EECS 16A Designing Information Devices and Systems I
Spring 2023 Homework 11

This homework is due April 7, 2023, at 23:59.
Self-grades are due April 14, 2023, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw11.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned)

Submit each file to its respective assignment on Gradescope.

1. **Reading Assignment**

For this homework, please read Note 17 and Note 18 to learn about comparators and op-amps. You are always encouraged to read beyond this as well.

(a) What is the purpose of a comparator? How can we use a comparator circuit to detect a touch for a capacitive touchscreen?

Solution: A comparator gives a binary output of either a logical 'high' or 'low' value depending on the difference of voltage between its two input terminals. Thus, comparators can be used as logical indicators, and in the case of a touchscreen a comparator can indicate whether or not a touch occurs. In the capacitive touchscreen, we connect an alternating current source to the equivalent capacitance, so that it can charge and discharge periodically. Then one terminal of the comparator is connected to the capacitor voltage and the other terminal is connected to a reference voltage source. The value of this reference voltage source needs to be between the peaks of the capacitor voltage with and without touch. Thus, when a touch occurs, the voltage difference between the input terminals will invert in sign, and the comparator will respond.

(b) If the op-amp supply voltages are $V_{DD} = 5\,\text{V}$ and $V_{SS} = 0\,\text{V}$, then what are the minimum and maximum value of V_{out}?

Solution: The minimum op-amp output will always be the value of the V_{SS} supply rail, and thus 0V in this case. The maximum op-amp output will always be the value of the V_{DD} supply rail, and thus 5V in this case.

(c) What does the internal gain of an op-amp, A, mean? What is its value for an ideal op-amp? What about for a non-ideal one?

Solution: The internal gain of an op-amp, A is the ratio of the output voltage to the error voltage, i.e. A is given by $\frac{V_{out}}{u_+ - u_-}$. For ideal op-amps, $A \to \infty$. For non-ideal op-amps, A is finite.

2. **It's finally raining!**

A lettuce farmer in the Salinas Valley has grown tired of imprecise online rainfall forecasts. They decide to take matters into their own hands by building a rain sensor. They place a square tank outside and attach two metal plates to two opposite sides in an effort to make a capacitor whose capacitance varies with the amount of water inside.

Note: In practice, water is conductive. However for this problem, assume the metal plates are properly insulated so that no current flows through the water and we can treat it like a dielectric material. In other words, the electric circuit is better modeled as a capacitance and not a resistance.

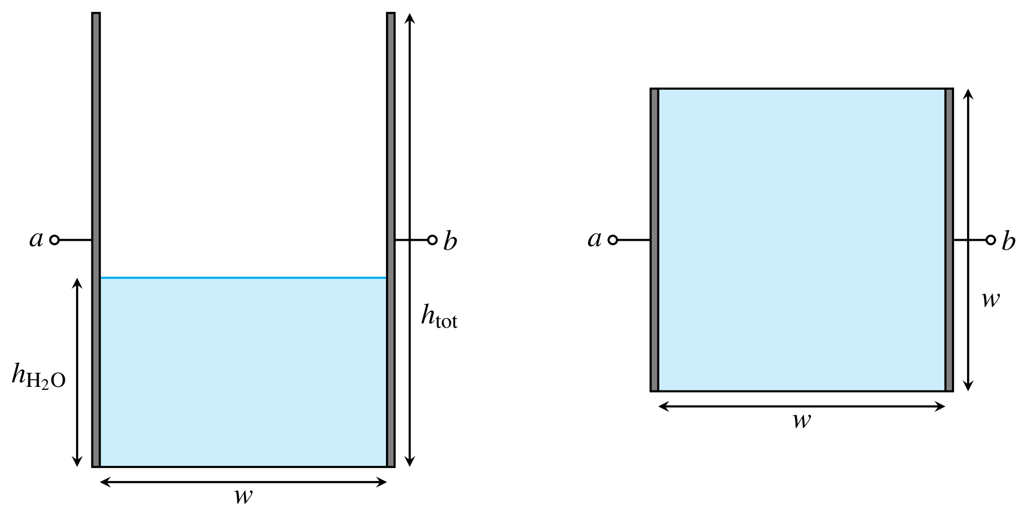


Figure 1: Tank side view (left) and top view (right).

The width and length of the tank are both w (i.e., the base is square) and the height of the tank is h_{tot}.

(a) What is the capacitance between terminals a and b when the tank is full? What about when it is empty? The permittivity of air is $\varepsilon_{air} = \varepsilon_0$, and the permittivity of rainwater is $\varepsilon_{H_2O} = 75\varepsilon_0$.

Solution:

Capacitance of parallel plates is governed by the equation:

$$C = \frac{\varepsilon A}{d},$$

where ε is the *permittivity* of the dielectric material, A is the area of the plates, and d is the distance between the plates. If we apply this to our physical structure, we find that the area of the plates is $h_{tot} \cdot w$, and the distance between the plates is w. The only difference here between a full and empty tank is the permittivity of the material between the two plates.

$$C_{empty} = \frac{\varepsilon_{air}\, h_{tot} w}{w} = \varepsilon_0\, h_{tot}$$

$$C_{full} = \frac{\varepsilon_{H_2O}\, h_{tot} w}{w} = 75\varepsilon_0\, h_{tot}$$

(b) Suppose the height of the water in the tank is h_{H_2O}. Model the tank as a pair of capacitors in parallel, where one capacitor has a dielectric of air, and one capacitor has a dielectric of water. Find the total capacitance C_{tank} between the two metal walls/plates using circuit equivalence.

Solution:

We can break the total capacitance into two parts. First, let's calculate the capacitance of the two plates separated by water:

$$C_{\text{water}} = \frac{\varepsilon_{H_2O} h_{H_2O} w}{w} = 75\varepsilon_0 h_{H_2O}$$

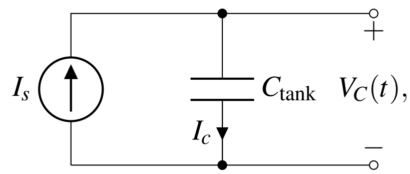
And now we can calculate the capacitance of the two plates separated by air:

$$C_{\text{air}} = \frac{\varepsilon_{\text{air}} (h_{\text{tot}} - h_{H_2O}) w}{w} = \varepsilon_0 \cdot (h_{\text{tot}} - h_{H_2O})$$

These two capacitors appear in parallel, as the result from the layer of water at the bottom of the tank, and the air above the water. Because these two capacitors appear in parallel, we can simply add our two previous results to find the total equivalent capacitance:

$$C_{\text{tank}} = C_{\text{water}} + C_{\text{air}} = \varepsilon_0 \cdot (h_{\text{tot}} + 74 h_{H_2O})$$

(c) After building this tank, the farmer consults the internet to assist them with a capacitance-measuring circuit. A fellow internet user recommends building the following circuit:



where C_{tank} is the total tank capacitance between terminals a and b calculated in part (b), and I_s is a known current supplied by a current source.

