EECS 16A Designing Information Devices and Systems I
Spring 2023 Homework 4

This homework is due February 17, 2023, at 23:59.
Self-grades are due February 24, 2023, at 23:59.

Submission Format
Your homework submission should consist of **one** file.
`hw4.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).
Submit each file to its respective assignment on Gradescope.

1. Reading Assignment

For this homework, please read Notes 5, 6, 7, and 8. Note 5 provides an overview of multiplication of matrices with vectors, by considering the example of water reservoirs and water pumps. Note 6 introduces matrix inversion. Notes 7 and 8 give an overview of matrix vector spaces and subspaces, as well as column spaces and nullspaces. You are always welcome and encouraged to read beyond this as well.

Please answer the following questions:

(a) You have seen in Note 5 that the pump system can be represented by a state transition matrix. What constraint must this matrix satisfy in order for the pump system to obey water conservation?

(b) From Note 8, what are the three necessary properties for a vector space to be a *subspace*?

Solution:

(a) Each column in the state transition matrix must sum to one.

(b) See Note 8 for definition of a subspace

i. Contains the zero vector: $\vec{0} \in \mathbb{U}$.

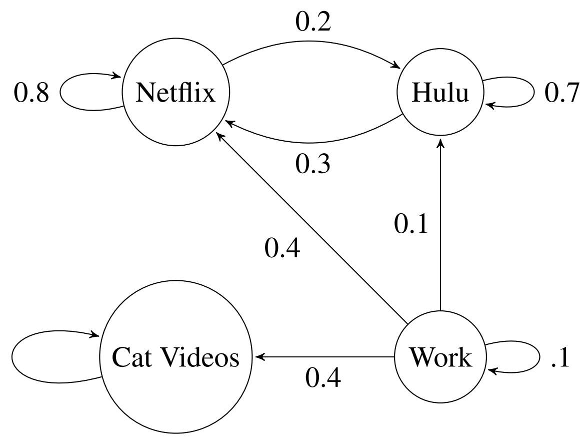
ii. Closed under vector addition: For any two vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{U}$, their sum $\vec{v}_1 + \vec{v}_2$ must also be in $\mathbb{U}$.

iii. Closed under scalar multiplication: For any vector $\vec{v} \in \mathbb{U}$ and scalar $\alpha \in \mathbb{R}$, the product $\alpha\vec{v}$ must also be in $\mathbb{U}$.

2. Social Media

Learning Objective: *Practice setting up transition matrices from a diagram and understand how to compute subsequent states of the system.*

As a tech-savvy Berkeley student, the distractions of streaming services are always calling you away from productive stuff like homework for your classes. You're curious—are you the only one who spends hours switching between Netflix or Hulu? How do other students manage to get stuff done and balance staying up to date with the Bachelor? You conduct an experiment, collect some data, and notice Berkeley students tend to follow a pattern of behavior similar to the figure below. So, for example, if $x = 100$ students are on Netflix, in the next timestep, 20 (i.e., $0.2 \cdot x$) of them will click on a link and move to Hulu, and 80 (i.e, $0.8 \cdot x$) will remain on Netflix.



(a) Let us define $x_N[n]$ as the number of students on Netflix at time-step n; $x_H[n]$ as the number of students on Hulu at time-step n; $x_C[n]$ as the number of students watching any kind of cat video at time-step n; and $x_W[n]$ as the number of students working at time-step n.

Let the state vector be: $\vec{x}[n] = \begin{bmatrix} x_N[n] \\ x_H[n] \\ x_C[n] \\ x_W[n] \end{bmatrix}$. Derive the corresponding transition matrix $\mathbf{A}$.

Hint: A transition matrix, $\mathbf{A}$, is the matrix that transitions $\vec{x}[n]$, the vector at time-step n to $\vec{x}[n+1]$, the vector at time-step $n+1$. In other words: $\vec{x}[n+1] = \mathbf{A}\vec{x}[n]$.

