EECS 16A Designing Information Devices and Systems I
Spring 2023 Lecture Notes Note 11B

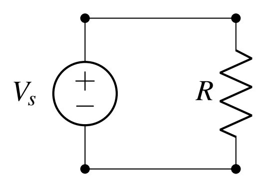
11.1 Node Voltage Analysis

In this course, we will learn how to take a real world system and build a circuit diagram that models the behavior of that system, and we will design our own circuits for specific real world tasks. In this note, however, we will assume that we already have an accurate circuit diagram, and will learn how to analyze the circuit.

For a given circuit, we would like to find all of the voltages and currents—sometimes we call this "solving" the circuit. Specifically, once you have or are given an accurate circuit diagram, you should be able to follow the given analysis algorithm to "solve" the circuit (i.e., compute all of the voltages and currents) correctly. Circuits are deterministic; if you follow the steps accurately and completely, the math will work.

Definition 11.1 (Node Voltage Analysis): **Node Voltage Analysis** is procedure for analyzing and solving a circuit using KCL equations to derive its node voltages.

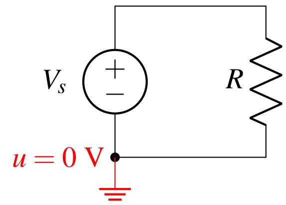
For now we will present **node voltage analysis** using *junctions* instead of *nodes* which we'll discuss more formally later in this note. Consider an example using the following diagram, which consists of four elements: a voltage source, a resistor, and two wires.



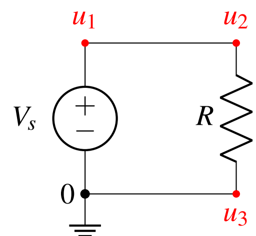
Definition 11.2 (Junction): A circuit **junction** is a place where two or more circuit elements meet, *including wires*. And in the above circuit there are four junctions (one at each corner).

For the sake of clarity, after each step of the analysis algorithm we show what the current circuit diagram looks like. When you perform the algorithm on your own, however, *you do not need to redraw the circuit each time*; instead you can simply label/annotate a single diagram.

Step 1: Pick the junction between the negative terminal of the voltage source and the bottom wire and label it as $u = 0\ V$ ("reference"), meaning that we will measure all of the voltages in the rest of the circuit relative to this point.

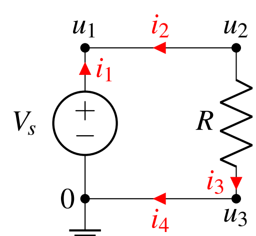


Step 2: Label all remaining junctions as some "u_i", representing the voltage at each junction relative to the zero/reference junction.



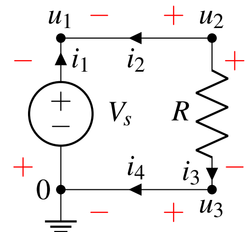
We can simplify our procedure by labeling nodes rather than junctions in the circuit. Once you have some familiarity with the procedure, there are simplifications (See 11.2) we can make to avoid analysis in every single wire, but here we describe here the most complete and rigorous version.

Step 3: Label the current through every element in the circuit "i_n". Every element in the circuit that was listed above should have a current label, including ideal wires. The direction of the arrow indicates which direction of current flow you are considering to be positive. At this stage of the algorithm, you can pick the direction of all the current arrows *arbitrarily*. As long as you are consistent with this choice and follow the rules described in the rest of this algorithm, the math will work out correctly.

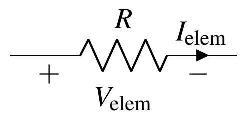


Note that we only label the current once for each element—for example, we can label i_3 as the current leaving the resistor (as is done in the diagram) *or* we can label it as the the current entering the resistor. These are equivalent because KCL also holds within the element itself—i.e., the current that enters an element must be equal to the current that exits that same element.

Step 4: Add $+/-$ labels on each element to indicate positive/negative voltage, following **passive sign convention** (defined below). These labels will indicate the direction with which voltage will be measured across that element.



Definition 11.3 (Passive Sign Convention): The **passive sign convention** dictates that positive current should *enter* the positive voltage terminal and *exit* the negative voltage terminal of an element. Below is an example for a resistor:



Labeling using passive sign convention ensures the values of voltage and current *as labeled* will either both be positive ($V_{elem} >$ and $I_{elem} > 0$) or both be negative ($V_{elem} <$ and $I_{elem} < 0$) for *passive* circuit elements. When we discuss the electrical quantity **power** later in the module, we will define the term "passive".