The user suggests measuring $V_C(t)$ for a brief interval of time, compute the rate of change of V_C, and determine C_{tank}.

Determine $V_C(t)$, where t is the number of seconds elapsed since the start of the measurement. You should assume that before any measurements are taken, the voltage across C_{tank} is initialized to 0 V, i.e. $V_C(0) = 0$.

Solution:

The element equation for the capacitor is:

$$I_C = C_{\text{tank}} \frac{dV_C}{dt}$$

We also know from KCL that:

$$I_C = I_s$$

Thus, we get the following differential equation for V_C:

$$\frac{dV_C}{dt} = \frac{I_s}{C_{\text{tank}}}$$

We recall that I_s and C_{tank} are constant values and the initial value of V_C is zero ($V_C(0) = 0$). Applying these facts and integrating the differential equation, we get the following equation for V_C:

$$V_C(t) = \frac{I_s}{C_{\text{tank}}} t$$

(d) Using the equation you derived for $V_C(t)$, describe how you can use this circuit to determine C_{tank} and $h_{\text{H}_2\text{O}}$.

Solution:

We connect the current source providing I_s to the capacitor C_{tank}. After a known amount time, t_f, passes, we measure the capacitor voltage, $V_C(t_f)$, and plug it into the following equation (assuming, as before, that $V_C(0) = 0$):

$$C_{\text{tank}} = \frac{I_s}{V_C(t_f)} t_f$$

If we know C_{tank}, we can determine $h_{\text{H}_2\text{O}}$. Using the equation derived in part (b), we see that

$$h_{\text{H}_2\text{O}} = \frac{C_{\text{tank}} - h_{tot}\varepsilon}{74\varepsilon}$$

3. Op-Amp in Negative Feedback

In this question, we analyze op-amp circuits that have finite op-amp gain A. We replace the op-amp with an equivalent circuit model with parameterized gain, A, and observe the gain's effect on the terminal and output voltages as the gain approaches infinity. **Note here that $V_{SS} = -V_{DD}$.**

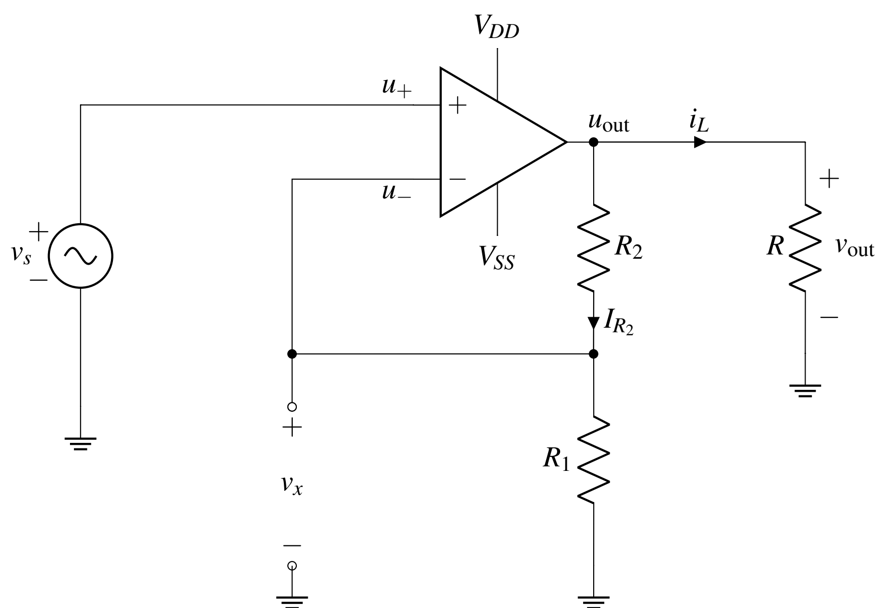


Figure 2: Non-inverting amplifier circuit using an op-amp for feedback

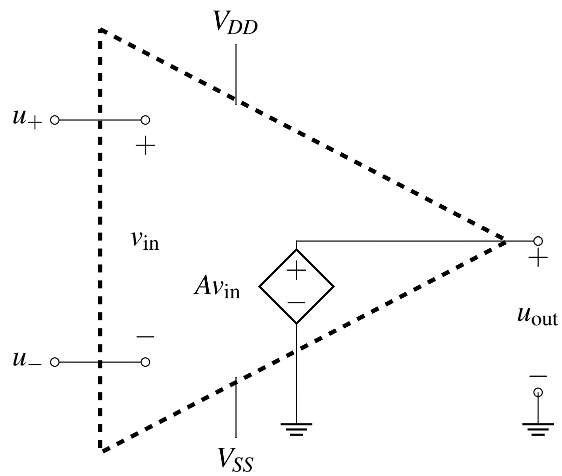
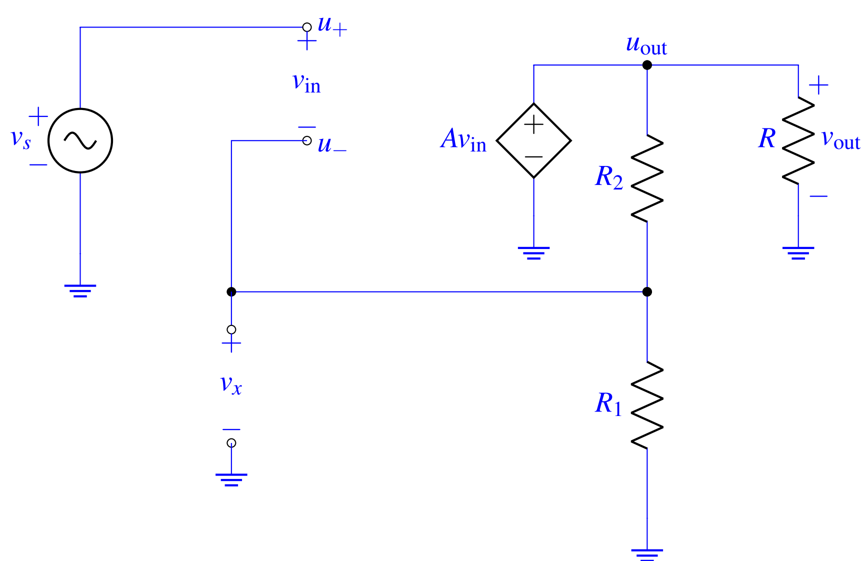


