
EECS 16A Designing Information Devices and Systems I

Fall 2022 Homework 8

This homework is due Friday, October 28, 2022 at 23:59. Self-grades are due Monday, October 31, 2022, at 23:59.

Submission Format

Your homework submission should consist of **one** file.

- `hw8.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

Submit the file to the appropriate assignment on Gradescope.

1. Reading Assignment

For this homework, please read Notes 15 and 16. Note 15 covers superposition and equivalence, two very helpful techniques to help simplify circuit analysis. Note 16 will provide an introduction to capacitors (a circuit element which stores charge), capacitive equivalence, and the underlying physics behind them.

- (a) Describe the key ideas behind how a capacitor works. How are capacitor equivalences calculated? Compare this with how we calculate resistor equivalences.

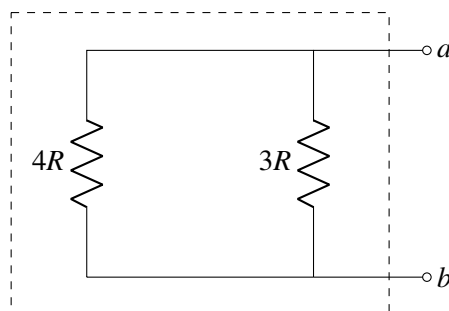
Solution:

- a) A capacitor is a device that can store charge (and hence, energy) by separating two conducting surfaces with a non-conducting material. This allows equal and opposite amounts of charge to build up on the surfaces, creating a potential difference. Capacitors in parallel can be combined into an equivalent capacitance that is the sum of the individual capacitance (just like resistors in **series**). Capacitors C_1, C_2 in series can be combined into an equivalent capacitance of $\frac{C_1 C_2}{C_1 + C_2}$ (just like resistors in **parallel**).

2. Equivalent Resistance

Learning Goal: The objective of this problem is to practice finding the equivalent to a series/parallel combination of resistors.

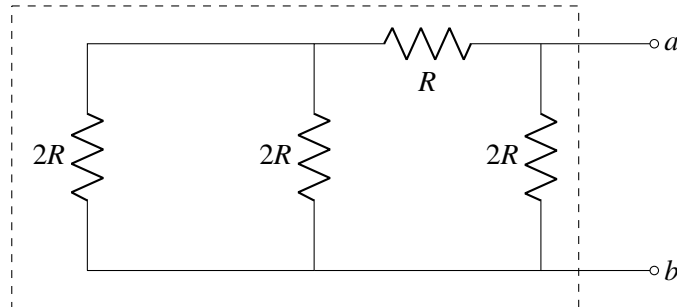
- (a) Find the equivalent resistance looking in from points a and b . In other words, express the resistive network in the dashed box as one resistor.



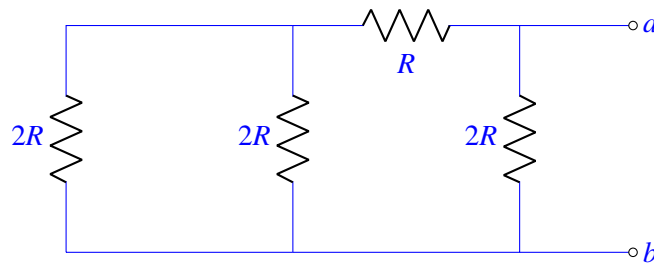
Solution:

$$R_{eq} = 4R \parallel 3R = \frac{4R \cdot 3R}{4R + 3R} = \frac{12}{7}R$$

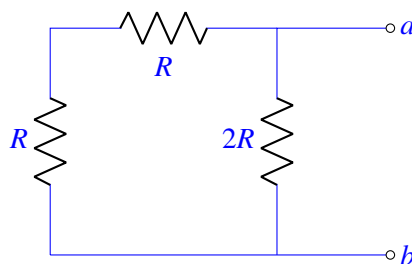
- (b) Find the equivalent resistance looking in from points a and b . In other words, express the resistive network in the dashed box as one resistor.



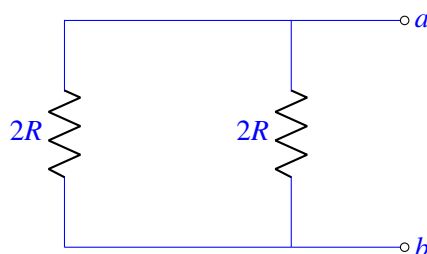
Solution: We find the equivalent resistance for the resistors from left to right. In general, it is easiest to start furthest away from the node voltages we wish to measure.



We can begin by combining the leftmost two resistors, which are in parallel and both have values $2R$. The equivalent resistor would have resistance $R_{eq} = 2R \parallel 2R = \frac{2R \cdot 2R}{2R + 2R} = R$, so we replace both resistors with a equivalent resistor of resistance R .

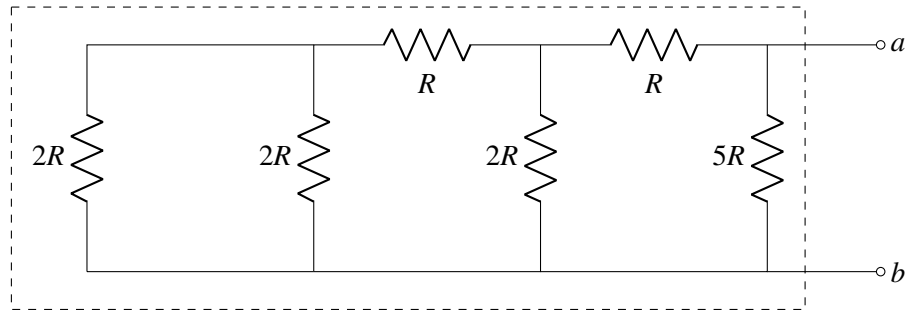


Now, the leftmost resistors are in series, both with values R , so the equivalent resistor would have resistance $R_{eq} = R + R = 2R$.

