definition of a conservative system. Then, from part (b), we know that $\mathbf{A}^T$ has the eigenvalue $\lambda = 1$. Furthermore, from part (a), we know that the $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalues, so $\mathbf{A}$ also has the eigenvalue $\lambda = 1$.

8. Traffic Flows (OPTIONAL Practice for Midterm 1)

Learning Objective: *The learning objective of this problem is to see how the concept of nullspaces can be applied to flow problems.*

Your goal is to measure the flow rates of vehicles along roads in a town. It is prohibitively (too) expensive to place a traffic sensor along every road. You realize, however, that the number of cars flowing into an intersection must equal the number of cars flowing out. You can use this "flow conservation" to determine the traffic along all roads in a network by measuring the flow along only some roads. In this problem, we will explore this concept.

(a) Let's begin with a network with three intersections, A, B and C. Define the flow t_1 as the rate of cars (cars/hour) on the road between B and A, flow t_2 as the rate on the road between C and B, and flow t_3 as the rate on the road between C and A.

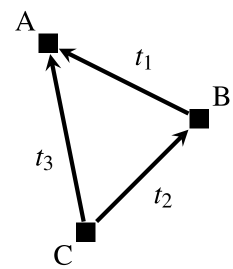


Figure 2: A simple road network.

Note: The directions of the arrows in the figure are the way that we define positive flow by convention. For example, if there were 100 cars per hour traveling from A to C, then $t_3 = -100$. The flows now are not fractions of water per timestep in reservoirs as in the reservoir pumps setting, but numbers of cars per timestep.

We assume the 'flow conservation' constraint: the net number of cars per hour flowing into each intersection is zero. For example at intersection B, we have the constraint $t_2 - t_1 = 0$. The full set of constraints (one per intersection) is:

$$\begin{cases} t_1 + t_3 = 0 \\ t_2 - t_1 = 0 \\ -t_3 - t_2 = 0 \end{cases}$$

As mentioned earlier, we can place sensors on a road to measure the flow through it, but we have a limited budget, and we would like to determine all of the flows with the smallest possible number of sensors.

Suppose for the network above we have one sensor reading, $t_1 = 10$. Can we figure out the flows along the other roads? (That is, the values of t_2 and t_3). If we can, find the values of t_2 and t_3.

Solution:

Yes, since we know that $t_1 = t_2 = -t_3$, so we must have $t_2 = 10$ and $t_3 = -10$.

(b) Now suppose we have a larger network, as shown in Figure 3.

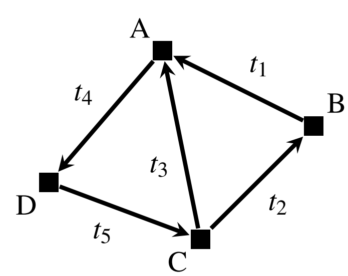


Figure 3: A larger road network.

We would again like to determine the traffic flows on all roads, using measurements from some sensors. A Berkeley student claims that we need two sensors placed on the roads CA (measuring t_3) and DC (measuring t_5). A Stanford student claims that we need two sensors placed on the roads CB (measuring t_2) and BA (measuring t_1). Write out the system of linear equations that represents this flow graph. Is it possible to determine all traffic flows, $\left[t_1, t_2, t_3, t_4, t_5\right]^T$, with the Berkeley student's suggestion? How about the Stanford student's suggestion? *Hint: This can be solved just writing out the relevant equations and reasoning about them.*

Solution: Since we have 4 intersections, we can write 4 linear equations describing the flows into and out of each intersection. We know that the flows into and out of an intersection must sum to 0. The set of linear equations that represents this flow graph is:

$$\begin{cases} t_1 + t_3 - t_4 = 0 \\ t_2 - t_1 = 0 \\ t_5 - t_2 - t_3 = 0 \\ t_4 - t_5 = 0 \end{cases}$$