Figure 3: Op-amp model with finite gain, A

(a) Let us first examine when gain A is finite (i.e. not ∞). To understand what happens in this case, first draw an equivalent circuit for the first op amp circuit in Fig. 2, **by replacing the ideal op-amp in the non-inverting amplifier with the op-amp model in Fig. 3.**

Solution:

This is the equivalent circuit of the op-amp:



(b) Now, using this setup, calculate v_{out} and v_x in terms of A, v_s, R_1, R_2, and R. Is the magnitude of v_x larger or smaller than the magnitude of v_s? Do these values depend on R? *Hint: Note that the currents into the input terminals of the op-amp are zero.*

Solution:

Since v_{out} is connected to the output of the op-amp, which is a voltage source, we can determine v_{out}:

$$v_{out} = A \cdot (u_+ - u_-) = A \cdot (v_s - v_x)$$

Since there is no current flowing into the op-amp input terminals from nodes u_+ and u_-, R_1 and R_2 form a voltage divider and $v_x = \frac{R_1}{R_1+R_2} v_{\text{out}}$. Thus, substituting and solving for v_{out}:

$$v_{\text{out}} = A \cdot \left(v_s - v_{\text{out}} \frac{R_1}{R_1+R_2} \right) = \frac{1}{\frac{R_1}{R_1+R_2} + \frac{1}{A}} v_s$$

Knowing v_{out}, we can find v_x:

$$v_x = \frac{R_1}{R_1+R_2} v_{\text{out}} = \frac{1}{1 + \frac{R_1+R_2}{AR_1}} v_s$$

Notice that v_x is slightly smaller than v_s, meaning that in equilibrium in the non-ideal case, u_+ and u_- are not equal. v_{out} and v_x do not depend on R, which means that we can treat v_{out} as a voltage source that supplies a constant voltage independent of the load R.

(c) Using your solution to the previous part, calculate the limits of v_{out} and v_x as $A \to \infty$.

Solution:

As $A \to \infty$, the fraction $\frac{1}{A} \to 0$, so

$$v_{\text{out}} = \frac{1}{\frac{R_1}{R_1+R_2} + \frac{1}{A}} v_s$$

converges to

$$\begin{aligned} \lim_{A\to\infty} v_{\text{out}} &= \left(\frac{1}{\frac{R_1}{R_1+R_2} + 0} \right) \cdot v_s \\ &= \frac{R_1+R_2}{R_1} v_s \end{aligned}$$

Additionally, v_s converges to:

$$\begin{aligned} \lim_{A\to\infty} v_x &= \lim_{A\to\infty} \frac{R_1}{R_1+R_2} v_{\text{out}} = \frac{R_1}{R_1+R_2} \left(\frac{R_1+R_2}{R_1} v_s \right) \\ &= v_s \end{aligned}$$

If we observe the op-amp is in negative feedback, we can apply the Golden Rule: $u_+ = u_-$. Thus $v_x = v_s$. Then the current flowing through R_1 to ground is $\frac{v_s}{R_1}$. By KCL, this same current flows through R_2 since no current flows into the negative input terminal of the op-amp (u_-). Thus, the voltage drop across R_2 is $v_{\text{out}} - v_x = i \cdot R_2 = \frac{R_2}{R_1} v_s$. Therefore, $v_{\text{out}} = v_s + \frac{R_2}{R_1} v_s = \frac{R_1+R_2}{R_1} v_s$. The answers are the same if you take the limit as $A \to \infty$.

(d) Are there maximum and minimum values v_{out} can be for this op-amp circuit? If so, determine these values.

Hint: Consider the voltage supply rails.

Solution:

For a valid op-amp circuit, the maximum value of v_{out} is V_{DD} and the minimum value of v_{out} is $V_{SS} = -V_{DD}$.

4. Transistor Equivalent

Consider the amplifier circuit in Fig. 4 which amplifies input V_{in} to output V_{out}. The circuit accomplishes this by using a bipolar junction transistor or BJT. The BJT is a three-terminal circuit element with nodes B, C, and E.

In some situations, the BJT can be modeled with an equivalent linear circuit containing a voltage-dependent current source as shown in Fig. 5.