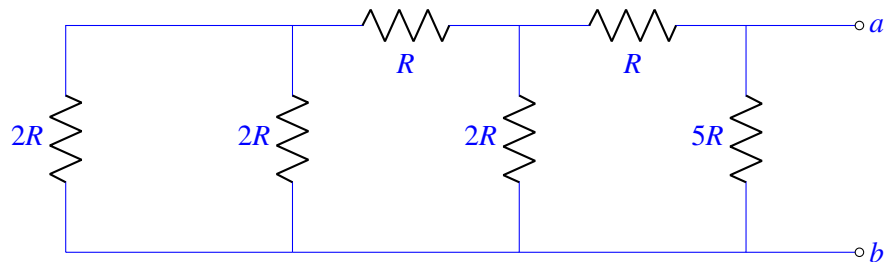


Finally, we are left with a parallel combination of resistors with values $2R$ and $2R$, so the equivalent resistor would have resistance $R_{eq} = 2R \parallel 2R = R$.

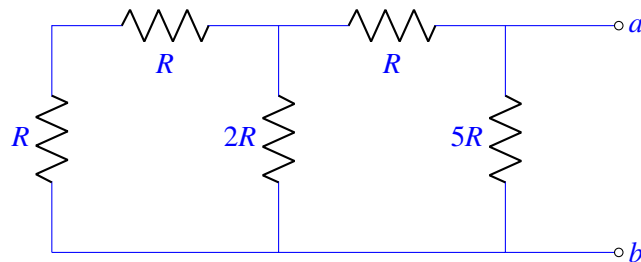
- (c) Find the equivalent resistance looking in from points a and b . In other words, express the resistive network in the dashed box as one resistor.



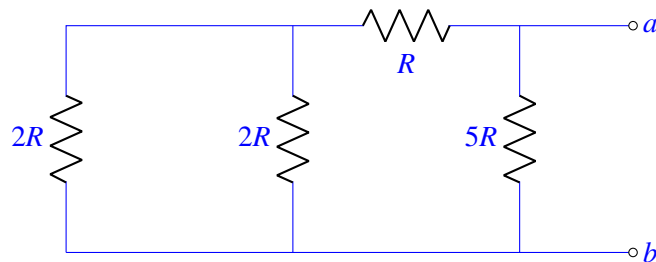
Solution: Again, we find the equivalent resistance for the resistors from left to right.



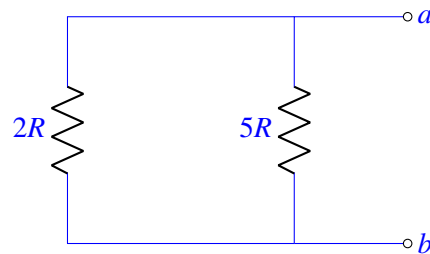
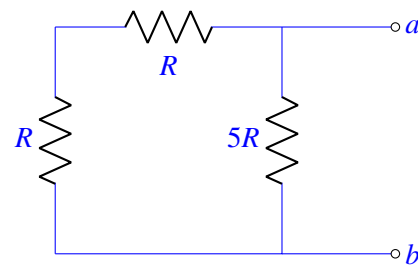
We first combine the two parallel resistors on the left:



Next, the two series resistors on the left:

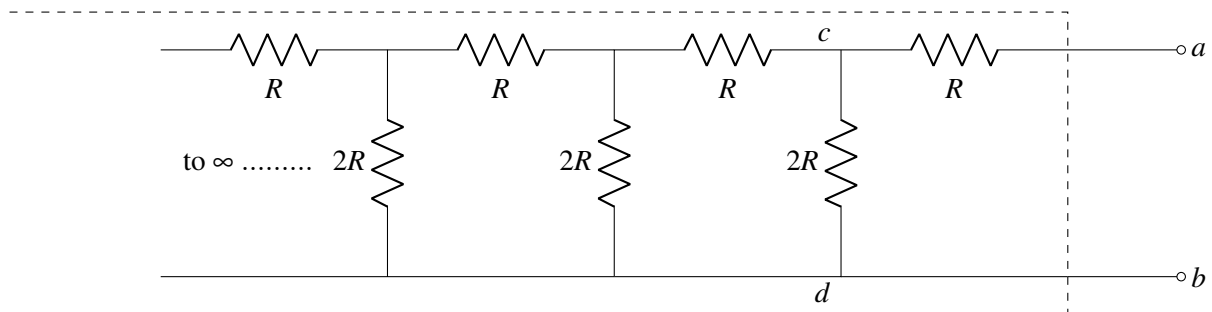


Following the process, we can eventually simplify down to a circuit with a single equivalent resistor.

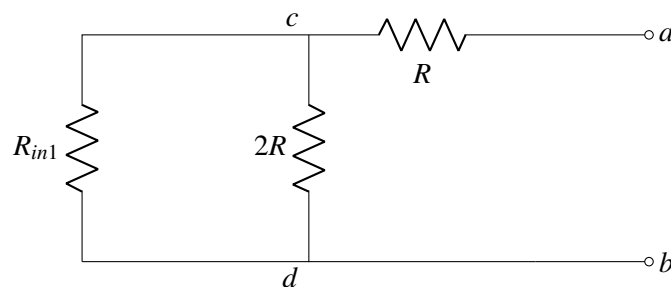


$$R_{eq} = 2R \parallel 5R = \frac{10}{7}R$$

- (d) **(OPTIONAL, CHALLENGE)** Find the equivalent resistance for the infinite ladder looking in from points a and b . In other words, express the resistive network in the dashed region as one resistor. (Hint: Let's call the resistance looking in from a and b as R_{in} , and the resistance looking to the left from points c and d as R_{in1} . Replace the entire circuit to the left of points c and d with a resistor whose value is given by R_{in1} . Find the relationship between R_{in} and R_{in1} using this circuit. Find another relationship between R_{in} and R_{in1} using the fact that the ladder is infinite. For an infinite ladder, adding another branch does not change the equivalent resistance. Think of this as a convergent infinite series.)



As a first step you can replace the circuit looking to the left from c and d by R_{in1} .



