EECS 16A Designing Information Devices and Systems I
Fall 2022 Homework 2

This homework is due September 16th, 2022, at 23:59.
Self-grades are due September 19th, 2022, at 23:59.

Submission Format

Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

1. Reading Assignment

For this homework, please read Note 2A, Note 2B, Note 3, and Note 4. Notes 2A and 2B provide an overview of vectors, matrices, and operations among them. Note 3 provides an overview of linear dependence (not yet covered) and span. Note 4 introduces mathematical thinking and writing proofs.

Please answer the following questions:

(a) What is the span of a set of vectors?

(b) How can you check if a particular vector is in the span of a set of vectors?

(c) Given that $\vec{b} \in \text{span}\{\ \vec{a}_1, \vec{a}_2, \vec{a}_3\ \}$ and $\vec{a}_1, \vec{a}_2, \vec{a}_3$ are column vectors of $\mathbf{A}$, which *one* of the following statements does not make sense:

i. $\vec{b}$ is in the span of matrix $\mathbf{A}$

ii. $\vec{b}$ is in the range of $\mathbf{A}$

iii. $\vec{b}$ is in the column space of $\mathbf{A}$

(d) Please write a few sentences about how you can use the strategies in the notes to tackle proof questions.

Solution:

(a) See Note 3.3 for definition of span of a set of vectors.

(b) We can check that a particular vector is in the span of a set of vectors if we can write that particular vector as a linear combination of the set of vectors.

(c) There is no such thing as a span of a matrix, only a span of a set of vectors (or column vectors within the matrix). Statement i is wrong.

(d) Give yourself credit for any reasonable answer based on Note 4.

2. Filtering Out The Troll

Learning Goal: *The goal of this problem is to explore the problem of sound reconstruction by solving a system of linear equations.*

You were attending the 16A lecture the day before the first exam, and decided to record it using two directional microphones (one microphone receives sound from the x direction and the other from the y direction).

However, someone (we have *no* idea who) in the audience was trolling around loudly, adding interference to the recording! The troll's interference dominates both of your microphones' recordings, so you cannot hear the recorded speech. Fortunately, since your recording device contained two microphones, you can combine the two individual microphone recordings to remove the troll's interference.

The diagram shown in Figure 1 shows the locations of the speaker, the troll, and you and your two microphones (at the origin).

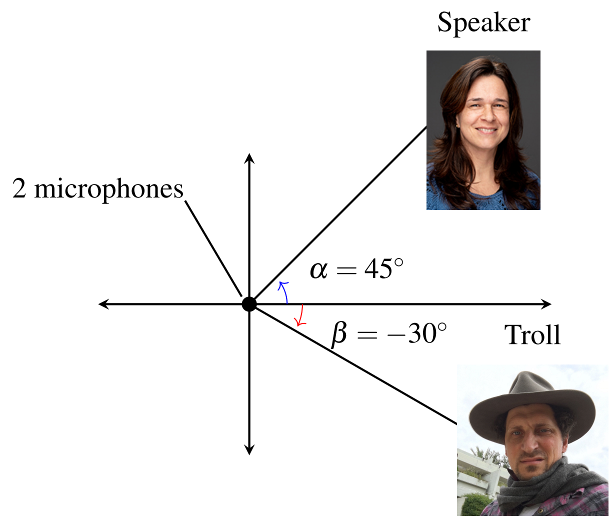


Figure 1: Locations of the speaker and the troll.

Since the microphones are directional, the strength of the recorded signal depends on the angle from which the sound arrives. Suppose that the sound arrives from an angle θ relative to the x-axis (in our case, these angles are 45°and $-30°$, labeled as α and β, respectively). The first microphone scales the signal by $\cos(\theta)$, while the second microphone scales the signal by $\sin(\theta)$. Each microphone records the weighted sum (or linear combination) of all received signals.

The speech signal can be represented as a vector, $\vec{s}$, and the troll's interference as vector $\vec{r}$, with each entry representing an audio sample at a given time. The recordings of the two microphones are given by $\vec{m}_1$ and $\vec{m}_2$:

$$\vec{m}_1 = \cos(\alpha) \cdot \vec{s} + \cos(\beta) \cdot \vec{r} \tag{1}$$

$$\vec{m}_2 = \sin(\alpha) \cdot \vec{s} + \sin(\beta) \cdot \vec{r} \tag{2}$$

where α and β are the angles at which the professor and the troll respectively are located with respect to the x-axis, and variables $\vec{s}$ and $\vec{r}$ are the audio signals produced by the professor and the troll respectively.

