EECS 16A Designing Information Devices and Systems I
Fall 2022 Homework 3

This homework is due September 23rd, 2022, at 23:59.
Self-grades are due September 26th, 2022, at 23:59.

Submission Format
Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned).

1. **Reading Assignment** For this homework, please read Notes 3,4, and 11A. Note 3 provides an overview of linear dependence and span, Note 4 gives an introduction to thinking about and writing proofs, and Note 11A gives an introduction to circuits.

   Please answer the following questions:

   (a) Why are there two definitions of linear dependence? What value does each definition provide?

   (b) Why is voltage "across" a circuit element?

   Solution:

   (a) See Note 3.1 for the two definitions. Definition (I) is more useful for mathematically proving linear dependence while Definition (II) provides a more intuitive understanding of linear dependence and formalizes the notion of redundancy.

   (b) See Note 11.2. Voltage, or electric potential, is only defined relative to another point. For convenience, we will frequently define a reference point, called ground, against which other voltages can be measured.

2. **Linear Dependence**

   Learning Goal: *Evaluate the linear dependency of a set of vectors.*

   State if the following sets of vectors are linearly independent or dependent. If the set is linearly dependent, provide a linear combination of the vectors that sum to the zero vector.

   (a) $\left\{ \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$

   Solution: The vectors are linearly independent. A set of two vectors can only be linearly dependent if one of the vectors is a scaled version of the other. $\begin{bmatrix} -5 \\ 2 \end{bmatrix} \neq \alpha \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ for $\alpha \in \mathbb{R}$.

   (b) $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Solution: This set of vectors is linearly dependent. To find a linear combination that yields $\vec{0}$, find the RREF of the following augmented matrix:

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ -1 & -2 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow -R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ -1 & -2 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow[R_4 \leftarrow R_4 + R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & -3 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 \leftarrow R_2/3]{R_5 \leftarrow R_5 - R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 3 & -1 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_4 \leftarrow R_4 + 3R_2]{R_3 \leftarrow R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The free variable can be chosen to be the variable of third column $[1\ -\frac{1}{3}\ 0\ 0\ 0]^T$. We can assign the value of the free variable to be 3α (chosen arbitrarily to make subsequent calculations easier). If we solve for the second column variable from the second row after plugging in 3α, $x_2 - \frac{1}{3}(3\alpha) = 0$,we obtain the variable of the second column to be α. We can do similarly with the first variable to obtain a value of -2α from the first row. This eventually gives the following linear combination that shows linear dependence:

$$-2\alpha \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix} + 3\alpha \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If you have a set of coefficients that match a specific value of α, give yourself full credit.

(c) $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

Solution: This set of vectors is linearly dependent. To find a specific linear combination that shows linear dependence find the RREF of the following augmented matrix:

$$\left[\begin{array}{cccc|c} 2 & 0 & 2 & 0 & 0 \\ 2 & 1 & 4 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1/2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 4 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & -3 & 2 & 0 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3/-3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 \end{array}\right] \xrightarrow[R_1 \leftarrow R_1 - R_3]{R_2 \leftarrow R_2 - 2R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 \end{array}\right]$$

The free variable can be chosen to be the variable of third column $[0\ 0\ 1\ -\frac{2}{3}]^T$. We can assign the value of the free variable to be 3α (chosen arbitrarily to make subsequent calculations easier). Once again, using back substitution into the other rows of the matrix, we can find $x_3 - \frac{2}{3}(3\alpha) = 0 \rightarrow x_3 = 2\alpha$, $x_2 + \frac{1}{3}(3\alpha) = 0 \rightarrow x_2 = -\alpha$, and $x_1 + \frac{2}{3}(3\alpha) = 0 \rightarrow x_1 = -2\alpha$:

$$-2\alpha \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \alpha \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 2\alpha \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} + 3\alpha \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If you have a set of coefficients that match a specific value of α, give yourself full credit.

Alternatively, since the subset of the first 3 vectors is linearly independent, they span $\mathbb{R}^3$. We know this because the 3 vectors have no linear combination that yields 0 other than the trivial solution where all coefficients are also 0.

