

EECS 16A Designing Information Devices and Systems I

Fall 2022 Discussion 6A

1. Steady and Unsteady States

You're given the matrix \mathbf{M} :

$$\mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

which generates the next state of a physical system from its previous state: $\vec{x}[k+1] = \mathbf{M}\vec{x}[k]$.

- (a) The eigenvalues of \mathbf{M} are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = \frac{1}{2}$. Define $\vec{x} = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$, a linear combination of the eigenvectors corresponding to the eigenvalues. For each of the cases in the table, determine if

$$\lim_{n \rightarrow \infty} \mathbf{M}^n \vec{x}$$

converges. If it does, what does it converge to?

α	β	γ	Converges?	$\lim_{n \rightarrow \infty} \mathbf{M}^n \vec{x}$
0	0	$\neq 0$		
0	$\neq 0$	0		
0	$\neq 0$	$\neq 0$		
$\neq 0$	0	0		
$\neq 0$	0	$\neq 0$		
$\neq 0$	$\neq 0$	0		
$\neq 0$	$\neq 0$	$\neq 0$		

Answer:

$$\begin{aligned} \mathbf{M}^n \vec{x} &= \mathbf{M}^n (\alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3) \\ &= \alpha\mathbf{M}^n \vec{v}_1 + \beta\mathbf{M}^n \vec{v}_2 + \gamma\mathbf{M}^n \vec{v}_3 \\ &= 1^n \alpha\vec{v}_1 + 2^n \beta\vec{v}_2 + \left(\frac{1}{2}\right)^n \gamma\vec{v}_3 \end{aligned}$$

α	β	γ	Converges?	$\lim_{n \rightarrow \infty} \mathbf{M}^n \vec{x}$
0	0	$\neq 0$	Yes	$\vec{0}$
0	$\neq 0$	0	No	-
0	$\neq 0$	$\neq 0$	No	-
$\neq 0$	0	0	Yes	$\alpha\vec{v}_1$
$\neq 0$	0	$\neq 0$	Yes	$\alpha\vec{v}_1$
$\neq 0$	$\neq 0$	0	No	-
$\neq 0$	$\neq 0$	$\neq 0$	No	-

- (b) (**Practice**) Find the eigenspaces associated with the eigenvalues:

- i. $\text{span}(\vec{v}_1)$, associated with $\lambda_1 = 1$

- ii. $\text{span}(\vec{v}_2)$, associated with $\lambda_2 = 2$
- iii. $\text{span}(\vec{v}_3)$, associated with $\lambda_3 = \frac{1}{2}$

Answer:

- i. $\lambda = 1$:

$$\left[\begin{array}{ccc|c} \mathbf{M} - \mathbf{I} & & & \vec{0} \end{array} \right] = \left[\begin{array}{ccc|c} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{G.E.} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\vec{v}_1 = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}$$

This means that

$$\text{span}\{\vec{v}_1\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

- ii. $\lambda = 2$:

$$\left[\begin{array}{ccc|c} \mathbf{M} - 2\mathbf{I} & & & \vec{0} \end{array} \right] = \left[\begin{array}{ccc|c} -\frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{G.E.} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\vec{v}_2 = \beta \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \beta \in \mathbb{R}$$

This means that

$$\text{span}\{\vec{v}_2\} = \text{span}\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

- iii. $\lambda = \frac{1}{2}$:

$$\left[\begin{array}{ccc|c} \mathbf{M} - \frac{1}{2}\mathbf{I} & & & \vec{0} \end{array} \right] = \left[\begin{array}{ccc|c} 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -2 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \end{array} \right] \xrightarrow{G.E.} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\vec{v}_3 = \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \gamma \in \mathbb{R}$$

This means that

$$\text{span}\{\vec{v}_3\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

2. Steady State Reservoir Levels

We have 3 reservoirs: A, B , and C . The pumps system between the reservoirs is depicted in Figure 1.

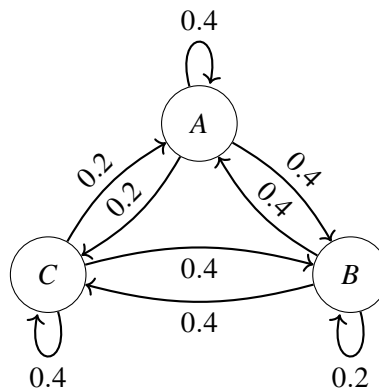


Figure 1: Reservoir pumps system.