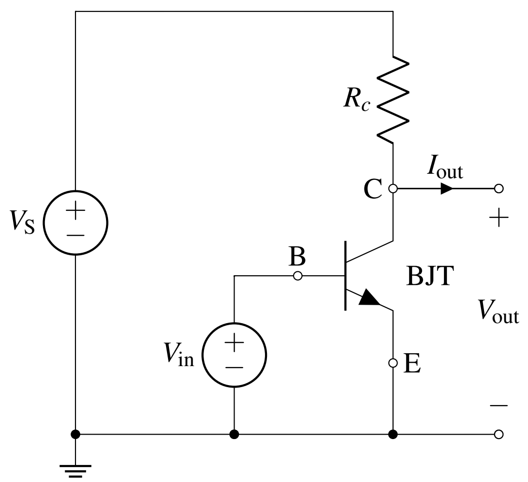


Figure 4: Amplifier circuit with BJT

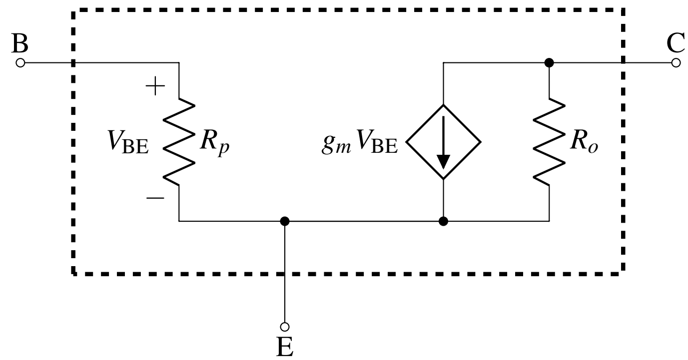


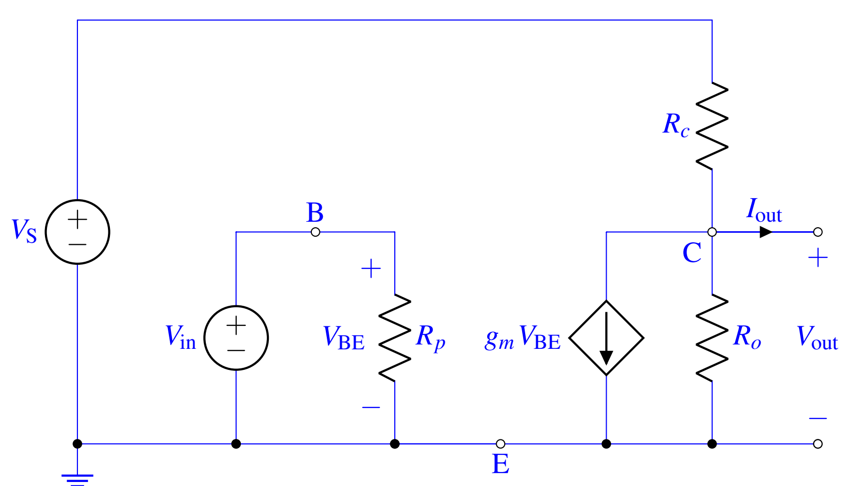
Figure 5: Equivalent circuit model for BJT

We want to find the Thevenin and Norton equivalent of the amplifier circuit across the terminals V_{out} (i.e., between nodes C and E).

Note: You can use the parallel operator (||) in your final answers.

(a) Redraw the original amplifier circuit in Fig. 4 but with the BJT equivalent circuit model in Fig. 5 substituted.

Solution:

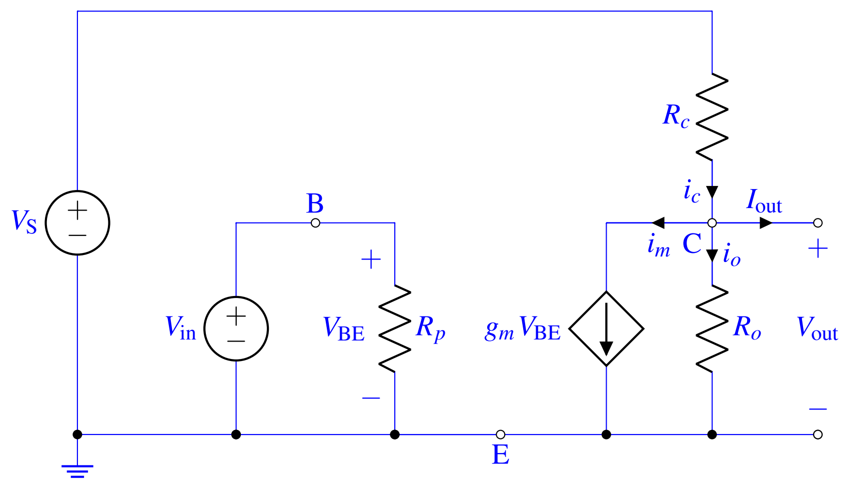


(b) Use the open circuit test to find the Thevenin voltage, V_{th}, between nodes C and E.

Recall the open circuit test finds $V_{out} = V_{oc}$ when an open circuit is connected across the terminals, then $V_{th} = V_{oc}$.

Solution:

Take the full equivalent circuit from part (a) and connect an open circuit across the output terminals.



Identify the KVL equation

$$V_{in} - V_{BE} = 0$$

which results in the dependent current source having a current of

$$g_m V_{BE} = g_m V_{in}$$

Next, identify a KCL equation at the output node and substitute the known node voltages

$$i_c - i_m - i_o - I_{out} = 0$$
$$\frac{V_S - V_{out}}{R_c} - g_m V_{in} - \frac{V_{out}}{R_o} - 0 = 0$$

Simplifying and rearranging this node voltage equation in terms of V_{out} is

$$V_{out} \cdot \left(\frac{1}{R_c} + \frac{1}{R_o}\right) = \frac{1}{R_c} V_S - g_m V_{in}$$
$$V_{out} = \left(\frac{1}{R_c} + \frac{1}{R_o}\right)^{-1} \cdot \left(\frac{1}{R_c} V_S - g_m V_{in}\right)$$
$$V_{out} = R_c || R_o \cdot \left(\frac{1}{R_c} V_S - g_m V_{in}\right)$$

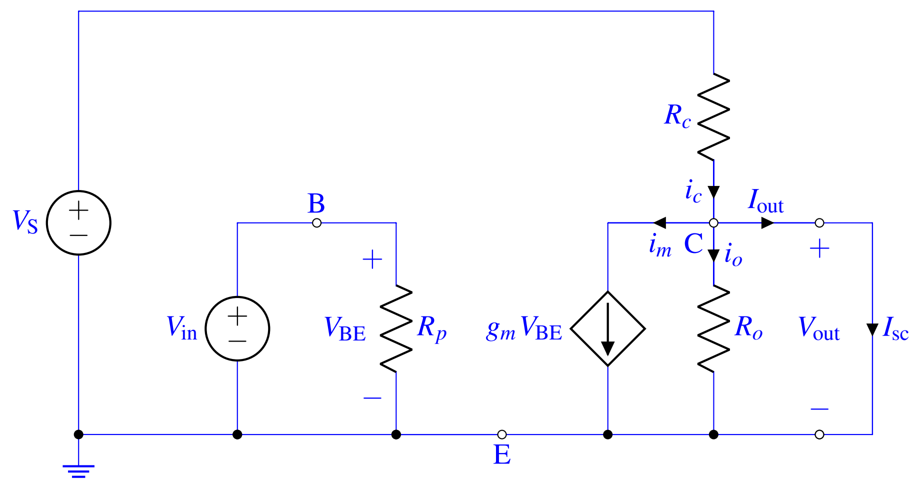
This concludes the open circuit test where the Thevenin voltage is

$$V_{th} = V_{oc} = V_{out} = R_c || R_o \cdot \left(\frac{1}{R_c} V_S - g_m V_{in}\right)$$

(c) Use the short circuit test to find the Norton current, I_{no}, between nodes C and E.
Recall the short circuit test finds $I_{out} = I_{sc}$ when a short circuit is connected between the terminals, then $I_{no} = I_{sc}$.

Solution:

Now for the short circuit test, connect a short circuit across the output terminals.