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_1/2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 1 & -4 & 0 \end{array}\right] \xrightarrow{(R_3-2R_1-R2)/3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The solution to this system has one unique solution, where all variables must be 0. Thus, there is no linear combination of the first three column vectors that can be equal to 0 (other than the trivial solution), so those vectors are linearly indepedent. Since the fourth vector is in $\mathbb{R}^3$, we are guaranteed some linear combination of the first three vectors that yields the fourth.

(d) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Solution: Since this set contains $\vec{0}$, it is linearly dependent as we can take the linear combination $\alpha \cdot \vec{0}$ where $\alpha \neq 0$ to get $\vec{0}$.

3. Linear Dependence in a Square Matrix

Learning Objective: *This is an opportunity to practice applying proof techniques. This question is specifically focused on linear dependence of rows and columns in a square matrix.*

Let A be a square $n \times n$ matrix, (i.e. both the columns and rows are vectors in $\mathbb{R}^n$). Suppose we are told that the columns of A are linearly dependent. Prove, then, that the rows of A must also be linearly dependent. You can use the following conclusion in your proof:

If Gaussian elimination is applied to a matrix A, and the resulting matrix (in reduced row echelon form) has at least one row of all zeros, this means that the rows of A are linearly dependent.

(**Hint**: Can you use the linear dependence of the columns to say something about the number of solutions to $A\vec{x} = \vec{0}$? How does the number of solutions relate to the result of Gaussian elimination?)

Solution:

Let $\vec{a}_1, \vec{a}_2, \dots \vec{a}_n$ be the columns of A. By the definition of linear dependence, there exist scalars, $c_1, c_2, \dots c_n$, not all zero, such that

$$c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n = \vec{0} \tag{1}$$

We define $\vec{c}$ to be a vector containing the c_i's as follows: $\vec{c} = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}^T$, where $\vec{c} \neq \vec{0}$ by the definition of linear dependence. We can write Eq. 1 in matrix vector form:

$$A\vec{c} = \vec{0} \tag{2}$$

Let's use the first hint: How many solutions are there to the equation $A\vec{x} = \vec{0}$? We know from Eq. 2 that $\vec{c}$ is a solution, but we can also show that $\alpha\vec{c}$ is a solution for any α:

$$A(\alpha\vec{c}) = \alpha\vec{0} = \vec{0} \tag{3}$$

Since $\vec{c}$ is not zero, every multiple of $\vec{c}$ is a different solution. Therefore there are infinite solutions to the equation $A\vec{x} = \vec{0}$.

What can we say about the result of Gaussian elimination if there are infinite solutions? We know that if there are infinite solutions, there must be a free variable after Gaussian elimination. In other words, there must be a column in the row reduced matrix with no leading entry. Therefore, there must be fewer leading entries than the number of columns. Since the matrix A is square, it has the same number of rows as columns, so there must be fewer leading entries than the number of rows. That means there is at least one row with no leading entry, which is equivalent to saying there must be one row that's all zeros in the row reduced matrix.

For example consider performing elimination on the following square matrix with infinite solutions:

$$\left[\begin{array}{ccc|c} 2 & 2 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 2 & 0 & 1 & 1 \end{array}\right]$$

Subtracting row 1 from row 3 ($R_3 - R_1 \to R_3$):

$$\left[\begin{array}{ccc|c} 2 & 2 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 0 & -2 & -2 & -6 \end{array}\right]$$

Dividing row 1 by 2 ($R_1/2 \to R_1$):

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 1 & 1 & 3 \\ 0 & -2 & -2 & -6 \end{array}\right]$$

Adding row 2 multiplied by 2 to row 3 ($R_3 + 2 * R_2 \to R_3$):

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

where we see that the last row is missing the leading entry and becomes all 0s.

Finally, we were given that if there is a row of all zeros in the row reduced matrix, then the rows of A must be linearly dependent.