In the about circuit example, it might seem peculiar that the voltage labeling on the voltage source is opposite to its 'internal' voltage polarity. This results from: 1) the arbitrary direction we picked for current i_1 and 2) following passive sign convention for voltage labeling. Since we consistently followed the above steps, this does not effect our final solution of junction voltages and element currents, as we shall see.

Note: however you label the element currents and voltages, ***as long as it is consistent****, the solution to the circuit voltages and currents will be correct with respect to the way you labeled them.*

At this stage in the circuit analysis algorithm, we find that there are several **unknowns** labeled on our circuit. These are: $i_1, i_2, i_3, i_4, u_1, u_2, u_3$.

Step 5: Use KCL to write equations with our unknowns.

Begin by writing KCL equations for every junction in the circuit.

$$
\begin{aligned}
i_1 + i_2 &= 0 \\
-i_2 - i_3 &= 0 \\
i_3 - i_4 &= 0 \\
i_4 - i_1 &= 0
\end{aligned}
$$

Notice the last equation we get is linearly dependent with the first three—you can see this by adding all three of the first equations to each other and multiplying the entire result by -1. We will therefore omit this equation.

In general, if you use KCL at every junction, you will get one linearly dependent equation, and so you can typically simply skip one junction; *skipping the junction that has been labeled as* ***reference*** *is a common choice*.

Step 6: Use the I-V relationships of each of the elements.

We know that the difference in potentials across the voltage source must be its voltage, V_s. We also know that the voltage across the resistor is equal to the current times the resistance, from Ohm's Law (i.e., $V = I \cdot R$). For the wires, we know the difference in potential is 0 V. Thus, we have the following equations:

$$\begin{aligned} 0 - u_1 &= -V_s \\ u_2 - u_1 &= 0 \\ u_2 - u_3 &= R\, i_3 \\ u_3 - 0 &= 0 \end{aligned}$$

Since u_3 is a junction connected to reference by only a wire, u_3 is simply 0 V. Again, this shows that it is not always necessary to label all junctions (see 11.2 below).

Step 7: Simplify your equations and solve. First take a look at all our equations:

$$\begin{aligned} i_1 + i_2 &= 0 \\ -i_2 - i_3 &= 0 \\ i_3 - i_4 &= 0 \\ i_4 - i_1 &= 0 \\ 0 - u_1 &= -V_s \\ u_2 - u_1 &= 0 \\ u_2 - u_3 &= R\, i_3 \\ u_3 - 0 &= 0 \end{aligned}$$

Recall that we noticed the fourth equation is linearly dependent with the first three, so we will omit this equation since it will not present any new information to solve our system. Simplifying our equations, we now have:

$$\begin{aligned} i_1 + i_2 &= 0 \\ -i_2 - i_3 &= 0 \\ i_3 - i_4 &= 0 \\ u_1 &= V_s \\ u_1 &= u_2 \\ u_2 - u_3 &= R\, i_3 \\ u_3 &= 0 \end{aligned}$$

Simplifying the last four equations, we get:

$$\begin{aligned} u_1 &= V_s \\ u_2 &= V_s \\ u_2 &= R\, i_3 \\ u_3 &= 0 \end{aligned}$$

These values make sense. Voltages u_1 and u_2 are both connected to V_s by a wire, thus $u_1 = u_2 = V_s$. The junction u_3 is connected to reference by a wire, thus $u_3 = 0$ V. Finally, we can find $i_3 = u_2/R = V_s/R$.

Knowing i_3 we can use substitution and find that $i_3 = i_4 = -i_2 = i_1 = V_s/R$ and we have found all unknowns.

Note that at this step, if your system of equations is too complex, you may also choose to use a different method of finding solutions to your unknowns (discussed in the next section).

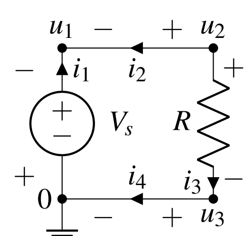
Circuit Analysis with Matrices

Alternatively, we can perform circuit analysis using linear algebra techniques we have learned. Note that this approach is essentially the same as the steps presented previously. However, here we are approaching the problem with the intent of using matrices to analyze our circuit. In more complex circuits, this method will be very useful (see 11.3 Example).