Similar to part (a), we identify the KVL equation

$$V_{\text{in}} - V_{\text{BE}} = 0$$

which results in the dependent current source having a current of

$$g_m V_{\text{BE}} = g_m V_{\text{in}}$$

Next, identify a KCL equation at the output node and substitute the known node voltages

$$i_c - i_m - i_o - I_{out} = 0$$

$$\frac{V_S - V_{\text{out}}}{R_c} - g_m V_{\text{in}} - \frac{V_{out}}{R_o} - I_{\text{out}} = 0$$

In this circuit, $V_{\text{out}} = 0$. Simplify and rearrange this node voltage equation in terms of I_{out} as

$$I_{\text{out}} = \frac{1}{R_c} V_S - g_m V_{\text{in}}$$

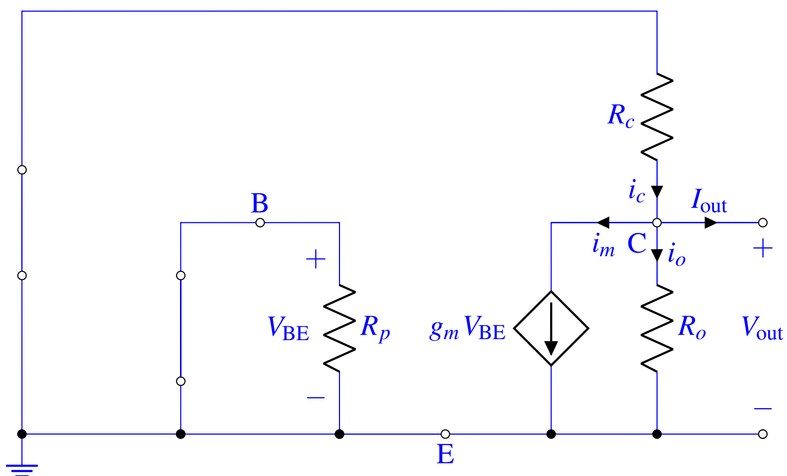
This concludes the short circuit test where the Norton current is

$$I_{\text{no}} = I_{\text{sc}} = \frac{1}{R_c} V_S - g_m V_{\text{in}}$$

(d) Find the Thevenin/Norton resistance $R_{\text{th}} = R_{\text{no}}$ using $R_{\text{th}} = \frac{V_{\text{th}}}{I_{\text{no}}}$.
Solution:

$$\begin{aligned} R_{\text{th}} = \frac{V_{\text{th}}}{I_{\text{no}}} &= \frac{R_c || R_o \cdot \left(\frac{1}{R_c} V_S - g_m V_{\text{in}}\right)}{\left(\frac{1}{R_c} V_S - g_m V_{\text{in}}\right)} \\ &= R_c || R_o \end{aligned}$$

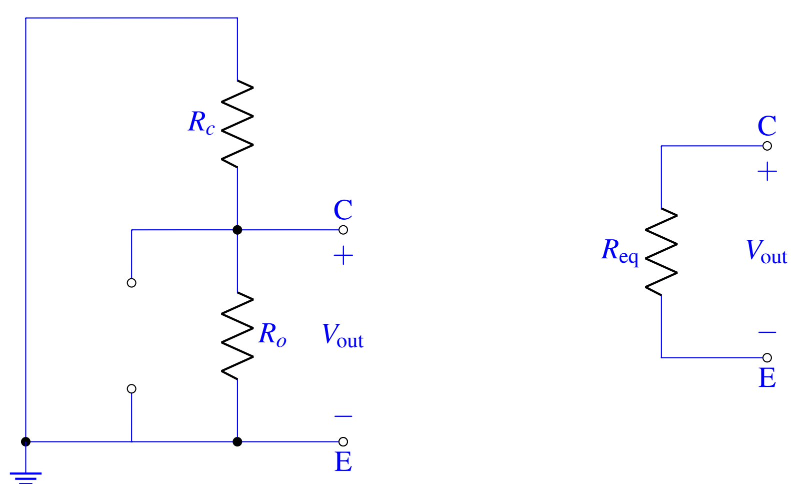
(e) We can also find R_{th} by turning off all of the independent sources (but *not* the dependent sources) and deriving the equivalent resistance seen from the terminals. Derive R_{th} with this method. Does it match your answer from part (c)?
Hint: To simplify the dependent source, focus on first finding V_{BE}.
Solution:



If we turn off the two independent voltage sources ($V = 0$), then

$$V_{BE} = 0 \quad \longrightarrow \quad g_m V_{BE} = 0$$

which results in the dependent current source being an equivalent open circuit ($I = 0$).



The final equivalent resistance R_{eq} is comprised of R_c in parallel with R_o.

$$R_{th} = R_{eq} = R_c || R_o$$

This Thevenin resistance R_{th} matches the value derived in part (c).

5. Digital to Analog Converter (DAC)

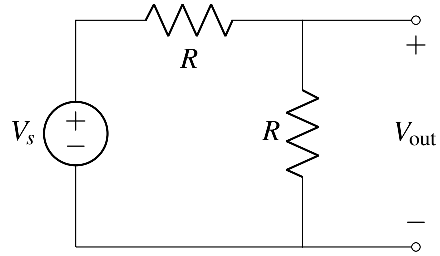
In many electronics applications, such as audio speakers, we need to produce an analog output, or any voltage between 0 to V_s. These analog voltages must be produced from digital voltages that can only be values of V_s or 0. A circuit that does this is known as a Digital to Analog Converter (DAC). It takes a binary representation of a number and turns it into an analog voltage.

The output of a DAC can be represented with the equation shown below:

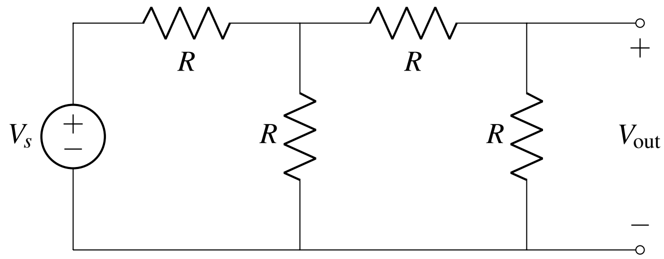
$$V_{out} = V_s \sum_{n=0}^{N} \frac{1}{2^n} b_n$$

where each binary digit b_n is multiplied by $\frac{1}{2^n}$.