4. Image Stitching

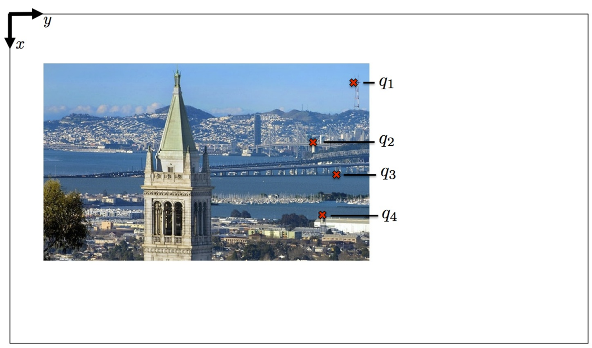
Learning Objective: *This problem is similar to one that students might experience in an upper division computer vision course. Our goal is to give students a flavor of the power of tools from fundamental linear algebra and their wide range of applications.*

Often, when people take pictures of a large object, they are constrained by the field of vision of the camera. This means that they have two options to capture the entire object:

- Stand as far away as they need to include the entire object in the camera's field of view (clearly, we do not want to do this as it reduces the amount of detail in the image)
- (This is more exciting) Take several pictures of different parts of the object and stitch them together like a jigsaw puzzle.

We are going to explore the second option in this problem. Daniel, who is a professional photographer, wants to construct an image by using "image stitching". Unfortunately, Daniel took some of the pictures from different angles as well as from different positions and distances from the object. While processing these pictures, Daniel lost information about the positions and orientations from which the pictures were taken. Luckily, you and your friend Marcela, with your wealth of newly acquired knowledge about vectors and matrices, can help him!

You and Marcela are designing an iPhone app that stitches photographs together into one larger image. Marcela has already written an algorithm that finds common points in overlapping images. **It's your job to figure out how to stitch the images together using Marcela's common points to reconstruct the larger image.**



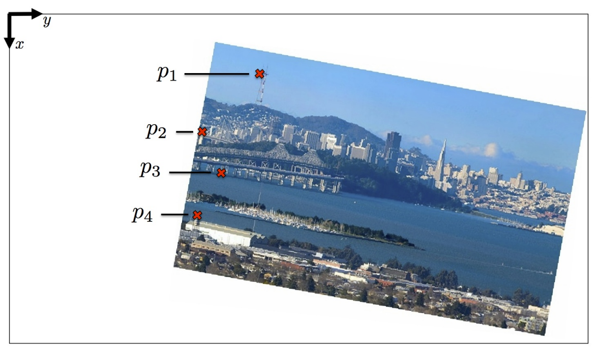


Figure 1: Two images to be stitched together with pairs of matching points labeled.

We will use vectors to represent the common points which are related by a affine transformation. Your idea is to find this affine transformation. For this you will use a single matrix, $\mathbf{R}$, and a vector, $\vec{t}$, that transforms every common point in one image to their corresponding point in the other image. Once you find $\mathbf{R}$ and $\vec{t}$ you will be able to transform one image so that it lines up with the other image.

Suppose $\vec{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$ is a point in one image , which is transformed to $\vec{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$ is the corresponding point in the other image (i.e., they represent the same object in the scene). For example, Fig. 1 shows how the points $\vec{p}_1$, $\vec{p}_2$ ... in the right image are transformed to points $\vec{q}_1$, $\vec{q}_2$ ... on the left image. You write down the following relationship between $\vec{p}$ and $\vec{q}$.

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \underbrace{\begin{bmatrix} r_{xx} & r_{xy} \\ r_{yx} & r_{yy} \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\vec{t}} \tag{4}$$

This problem focuses on finding the unknowns (i.e. the components of $\mathbf{R}$ and $\vec{t}$), so that you will be able to stitch the image together. *Note that this is the opposite from our usual setting in which we would solve for $\vec{p}$ given all other variables.*

(a) To understand how the matrix $\mathbf{R}$ and vector $\vec{t}$ transforms any vector representing a point on a image, Consider this example equation similar to Equation (4),

$$\vec{v} = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \vec{u} + \vec{w} = \vec{v}_1 + \vec{w}. \tag{5}$$

Use $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for this part.
We want to find out what geometric transformation(s) can be applied on $\vec{u}$ to give $\vec{v}$.