(a) Plug in $\alpha = 45^\circ = \frac{\pi}{4}$ and $\beta = -30^\circ = -\frac{\pi}{6}$ to Equations 1 and 2 to write the recordings of the two microphones $\vec{m}_1$ and $\vec{m}_2$ as a linear combination (i.e. a weighted sum) of $\vec{s}$ and $\vec{r}$.

Solution:

$$\begin{aligned}\vec{m}_1 &= \cos\left(\frac{\pi}{4}\right)\cdot\vec{s}+\cos\left(-\frac{\pi}{6}\right)\cdot\vec{r}\\ &= \frac{1}{\sqrt{2}}\cdot\vec{s}+\frac{\sqrt{3}}{2}\cdot\vec{r}\\ \vec{m}_2 &= \sin\left(\frac{\pi}{4}\right)\cdot\vec{s}+\sin\left(-\frac{\pi}{6}\right)\cdot\vec{r}\\ &= \frac{1}{\sqrt{2}}\cdot\vec{s}-\frac{1}{2}\cdot\vec{r}\end{aligned}$$

(b) Solve the system from the earlier part using any convenient method you prefer to recover the important speech $\vec{s}$ as a weighted combination of $\vec{m}_1$ and $\vec{m}_2$. In other words, write $\vec{s} = c\cdot\vec{m}_1 + k\cdot\vec{m}_2$ (where c and k are scalars). What are the values of c and k?

Solution: Solving the system of linear equations yields

$$\vec{s} = \frac{\sqrt{2}}{1+\sqrt{3}}\cdot\left(\vec{m}_1+\sqrt{3}\vec{m}_2\right).$$

Therefore, the values are $c = \frac{\sqrt{2}}{1+\sqrt{3}}$ and $k = \frac{\sqrt{6}}{1+\sqrt{3}}$.

It is fine if you solved this either using IPython or by hand using any valid technique. The easiest approach is to subtract either of the two equations from the other and immediately see that $\vec{r} = \frac{2}{\sqrt{3}+1}(\vec{m}_1-\vec{m}_2)$. Substituting b back into the second equation and multiplying through by $\sqrt{2}$ gives that $\vec{s} = \sqrt{2}(\vec{m}_2 + \frac{1}{\sqrt{3}+1}(\vec{m}_1-\vec{m}_2))$, which simplifies to the expression given above.

Notice that subtracting one equation from the other is natural given the symmetry of the microphone patterns and the fact that the patterns intersect at the 45 degree line where the important speech is happening, and the fact that $\sin(45°) = \cos(45°)$. So we know that the result of subtracting one microphone recording from the other results in only the troll's contribution. Once we have the troll contribution, we can remove it and obtain the professor's sole content.

(c) Partial IPython code can be found in `prob2.ipynb`, which you can access through the Datahub link associated with this assignment on the course website. Complete the code to get the signal of the important speech. Write out what the speaker says. (Optional: Where is the speech taken from?)

Note: You may have noticed that the recordings of the two microphones sound remarkably similar. This means that you could recover the real speech from two "trolled" recordings that sound almost identical! Leave out the fact that the recordings are actually different, and have some fun with your friends who aren't lucky enough to be taking EECS16A.

Solution:

The solution code can be found in `sol2.ipynb`. The speaker (Professor Arias) is discussing Node Voltage Analysis, a circuit analysis algorithm you will learn in Module 2, and the audio is taken from a lecture in which Professor Lustig was being *particularly* disruptive.

The idea of using multiple microphones to isolate speech is interesting and is increasingly used in practice. Furthermore, similar techniques are used in wireless communication both by cellular systems like LTE and increasingly by WiFi hotspots. (This is why they often have multiple antennas).