Goal: The goal is to set up the relationship $\mathbf{A}\vec{x} = \vec{b}$, where $\vec{x}$ is comprised of the unknown circuit variables we want to solve for (currents and node potentials—that is, the i's and u's). Matrix $\mathbf{A}$ will be an $n \times n$ matrix where n is equal to the number of unknown variables. For the circuit above, we have 3 unknown potentials (u) and 4 unknown currents (i), therefore we form a 7×7 matrix.

$$\begin{bmatrix} ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix}$$

Here is the circuit again after Step 4:



Step 5 (Alt.): Use KCL to fill in as many **linearly independent** rows of $\mathbf{A}$ and $\vec{b}$ as possible.

Begin by writing KCL equations for every junction in the circuit.

$$\begin{aligned} i_1 + i_2 &= 0 \\ -i_2 - i_3 &= 0 \\ i_3 - i_4 &= 0 \\ i_4 - i_1 &= 0 \end{aligned}$$

As previously mentioned, the first three equations are linearly independent and the fourth is linearly dependent. In order to end up with a square and invertible $\mathbf{A}$ matrix, we will therefore omit the fourth equation.

In general, if you use KCL at every junction, you will get one linearly dependent equation, and so you can typically simply skip one junction; *skipping the junction that has been labeled as* ***reference*** *is a common choice*.

Now we put these equations in matrix form:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ ? \\ ? \\ ? \\ ? \end{bmatrix}$$

Step 6 (Alt.): Use the I-V relationships of each of the elements to fill in the remaining equations (rows of **A** and values of $\vec{b}$).

In this example, we need four more linearly independent equations, and there are four circuit elements, each with their own I-V relationship (this is not a coincidence, as will be explained shortly). We use what we know about each element to form four more equations.

We know that the difference in potentials across the voltage source must be its voltage, V_s. We also know that the voltage across the resistor is equal to the current times the resistance, from Ohm's Law (i.e., $V = I \cdot R$). For the wires, we know the difference in potential is 0 V. Thus, we have the following equations:

$$\begin{aligned} 0 - u_1 &= -V_s \\ u_2 - u_1 &= 0 \\ u_2 - u_3 &= Ri_3 \\ u_3 - 0 &= 0 \end{aligned}$$

After filling in these equations, our matrix is:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -R & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -V_s \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Step 7 (Alt.): Solve.

At this point the analysis procedure is effectively complete - all that's left to do is solve the system of linear equations (by applying Gaussian Elimination, inverting **A** computationally, etc.) to find the values for the u's and i's.

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} V_s/R \\ -V_s/R \\ V_s/R \\ V_s/R \\ V_s \\ V_s \\ 0 \end{bmatrix}$$

Before we move on, it is worth pausing at this point to highlight why the procedure always works, and in particular, *will we always have as many (linearly independent) equations as we do unknowns?*

– *Unknowns:* If a circuit has m elements in it and n junctions, there will be $(n-1)$ voltages u (since we have defined one of them as reference/zero), and m currents i (one for each element).

– *Equations:* With n junctions, we will get $(n-1)$ linearly independent KCL equations from Step 5. Similarly, since each element has a defining I-V relationship, Step 6 will provide us with m equations.

Thus, this circuit analysis procedure will always provide as many $m+(n-1)$ linearly independent equations as unknowns.

11.2 Simplifying the Circuit Analysis Procedure

The analysis procedure we described in the previous section will always work, and introducing the procedure at this level of comprehensiveness is necessary to ensure that one can always follow it successfully. However as is most likely clear, even for very simple circuits the procedure will quickly involve a large number of variables and hence large matrices. Fortunately, we can substantially reduce the number of variables by noticing two things:

1. There is no voltage drop across ideal wires. Therefore, the junction voltage potentials at two ends of a wire (relative to reference) are always equal (i.e., **equipotential**).

2. When a junction involves only two elements, KCL tells us that the current flowing in through the first element must equal the current flowing out through the second element.

The next two sections describe in more detail how we can use these observations to simplify solving a circuit.