The Stanford student is wrong. Observing t_1 and t_2 is not sufficient, as t_3, t_4 and t_5 can still not be uniquely determined. Specifically, for any $\alpha \in \mathbb{R}$, the following flow satisfies the constraints and the measurements:

$$t_4 = \alpha$$
$$t_5 = \alpha$$
$$t_3 = \alpha - t_1$$

On the other hand, if we're given t_3 and t_5, we can uniquely solve for all the traffic densities as follows since we know the flow conservation constraints. From the set of linear equations we obtain:

$$t_1 = t_5 - t_3$$
$$t_2 = t_5 - t_3$$
$$t_4 = t_5$$

This is related to the fact that t_3 and t_5 are parts of different loops in the graph, whereas t_1 and t_2 are in the same loop, so measuring both of them would not give additional information.

(c) We would like a more general way of determining the possible traffic flows in a network. Suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. As a first step, let us try to write all the flow conservation constraints (one per intersection) i.e. the system of equations from part (b) as a matrix equation.

Construct a 4×5 matrix $\mathbf{B}$ such that the equation $\mathbf{B}\vec{t} = \vec{0}$:

$$\begin{bmatrix} & & & & \\ & & \mathbf{B} & & \\ & & & & \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

represents the flow conservation constraints for the network in Figure 3.

*Hint: You can construct **B** using only $0, 1,$ and -1 entries. Each row represents the inflow/outflow of an intersection. This matrix is called the **incidence matrix**.*

Solution:

$$\mathbf{B} = \begin{array}{c} \\ \end{array}\begin{array}{cc} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \\ D \end{array} \\ \begin{array}{ccccc} t_1 & t_2 & t_3 & t_4 & t_5 \end{array} & \end{array}$$

The rows of this matrix can be in any order and your solution can differ by a factor of -1. However, the order of the elements within the row is still important and it must match the order of the elements of $\vec{t}$.

Each row represents an intersection, and each column represents a road between two intersections. Each 1 on a row represents a road flowing into an intersection, and each -1 represents a road flowing out of an intersection. Each -1 in a column represents the source intersection of a road (where the arrow starts), and each 1 in a column represents the destination intersection of a road (where the arrow ends).

We expect each column of $\mathbf{B}$ to sum to 0 (and actually have exactly one -1 and one 1) because each road flows into one intersection and out of another.

(d) Again, suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. Then, determine the subspace of all valid traffic flows for the network of Figure 3. Notice that the set of all vectors $\vec{t}$ that satisfy $\mathbf{B}\vec{t} = \vec{0}$ is exactly the nullspace of the matrix $\mathbf{B}$. That is, we can find all valid traffic flows by computing the nullspace of $\mathbf{B}$. Find the nullspace of $\mathbf{B}$. What is its dimension?

Solution:

We use Gaussian Elimination to find the nullspace, i.e. solve the system of equations $\mathbf{B}\vec{t} = \vec{0}$. Because the right hand side of the equation is all zeros, we will omit it from our augmented matrix since Gaussian elimination will never make it nonzero.

$$\begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \overbrace{\Rightarrow}^{R_2 \leftarrow R_2 + R_1} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \overbrace{\Rightarrow}^{R_3 \leftarrow R_2 + R_3} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\overbrace{\Rightarrow}^{R_5 \leftarrow R_4 + R_5} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overbrace{\Rightarrow}^{R_3 \leftarrow (-1) R_3} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overbrace{\Rightarrow}^{R_2 \leftarrow R_2 + R_3} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overbrace{\Rightarrow}^{R_1 \leftarrow R_1 + R_3} \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that t_3 and t_5 are free variables, so we let $t_3 = \alpha$ and $t_5 = \beta$. Then, remembering that the right hand side is all zeros, the equations are:

$$\begin{aligned} t_1 &= \beta - \alpha \\ t_2 &= \beta - \alpha \\ t_3 &= \alpha \\ t_4 &= \beta \\ t_5 &= \beta \end{aligned}$$

So the solutions to $\mathbf{B}\vec{t} = \vec{0}$ can be written as:

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Another way to write the set of solutions would be $span\{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}\}$.