3. Gaussian Elimination

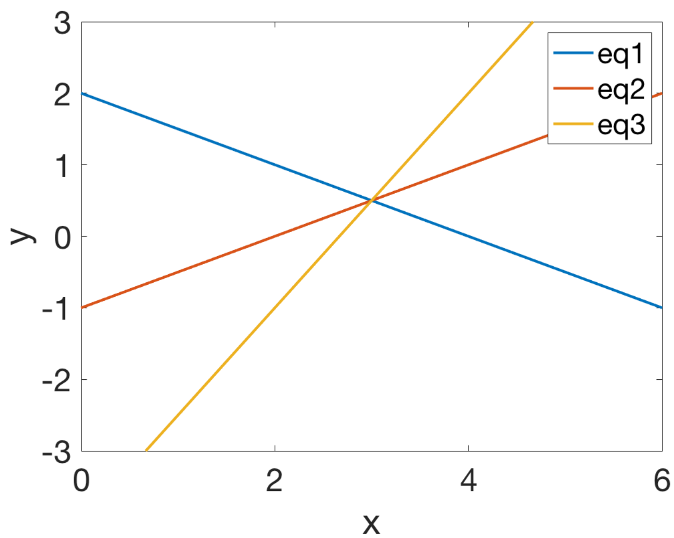
Learning Goal: *Understand the relationship between Gaussian elimination and the graphical representation of linear equations, and explore different types of solutions to a system of equations. You will also* *practice determining the parametric solutions when there are infinitely many solutions.*

(a) In this problem we will investigate the relationship between Gaussian elimination and the geometric interpretation of linear equations. You are welcome to draw plots by hand or using software. Please be sure to label your equations with a legend on the plot.

i. Plot the following set of linear equations in the x-y plane. If the lines intersect, write down the point or points of intersection.

$$x+2y=4 \tag{1}$$
$$2x-4y=4 \tag{2}$$
$$3x-2y=8 \tag{3}$$

Solution:
The three lines intersect at the point (3,0.5).



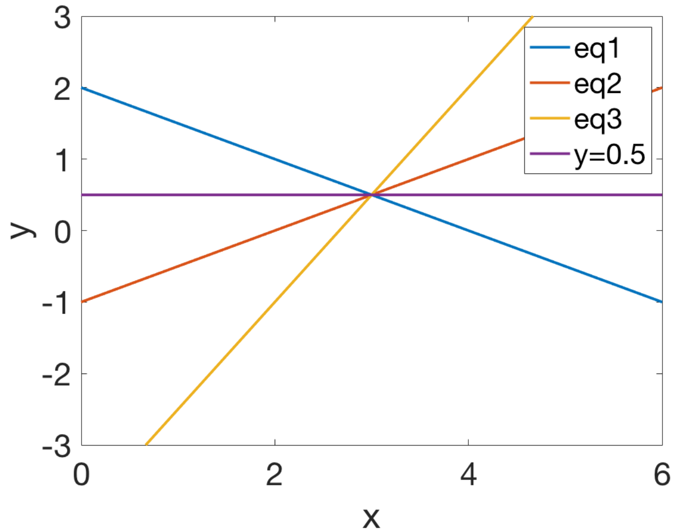
ii. Write the above set of linear equations in augmented matrix form and do the first step of Gaussian elimination to eliminate the x variable from equation 2. Now, the second row of the augmented matrix has changed. Plot the corresponding new equation created in this step on the same graph as above. What do you notice about the new line you draw?
Solution: We start with the following augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & -4 & 4 \\ 3 & -2 & 8 \end{array}\right]$$

We then eliminate x from the second equation by subtracting $2\times$Row 1 from Row 2:

$$\text{Row 2: subtract } 2\times\text{Row 1} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -8 & -4 \\ 3 & -2 & 8 \end{array}\right]$$

So equation 2 becomes $-8y = -4$, which is equivalent to $y = 0.5$. You will notice that the line $y = 0.5$ intersects with the three lines you drew previously.



iii. Complete all of the steps of Gaussian elimination including back substitution. Now plot the new equations represented by the rows of the augmented matrix in the last step (after completing back substitution) on the same graph as above. What do you notice about the new line you draw?

Solution:

We continue from the previous part, where we had the following augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -8 & -4 \\ 3 & -2 & 8 \end{array}\right]$$

and take the following steps to complete Gaussian elimination:

