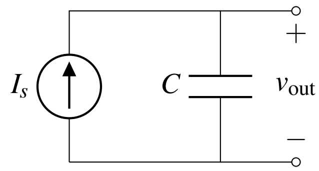
EECS 16A Designing Information Devices and Systems I
Spring 2023 Discussion 8B

1. Current Sources And Capacitors

Given the circuit below, find an expression for $v_{\text{out}}(t)$ in terms of I_s, C, V_0, and t, where V_0 is the initial voltage across the capacitor at $t = 0$.



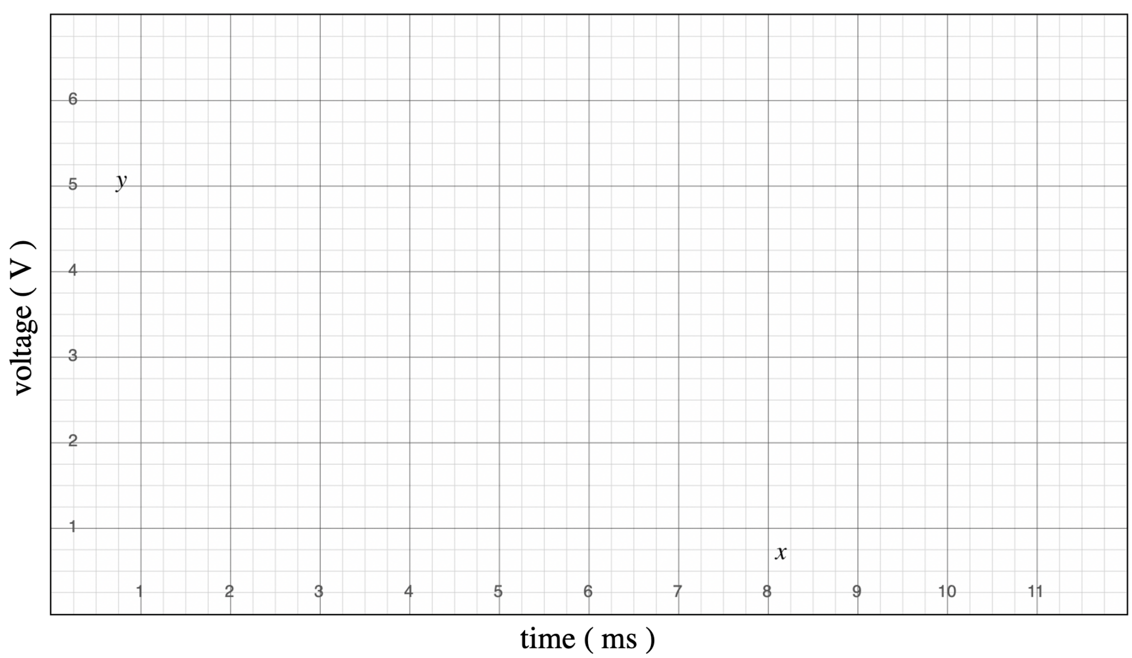
Then plot the function $v_{\text{out}}(t)$ over time on the graph below for the following conditions detailed below.
Use the values $I_s = 1\text{mA}$ and $C = 2\,\mu\text{F}$.

(a) Capacitor is initially uncharged, with $V_0 = 0$ at $t = 0$.

(b) Capacitor has been charged with $V_0 = +1.5\text{V}$ at $t = 0$.

(c) **Practice:** Swap this capacitor for one with half the capacitance $C = 1\,\mu\text{F}$, which is initially uncharged, with $V_0 = 0$ at $t = 0$.

HINT: Recall the calculus identity $\int_a^b f'(x)dx = f(b) - f(a)$, where $f'(x) = \frac{df}{dx}$.



Answer:
The key here is to exploit the capacitor equation by taking its time-derivative

$$Q = C\,V_{out} \quad \longrightarrow \quad \frac{dQ}{dt} \equiv I_s = C\frac{dV_{out}}{dt}\,.$$

From here we can rearrange and show that $\frac{dV_{out}}{dt} = \frac{I_s}{C}$.
Thus the voltage has a constant slope!
Our solution is

$$V_{out}(t) = V_0 + \left(\frac{I_s}{C}\right) t \qquad \square$$

To be more mathematically formal, we are solving a differential equation that happens to return a linear function for $v_{out}(t)$:

$$\frac{dV_{out}}{dt} = \frac{I_s}{C} \quad \longrightarrow \quad \int_0^t \frac{dV_{out}}{dt}dt \equiv V_{out}(t) - V_{out}(0) = \int_0^t \frac{I_s}{C}dt \equiv \frac{I_s}{C}\int_0^t 1dt \equiv \frac{I_s}{C}\,t$$

Thus we arrive at the same statement as seen earlier $V_{out}(t) = V_{out}(0) + \left(\frac{I_s}{C}\right)t$.