Step 1: Find out how $\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$ is transforming $\vec{u}$. Evaluate $\vec{v}_1 = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \vec{u}$.
What **geometric transformation(s)** might be applied to $\vec{u}$ to get $\vec{v}_1$? Choose the options that answers the question and explain your choice.

(i) Rotation
(ii) Scaling
(iii) Shifting/Translation

Drawing the vectors $\vec{u}$, and $\vec{v}_1$ in two dimensions on a single plot might help you to visualize the transformations.

Step 2: Find out $\vec{v} = \vec{v}_1 + \vec{w}$. Find out how **addition of $\vec{w}$ is geometrically transforming** $\vec{v}_1$. Choose the option that answers the question and explain your choice.

(i) Rotation
(ii) Scaling
(iii) Shifting/Translation

Drawing the vectors $\vec{v}$, $\vec{w}$, and $\vec{v}_1$ in two dimensions on a single plot might help you to visualize the transformations.

Solution: Plugging in the given vectors and performing the matrix vector multiplication,

$$\vec{v}_1 = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}. \tag{6}$$

It is observable that $\vec{v}_1$ is a scaled, rotated version of $\vec{u}$.
We get $\vec{v} = \vec{v}_1 + \vec{w} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. We can also see that $\vec{v}$ is a shifted version of $\vec{v}_1$.
Hence $\vec{u}$ is scaled, rotated and shifted to get $\vec{v}$.
Note: We can only use matrix transformations to scale and/rotate a vector. We cannot translate a vector through matrix transformations; instead we must use vector addition for this.

(b) Now back to the main problem. First, multiply Equation (4) out into **two equations**.

(i) What are the known values and what are the unknown values in each equation (recall what we are trying to solve for in Equation (4))?
(ii) How many unknown values are there?
(iii) How many independent equations do you need to solve for all the unknowns?
(iv) How many pairs of common points $\vec{p}$ and $\vec{q}$ will you need in order to write down a system of equations that you can use to solve for the unknowns? ***Hint**: Remember that each pair of $\vec{p}$ and $\vec{q}$ is related by two equations, one for each coordinate.*

Solution:

We can rewrite the above matrix equation as the following two scalar equations:

$$q_x = p_x r_{xx} + p_y r_{xy} + t_x$$
$$q_y = p_x r_{yx} + p_y r_{yy} + t_y$$

Here, the known values are each pair of points' elements: q_x, q_y, p_x, p_y, and the scaling factor of the $\vec{t}$ vector (1). The unknowns are elements of $\mathbf{R}$ and $\vec{t}$: r_{xx}, r_{xy}, r_{yx}, r_{yy}, t_x, and t_y. There are 6 unknowns, so we need a total of 6 equations to solve for them. For every pair of points we add, we get two more equations. Thus, we need 3 pairs of common points to get 6 equations.

(c) Use what you learned in the above two subparts to explicitly write out **just enough** equations of these transformations as you need to solve the system. Assume that the four pairs of points from Fig. 1 are labeled as:

$$\vec{q}_1 = \begin{bmatrix} q_{1x} \\ q_{1y} \end{bmatrix}, \vec{p}_1 = \begin{bmatrix} p_{1x} \\ p_{1y} \end{bmatrix} \qquad \vec{q}_2 = \begin{bmatrix} q_{2x} \\ q_{2y} \end{bmatrix}, \vec{p}_2 = \begin{bmatrix} p_{2x} \\ p_{2y} \end{bmatrix} \qquad \vec{q}_3 = \begin{bmatrix} q_{3x} \\ q_{3y} \end{bmatrix}, \vec{p}_3 = \begin{bmatrix} p_{3x} \\ p_{3y} \end{bmatrix} \qquad \vec{q}_4 = \begin{bmatrix} q_{4x} \\ q_{4y} \end{bmatrix}, \vec{p}_4 = \begin{bmatrix} p_{4x} \\ p_{4y} \end{bmatrix}.$$