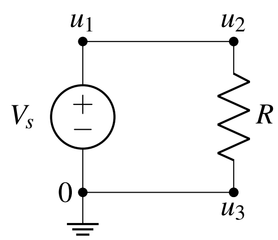
11.2.1 Labeling Nodes Instead of Junctions

Since wires always have zero voltage drop across them, there is no specific need for us to keep track of the voltage (relative to reference) on the two sides of a wire separately. In other words, *all of the junctions that are connected to each other by wires can be labeled with a single voltage variable* '*u*'. A set of such junctions connected to each other only via wires is defined as a **node**.

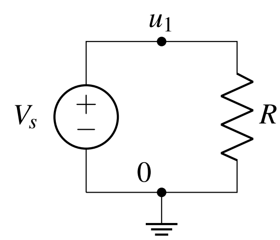
Definition 11.4 (Node): A circuit **node** describes a region of the circuit which is *equipotential* (i.e., at the same/equiuvalent voltage throughout). Alternatively, it is a place where two or more circuit elements meet, *excluding wires* since wires have no voltage across them and are thus equipotential.

Definition 11.5 (Branch): A circuit **branch** describes the circuit's path between any two **nodes**. A branch can contain one or more circuit elements.

As an example, consider the circuit we were analyzing, but return to Steps 1 and 2. As shown below, the junctions previously labeled as u_3 and reference are connected by a wire and are therefore a single node. We can label that entire node as reference. Similarly, the junctions previously labeled as u_1 and u_2 are also connected by a wire, so are also a single node. We can label that entire node as u_1.



Original procedure:
Labeling junctions



Simplified procedure:
Labeling nodes

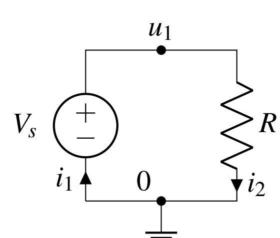
When we followed the original analysis procedure where we labeled *junctions*, we ended up with three unknown u's; by labeling only the *nodes*, we have simplified down to a single unknown voltage potential (u_1). In general, since wires are abundant in circuit diagrams, labeling only the nodes (instead of the junctions) will substantially reduce the number of variables.

11.2.2 Trivial Junctions

Definition 11.6 (Trivial Junction): A **trival junction** is a junction connecting only two elements. KCL dictates that the current entering the junction must be equal to the current exiting. Since there are only two elements, it follows that the two currents must be equal (as long as we label the direction of current flow to be the same—if not, the currents will simply be opposite in sign).

Therefore, another simplification to our analysis procedure is to *label the currents only in the non-wire elements* in our circuit (sometimes these currents are called **branch currents**). We can later find the current in any given wire by looking for a trivial junction between the wire and a non-wire element. When we use KCL, we can now consider *nodes* (instead of junctions)—i.e. the current flowing into the node is equal to the current leaving the node.

Returning to our example, if we repeat Step 3 (and assume labeled nodes rather than junctions, as explained in the previous section), we would now label only the current through the two non-wire elements: the voltage source and the resistor.



With this simplified approach, when we get to Step 5 (KCL), we would apply KCL at the node u_1, which would result in the single equation:

$$i_1 - i_2 = 0$$

11.2.3 Summary of Simplified Procedure

By labeling nodes instead of junctions and labeling currents in non-wire elements only, we can greatly reduce the number of variables in our circuit analysis procedure, so this is what we will do in the future. Here's a summary of the steps:

Step 1: Pick a reference **node** and label it as $u = 0$ V, meaning that we will measure all of the node voltages in the rest of the circuit relative to this node.

Step 2: Label all remaining **nodes** as some "u_i", representing the voltage at each node relative to the reference node.

Step 3: Label the current through every **non-wire** element in the circuit "i_n".

Step 4: Add $+/-$ labels (indicating direction of voltage measurement) on each **non-wire** element by following the passive sign convention.

Step 5: Use KCL to write equations at the labeled **nodes**, excluding the 'reference' node.

Step 6: Write the I-V relationships of non-wire elements. Use Ohm's Law for resistors.

Step 7: Solve system of equations using substitution.

11.2.4 Summary of Matrix-Oriented Procedure

Goal: Set up the relationship $\mathbf{A}\vec{x} = \vec{b}$, where $\vec{x}$ is comprised of the u_i's and i_n's defined in the previous steps.

Step 1-4: Same as the simplified procedure above.