Solution: Let us explicitly write the transition equation for each state and then we can use these to identify the state transition matrix.

$$\begin{aligned} x_N[n+1] &= 0.8 \cdot x_N[n] + 0.3 \cdot x_H[n] + 0.4 \cdot x_W[n] \\ x_H[n+1] &= 0.2 \cdot x_N[n] + 0.7 \cdot x_H[n] + 0.1 \cdot x_W[n] \\ x_C[n+1] &= 1 \cdot x_C[n] + 0.4 \cdot x_W[n] \\ x_W[n+1] &= 0.1 \cdot x_W[n] \end{aligned}$$

If $\vec{x}[n] = \begin{bmatrix} x_N[n] \\ x_H[n] \\ x_C[n] \\ x_W[n] \end{bmatrix}$, then for $\vec{x}[n+1] = \mathbf{A}\vec{x}[n]$ we can identify the state transition matrix $\mathbf{A}$ as

$$\begin{bmatrix} 0.8 & 0.3 & 0 & 0.4 \\ 0.2 & 0.7 & 0 & 0.1 \\ 0 & 0 & 1 & 0.4 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}$$

(b) There are 320 of you in the class. Suppose on a given Friday evening (the day when HW is due), there are 110 EECS16A students on Netflix, 60 on Hulu, 10 watching Cat Videos, and 140 actually doing work. In the next timestep, how many people will be doing each activity? In other words, after you apply the matrix once to reach the next timestep, what is the state vector?

Solution:

In order to calculate the state vector at the next timestep, we can use the equation $\vec{x}[n+1] = \mathbf{A}\vec{x}[n]$. Substituting the values for A and $\vec{x}[n]$, we get the following:

$$\begin{bmatrix} 0.8 & 0.3 & 0 & 0.4 \\ 0.2 & 0.7 & 0 & 0.1 \\ 0 & 0 & 1 & 0.4 \\ 0 & 0 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} 110 \\ 60 \\ 10 \\ 140 \end{bmatrix} = \begin{bmatrix} 162 \\ 78 \\ 66 \\ 14 \end{bmatrix}$$

(c) Compute the sum of each column in the state transition matrix. What is the interpretation of this?

Solution:

Since each column's sum is equal to 1, the system is conservative. This means that we aren't losing (or gaining) students after each time step and the total number of students remains constant.

3. Inverse Transforms

Learning Objectives: *Matrices represent linear transformations, and their inverses (if they exist) represent the opposite transformation. Here we practice inversion, but are also looking to develop an intuition. Visualizing the transformations might help develop this intuition.*

For each of the following choices of matrix A:

i. Find the inverse, $\mathbf{A}^{-1}$, if it exists. If you find that the inverse does not exist, mention how you decided that. Solve this by hand.

ii. **For parts (a)-(d) only**, in addition to finding the inverse (if it exists), describe how the matrix $\mathbf{A}$ geometrically transforms an arbitrary vector $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in \mathbb{R}^2$.

For example, if $\mathbf{A} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 2x_0 \\ 2y_0 \end{bmatrix}$, then $\mathbf{A}$ could scale $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ by 2 to get $\begin{bmatrix} 2x_0 \\ 2y_0 \end{bmatrix}$. If $\mathbf{A} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ -y_0 \end{bmatrix}$, then $\mathbf{A}$ could reflect $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ across the x-axis, etc. *Hint: It may help to plot a few examples to recognize the pattern.*

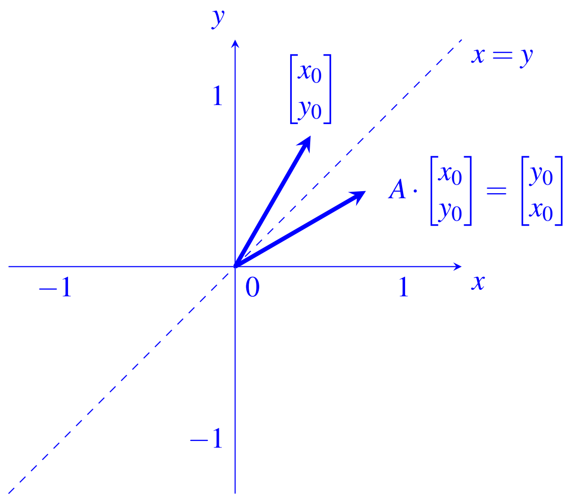
iii. **Again, for parts (a)-(d) only**, if we use $\mathbf{A}$ to geometrically transform $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ to get $\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, **is it possible to reverse the transformation geometrically,** i.e. is it possible to retrieve $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ from $\begin{bmatrix} u \\ v \end{bmatrix}$ geometrically?