Solution: From the previous part, we know that we will need six equations, as we have six unknowns. Recalling that each point provides us with two equations, we arbitrarily select the first three points:

$$\vec{q}_1 = \begin{bmatrix} q_{1x} \\ q_{1y} \end{bmatrix}, \quad \vec{p}_1 = \begin{bmatrix} p_{1x} \\ p_{1y} \end{bmatrix} \qquad \vec{q}_2 = \begin{bmatrix} q_{2x} \\ q_{2y} \end{bmatrix}, \quad \vec{p}_2 = \begin{bmatrix} p_{2x} \\ p_{2y} \end{bmatrix} \qquad \vec{q}_3 = \begin{bmatrix} q_{3x} \\ q_{3y} \end{bmatrix}, \quad \vec{p}_3 = \begin{bmatrix} p_{3x} \\ p_{3y} \end{bmatrix}$$

which yield the following system:

$$r_{xx}p_{1x} + r_{xy}p_{1y} + t_x = q_{1x} \tag{7}$$
$$r_{yx}p_{1x} + r_{yy}p_{1y} + t_y = q_{1y} \tag{8}$$
$$r_{xx}p_{2x} + r_{xy}p_{2y} + t_x = q_{2x} \tag{9}$$
$$r_{yx}p_{2x} + r_{yy}p_{2y} + t_y = q_{2y} \tag{10}$$
$$r_{xx}p_{3x} + r_{xy}p_{3y} + t_x = q_{3x} \tag{11}$$
$$r_{yx}p_{3x} + r_{yy}p_{3y} + t_y = q_{3y} \tag{12}$$

(d) Remember that we are ultimately solving for the components of the $\mathbf{R}$ matrix and the vector $\vec{t}$. This is different from our usual setting and so we need to reformulate the problem into something we are more used to (i.e., $A\vec{x} = \vec{b}$ where x is the unknown). In order to do this, let's view the components of $\mathbf{R}$ and $\vec{t}$ as the unknowns in the equations from from part c). We then have a system of linear equations which we should be able to write in the familiar matrix-vector form. Specifically, we can store the unknowns in a vector $\vec{\alpha} = \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{yx} \\ r_{yy} \\ t_x \\ t_y \end{bmatrix}$ and specify 6×6 matrix $\mathbf{A}$ and vector $\vec{b}$ such that $A\vec{\alpha} = \vec{b}$. Please write out the entries of $\mathbf{A}$ and $\vec{b}$ to match your equations from part c). To get you started, we provide the first row of $\mathbf{A}$ and first entry of $\vec{b}$ which corresponds to one possible equation from part c):

$$\begin{bmatrix} p_{1x} & p_{1y} & 0 & 0 & 1 & 0 \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{yx} \\ r_{yy} \\ t_x \\ t_y \end{bmatrix} = \begin{bmatrix} q_{1x} \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Your job is the fill in the remaining entries according to the other equations.

Solution: We write the system of linear equations from the previous part in matrix form.

$$\begin{bmatrix} p_{1x} & p_{1y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{1x} & p_{1y} & 0 & 1 \\ p_{2x} & p_{2y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{2x} & p_{2y} & 0 & 1 \\ p_{3x} & p_{3y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{3x} & p_{3y} & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{yx} \\ r_{yy} \\ t_x \\ t_y \end{bmatrix} = \begin{bmatrix} q_{1x} \\ q_{1y} \\ q_{2x} \\ q_{2y} \\ q_{3x} \\ q_{3y} \end{bmatrix}$$

(e) In the IPython notebook `prob4.ipynb`, you will have a chance to test out your solution. Plug in the values that you are given for p_x, p_y, q_x, and q_y for each pair of points into your system of equations to solve for the matrix, $\mathbf{R}$, and vector, $\vec{t}$. The notebook will solve the system of equations, apply your transformation to the second image, and show you if your stitching algorithm works. **You are NOT responsible for understanding the image stitching code or Marcela's algorithm.** What are the values for $\mathbf{R}$ and $\vec{t}$ which correctly stitch the images together?