Solution: We wish to compute the equivalent resistance R_{in} looking to the left from nodes a and b . The equivalent resistance looking to the left from nodes c and d is given by R_{in1} . Clearly,

$$R_{in} = (R_{in1} || 2R) + R$$

Additionally, since this is an infinite ladder, the equivalent resistance does not change by addition of an extra branch to the right, since having infinity + 1 steps is the same thing as having infinite steps. Therefore, $R_{in1} = R_{in}$. Using this result in the previous equation, we have,

$$R_{in} = (R_{in} || 2R) + R$$

$$R_{in} = \frac{2R \cdot R_{in}}{2R + R_{in}} + R$$

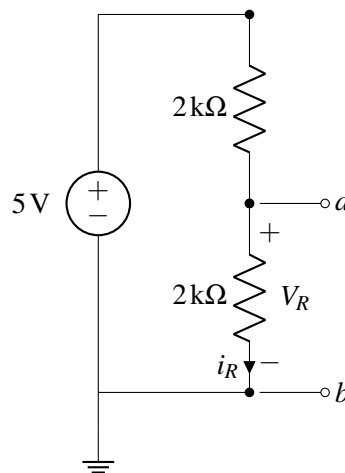
$$R_{in}^2 - RR_{in} - 2R^2 = 0$$

$$(R_{in} - 2R)(R_{in} + R) = 0$$

Clearly, $R_{in} = -R$ is not a physically realizable solution. The equivalent resistance looking into this infinite ladder is given by $R_{in} = 2R$.

3. Why Bother With Thévenin Anyway?

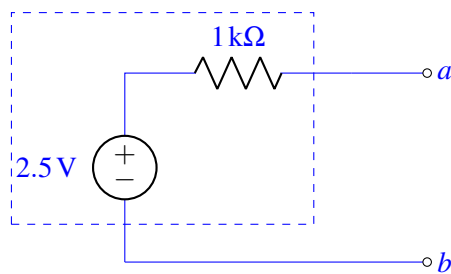
- (a) Find a Thévenin equivalent for the circuit shown below looking from the terminals a and b . (Hint: That is, find the open circuit voltage V_R across the terminals a and b . Also, find the equivalent resistance looking from the terminals a and b when the input voltage source is zeroed.)



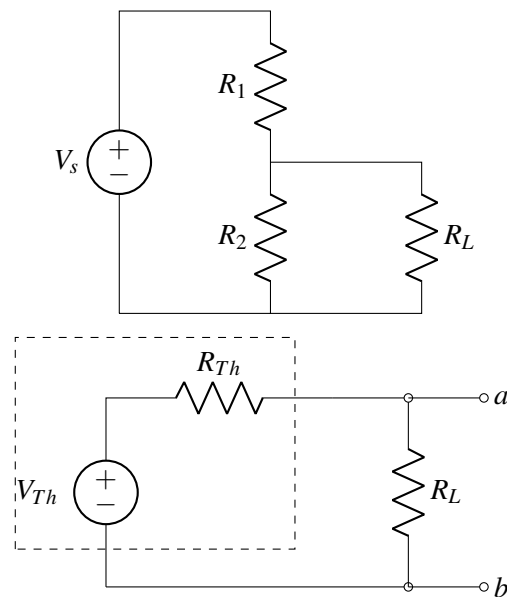
Solution: To find the voltage across terminals a and b , we notice that the circuit is a voltage divider. Therefore, we can use the voltage divider formula to find the voltage across a and b . Then for the equivalent resistance, we zero out the voltage source and notice that the resistors are in parallel with respect to the terminals a and b so we can use the parallel resistor equation to find the equivalent resistance. Be careful! It looks like the resistors are in series but if we combine them that way, we would be destroying node a !

$$V_{Th} = \frac{2\text{k}\Omega}{2\text{k}\Omega + 2\text{k}\Omega} \cdot 5\text{V} = 2.5\text{V}$$

$$R_{Th} = 2\text{k}\Omega || 2\text{k}\Omega = 1\text{k}\Omega$$

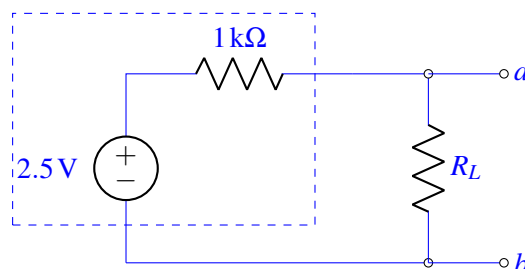


- (b) Now consider the circuit shown below where a load resistor of resistance R_L is attached across the terminals a and b . Compute the voltage drop V_R across the terminals a and b in this new circuit with the attached load. Express your answer in terms of R_L . (Hint: We have already computed the Thévenin equivalent of the unloaded circuit in part (a). To analyze the new circuit, attach R_L as the load resistance across the Thévenin equivalent computed in part (a), as shown in the figure below. One of the main advantages of using Thévenin (and Norton) equivalents is to avoid re-analyzing different circuits which differ only by the amount of loading.)



Solution:

We just attach the R_L resistor to our Thévenin equivalent circuit that we found part (a) and calculate the voltage across it.



$$V_R = \frac{R_L}{1\text{ k}\Omega + R_L} \cdot 2.5\text{ V}$$

- (c) Now compute the voltage drop V_R for three different values of R_L equal to $5/3 \text{ k}\Omega$, $5 \text{ k}\Omega$, and $50 \text{ k}\Omega$? What can you comment on the value of R_L needed to ensure that the loading does not reduce the voltage drop V_R compared to the unloaded voltage V_R computed in part (a)? **Solution:**

$$R_L = \frac{5}{3} \text{ k}\Omega:$$

$$V_R = \frac{\frac{5}{3} \text{ k}\Omega}{1 \text{ k}\Omega + \frac{5}{3} \text{ k}\Omega} \cdot 2.5 \text{ V} = 1.56 \text{ V}$$

$$R_L = 5 \text{ k}\Omega:$$

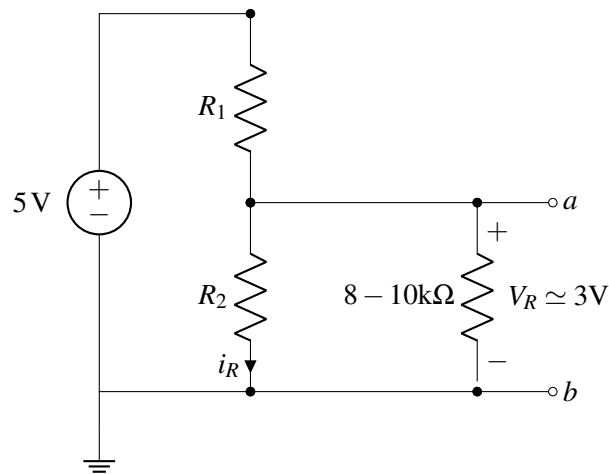
$$V_R = \frac{5 \text{ k}\Omega}{1 \text{ k}\Omega + 5 \text{ k}\Omega} \cdot 2.5 \text{ V} = 2.08 \text{ V}$$

$$R_L = 50 \text{ k}\Omega:$$

$$V_R = \frac{50 \text{ k}\Omega}{1 \text{ k}\Omega + 50 \text{ k}\Omega} \cdot 2.5 \text{ V} = 2.45 \text{ V}$$