(a) We know how to take an input voltage and divide it by 2:



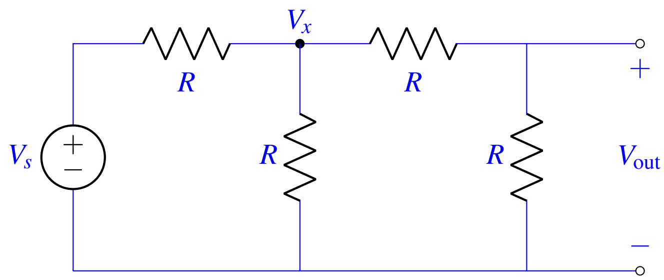
To divide by larger powers of two, we might hope to just "cascade" the above voltage divider. For example, consider:



Calculate V_{out} in the above circut. Is $V_{out} = \frac{1}{4}V_s$?

Solution:

We first find the potential V_x.



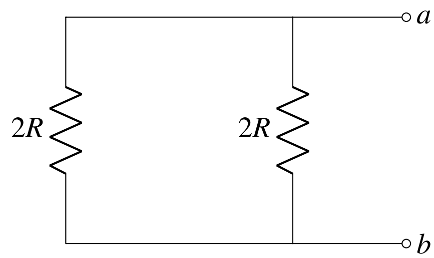
$$V_x = \frac{R \parallel 2R}{R + R \parallel 2R} V_s = \frac{\frac{2}{3}R}{R + \frac{2}{3}R} V_s = \frac{2}{5} V_s$$

$$V_{out} = \frac{R}{R+R} V_x = \frac{1}{2} \cdot \frac{2}{5} V_s = \frac{1}{5} V_s \neq \frac{1}{4} V_s$$

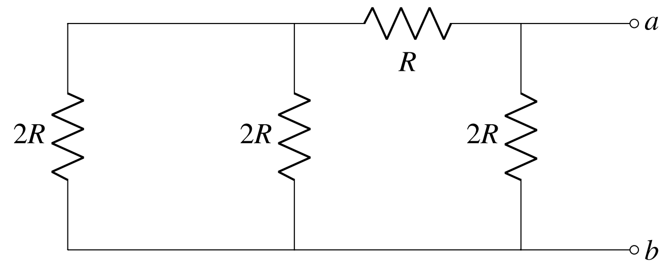
No, V_{out} does not equal $\frac{1}{4}V_s$.

(b) The R-$2R$ ladder, shown below, has a very nice property. For each of the circuits shown below, find the equivalent resistance looking in from points a and b. Do you see a pattern?

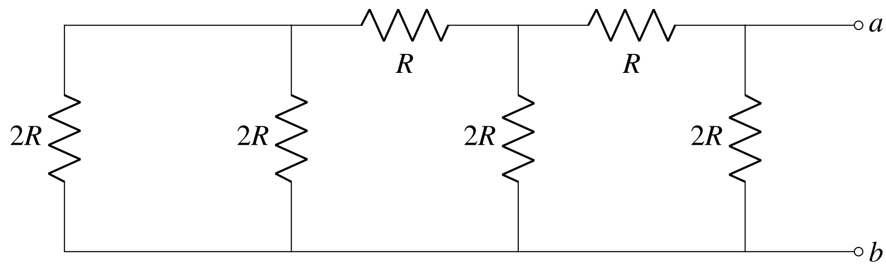
i.



ii.



iii.

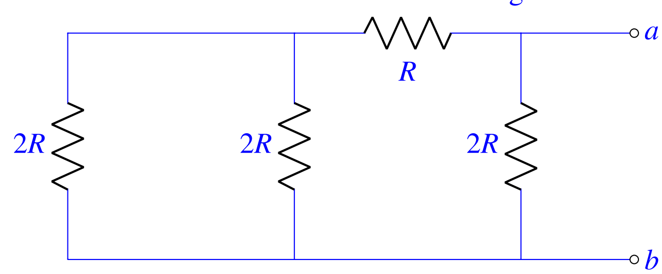


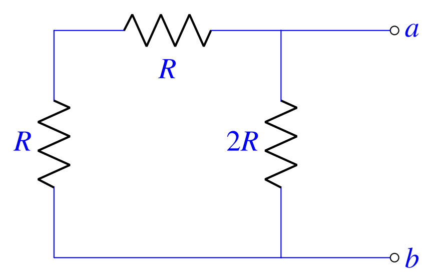
Solution:

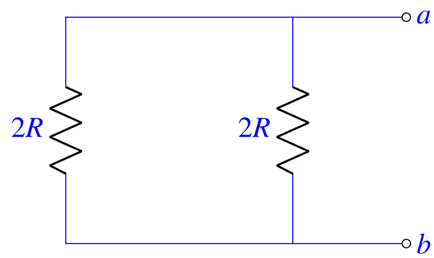
i.

$$R_{eq} = 2R \parallel 2R = R$$

ii. We find the equivalent resistance for the resistors from left to right.