Solution:

The parameters for the transformation from the coordinates of the first image to those of the second image are $\mathbf{R} = \begin{bmatrix} 1.1954 & .1046 \\ -.1046 & 1.1954 \end{bmatrix}$ and $\vec{t} = \begin{bmatrix} -150 \\ -250 \end{bmatrix}$.

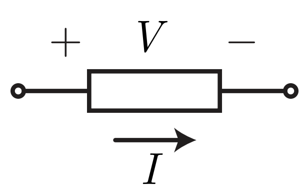
5. Basic Circuit Components

Learning Objectives: *Review basics of cucuit components and current-voltage relationships*

In the laboratory, you are tasked with identifying a single unknown component within a circuit. You can use a piece of electrical equipment called a multimeter to measure either voltage (voltmeter) across or the current (ammeter) through the component (you can measure both quantities simultaneously using two multimeters).

For each part of the problem, deduce the most likely type of circuit component based on the provided voltage and current measurements. Also draw the component circuit symbol and sketch the IV curve.

Hint: You are told the possible choices are ***short circuit (wire)***, ***open circuit***, ***resistor***, ***voltage source***, *and* ***current source***. *Moreover, each part in this problem has a unique component, there are no repeats.*



(a) First, to familiarize yourself with common quantities of *voltage*, *current*, and *resistance*, fill in the unit name and unit symbol for each:

Quantity	Symbol	Unit Name	Unit Symbol
Voltage	V		
Current	I		
Resistance	R		

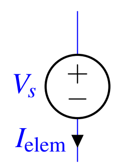
Solution:

Quantity	Symbol	Unit Name	Unit Symbol
Voltage	V	Volts	V
Current	I	Amperes	A
Resistance	R	Ohms	Ω

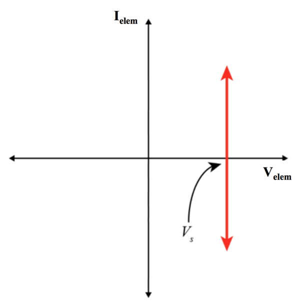
(b) You take one measurement and find the voltage is $V = 10$ V and current is $I = 1$ A. After changing a part of the circuit (not the part you are measuring), you take another measurement and find $V = 10$ V and $I = 2$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

Solution: The component is most likely a **voltage source** with $V_s = 10$ V. The voltage across the voltage source is always equal to the source value, V_s. The current through a voltage source is determined by the rest of the circuit.

Symbol



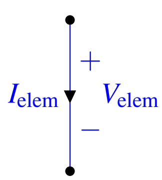
IV Relationship



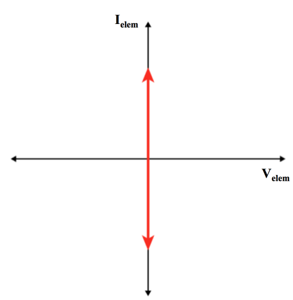
(c) You take one measurement and find the voltage is $V = 0$ V and current is $I = 1$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

Solution: The component is most likely a **short circuit (wire)**. A wire is an ideal connection with zero voltage across it. The current through the wire is determined by the rest of the circuit.