(a) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Solution:

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{swap } R_1, R_2} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

The inverse does exist: $\mathbf{A}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



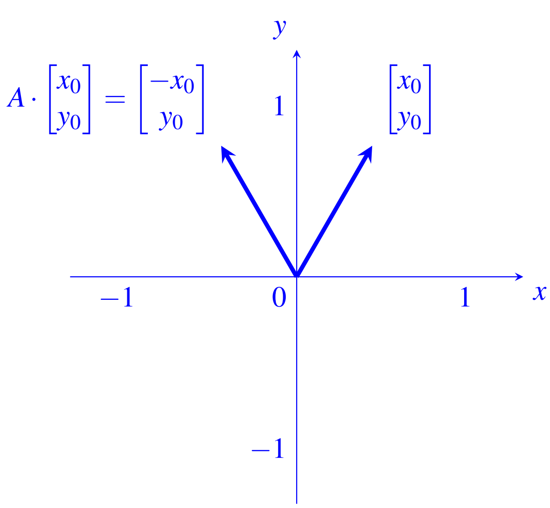
The original matrix $\mathbf{A}$ flips the x and y components of the vector. Any correct equivalent sequence of operations (such as reflecting the vector across the $x = y$ line) warrants full credit. Notice how the inverse does the exact same thing–that is, it switches the x and y components of the vector it's applied to. This makes sense—switching x and y twice on a vector $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ gives us the same vector $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$. So the transformation done by $\mathbf{A}$ is reversible.

(b) $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{cc|cc} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-R_1 \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

The inverse does exist: $\mathbf{A}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$



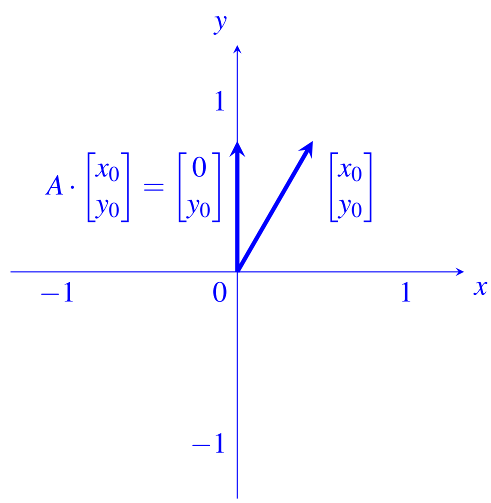
The original matrix $\mathbf{A}$ reflects the vector across the y-axis, i.e. it multiplies the vector's x-component by a factor of -1. Reflecting the vector across the y-axis again with $\mathbf{A}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ will give you the original vector, i.e. the transformation done by $\mathbf{A}$ is reversible.

(c) $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{\text{swap } R_1, R_2} \left[\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

We see here that *the inverse does <u>not</u> exist* because the second row represents an inconsistent equation. Another way to see that the inverse does not exist is by realizing that the first column (and first row) of the original matrix are the zero vector, so the columns are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.



The original matrix $\mathbf{A}$ removes the x-component of the vector it's applied to and keeps the same y-component. Graphically speaking, this matrix can be thought of as taking the "shadow" of the vector on the y-axis if you were to shine a light perpendicular to the y-axis.

Since the x-component of the vector is completely lost after the transformation, the process is not reversible.

(d) $\mathbf{A}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ assuming $\cos\theta \neq 0$. *Hint:* Recall $\cos^2\theta + \sin^2\theta = 1$.

In addition to answering subparts i., ii., and iii. (using three-dimensional vector $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \in \mathbb{R}^3$), also prove $\mathbf{A}(\theta)^{-1} = \mathbf{A}(-\theta)$.

Solution:

$$\left[\begin{array}{ccc|ccc} \cos\theta & -\sin\theta & 0 & 1 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{\cos\theta}R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & -\frac{\sin\theta}{\cos\theta} & 0 & \frac{1}{\cos\theta} & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 - \sin\theta \cdot R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & -\frac{\sin\theta}{\cos\theta} & 0 & \frac{1}{\cos\theta} & 0 & 0 \\ 0 & \cos\theta + \frac{\sin^2\theta}{\cos\theta} & 0 & -\frac{\sin\theta}{\cos\theta} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\cos^2\theta + \sin^2\theta = 1} \left[\begin{array}{ccc|ccc} 1 & -\frac{\sin\theta}{\cos\theta} & 0 & \frac{1}{\cos\theta} & 0 & 0 \\ 0 & \frac{1}{\cos\theta} & 0 & -\frac{\sin\theta}{\cos\theta} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\cos\theta \cdot R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & -\frac{\sin\theta}{\cos\theta} & 0 & \frac{1}{\cos\theta} & 0 & 0 \\ 0 & 1 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