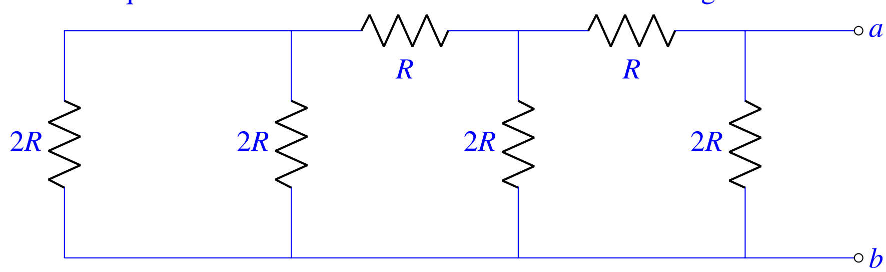


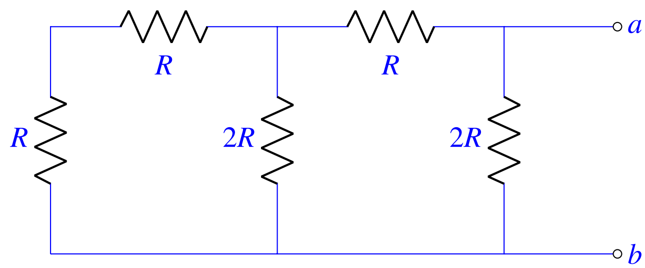


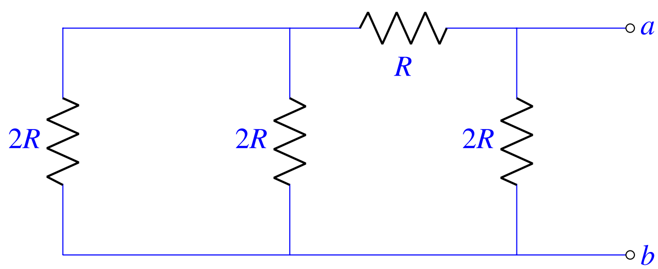


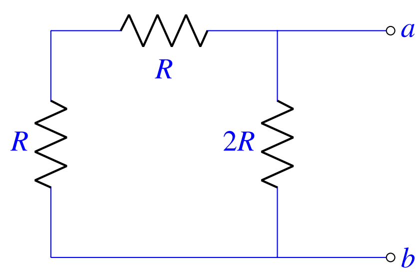
$$R_{eq} = 2R \parallel 2R = R$$

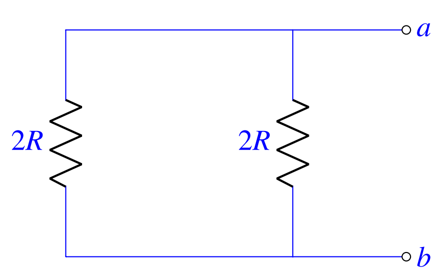
iii. Again, we find the equivalent resistance for the resistors from left to right.







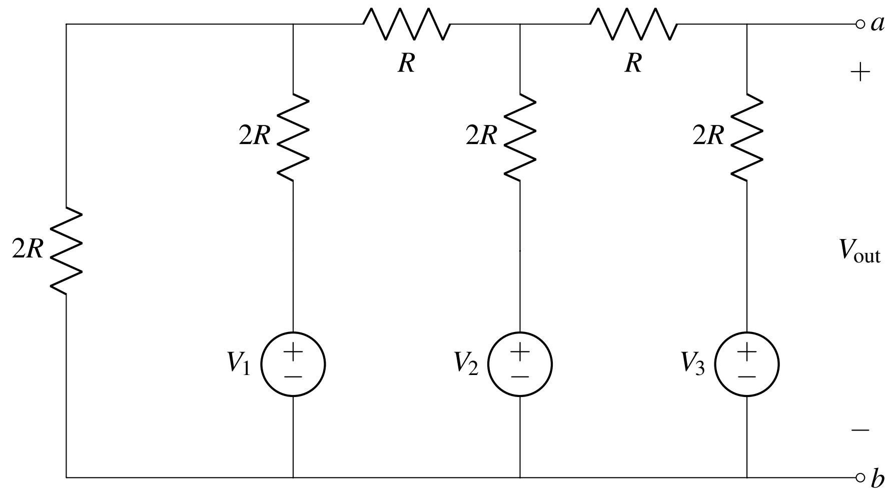




$$R_{eq} = 2R \parallel 2R = R$$

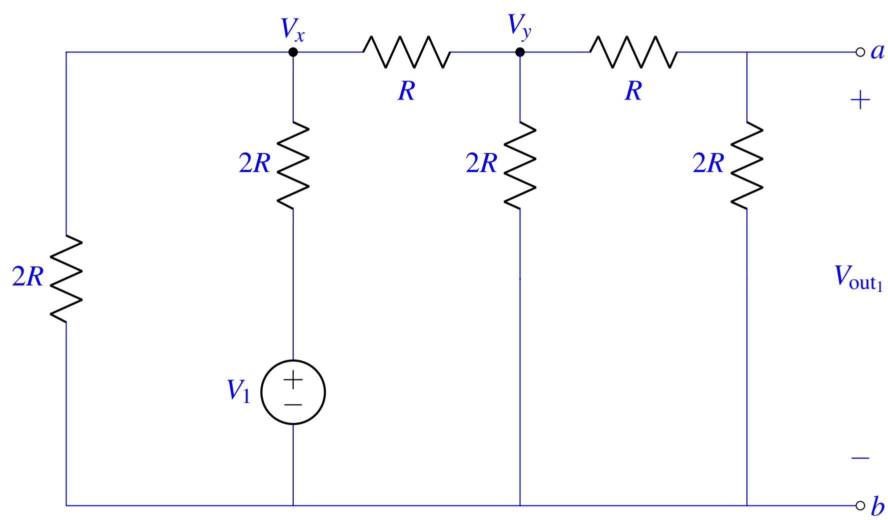
The equivalent resistance is always $R_{eq} = R$.

(c) The following circuit is an R-$2R$ DAC. To understand its functionality, use superposition to find V_{out} in terms of each V_1, V_2, and V_3.

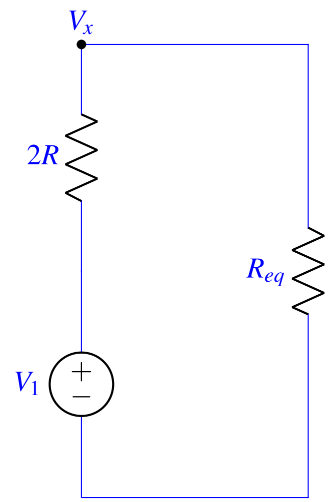


Solution:

V_1:



We first find the potential V_x. To do this, we can simplify the circuit.



$$R_{eq} = 2R \parallel (R + (2R \parallel (R + 2R))) = \frac{22}{21}R$$

We can then find V_x using the voltage divider formula.

$$V_x = \frac{\frac{22}{21}R}{2R + \frac{22}{21}R}V_1 = \frac{11}{32}V_1$$

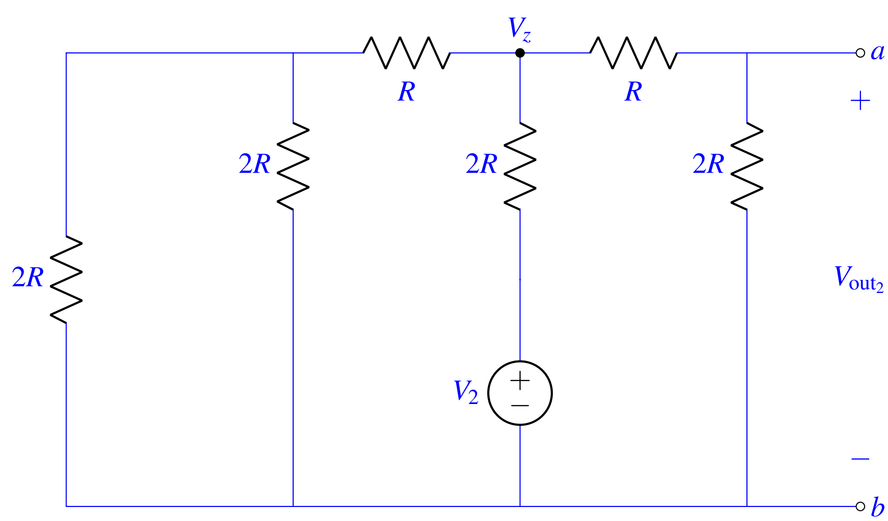
Similarly, we use the voltage divider formula to find V_y in terms of V_x.