Symbol

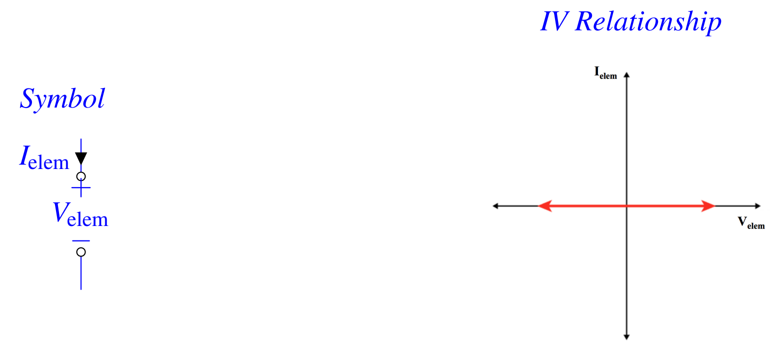


IV Relationship



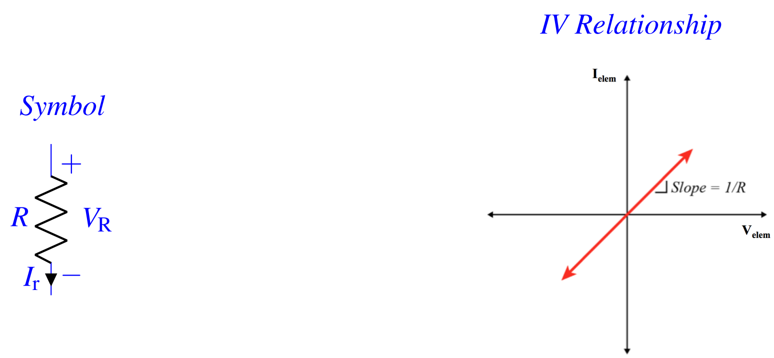
(d) You take one measurement and find the voltage is $V = 5$ V and current is $I = 0$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

Solution: The component is most likely an **open circuit**. There is no current going through an open circuit. The voltage potential across an open circuit is determined by the rest of the circuit.



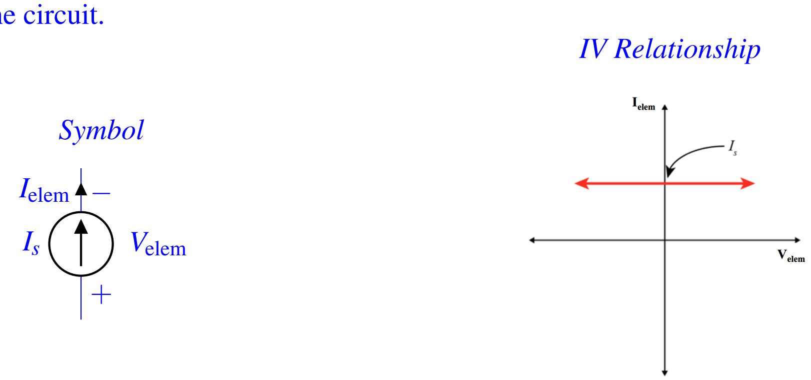
(e) You take one measurement and find the voltage is $V = 10$ V and current is $I = 2$ A. After changing a part of the circuit (not the part you are measuring), you take another measurement and find $V = -5$ V and $I = -1$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

Solution: The component is most likely an **resistor**. The relationship is described by Ohm's Law: $V_R = I_R R$



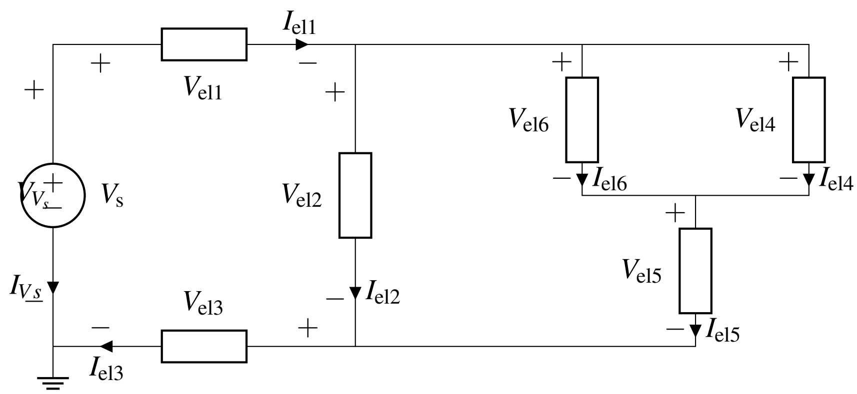
(f) You take one measurement and find the voltage is $V = 10$ V and current is $I = 2$ A. After changing a part of the circuit (not the part you are measuring), you take another measurement and find $V = -3$ V and $I = 2$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

Solution: The component is most likely a **current source** with $I_s = 2$ A. The current through a current source is always equal to the source value I_s. The voltage across a current source is determined by the rest of the circuit.