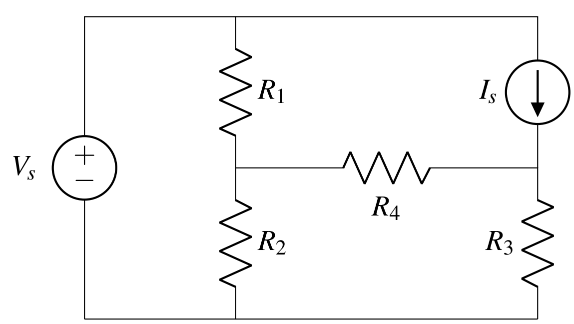
Step 5: If there are n **nodes** (including the reference node), use KCL on $(n-1)$ nodes to fill in $(n-1)$ rows of $\mathbf{A}$ and $\vec{b}$.

Step 6: If there m **non-wire** elements, use the I-V relationships of each non-wire element to fill in the remaining m equations (rows of $\mathbf{A}$ and values of $\vec{b}$).

Step 7: Solve with your favorite technique from linear algebra!

11.3 Node Voltage Example

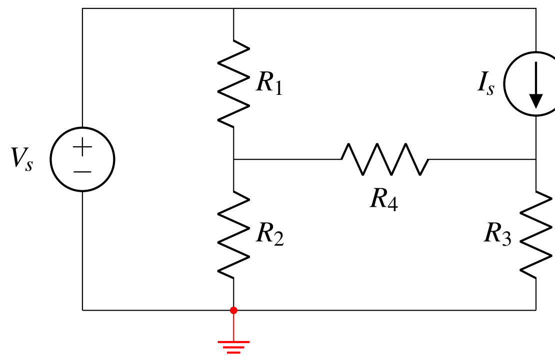
Example 11.1 (Node Voltage): Find all voltages (and currents) in the following electronic circuit. The circuit parameters are as follows: $R_1 = 1\ \Omega$, $R_2 = 2\ \Omega$, $R_3 = 3\ \Omega$, $R_4 = 4\ \Omega$, $V_s = 1$ V, and $I_s = 0.5$ A.



The following method proceeds through the above steps in the *Simplified Procedure* of 11.2.

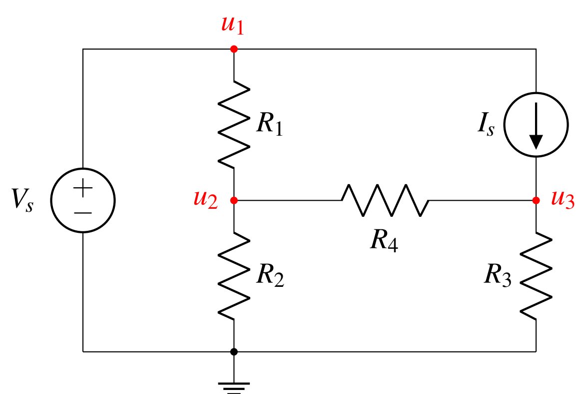
Step 1: Choose reference node

Select a reference node. Any node can be chosen for this purpose. In this example, we choose the node at the bottom of the circuit diagram.



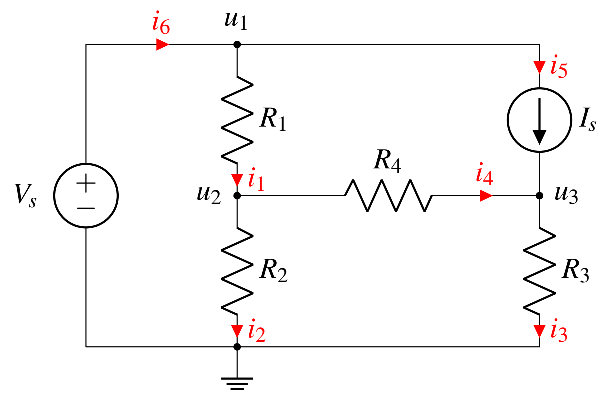
Step 2: Label all nodes

After choosing the reference node, label all the other nodes in the circuit. In this example there are three unknown nodes: u_1, u_2 and u_3. Notice the node voltage u_1 is equivalent to the voltage of the voltage source V_s.