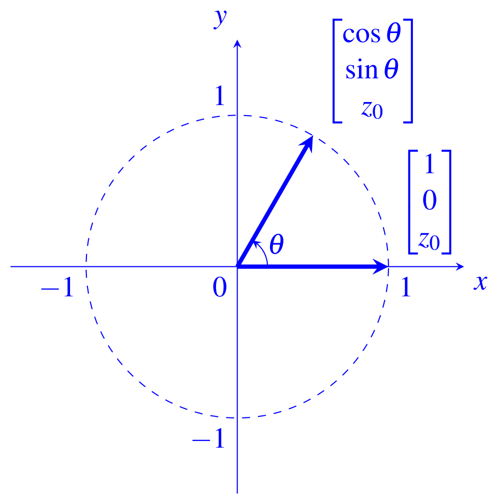
$$\xrightarrow{R_1 + \frac{\sin\theta}{\cos\theta} R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{\cos\theta} - \frac{\sin^2\theta}{\cos\theta} & \sin\theta & 0 \\ 0 & 1 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\cos^2\theta + \sin^2\theta = 1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 1 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

The inverse does exist: $\mathbf{A}(\theta)^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The original matrix $\mathbf{A}(\theta)$ is similar to the two-dimensional rotation matrix seen in discussion 3B. This three-dimensional rotation matrix rotates a vector in the counter-clockwise direction in the xy-plane, and its inverse rotates a vector in the clockwise direction in the xy-plane. So the transformation done by $\mathbf{A}(\theta)$ is a reversible process. Take the vector $\begin{bmatrix} 1 \\ 0 \\ z_0 \end{bmatrix}$ for example:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ z_0 \end{bmatrix}$$



Let's prove the inverse matrix $\mathbf{A}(\theta)^{-1}$ can also be found from the rotation matrix that rotates a vector by an angle $-\theta$.

$$\mathbf{A}(-\theta) = \begin{bmatrix}\cos(-\theta) & -\sin(-\theta) & 0\\ \sin(-\theta) & \cos(-\theta) & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} = \mathbf{A}(\theta)^{-1}$$

(e) $\mathbf{A} = \begin{bmatrix}1 & 2\\ 2 & 0\end{bmatrix}$

Solution:

We can use Gaussian elimination to find the inverse of the matrix.

$$\left[\begin{array}{cc|cc}1 & 2 & 1 & 0\\ 2 & 0 & 0 & 1\end{array}\right] \xrightarrow{R_1\to R_2, R_2\to R_1} \left[\begin{array}{cc|cc}2 & 0 & 0 & 1\\ 1 & 2 & 1 & 0\end{array}\right]$$
$$\xrightarrow{\frac{1}{2}R_1\to R_1} \left[\begin{array}{cc|cc}1 & 0 & 0 & \frac{1}{2}\\ 1 & 2 & 1 & 0\end{array}\right]$$
$$\xrightarrow{R_2-R_1\to R_2} \left[\begin{array}{cc|cc}1 & 0 & 0 & \frac{1}{2}\\ 0 & 2 & 1 & -\frac{1}{2}\end{array}\right]$$
$$\xrightarrow{\frac{1}{2}R_2\to R_2} \left[\begin{array}{cc|cc}1 & 0 & 0 & \frac{1}{2}\\ 0 & 1 & \frac{1}{2} & -\frac{1}{4}\end{array}\right]$$

The inverse does exist: $\mathbf{A}^{-1} = \begin{bmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{4}\end{bmatrix}$.

(f) $\mathbf{A} = \begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 2\\ 1 & 4 & 4\end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 1 & 4 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_1 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 4 & 4 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right]$$

The inverse does not exist because the last equation is inconsistent. That is, we have a row of zeros on the left hand side, corresponding to which there is no row of zeros on the right hand side. An alternative reason is that the second and third columns are equal, i.e., they are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.

(g) $\mathbf{A} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

Solution:

We can use Gaussian elimination to find the inverse of the matrix.

$$\left[\begin{array}{ccc|ccc} -1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 2 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_1 + R_2 \to R_1, R_2 + R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{array}\right]$$

The inverse does exist: $\mathbf{A}^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$

(h) **(OPTIONAL)** $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3+R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

The inverse does not exist because the last equation is inconsistent. That is, we have a row of zeros on the left hand side, corresponding to which there is no row of zeros on the right hand side. An alternative reason is that the third column is the negative of the second column, i.e., they are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.

(i) **(OPTIONAL)** $\mathbf{A} = \begin{bmatrix} 3 & 0 & -2 & 1 \\ 0 & 2 & 1 & 3 \\ 3 & 1 & 0 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Hint 1: What do the linear (in)dependence of the rows and columns tell us about the invertibility of a matrix?