For all parts, we have $I_s = 1\text{mA}$. For part (a), we have $V_{out}(0) = 0\text{V}$ and $C = 2\mu\text{F}$. Plugging this into our equation, we get:

$$V_{out}(t) = 0\text{V} + \frac{1\text{mA}}{2\mu\text{F}} = \left(\frac{1}{2}\frac{\text{V}}{\text{ms}}\right)t$$

.

For part (b), this only changes by an intercept / initial condition $V_{out}(0) = 1.5\text{V}$:

$$V_{out}(t) = 1.5\text{V} + \frac{1\text{mA}}{2\mu\text{F}} = 1.5\text{V} + \left(\frac{1}{2}\frac{\text{V}}{\text{ms}}\right)t$$

.

For part (c), we have $V_{out}(0) = 0\text{V}$ and $C = 1\mu\text{F}$.

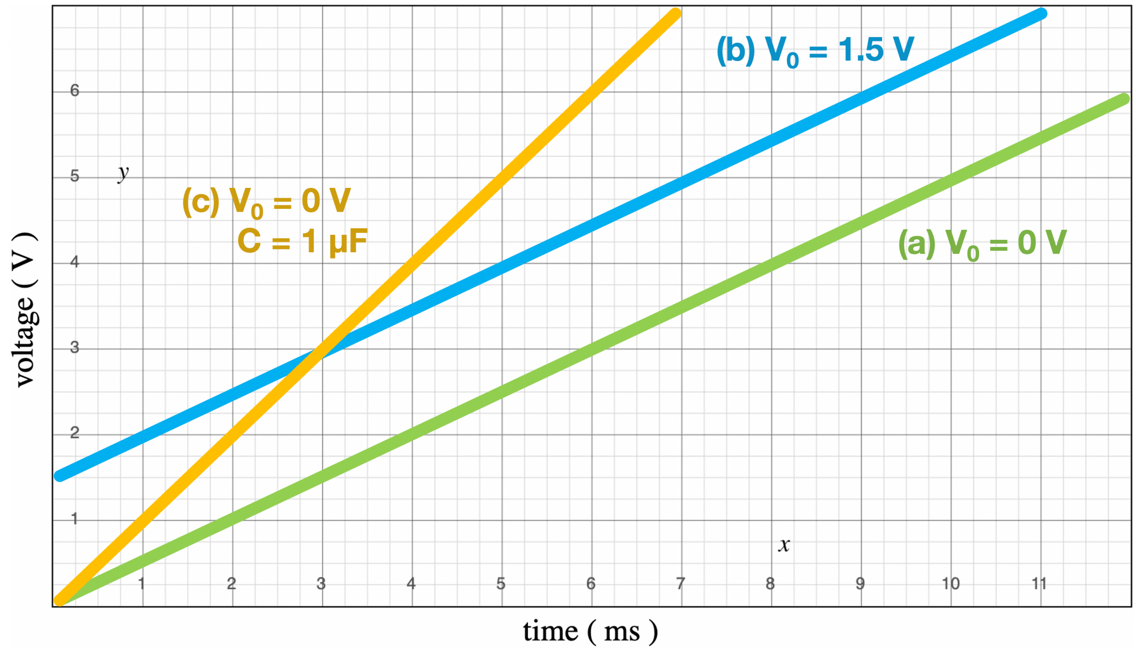
$$V_{out}(t) = 0\text{V} + \frac{1\text{mA}}{1\mu\text{F}} = \left(1\frac{\text{V}}{\text{ms}}\right)t$$

.

With half the capacitance and the same current, we get that the slope is twice as large. Some physical intuition: capacitors are like buckets, and charge is like the total water in the bucket. Voltage is a measure of the height of the water in the bucket. Capacitance is the cross-sectional area of the bucket, i.e. the area of the top hole. If the bucket's cross-sectional area (capacitance) is halved, then a hose filling the bucket (current source) would cause the water level (voltage) to rise twice as fast.