As the value of R_L is increased, the voltage drop V_R approaches the unloaded Thévenin voltage computed in part (a).

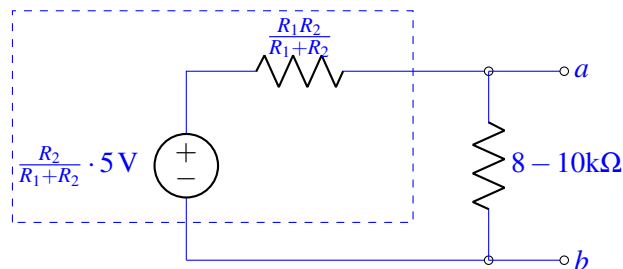
- (d) Say that we want to support loads in the range of $8 \text{ k}\Omega$ to $10 \text{ k}\Omega$. We would like to maintain 3 V across these loads. How can we approximately achieve this by setting R_1 and R_2 in the following circuit?



Solution:

$$V_{Th} = \frac{R_2}{R_1 + R_2} \cdot 5 \text{ V}$$

$$R_{Th} = R_1 \parallel R_2 = \frac{R_1 R_2}{R_1 + R_2}$$

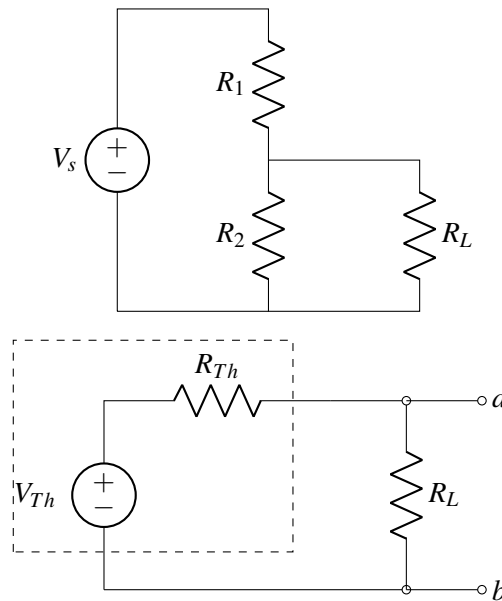


$$V_R = \frac{R}{R + \frac{R_1 R_2}{R_1 + R_2}} \cdot \frac{R_2}{R_1 + R_2} \cdot 5 \text{ V} \simeq 3 \text{ V}$$

$$\frac{RR_2}{R(R_1 + R_2) + R_1 R_2} = \frac{3}{5}$$

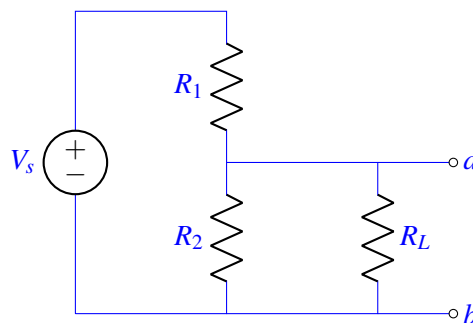
If we set $R_1, R_2 \ll R$, then $\frac{RR_2}{R(R_1 + R_2) + R_1 R_2} \approx \frac{RR_2}{R(R_1 + R_2)} = \frac{R_2}{R_1 + R_2}$. Therefore, we can just choose two small resistors $R_1, R_2 \ll 8 \text{ k}\Omega$, such that $R_2 = \frac{3}{2}R_1$.

- (e) Thus far, we have seen how to use Thévenin equivalents to compute the voltage drop across a load without re-analyzing the entire circuit. We would like to see if we can use the Thévenin equivalent for power computations. Consider the case where the load resistance $R_L = 8 \text{ k}\Omega$, $V_S = 5 \text{ V}$, $R_1 = R_2 = 2 \text{ k}\Omega$. Compute the power dissipated across the load resistor R_L both using the original circuit and the Thévenin equivalent. Are they equal? Now, compute the power dissipated by the voltage source V_S in the original circuit. Also, compute the power dissipated by the Thévenin voltage source V_{Th} in the Thévenin equivalent circuit. Is the power dissipated by the two sources equal?

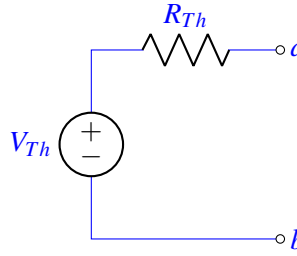


Solution:

We will compare the power dissipation in V_S vs. V_{Th} and R_L in either case. This could be done for the specific example above (with $R_L = 8 \text{ k}\Omega$), but it's more useful to go through this exercise generally. Thus, we will use the circuit shown below:



Recall that the Thévenin equivalent for the circuit above looks as follows:



where $R_{Th} = \frac{R_1 R_2}{R_1 + R_2}$ and $V_{Th} = \frac{R_2}{R_1 + R_2} V_s$.