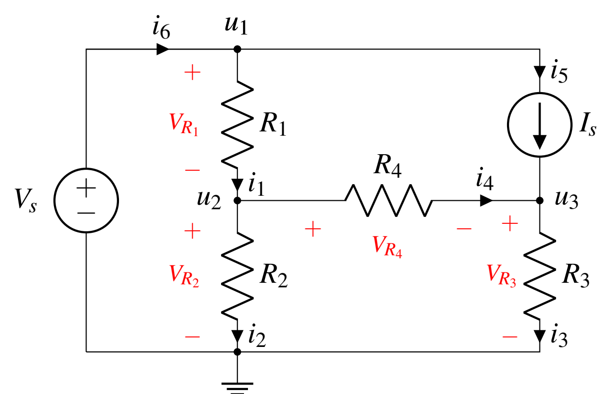
Step 3: Label currents through non-wire elements

The direction is arbitrary (top to bottom, bottom to top, it won't matter, but stick with your choice in subsequent steps).



Step 4: Label element potentials based on passive sign convention.

The element voltage for I_s is not marked in the example since it will not be needed in the calculations below. Same for the voltage source. There is no harm in marking those, too.



Step 5: KCL Equations

Write KCL equations for all nodes with unknown voltages: u_1, u_2, and u_3. Refer to Note 11A for a reminder of how to write KCL equations.

At node u_1 we get (sum of all currents entering the node equals sum of all currents exiting):

$$i_6 - i_1 - i_5 = 0$$

Similarly at node u_2 we get:

$$i_1 - i_2 - i_4 = 0$$

Finally, for node u_3:

$$i_5 + i_4 - i_3 = 0$$

Step 6: Element I-V relationships

Find expressions for all element voltages in terms of the currents and element characteristics (e.g. Ohm's law) for all circuit elements except the current source. In this example there are six element characteristics: R_1, R_2, R_3, R_4, V_s, and I_s. We can also represent the element voltages as linear combinations of the node voltages u_1, u_2, and u_3.

$$\begin{aligned} R_1 i_1 &= V_{R_1} = u_1 - u_2 \\ R_2 i_2 &= V_{R_2} = u_2 \\ R_3 i_3 &= V_{R_3} = u_3 \\ R_4 i_4 &= V_{R_4} = u_2 - u_3 \\ V_s &= u_1 \end{aligned}$$

Directly express the current through the current source i_5 as I_s:

$$i_5 = I_s$$

Now we have nine unique equations (including the three KCL equations in Step 5) and nine unknowns: u_1, u_2, u_3, i_1, i_2, i_3, i_4, i_5, and i_6.

Step 7: Solve

We will consider two potential methods for solving this system.

(a) *Method 1:* The first is the matrix-oriented approach introduced above where all of the node voltages and branch currents are solved simultaneously using linear algebra.

(b) *Method 2:* The second method focuses on deriving only the unknown node voltages using substitution and/or linear algebra. After the node voltages are found, the branch currents can be independently derived as needed.

In this course it will often be much easier (especially when solving problems by hand) to perform *Method 2*. This is because often a circuit is considered 'solved' once all of the node voltages are known. This is the practical utility of **node voltages** and **Node Voltage Analysis**.

Method 1: Using our linear algebra techniques, we can formulate a matrix-vector representation for all circuit node voltages and element/branch currents as $\mathbf{A}\vec{x} = \vec{b}$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -R_1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -R_2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -R_3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -R_4 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ V_s \\ I_s \end{bmatrix}$$

With the given circuit parameters: $R_1 = 1\ \Omega$, $R_2 = 2\ \Omega$, $R_3 = 3\ \Omega$, $R_4 = 4\ \Omega$, $V_s = 1$ V, $I_s = 0.5$ A; this provides enough information for us to populate matrix $\mathbf{A}$ and vector $\vec{b}$ and solve for the unknowns

$$\vec{x} = \mathbf{A}^{-1}\vec{b} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.261 \text{ A} \\ 0.370 \text{ A} \\ 0.391 \text{ A} \\ -0.109 \text{ A} \\ 0.500 \text{ A} \\ 0.761 \text{ A} \\ 1.000 \text{ V} \\ 0.739 \text{ V} \\ 1.174 \text{ V} \end{bmatrix}$$

Let us consider this particular solution more carefully:

- As expected, the branch current i_5 is equal to $I_s = 0.5$ A.
- Also, the node voltage u_1 is equal to the voltage source, $V_s = 1$ V.
- *What does a negative current in* i_4 *represent?* For these values of circuit parameters, positive current is flowing in the direction opposite to the (arbitrary) direction we defined it Step 3. This is not a issue in our analysis because we remained consistent in our labeling and when using KCL in Step 5.