Hint 2: We're reasonable people!

Solution:

Inverse does not exist because $\text{column}_1 + \text{column}_2 + \text{column}_3 = \text{column}_4$, which means that the columns are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.

4. Subspaces, Bases and Dimension

Learning Objective*: Explore how to recognize and show if a subset of a vector space is or is not a subspace. Further practice identifying a basis for (i.e., a minimal set of vectors which span) an arbitrary subspace.*

For each of the sets $\mathbb{U}$ (which are *subsets* of $\mathbb{R}^3$) defined below, state whether $\mathbb{U}$ is a *subspace* of $\mathbb{R}^3$ or not. If $\mathbb{U}$ is a subspace, find a basis for it and state the dimension.

Note:

- To show $\mathbb{U}$ is a subspace, you have to show that all three properties of a subspace hold.
- To show $\mathbb{U}$ is not a subspace, you only have to show at least one property of a subspace does not hold.

(a) $\mathbb{U} = \left\{ \begin{bmatrix} 2(x+y) \\ x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

Solution: We test the three properties of a subspace:

i. Let $\vec{v}_1 = \begin{bmatrix} 2(x_1+y_1) \\ x_1 \\ y_1 \end{bmatrix}$ be a member of the set $\mathbb{U}$. Assume $\vec{v}_2 = \alpha \vec{v}_1$, where α is a scalar. Here

$$\vec{v}_2 = \alpha \vec{v}_1 = \alpha \begin{bmatrix} 2(x_1+y_1) \\ x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2(\alpha x_1+\alpha y_1) \\ \alpha x_1 \\ \alpha y_1 \end{bmatrix} = \begin{bmatrix} 2(x_i+y_i) \\ x_i \\ y_i \end{bmatrix},$$

where $x_i = \alpha x_1$ and $y_i = \alpha y_1$. Hence, $\vec{v}_2 = \alpha \vec{v}_1$ is a member of the set as well and the set is closed under scalar multiplication.

ii. Let $\vec{v_1} = \begin{bmatrix} 2(x_1+y_1) \\ x_1 \\ y_1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 2(x_2+y_2) \\ x_2 \\ y_2 \end{bmatrix}$ be members of the set $\mathbb{U}$. Now, let us assume $\vec{v_3} = \vec{v_1} + \vec{v_2}$:

$$\vec{v_3} = \vec{v_1} + \vec{v_2} = \begin{bmatrix} 2(x_1+y_1)+2(x_2+y_2) \\ x_1+x_2 \\ y_1+y_2 \end{bmatrix} = \begin{bmatrix} 2(x_1+x_2+y_1+y_2) \\ x_1+x_2 \\ y_1+y_2 \end{bmatrix} = \begin{bmatrix} 2(x_3+y_3) \\ x_3 \\ y_3 \end{bmatrix},$$

where $x_3 = x_1+x_2$ and $y_3 = y_1+y_2$ Hence, $\vec{v_3}$ is a member of the set as well and the set is closed under vector addition.

iii. Let $\vec{v_0} = \begin{bmatrix} 2(x_0+y_0) \\ x_0 \\ y_0 \end{bmatrix}$ be a member of the set, where we choose $x_0 = 0$ and $y_0 = 0$. So the vector $\vec{v_0} = \begin{bmatrix} 2(0+0) \\ 0 \\ 0 \end{bmatrix} = \vec{0}$. So the zero vector is contained in this set.

Hence we can decide that $\mathbb{U}$ is a subspace of $\mathbb{R}^3$. Any vector in the subspace can be written as:

$$\begin{bmatrix} 2(x+y) \\ x \\ y \end{bmatrix} = x\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},$$

where x and y are free variables. So $\mathbb{U}$ can be expressed as span $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$. Hence the basis is given by the set: $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$. Dimension = 2.