Because we are going to end up writing a few expressions multiple times, we are going to define a new variable:

$$\beta = R_1 R_2 + R_L R_1 + R_L R_2$$

Let's start with our equivalent circuit. In the equivalent circuit, the current through the load resistor and equivalently every other element in the circuit is:

$$I = \frac{V_R}{R_L} = \frac{V_{Th}}{R_L + R_{Th}}$$

With this current, we find the power dissipated across the source and the load resistor.

$$P_{V_{Th}} = -IV = -\frac{V_{Th}^2}{R_L + R_{Th}} = -\frac{V_{Th}^2 (R_1 + R_2)}{\beta} = -\frac{V_s^2 R_2^2}{\beta (R_1 + R_2)} = -0.694 \text{ mW}$$

$$P_{R_L} = I^2 R = \frac{V_{Th}^2}{(R_L + R_{Th})^2} \cdot R_L = \frac{V_{Th}^2 (R_1 + R_2)^2}{\beta^2} \cdot R_L = \frac{V_s^2 R_2^2}{\beta^2} \cdot R_L = 0.617 \text{ mW}$$

Let's try to find the answer from the original circuit. We will begin by calculating the current through the source.

$$I_s = \frac{V_s}{R_{eq}} = \frac{V_s}{R_1 + R_2 \parallel R_L} = \frac{V_s (R_1 + R_2)}{\beta}$$

Now, we can calculate the power through the source.

$$P_{V_s} = -I_s V_s = -\frac{V_s^2 (R_2 + R_L)}{\beta} = -6.94 \text{ mW}$$

The power dissipated by the source in the original circuit is not the same as the power dissipated in the new circuit. What about the load resistor? We will first calculate the voltage across the load resistor.

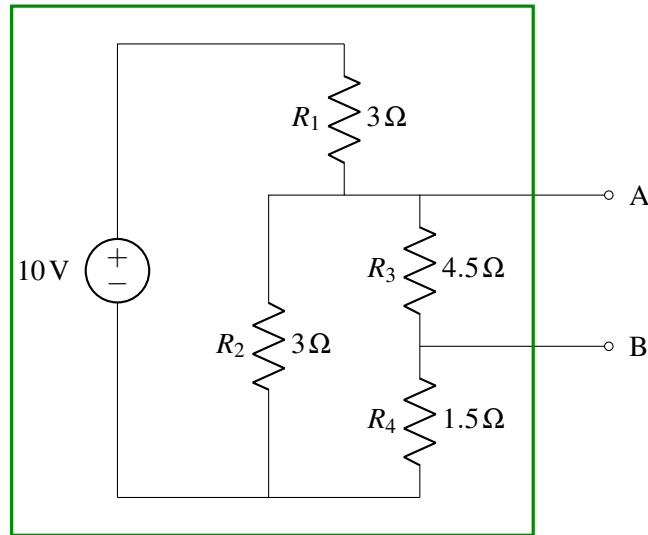
$$V_L = \frac{R_2 \parallel R_L}{R_1 + R_2 \parallel R_L} \cdot V_s = \frac{\frac{R_2 R_L}{R_2 + R_L}}{R_1 + \frac{R_2 R_L}{R_2 + R_L}} \cdot V_s = \frac{R_2 R_L}{\beta} \cdot V_s$$

$$P_L = \frac{V_L^2}{R_L} = \frac{V_s^2 R_2^2}{\beta^2} R_L = 0.617 \text{ mW}$$

The power through the load is the same! Thévenin equivalents can be used to calculate the power through elements that are not part of the circuit that was transformed.

4. Thévenin and Norton Equivalent Circuits

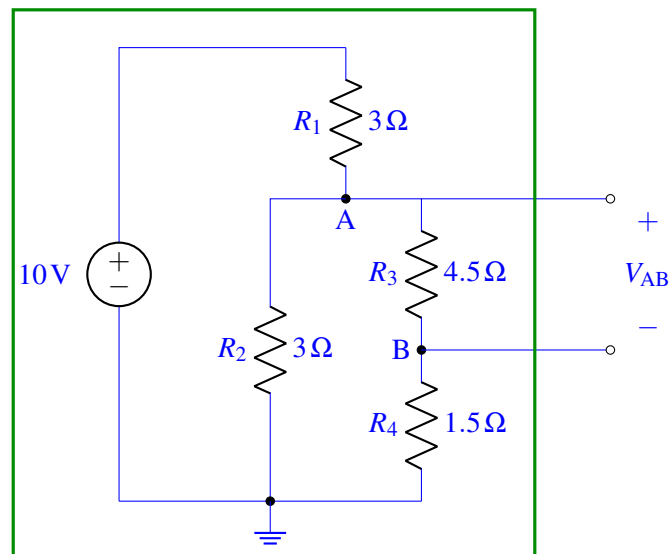
Find the Thévenin and Norton equivalent circuits seen from terminals A and B.



Solution:

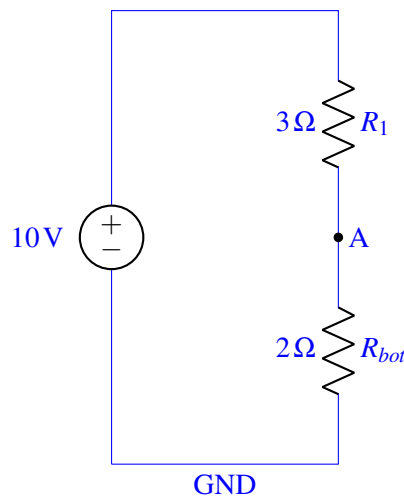
To find the Thévenin and Norton equivalent circuits, we are going to find (1) the open circuit voltage between the output ports and (2) the current flowing through the output ports when the ports are shorted.

For finding the open circuit voltage between the output ports, let us label the nodes as shown in the figure below.



First, let us begin by calculating the effective resistance between nodes A and Ground looking down from A. We have the 3Ω resistor in parallel to the $4.5\Omega + 1.5\Omega$ resistance. This allows us to express the behavior of R_2 , R_3 , and R_4 together as an equivalent resistance of

$$R_{bot} = \frac{1}{\frac{1}{R_2} + \frac{1}{R_3 + R_4}} = \frac{1}{\frac{1}{3\Omega} + \frac{1}{4.5\Omega + 1.5\Omega}} = 2\Omega.$$



Notice how in this reduced diagram, node B isn't there! That's OK, as we're just finding the node voltage at A for now; we'll refer to the unsimplified diagram when we're calculating the voltage at node B.