The dimension of the nullspace is 2 because a minimum of 2 vectors are required to span the entire nullspace.

Note: We show here, for your reference, that the space of all possible traffic flows is a subspace. You don't need to include this proof in your solution. Suppose we have a set of valid flows $\vec{t}$. Then, for any intersection, the total flow into it is the same as the total flow out of it. If we scale $\vec{t}$ by a constant a, each t_i will also get scaled by a. The total flows into and out of the intersection would be scaled by the same amount and remain equal to each other. Thus any scaling of a valid flow is still a valid flow. Suppose now we add valid flows $\vec{f}_1$ and $\vec{f}_2$ to get $\vec{t} = \vec{f}_1 + \vec{f}_2$. For any intersection I,

$$\text{total flow into } I = \text{total flow into } I \text{ from } \vec{f}_1 + \text{total flow into } I \text{ from } \vec{f}_2$$
$$\text{total flow out of } I = \text{total flow out of } I \text{ from } \vec{f}_1 + \text{total flow out of } I \text{ from } \vec{f}_2$$

Since the total flow into I from $\vec{f}_1$ is the same as the total flow out of I from $\vec{f}_1$ and similarly for $\vec{f}_2$, the total flow into I is the same as the total flow out of I. Therefore, the sum of any two valid flows is still a valid flow. Also, $\vec{t} = \vec{0}$ is a valid flow. Therefore the set of valid flows forms a subspace.

(e) **[Challenge]** Now let us try to draw some conclusions about more general road networks. Say there is a road network graph G, with incidence matrix $\mathbf{B}_G$. If $\mathbf{B}_G$ has a k-dimensional null space, does this mean measuring the flows along ***any*** k **roads** is always sufficient to recover all of the true flows? In other words, is there ever a possibility of being unable to recover the true flows depending on which k roads you choose?

Hint: Consider the Stanford student's measurement from part (b).

Solution:

No, consider the network of Figure 3. The corresponding incidence matrix has a $k = 2$ dimensional null space, as you showed in part (e). However, measuring t_1 and t_2 (as the Stanford student suggested) is not sufficient, as you showed in part (b).

(f) **[Challenge]** If the incidence matrix $\mathbf{B}_G$ has a k-dimensional null space, does this mean we can **always pick a set of k roads** such that measuring the flows along these roads is sufficient to recover the exact flows? If this is true, explain how you would pick these k roads to guarantee that you could recover the missing information. Otherwise, give a counterexample.

Solution:

Yes.

Let $\mathbf{U}$ be a matrix whose columns form a basis of the null space of $\mathbf{B}_G$, as above. The k columns of $\mathbf{U}$ are linearly independent since they form a basis. Since there are k linearly independent columns, when we run Gaussian elimination on $\mathbf{U}$, we must get k pivots. (Recall that "pivot" is the technical term for being able to row-reduce and turn a column into something that has exactly one 1 in it. The pivot is the entry that we found and turned into that 1.)

Therefore, the row space of $\mathbf{U}$ is k dimensional since there are some k linearly independent rows in $\mathbf{U}$ — namely the ones where we found pivots. Choose to measure the roads corresponding to these rows.

This will work because:

For a given valid flow $\vec{t} = \mathbf{U}\vec{x}$, the results of measuring this flow vector are $\mathbf{U}^{(k)}\vec{x}$, where the matrix $\mathbf{U}^{(k)}$ is some k linearly independent rows of $\mathbf{U}$. By construction, the $k \times k$ matrix $\mathbf{U}^{(k)}$ has all linearly independent rows, so we can invert $\mathbf{U}^{(k)}$ to find $\vec{x}$ from $\mathbf{U}^{(k)}\vec{x}$ and then recover the flows along all the edges as $\mathbf{U}\vec{x}$.

This isn't the only set of k roads that will work. But it does provide a set of k roads that are guaranteed to work.

9. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.