(b) $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ z+1 \end{bmatrix} \mid x,y,z \in \mathbb{R} \right\}$

Solution:

Again we check the three properties of a subspace:

i. Now let $\vec{v_1} = \begin{bmatrix} x_1 \\ y_1 \\ z_1+1 \end{bmatrix}$ be a member of the set $\mathbb{U}$. Assume $\vec{v_2} = \alpha\vec{v_1}$, where α is a scalar. Here

$$\vec{v_2} = \alpha\vec{v_1} = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha z_1 + \alpha \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ (\alpha z_1 + \alpha - 1) + 1 \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \\ z_i + 1 \end{bmatrix},$$

where $x_i = \alpha x_1$, $y_i = \alpha y_1$ and $z_i = \alpha z_1 + \alpha - 1$. Hence, $\vec{v_2} = \alpha\vec{v_1}$ is a member of the set as well and the set is closed under scalar multiplication.

ii. Let $\vec{v_1} = \begin{bmatrix} x_1 \\ y_1 \\ z_1+1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} x_2 \\ y_2 \\ z_2+1 \end{bmatrix}$ be members of the set $\mathbb{U}$. Now, let us assume $\vec{v_3} = \vec{v_1} + \vec{v_2}$:

$$\vec{v_3} = \vec{v_1} + \vec{v_2} = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2+2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ (z_1+z_2+1)+1 \end{bmatrix} = \begin{bmatrix} x_3 \\ y_3 \\ z_3+1 \end{bmatrix},$$

where $x_3 = x_1 + x_2$, $y_3 = y_1 + y_2$ and $z_3 = z_1 + z_2 + 1$. Hence, $\vec{v_3}$ is a member of the set as well and the set is closed under vector addition.

iii. Let $\vec{v_0} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 + 1 \end{bmatrix}$ be a member of the set, where we choose $x_0 = 0$, $y_0 = 0$ and $z_0 = -1$. So the vector $\vec{v_0} = \begin{bmatrix} 0 \\ 0 \\ -1+1 \end{bmatrix} = \vec{0}$. So the zero vector is contained in this set.

Hence we can decide that $\mathbb{U}$ is a subspace of $\mathbb{R}^3$. Any vector in the subspace can be written as:

$$\begin{bmatrix} x \\ y \\ z+1 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (z+1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z_{new} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where x, y and $z_{new} = z + 1$ are free variables. So $\mathbb{U}$ can be expressed as span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Hence the basis is given by the set: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. The dimension is 3, which makes $\mathbb{U}$ the same as $\mathbb{R}^3$.

(c) $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ x+1 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

Solution: Again we check the three properties of a subspace:

i. Now let $\vec{v_1} = \begin{bmatrix} x_1 \\ y_1 \\ x_1 + 1 \end{bmatrix}$ be a member of the set $\mathbb{U}$. Assume $\vec{v_2} = \alpha \vec{v_1}$, where α is a scalar. Here

$$\vec{v_2} = \alpha \vec{v_1} = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha x_1 + \alpha \end{bmatrix} \neq \begin{bmatrix} x_i \\ y_i \\ x_i + 1 \end{bmatrix},$$

where $x_i = \alpha x_1$ and $y_i = \alpha y_1$. Hence, $\vec{v_2} = \alpha \vec{v_1}$ is not a member of the set and the set is not closed under scalar multiplication.

ii. Let $\vec{v_1} = \begin{bmatrix} x_1 \\ y_1 \\ x_1 + 1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} x_2 \\ y_2 \\ x_2 + 1 \end{bmatrix}$ be members of the set $\mathbb{U}$. Now, let us assume $\vec{v_3} = \vec{v_1} + \vec{v_2}$:

$$\vec{v_3} = \vec{v_1} + \vec{v_2} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ x_1 + x_2 + 2 \end{bmatrix} \neq \begin{bmatrix} x_3 \\ y_3 \\ x_3 + 1 \end{bmatrix},$$

where $x_3 = x_1 + x_2$, and $y_3 = y_1 + y_2$. Hence, $\vec{v_3}$ is not a member of the set and the set is not closed under vector addition.

iii. Let $\vec{v_0} = \begin{bmatrix} x_0 \\ y_0 \\ x_0 + 1 \end{bmatrix}$ be a member of the set. The first and third elements cannot both be zero regardless of the value chosen for x_0. So the zero vector is not contained in this set.

Hence we can decide that $\mathbb{U}$ is not a subspace of $\mathbb{R}^3$. **Note that for full credit you only have to show that one of the properties is violated, you don't have to show all three.**

5. Finding Null Spaces and Column Spaces

Learning Objectives: *Null spaces and column spaces are two fundamental vector spaces associated with matrices and they describe important attributes of the transformations that these matrices represent. This problem explores how to find and express these spaces.*

Definition (Null space): The null space of a matrix, $\mathbf{A} \in \mathbb{R}^{m \times n}$, is the set of all vectors $\vec{x} \in \mathbb{R}^n$ such that $\mathbf{A}\vec{x} = \vec{0}$. The null space is notated as $Null(\mathbf{A})$ and the definition can be written in set notation as:

$$Null(\mathbf{A}) = \{\vec{x} \mid \mathbf{A}\vec{x} = \vec{0}, \vec{x} \in \mathbb{R}^n\}$$