Then we see that we have a voltage divider from the positive terminal of the 10 V supply. The voltage divider is made up of two resistances in series, where the resistances are 3Ω and 2Ω . This gives the voltage at node A equal to

$$V_A = 10\text{ V} \frac{R_{\text{bot}}}{R_{\text{top}} + R_{\text{bot}}} = 10\text{ V} \frac{2\Omega}{3\Omega + 2\Omega} = 4\text{ V}$$

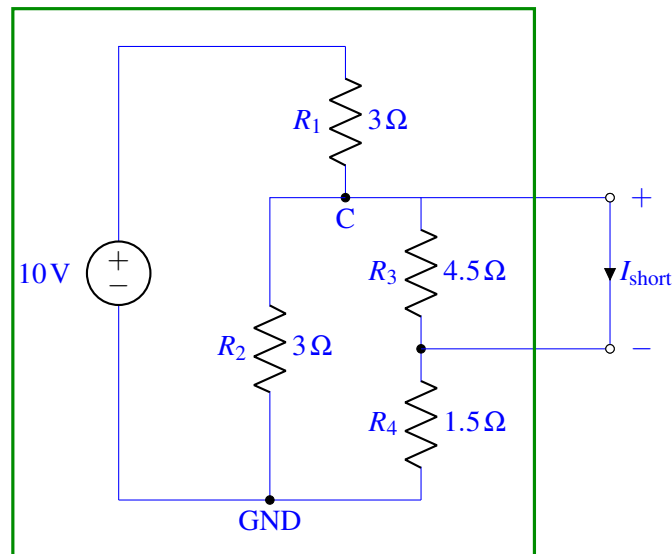
To find V_B , we have to look back at the *original* diagram given in the problem statement. To find the voltage at node B, note that we have another voltage divider between nodes A and Ground. Hence, we can find the voltage at node B as

$$V_B = V_A \frac{1.5\Omega}{4.5\Omega + 1.5\Omega} = 4\text{ V} \cdot \frac{1}{4} = 1\text{ V}$$

Hence the open circuit voltage seen between the output ports is equal to

$$\begin{aligned} V_{AB} &= V_A - V_B \\ &= 4\text{ V} - 1\text{ V} \\ &= 3\text{ V} \end{aligned}$$

Now let us find the short circuit current flowing through the output ports. When doing this, we get the following circuit.



Note that when we short the output terminals, the voltages at the nodes change, this is why we changed the label of the node below the resistor R_1 . Since there is a short circuit parallel to the resistor R_3 , there will be no current flowing through it, hence we have

$$I_{\text{short}} = I_{R_4}$$

To find this current, let us find the equivalent resistance due to R_2 being connected in parallel to R_4 when we short the output ports. We have 3Ω parallel to 1.5Ω , which gives an equivalent resistance

$$R_{\text{bot}} = \frac{1}{\frac{1}{R_2} + \frac{1}{R_4}} = \frac{1}{\frac{1}{3\Omega} + \frac{1}{1.5\Omega}} = 1\Omega$$

We again have a voltage divider between the positive side of the 10V supply and the ground. Using this voltage divider, we calculate the voltage at node C as

$$V_C = 10\text{V} \frac{1\Omega}{3\Omega + 1\Omega} = 2.5\text{V}$$

Hence, we see that the voltage across the resistor R_4 is equal to 2.5V . Using Ohm's law, we get

$$I_{R_4} = \frac{2.5\text{V}}{1.5\Omega} = 5/3\text{A}$$

Since we have $I_{\text{short}} = I_{R_4}$, we have

$$I_{\text{short}} = I_{R_4}$$

Summarizing the results, we have

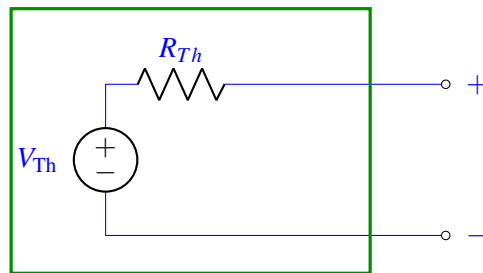
$$\begin{aligned} V_{\text{AB}} &= 3\text{V} \\ I_{\text{short}} &= 5/3\text{A} \end{aligned}$$

This gives

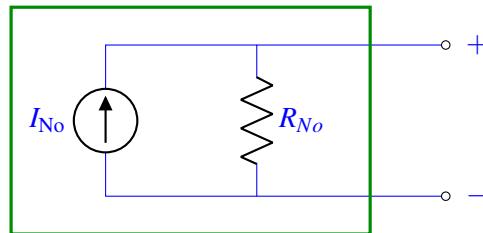
$$R_{\text{Th}} = \frac{V_{\text{AB}}}{I_{\text{short}}} = 9/5\Omega$$

Note that $R_{\text{Th}} = R_{\text{No}}$ across the two equivalent circuits.

Hence the Thévenin equivalent circuit is given by



where $R_{Th} = 9/5\Omega$ and $V_{Th} = V_{AB} = 3\text{ V}$, and the Norton equivalent circuit is given by

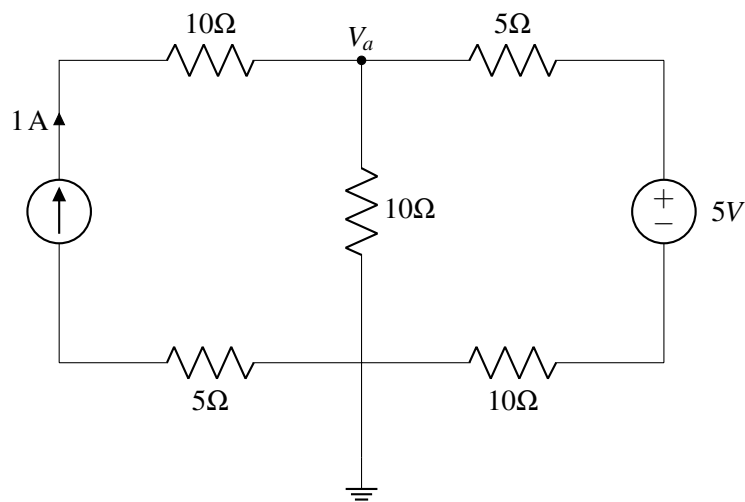


where $R_{No} = 9/5\Omega$ and $I_{No} = I_{short} = 5/3\text{ A}$.