When plotting, make sure to recall (a) and (c) start at the origin, while (b) has initially charged plates by

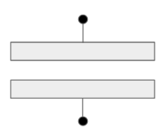
$V_0 = 1.5V$. Results are shown below



2. Capacitance Equivalence

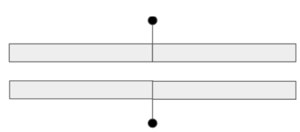
For the structures shown below, assume that the plates have a depth L into the page and a width W and are always a distance d apart. The dielectric between the plates has absolute permitivity ε. For the following calculations, assume the capacitance is purely parallel plate, i.e. ignore fringing field effects.

(a) What is the capacitance of the structure shown below?



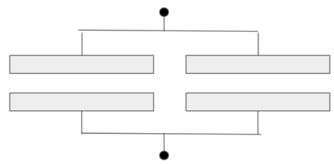
Answer:
The capacitance of two parallel plate conductors is given by $C = \varepsilon \frac{A}{d}$. The cross-sectional area A is WL, so the capacitance is $C = \varepsilon \frac{WL}{d}$.

(b) Suppose that we take two such structures and put them next to each other as shown below. What is the capacitance of this new structure?



Answer:
Here, we have just doubled the width of the capacitor plates. The new capacitance is $C = \varepsilon \frac{2WL}{d}$. Notice that this is just double the capacitance from the first part.

(c) Now suppose that rather than connecting them together as shown above, we connect them with an ideal wire as shown below. What is the capacitance of this structure?

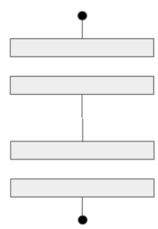


Answer:

Even though we use a wire to connect the plates (vs in part (b) where they are abutted), they are still connected to the same nodes and so can be treated as one capacitor. We have to look at the area with overlapping parallel plates. Hence the total plate area is still $2WL$, and we get $C = \varepsilon \frac{2WL}{d}$.

In terms of the physical intuition of capacitors as buckets that fill up with charge, we can see that having two buckets with the same water level (voltage) is equivalent to one larger bucket with double the cross-sectional area (capacitance).

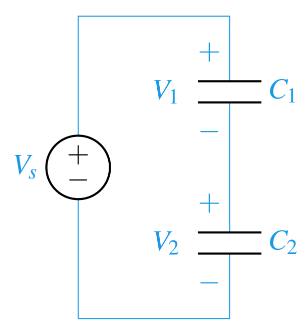
(d) Suppose that we now take two capacitors and connect them as shown below. What is the capacitance of the structure?



Answer:

Physically, if we were to charge these capacitors, say with positive current coming in from the top, and the top plate was charged to some $+Q$, the second plate would end up with charge $-Q$ because like charges repel and opposite ones attract. Notice that the second and third plates are not connected (via any conductive material) to the rest of the circuit, meaning that together they are a neutral piece of metal that must have net zero charge. Therefore, if the second plate has $-Q$ charge, the third plate must have $+Q$. Finally, this induces the bottommost plate to have $-Q$ charge. So we see that two capacitors linked up like this must have the same charge if they started at the same charge.

To solve for the equivalent capacitance of the structure, we apply a voltage source and calculate what C_{eq} would make $Q_{tot} = C_{eq}V_s$, where Q_{tot} is the total charge building up in the circuit.



where $C_1 = C_2 = C = \varepsilon \frac{A}{d}$. Then KVL says:

$$V_s = V_1 + V_2$$
$$\frac{Q_{tot}}{C_{eq}} = \frac{Q_1}{C_1} + \frac{Q_2}{C_2}$$

If the capacitors were initially discharged, then we know that the charge across them is equal: $Q_1 = Q_2 = Q$. Furthermore, $Q_{tot} = Q$ also because we only count the charge that built up on the topmost and bottommost plates. The charge in the middle node/on the middle two plates could never leave the node, and could not be discharged to do electrical work, so it does not count.

$$\frac{Q}{C_{eq}} = \frac{Q}{C_1} + \frac{Q}{C_2}$$
$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

This gives an equivalent capacitor to the two in series. This last equation is the same result as the "parallel" rule. In this case, $C_1 = C_2 = C$, so $C_{eq} = \frac{C}{2}$.