Definition (Column space): The column space of a matrix, $\mathbf{A} \in \mathbb{R}^{m \times n}$, is the set of all vectors $\mathbf{A}\vec{x} \in \mathbb{R}^m$ for all choices of $\vec{x} \in \mathbb{R}^n$. Equivalently, it is also the span of the column vector of $\mathbf{A}$. The column space can be notated as $Col(\mathbf{A})$or $Range(\mathbf{A})$ and the definition can be written in set notation as:

$$Col(\mathbf{A}) = \{\mathbf{A}\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

Definition (Dimension): The dimension of a vector space is the number of basis vectors — i.e. the minimum number of vectors required to span the vector space.

(a) Consider a matrix $\mathbf{A} \in \mathbb{R}^{3 \times 5}$. What is the maximum possible number of linearly independent column vectors (i.e. the maximum possible dimension) of $Col(\mathbf{A})$?

Solution:

There are a maximum of 3 linearly independent columns of A.

If you are stuck solving a problem like this, consider concrete examples. We want to find the maximum possible number of linearly independent column vectors, so we look for examples and check if we can exceed certain values.

Consider the following example matrix, where the entries marked with $*$ are arbitrary values:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \end{bmatrix}$$

Here all 5 columns are $\in \mathbb{R}^3$. The first three columns are linearly independent, so at least three linearly independent columns are achievable. The first three columns span $\mathbb{R}^3$, therefore any choice of fourth and fifth columns, also in $\mathbb{R}^3$, can be written as a linear combination of the first three columns. This means that we cannot exceed three linearly independent columns. Thus the maximum number of linearly independent column vectors is 3. In general, if $m < n$, then the columns of $\mathbf{A} \in \mathbb{R}^{m \times n}$ will always be linearly dependent, since you cannot have more than m linearly independent columns in $\mathbb{R}^m$.

(b) You are given the following matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a *minimum* set of vectors that span $Col(\mathbf{A})$ (i.e. a basis for $Col(\mathbf{A})$). (This problem does not have a unique answer, since you can choose many different sets of vectors that fit the description here.) What is the dimension of $Col(\mathbf{A})$?

Hint: You can do this problem by observation. Alternatively, use Gaussian Elimination on the matrix to identify how many columns of the matrix are linearly independent. The columns with pivots (leading ones) in them correspond to the columns in the original matrix that are linearly independent.

Solution: $Col(\mathbf{A})$ is the space spanned by its columns, so the set of all columns is a valid span for $Col(\mathbf{A})$. However, we are asking you to choose a subset of the columns and still span $Col(\mathbf{A})$, as we showed in part (a). To find the minimum number of columns needed and determine the dimension of $Col(\mathbf{A})$, we can remove vectors from the set of columns until we are left with a linearly independent set.

By inspection, the second, fourth, and fifth columns can be omitted from a set of columns as they can be expressed as linear combinations of the first and third columns. Thus the dimension of $\mathbf{A}$ is 2.

One set spanning $Col(\mathbf{A})$ is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Another valid set of vectors which span $Col(\mathbf{A})$ is:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

Note with this second set, none of the columns of $\mathbf{A}$ appear. Despite this, the span of this set will still be equal to $Col(\mathbf{A})$, which for this matrix is the set of all vectors in $\mathbb{R}^3$ with zero third entry. Geometrically, both of these solutions span the same plane, i.e. the xy-plane in the 3D space.

Give yourself full credit if you recognized that the dimension was 2, and if you had a *minimum* set of vectors that spans $Col(\mathbf{A})$.

(c) Find a *minimum* set of vectors that span $Null(\mathbf{A})$ (i.e. a basis for $Null(\mathbf{A})$), where $\mathbf{A}$ is the same matrix as in part (b). What is the dimension of $Null(\mathbf{A})$?

Solution:

Finding $Null(\mathbf{A})$ is the same as solving the following system of linear equations:

$$\begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{aligned} x_1 + x_2 - 2x_4 + 3x_5 &= 0 \\ x_3 - x_4 + x_5 &= 0 \end{aligned}$$

We observe that x_2, x_4, and x_5 are free variables, since they correspond to the columns with no pivots. Thus, we let $x_2 = a$, $x_4 = b$, and $x_5 = c$. Now we rewrite the equations as:

$$\begin{aligned} x_1 &= -a + 2b - 3c \\ x_2 &= a \\ x_3 &= b - c \\ x_4 &= b \\ x_5 &= c \end{aligned}$$

We can then write this in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, $Null(\mathbf{A})$ is spanned by the vectors:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The dimension of $Null(\mathbf{A})$ is 3, as it is the minimum number of vectors we need to span it.