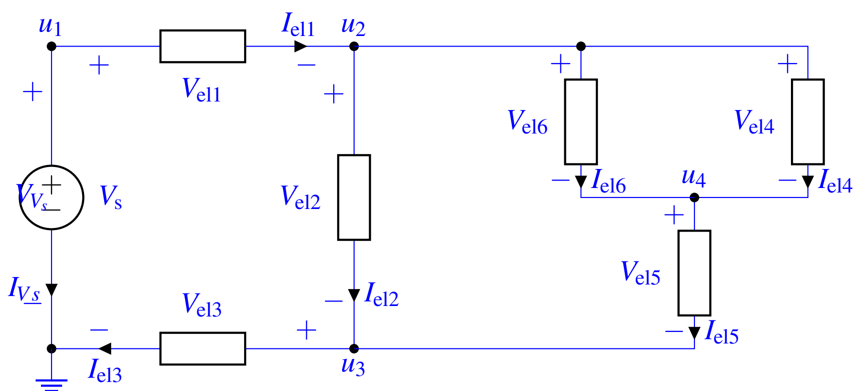
6. Intro to Circuits

Learning Goal: *This problem will help you practice labeling circuit elements and setting up KVL equations.*



(a) How many nodes does the above circuit have? Label them.
Note: The reference/0V (ground) node has been selected for you, so you don't need to label that, but you need to include it in your node count.

Solution: There are a total of 5 nodes in the circuit, including the reference node. They are labeled u_1 - u_4 below:



(b) Express all element voltages (including the element voltage across the source, V_s) as a function of node voltages. This will depend on the node labeling you chose in part (a).

Solution: For our specific node labeling we can write:

$$V_{V_s} = u_1 - 0 = u_1 (= V_s)$$

$$V_{el1} = u_1 - u_2$$

$$V_{el2} = u_2 - u_3$$

$$V_{el3} = u_3 - 0 = u_3$$

$$V_{el4} = u_2 - u_4$$

$$V_{el5} = u_4 - u_3$$

$$V_{el6} = u_2 - u_4$$

Notice that the element voltage is always of the form: $V_{el} = u_+ - u_-$.

(c) Write a KVL equation for all the loops that contain the voltage source V_s. These equations should be a function of element voltages and the voltage source V_s.

Solution: Notice that there are in fact 3 loops that contain the voltage source V_s, for which we can write the following equations, starting each time from the reference node and ending at the reference node:

$$V_s - V_{\text{el1}} - V_{\text{el2}} - V_{\text{el3}} = 0$$
$$V_s - V_{\text{el1}} - V_{\text{el6}} - V_{\text{el5}} - V_{\text{el3}} = 0$$
$$V_s - V_{\text{el1}} - V_{\text{el4}} - V_{\text{el5}} - V_{\text{el3}} = 0$$

The reason this is not specific to our labeling is that the polarity of all elements is either given or set through the passive sign convention.

7. Prelab Questions These questions pertain to the Pre-Lab reading for the Imaging 2 lab. You can find the reading under the Imaging 2 Lab section on the 'Schedule' page of the website. We do not expect in-depth answers for the questions. Please limit your answers to a maximum of 2 sentences.

(a) Briefly explain (in 1-2 sentences) what the H matrix signifies.

(b) How will we get the image vector $\vec{i}$ from $\vec{s} = H\vec{i}$, the equation representing our imaging system?

Solution:

(a) The H matrix is also known as the mask matrix. It allows us to selectively choose what pixels we want to read (scan) at a given time.

(b) $\vec{i} = H^{-1}\vec{s}$. Multiplying by H^{-1} on both sides of the equation gives us the image vector $\vec{i}$

8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.