$$V_y = \frac{2R \parallel (R + 2R)}{R + 2R \parallel (R + 2R)}V_x = \frac{\frac{6}{5}R}{R + \frac{6}{5}R}V_x = \frac{6}{11} \cdot \frac{11}{32}V_1 = \frac{3}{16}V_1$$

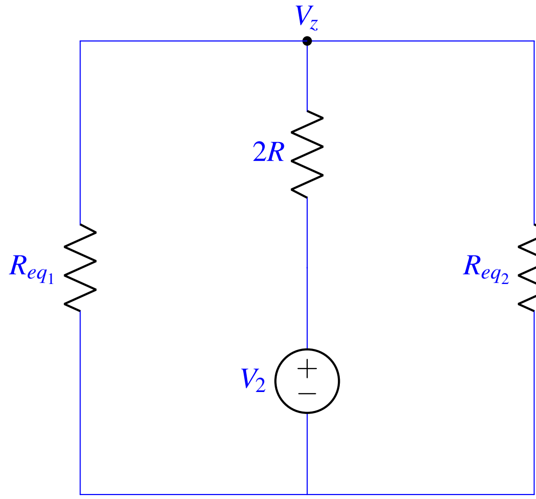
Applying the voltage divider formula again gives us V_{out_1}.

$$V_{\text{out}_1} = \frac{2R}{R + 2R}V_y = \frac{2}{3} \cdot \frac{3}{16}V_1 = \frac{1}{8}V_1$$

V_2:



We first find the potential V_z. To do this, we can simplify the circuit.



$$R_{eq_1} = R + (2R \parallel 2R) = R + R = 2R$$

$$R_{eq_2} = R + 2R = 3R$$

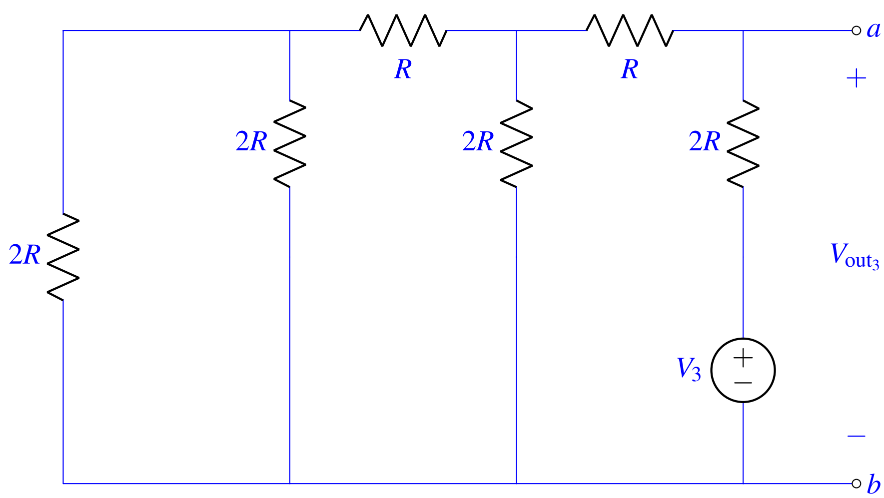
We can then find V_z using the voltage divider formula.

$$V_z = \frac{2R \parallel 3R}{2R + (2R \parallel 3R)} V_2 = \frac{\frac{6}{5}R}{2R + \frac{6}{5}R} V_2 = \frac{3}{8} V_2$$

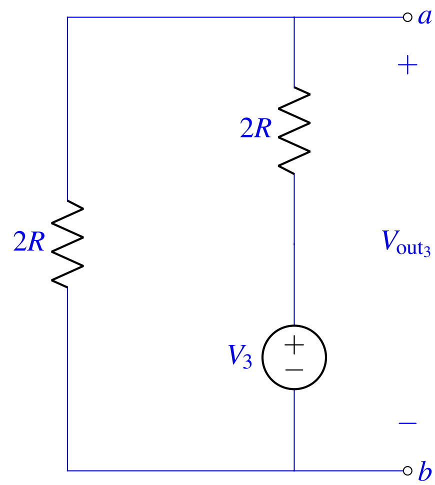
Applying the voltage divider formula again gives us V_{out_2}.

$$V_{\text{out}_2} = \frac{2R}{R + 2R} V_z = \frac{2}{3} \cdot \frac{3}{8} V_2 = \frac{1}{4} V_2$$

V_3:



We can simplify this circuit.



$$V_{\text{out}_3} = \frac{2R}{2R+2R}V_3 = \frac{1}{2}V_3$$

$$V_{\text{out}} = V_{\text{out}_1} + V_{\text{out}_2} + V_{\text{out}_3} = \frac{1}{8}V_1 + \frac{1}{4}V_2 + \frac{1}{2}V_3$$

(d) We've now designed a 3-bit R-$2R$ DAC. What is the output voltage V_{out} if $V_2 = 1\,\text{V}$ and $V_1 = V_3 = 0\,\text{V}$?

Solution:

$$V_{\text{out}} = \frac{1}{8}\cdot 0\,\text{V} + \frac{1}{4}\cdot 1\,\text{V} + \frac{1}{2}\cdot 0\,\text{V} = 0.25\,\text{V}$$

6. Pre-Lab Questions

These questions pertain to the Pre-Lab reading for the Touch 3B lab. You can find the reading under the Touch 3B Lab section on the 'Schedule' page of the website.

(a) Why can't we have an ideal current source?

Solution: We need an infinite parallel resistance across the ends of a current source for it to be ideal. An infinite resistance is impossible and thus, an ideal current source is not possible to create in the real world.

(b) For the integrator circuit, if $V_{in}(t)$ is a triangle wave, what kind of wave will $I_{in}(t)$ be?

Solution: Triangle wave.

(c) Does V_{out} increase or decrease when you touch the touchscreen?

Solution: C_{pixel} increases with touch so V_{out} will decrease since $V_{out}(t) = -\frac{1}{R_{in}C_{pixel}}\int_{t_0}^{t} V_{in}(t)\mathrm{d}t$

7. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.