(d) For the following matrix $\mathbf{D}$, find $Col(\mathbf{D})$ and its dimension, and $Null(\mathbf{D})$ and its dimension. Using inspection or Gaussian elimination are both valid methods to solve the problem.

$$\mathbf{D} = \begin{bmatrix} 1 & -1 & -3 & 4 \\ 3 & -3 & -5 & 8 \\ 1 & -1 & -1 & 2 \end{bmatrix}$$

Solution:

To find $Col(\mathbf{D})$, we identify the linearly independent columns of $\mathbf{D}$ by inspection. The second column is a scaled version of the first column. The third column is linearly independent from the first and second columns, since it is not a scaled version of the first column. Finally, the fourth column is the first column minus the third column and thus is linearly dependent with respect to prior columns. If the linear dependence of the fourth column is not clear by inspection, we can instead perform Gaussian elimination. By doing so, we would find that the second and fourth column lack pivots, which also indicates their linear dependence.

So we conclude that the linearly independent columns of $\mathbf{D}$ are the first and third columns so that a basis for $Col(\mathbf{D})$ is:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -1 \end{bmatrix} \right\}$$

and thus the dimension of $Col(\mathbf{D})$ is 2.

To find $Null(\mathbf{D})$, we can row reduce the matrix to find solutions to $\mathbf{D}\vec{x} = \vec{0}$.

$$\left[\begin{array}{cccc|c} 1 & -1 & -3 & 4 & 0 \\ 3 & -3 & -5 & 8 & 0 \\ 1 & -1 & -1 & 2 & 0 \end{array}\right] \xrightarrow[\text{rref}]{} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since we only have pivots in the first and third columns, we can assign the free variables $x_2 = s$ and $x_4 = t$. We can write all solutions to $\mathbf{D}\vec{x} = \vec{0}$ as:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s-t \\ s \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} t$$

A basis for $Null(\mathbf{D})$ is:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and thus the dimension of $Null(\mathbf{D})$ is 2.

(e) Find the sum of the dimensions of $Null(\mathbf{A})$ and $Col(\mathbf{A})$. Also find the sum of the dimensions of $Null(\mathbf{D})$ and $Col(\mathbf{D})$. What do you notice about these sums in relation to the dimensions of $\mathbf{A}$ and $\mathbf{D}$, respectively?

Solution:

The dimensions of $Col(\mathbf{A})$ and $Null(\mathbf{A})$ add up to the number of columns in $\mathbf{A}$. The same is true of $\mathbf{D}$. This is true of all matrices and relates to what is known as the 'rank-nullity theorem'; however we will not be covering this in 16A. You'll get to explore this in 16B.

6. Prelab Questions

These questions pertain to the Pre-Lab reading for the Imaging 3 lab. You can find the reading under the Imaging 3 Lab section on the 'Schedule' page of the website. We do not expect in-depth answers for the questions. Please limit your answers to a maximum of 2 sentences.

(a) What properties does the mask matrix H need to have for us to reconstruct the image?

(b) Briefly describe why averaging multiple signals/measurements is a good idea.

(c) How do we get the image back from the new equation that models our system? Note that your answer cannot contain the image vector, $\vec{i}$ (since it is an unknown). You may, however, give an answer that includes $\vec{i}_{est}$.

(d) What term allows us to control the effect of noise in our system? *Hint*: Look at the terms in the equation that contains $\vec{i}_{est}$.

Solution:

(a) Invertible, linearly independent columns, a trivial nullspace, non-zero determinant, unique solution for $\mathbf{A}\vec{x} = \vec{b}$. All these properties are equivalent!

(b) Averaging is good because it reduces noise over multiple measurements especially if we have one or more bad measurements. (can talk about the given single-pixel vs multi-pixel example in the reading)

(c) $\vec{i}_{est} = H^{-1}\vec{s}_{ideal} + H^{-1}\vec{w}$

(d) The term $H^{-1}\vec{w}$ allows us to control the noise term. [OPTIONAL: A smaller $H^{-1}\vec{w}$ reduces the effect of noise on our system.]

7. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.