- (a) Write out the transition matrix \mathbf{T} representing the pumps system.

Answer:

$$\mathbf{T} = \begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.4 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

- (b) You are told that $\lambda_1 = 1$, $\lambda_2 = \frac{1}{5}$, $\lambda_3 = -\frac{1}{5}$ are the eigenvalues of \mathbf{T} . Find a steady state vector \vec{x} , i.e. a vector such that $T\vec{x} = \vec{x}$.

Answer:

We know $\lambda_1 = 1$ is the eigenvalue corresponding to the steady state eigenvector. Therefore,

$$\begin{aligned} T\vec{x} &= 1\vec{x} \\ &= \lambda_1\vec{x} \\ \Rightarrow \vec{x} &\in N(\mathbf{T} - 1 \cdot \mathbf{I}) \\ \vec{x} &\in N\left(\begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.4 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \\ \vec{x} &\in N\left(\begin{bmatrix} -0.6 & 0.4 & 0.2 \\ 0.4 & -0.8 & 0.4 \\ 0.2 & 0.4 & -0.6 \end{bmatrix}\right). \end{aligned}$$

In order to row reduce $\mathbf{T} - 1 \cdot \mathbf{I}$ we use Gaussian elimination. We multiply everything by 10 to make it

easier

$$\begin{aligned}
 \begin{bmatrix} -6 & 4 & 2 \\ 4 & -8 & 4 \\ 2 & 4 & -6 \end{bmatrix} &\xrightarrow[R_3 \leftarrow 1/2 R_3]{R_1 \leftarrow 1/2 R_1, R_2 \leftarrow 1/4 R_2} \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \\
 &\xrightarrow{R_1 \leftarrow R_3, R_3 \leftarrow R_1} \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ -3 & 2 & 1 \end{bmatrix} \\
 &\xrightarrow[R_3 \leftarrow R_3 + 3R_1]{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 8 & -8 \end{bmatrix} \\
 &\xrightarrow{R_3 \leftarrow R_3 + 2R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \\
 &\xrightarrow{R_2 \leftarrow 1/4 R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\
 &\xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is a vector describing the steady state, then we can set x_3 to be the free variable. Thus we can write the form any steady state vector should take using the first two equations represented by the row reduced matrix:

$$\begin{aligned}
 x_1 - x_3 &= 0 \\
 x_2 - x_3 &= 0 \\
 x_3 &= \alpha \in \mathbb{R}
 \end{aligned} \implies \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \alpha$$

- (c) What does the magnitude of the other two eigenvalues λ_2 and λ_3 say about the steady state behavior of their associated eigenvectors?

Answer: The magnitude of both eigenvalues is less than 1, so in steady state, the components associated with those eigenvectors \vec{v}_2 and \vec{v}_3 will trend toward $\vec{0}$. Additionally, since $\lambda_3 < 0$, its associated eigenvector will oscillate / flip signs back and forth.

- (d) Assuming that you start the pumps with the water levels of the reservoirs at $A_0 = 150, B_0 = 250, C_0 = 200$ (in kiloliters), what would be the steady state water levels (in kiloliters) according to the pumps system described above?

Answer:

From the previous sub-parts we know the steady-state solution should have the form $\vec{x}_{ss} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for any α .

But after inspecting the transition matrix we recognize that the columns each sum to one, thus, we have a conservative system, meaning that the total volume across all three reservoirs ($A_0 + B_0 + C_0$) must remain constant at all iterations. This gives us a sufficient condition to identify α .

So far the sum, with $\alpha = 1$ of \vec{x}_{ss} is $1 + 1 + 1 = 3$ (kiloliters), while the initial state starts with $A_0 + B_0 + C_0 = 150 + 250 + 200 = 600$ kiloliters. By inspection we see that $\alpha = 200$ is the proper rescaling of the steady-state eigenvector to satisfy this condition. Thus

$$\vec{x}_{ss} = \begin{bmatrix} 200 \\ 200 \\ 200 \end{bmatrix}. \quad \square$$