Again, it turns out that if we know all the node voltages, we can quickly use Ohm's Law (e.g., $i_{1-2} = \frac{u_1-u_2}{R}$) to derive the currents through each and every resistor and by extension every circuit **branch**. The current through the voltage source however must be obtained with a KCL equation at a common node.

Method 2: A more streamlined method involves solving a system of KCL equations with respect to the node voltages and known quantities only.

Substitute the derived element expressions into the KCL equations (from Step 5) at circuit nodes with unknown voltages (in this case u_2 and u_3). Also substitute known variables such as $u_1 = V_s$ and $i_5 = I_s$.

$$\begin{aligned} i_1 - i_2 - i_4 = 0 \quad &\Longrightarrow \quad \frac{V_s - u_2}{R_1} - \frac{u_2}{R_2} - \frac{u_2 - u_3}{R_4} = 0 \\ i_5 + i_4 - i_3 = 0 \quad &\Longrightarrow \quad I_s + \frac{u_2 - u_3}{R_4} - \frac{u_3}{R_3} = 0 \end{aligned}$$

Now we have a system of two linearly independent equations and two unknowns (u_2 and u_3). Further substitution can be used, but linear algebra (with much fewer unknowns) can also be a useful tool here. Let us first reorganize these equations by grouping the unknowns (u_2 and u_3) on the left side and the known terms on the right:

$$\begin{aligned} u_2 \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_4}\right) R_1 + u_3 \cdot \left(-\frac{R_1}{R_4}\right) &= V_s \\ u_2 \cdot \left(-\frac{1}{R_4}\right) + u_3 \cdot \left(\frac{1}{R_3} + \frac{1}{R_4}\right) &= I_s \end{aligned}$$

Then cast the equations into a matrix-vector representation ($\mathbf{A}'\vec{x}' = \vec{b}'$):

$$\begin{bmatrix} \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_4}\right) R_1 & -\frac{R_1}{R_4} \\ -\frac{1}{R_4} & \frac{1}{R_3} + \frac{1}{R_4} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} V_s \\ I_s \end{bmatrix}$$

Finally compute the solution using Gaussian Elimination (or let the computer do the work, here using sympy):

```
from sympy import *
init_printing(use_unicode=True)

R1, R2, R3, R4 = symbols('R1 R2 R3 R4')
Y = Matrix([[ 1/R1+1/R2+1/R4, -1/R4], [-1/R4, 1/R3+1/R4]])

V1, I1 = symbols('V1 I1')
b = Matrix([ V1/R1, I1 ])

Vn1, Vn2 = linsolve((Y, b)).args[0]
```

Algebraic result:

```
>>> Vn1
    R₂⋅(I₁⋅R₁⋅R₃ + R₃⋅V₁ + R₄⋅V₁)
─────────────────────────────────────
R₁⋅R₂ + R₁⋅R₃ + R₁⋅R₄ + R₂⋅R₃ + R₂⋅R₄
>>> Vn2
  R₃⋅R₄⋅(I₁⋅(R₁⋅R₂ + R₁⋅R₄ + R₂⋅R₄) + R₂⋅V₁)
─────────────────────────────────────────────
-R₁⋅R₂⋅R₃ + (R₃ + R₄)⋅(R₁⋅R₂ + R₁⋅R₄ + R₂⋅R₄)
```

Numerical result:

```
>>> values = {R1:1, R2:2, R3:3, R4:4, I1:0.5, V1:1}
>>>
>>> f"Vn1 = {Vn1.evalf(3, subs=values)} V"
Vn1 = 0.739 V
>>> f"Vn2 = {Vn2.evalf(3, subs=values)} V"
Vn2 = 1.17 V
```

And the solution to $\vec{x}'$ yields

$$\vec{x}' = \mathbf{A}'^{-1}\vec{b}' = \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.739 \text{ V} \\ 1.174 \text{ V} \end{bmatrix}$$

From here, knowing the three node voltages u_1, u_2, and u_3 is sufficient to quickly derive any of the branch/element currents using the element equations in Step 6. For example, the current i_4 through resistor R_4 is

$$i_4 = \frac{V_{R_4}}{R_4} = \frac{u_2 - u_3}{R_4} = -0.108 \text{ A}$$

which matches the derived result from the prior system of nine linear equations in *Method 1*.

Additional Resources For more on node voltage analysis, read *Schaum*'s Outline of Electric Circuits, Seventh Edition, Section 4.4.