5. Superposition

Learning Goal: The objective of this problem is to help you practice solving circuits using the principles of superposition.

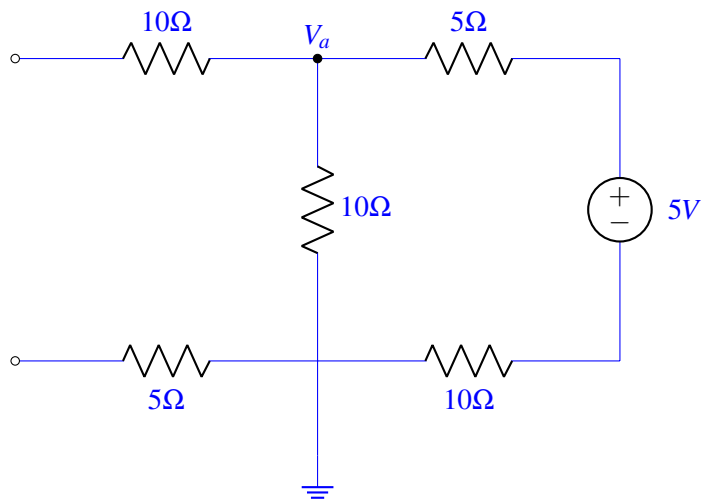
Find the node potential V_a indicated in the diagram using superposition. Be careful when solving to take into account where the reference potential is.



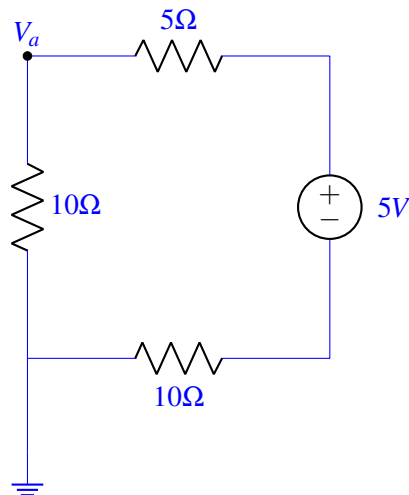
Solution:

Consider the circuits obtained by:

- (a) Zeroing out the 1 A current source:

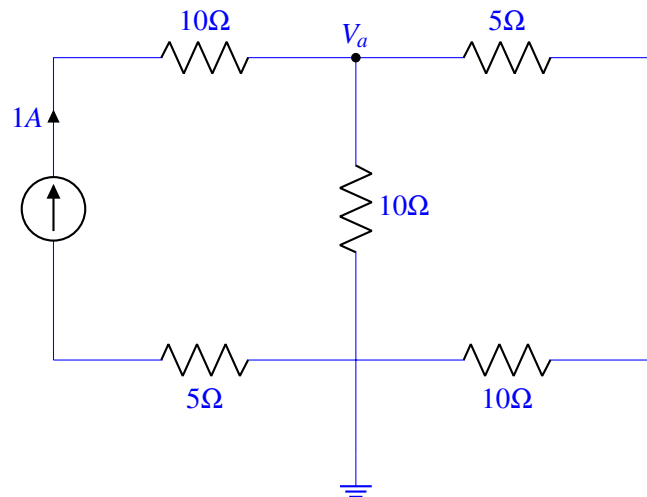


In the above circuit, no current flows through the two resistors in the top left and bottom left, so we can remove them and get the following circuit:

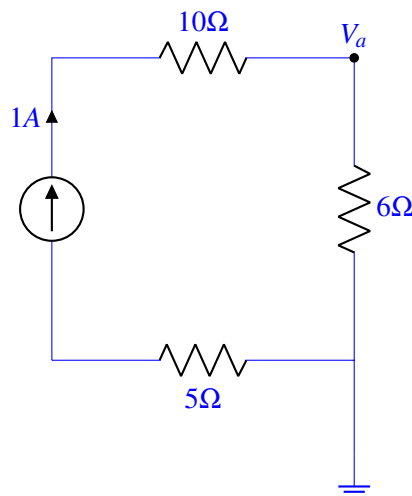


Using NVA, we can find that the voltage drop across the $5\ \Omega$ resistor is 1 V and the voltage drop across the $10\ \Omega$ resistor is 2 V . (You can also do this by combining resistors and noticing that it is a voltage divider). Note that the reference node is not right under the voltage source like we typically see, so V_a is just the voltage drop across the $10\ \Omega$ resistor which is 2 V .

(b) Zeroing out the 5 V voltage source:



We can reduce this circuit using resistor equivalences to make it easier to solve. We note that the top right and bottom right resistors are in series, so combined we get 15Ω . Then we can combine this 15Ω resistor with the 10Ω resistor in the middle as they are in parallel to get $\frac{(10\Omega)(15\Omega)}{10\Omega+15\Omega} = 6\Omega$. Our reduced circuit then looks like this:

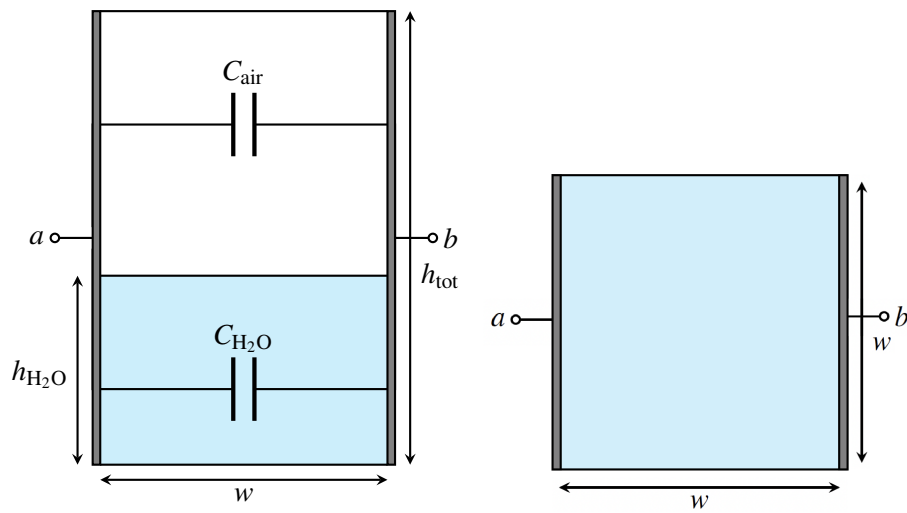


We can then just use Ohm's law to find the voltage drop across the 6Ω to find our node potential at V_a , which is just $(1A)(6\Omega) = 6V$

