EECS 16A Designing Information Devices and Systems I
Spring 2023 Discussion 4B

1. Identifying a Subspace: Proof

Is the set

$$V = \left\{ \vec{v} \;\middle|\; \vec{v} = c \begin{bmatrix}1\\1\\1\end{bmatrix} + d \begin{bmatrix}1\\0\\1\end{bmatrix}, \text{ where } c, d \in \mathbb{R} \right\}$$

a subspace of $\mathbb{R}^3$? Why or why not?

Answer:

Yes, V is a subspace of $\mathbb{R}^3$. We will *prove this* by using the definition of a subspace.

First of all, note that V is a subset of $\mathbb{R}^3$ – all elements in V are of the form $\begin{bmatrix}c+d\\c\\c+d\end{bmatrix}$, which is a 3-dimensional real vector.

Now, consider two elements $\vec{v}_1, \vec{v}_2 \in V$ and $\alpha \in \mathbb{R}$.

This means that there exists $c_1, d_1 \in \mathbb{R}$, such that $\vec{v}_1 = c_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + d_1 \begin{bmatrix}1\\0\\1\end{bmatrix}$. Similarly, there exists $c_2, d_2 \in \mathbb{R}$, such that $\vec{v}_2 = c_2 \begin{bmatrix}1\\1\\1\end{bmatrix} + d_2 \begin{bmatrix}1\\0\\1\end{bmatrix}$.

Now, we can see that

$$\vec{v}_1 + \vec{v}_2 = (c_1 + c_2) \begin{bmatrix}1\\1\\1\end{bmatrix} + (d_1 + d_2) \begin{bmatrix}1\\0\\1\end{bmatrix},$$

so $\vec{v}_1 + \vec{v}_2 \in V$.

Also,

$$\alpha\vec{v}_1 = (\alpha c_1) \begin{bmatrix}1\\1\\1\end{bmatrix} + (\alpha d_1) \begin{bmatrix}1\\0\\1\end{bmatrix},$$

so $\alpha\vec{v}_1 \in V$.

Furthermore, we observe that the zero vector is contained in V, when we set $c = 0$ and $d = 0$.

We have thus identified V as a subset of $\mathbb{R}^3$, shown both of the no escape (closure) properties (closure under vector addition and closure under scalar multiplication), as well as the existence of a zero vector, so V is a subspace of $\mathbb{R}^3$.

It's important to note that satisfying the subset property and the two forms of closure additionally implies that subspace V also satisfies the axioms of a vector space, and therefore is also a vector space.

2. Mechanical Determinants

(a) Compute the determinant of $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Answer:

We can use the form of a 2×2 determinant from lecture:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Therefore,

$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right) = 2 \cdot 3 - 0 \cdot 0 = 6$$

(b) Compute the determinant of $\begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix}$.

Answer:

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a \cdot \det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b \cdot \det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c \cdot \det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

Therefore,

$$\begin{aligned}\det\left(\begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix}\right) &= 2 \cdot \det\left(\begin{bmatrix} 0 & -1 \\ 4 & 5 \end{bmatrix}\right) + 3 \cdot \det\left(\begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix}\right) + 1 \cdot \det\left(\begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}\right) \\ &= 2 \cdot (0 - (-4)) + 3 \cdot (10 - (-1)) + 1 \cdot (8 - 0) \\ &= 8 + 33 + 8 \\ &= 49\end{aligned}$$

(c) Recall from lecture that the determinant of a matrix represents the multi-dimensional volume formed by the column vectors. Explain geometrically why the determinant of a matrix with linearly dependent column vectors is always 0.

Answer: Consider an example in $\mathbb{R}^2$. If we have two vectors that are linearly independent, we can form a parallelogram with them and calculate its area, which will be a nonzero value. Thus, we'll have a nonzero determinant as well. However, if the vectors are linearly dependent, we only have one dimension (since the other dimension would have been compressed to zero), so our "parallelogram" would have an area of 0, corresponding to a determinant with a value of 0.