Now, applying the principle of superposition, we have $V_a = 2V + 6V = 8V$

6. It's finally raining!

A lettuce farmer in Salinas Valley has grown tired of weather.com's imprecise rain measurements. Therefore, they decided to take matters into their own hands by building a rain sensor. They placed a square tank outside and attached two metal plates to two opposite sides in an effort to make a capacitor whose capacitance varies with the amount of water inside.



Tank side view (left) and top view (right).

The width and length of the tank are both w (i.e., the base is square) and the height of the tank is h_{tot} .

- (a) What is the capacitance between terminals a and b when the tank is full? What about when it is empty?
Note: the permittivity of air is ϵ , and the permittivity of rainwater is 71ϵ .

Solution:

Capacitance of parallel plates is governed by the equation:

$$C = \frac{\epsilon A}{d},$$

where ϵ is the *permittivity* of the dielectric material, A is the area of the plates, and d is the distance between the plates. If we apply this to our physical structure, we find that the area of the plates is $h_{\text{tot}} \cdot w$, and the distance between the plates is w . The only difference here between a full and empty tank is the permittivity of the material between the two plates.

$$C_{\text{empty}} = \frac{\epsilon_{\text{air}} h_{\text{tot}} w}{w} = \epsilon h_{\text{tot}}$$

$$C_{\text{full}} = \frac{\epsilon_{\text{H}_2\text{O}} h_{\text{tot}} w}{w} = 71\epsilon h_{\text{tot}}$$

- (b) Suppose the height of the water in the tank is $h_{\text{H}_2\text{O}}$. Model the tank as a pair of capacitors in parallel, where one capacitor has a dielectric of air, and one capacitor has a dielectric of water. Find the total capacitance between the two plates using equivalence. Call this capacitance C_{tank} .

Solution:

We can break the total capacitance into two parts. First, let's calculate the capacitance of the two plates separated by water:

$$C_{\text{water}} = \frac{\epsilon_{\text{H}_2\text{O}} h_{\text{H}_2\text{O}} w}{w} = 71\epsilon h_{\text{H}_2\text{O}}$$

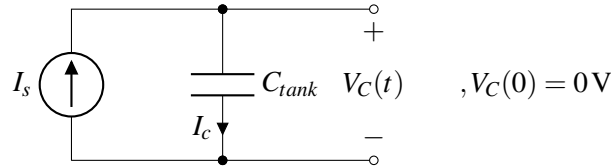
And now we can calculate the capacitance of the two plates separated by air:

$$C_{\text{air}} = \frac{\epsilon_{\text{air}} (h_{\text{tot}} - h_{\text{H}_2\text{O}}) w}{w} = \epsilon (h_{\text{tot}} - h_{\text{H}_2\text{O}})$$

These two capacitors appear in parallel, as the result from the layer of water at the bottom of the tank, and the air above the water. Because these two capacitors appear in parallel, we can simply add our two previous results to find the total equivalent capacitance:

$$C_{\text{tank}} = C_{\text{water}} + C_{\text{air}} = \epsilon (h_{\text{tot}} + 70h_{\text{H}_2\text{O}})$$

- (c) After building this capacitor, the farmer consults the internet to assist them with a capacitance-measuring circuit. A fellow internet user recommends the following:



In this circuit, C_{tank} is the total tank capacitance that you calculated earlier. I_s is a known current supplied by a current source.

The suggestion is to measure V_C for a brief interval of time, and then use the difference to determine C_{tank} .

Determine $V_C(t)$, where t is the number of seconds elapsed since the start of the measurement. You should assume that before any measurements are taken, the voltage across C_{tank} , i.e. V_C , is initialized to 0 V, i.e. $V_C(0) = 0$.

Solution: The element equation for the capacitor is:

$$I_C = C_{\text{tank}} \frac{dV_C}{dt}$$

We also know from KCL that:

$$I_C = I_s$$

Thus, we get the following differential equation for V_C :

$$\frac{dV_C}{dt} = \frac{I_s}{C_{\text{tank}}}$$

We recall that I_s and C_{tank} are constant values and the initial value of V_C is zero ($V_C(0) = 0$). Applying these facts and integrating the differential equation, we get the following equation for V_C :

$$V_C(t) = \frac{I_s}{C_{\text{tank}}} t$$

- (d) Using the equation you derived for $V_C(t)$, describe how you can use this circuit to determine C_{tank} and $h_{\text{H}_2\text{O}}$.

Solution: We connect the current source providing I_s to the capacitor C_{tank} . At the same time, we can measure $V_C(t)$. After some time passes, we measure $V_C(t)$ and plug it into the following equation (assuming, as before, that $V_C(0) = 0$):

$$C_{\text{tank}} = \frac{I_s}{V_C(t)} t$$

If we know C_{tank} , we can determine $h_{\text{H}_2\text{O}}$. Using the equation derived in part (b), we see that

$$h_{\text{H}_2\text{O}} = \frac{C_{\text{tank}} - h_{\text{tot}}\epsilon}{70\epsilon}$$

7. Prelab Questions

These questions pertain to the Pre-Lab reading for the Touch 3A lab. You can find the reading under the Touch 3A Lab section on the 'Schedule' page of the website.

- (a) What basic principle governs the working of the capacitive touchscreen?
- (b) What is the equation that relates the Current (I), Voltage (V) and Capacitance (C) of a capacitor?

Solution:

- (a) The capacitive touchscreen works on the simple principle that touching the screen induces a change in capacitance due to the addition of our finger(s).
- (b) $I = C \frac{dV}{dt}$

8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.