This idea generalizes to N dimensions. If we have fewer than N linearly independent vectors, then the multi-dimensional volume will have at least 1 dimension compressed to 0, giving us 0 volume and 0 determinant.

3. Mechanical Eigenvalues and Eigenvectors

In each part, find the eigenvalues of the matrix $\mathbf{M}$ and their associated eigenvectors.

(a) $\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

Do you observe anything about the eigenvalues and eigenvectors?

Answer:

Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix}1 & 0\\ 0 & 9\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right) = \det\left(\begin{bmatrix}1-\lambda & 0\\ 0 & 9-\lambda\end{bmatrix}\right) = 0$$

The determinant of a diagonal matrix is the product of the entries.

$$(1-\lambda)(9-\lambda) = 0$$

From the above equation, we know that the eigenvalues are $\lambda = 1$ and $\lambda = 9$.
For the eigenvalue $\lambda = 1$:

$$\begin{aligned}(\mathbf{M} - 1\mathbf{I})\vec{x} &= \vec{0}\\ \left(\begin{bmatrix}1 & 0\\ 0 & 9\end{bmatrix} - 1\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\vec{x} &= \vec{0}\\ \begin{bmatrix}0 & 0\\ 0 & 8\end{bmatrix}\vec{x} &= \vec{0}\end{aligned}$$

From the second equation in the system, $x_2 = 0$, with any solution having the form $\begin{bmatrix}1\\0\end{bmatrix} t$ for $t \in \mathbb{R}$. The eigenspace is thus span $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$.
For the eigenvalue $\lambda = 9$:

$$\begin{aligned}(\mathbf{M} - 9\mathbf{I})\vec{x} &= \vec{0}\\ \left(\begin{bmatrix}1 & 0\\ 0 & 9\end{bmatrix} - 9\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\vec{x} &= \vec{0}\\ \left(\begin{bmatrix}1 & 0\\ 0 & 9\end{bmatrix} - \begin{bmatrix}9 & 0\\ 0 & 9\end{bmatrix}\right)\vec{x} &= \vec{0}\\ \begin{bmatrix}-8 & 0\\ 0 & 0\end{bmatrix}\vec{x} &= \vec{0}\end{aligned}$$

From the first equation in the system, $x_1 = 0$, so any solution must take the form $\begin{bmatrix}0\\1\end{bmatrix} t$ for $t \in \mathbb{R}$. The eigenspace is span $\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$.
We observe that the eigenvalues are just the diagonal entries. Since the matrix is diagonal, multiplying the diagonal matrix $\mathbf{D}$ with any standard basis vector $\vec{e}_i$ produces $d_i\vec{e}_i$, that is, $\mathbf{D}\vec{e}_i = d_i\vec{e}_i$. Therefore, the eigenvalues are the diagonal entries d_i of $\mathbf{D}$, and the corresponding eigenvector associated with $\lambda = d_i$ is the standard basis vector $\vec{e}_i$.

(b) $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}\right) = 0$$

$$\begin{aligned} -\lambda(-3-\lambda)+2 &= 0 \\ \lambda^2 + 3\lambda + 2 &= 0 \\ (\lambda+2)(\lambda+1) &= 0 \\ \lambda &= -1, -2 \end{aligned}$$

$\lambda = -1$:

$$\left[\begin{array}{cc|c} 0-(-1) & 1 & 0 \\ -2 & -3-(-1) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right] \xrightarrow{G.E.} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 &= t \end{aligned} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t$$

The eigenspace for $\lambda = -1$ is $\text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

$\lambda = -2$:

$$\left[\begin{array}{cc|c} 0-(-2) & 1 & 0 \\ -2 & -3-(-2) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -2 & -1 & 0 \end{array}\right] \xrightarrow{G.E.} \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 + x_2/2 &= 0 \\ x_2 &= t \end{aligned} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t/2 \\ t \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} t$$

The eigenspace for $\lambda = -2$ is $\text{span}\left\{\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}\right\}$.