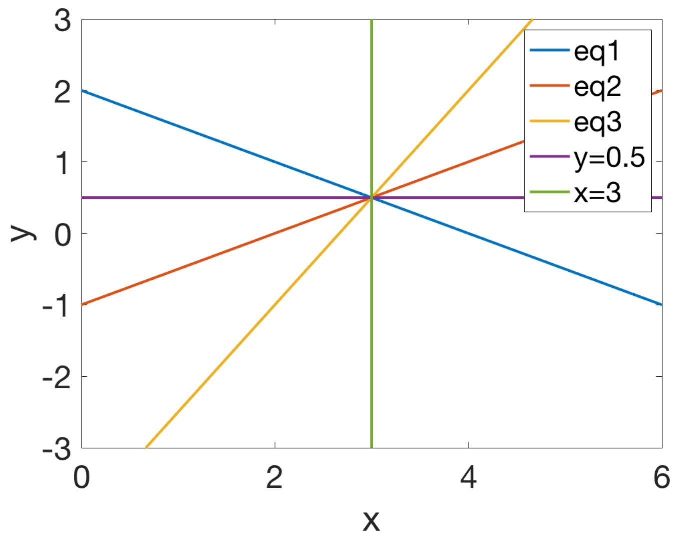
$$\text{Row 3: subtract } 3\times\text{Row 1} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -8 & -4 \\ 0 & -8 & -4 \end{array}\right]$$

$$\text{Row 2: divide by } -8 \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 0.5 \\ 0 & -8 & -4 \end{array}\right]$$

$$\text{Row 3: subtract } -8\times\text{Row 2} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{array}\right]$$

$$\text{Row 1: subtract } 2\times\text{Row 2} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, we end up with the solution $x = 3$ and $y = 0.5$.
Plotting the new equation $x = 3$ on the same graph as before, we see that all five lines intersect at the same point (3,0.5).



(b) Write the following set of linear equations in augmented matrix form and use Gaussian elimination to determine if there are no solutions, infinite solutions, or a unique solution. If any solutions exist, determine what they are. You may do this problem by hand or use a computer. We encourage you to try it by hand to ensure you understand Gaussian elimination. Remember that it is possible to end up with fractions during Gaussian elimination.

$$\begin{aligned} x + 2y + 5z &= 3 \\ x + 12y + 6z &= 1 \\ 2y + z &= 4 \\ 3x + 16y + 16z &= 7 \end{aligned}$$

Solution:
Writing the system in augmented matrix form we get the following:

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 1 & 12 & 6 & 1 \\ 0 & 2 & 1 & 4 \\ 3 & 16 & 16 & 7 \end{array}\right]$$

We eliminate the x variables from the second and fourth equations:

$$\begin{array}{c} \text{Row 2: subtract Row 1} \\ \text{Row 4: subtract } 3\times\text{Row 1} \end{array} \implies \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 10 & 1 & -2 \\ 0 & 2 & 1 & 4 \\ 0 & 10 & 1 & -2 \end{array}\right]$$

We then divide Row 2 by 10 to get a 1 in the pivot position:

$$\text{Row 2: divide by 10} \quad \Longrightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 1 & 0.1 & -0.2 \\ 0 & 2 & 1 & 4 \\ 0 & 10 & 1 & -2 \end{array}\right]$$

Next, we eliminate the y variables from the third and fourth equations:

$$\begin{array}{c}\text{Row 3: subtract } 2\times\text{Row 2} \\ \text{Row 4: subtract } 10\times\text{Row 2}\end{array} \quad \Longrightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 1 & 0.1 & -0.2 \\ 0 & 0 & 0.8 & 4.4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We divide Row 3 by 0.8 to get a 1 in the pivot position:

$$\text{Row 3: divide by 0.8} \quad \Longrightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 1 & 0.1 & -0.2 \\ 0 & 0 & 1 & 5.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We then proceed with back-substitution:

$$\begin{array}{c}\text{Row 2: subtract } 0.1\times\text{Row 3} \\ \text{Row 1: subtract } 5\times\text{Row 3}\end{array} \quad \Longrightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & 0 & -24.5 \\ 0 & 1 & 0 & -0.75 \\ 0 & 0 & 1 & 5.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{Row 1: subtract } 2\times\text{Row 2} \quad \Longrightarrow \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -23 \\ 0 & 1 & 0 & -0.75 \\ 0 & 0 & 1 & 5.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This final matrix is in reduced row echelon form. The first three rows of the matrix have non-zero elements in pivot position, for a system with three unknowns, and the fourth row is a row of zeros, so we can conclude there is a unique solution: $x = -23$, $y = -0.75$, and $z = 5.5$.

(c) Consider the following system:

$$\begin{aligned} 4x+4y+4z+w+v &= 1 \\ x+y+2z+4w+v &= 2 \\ 5x+5y+5z+w+v &= 0 \end{aligned}$$

If you were to write the above equations in augmented matrix form and use Gaussian elimination to solve the system, you would get the following (for extra practice, you can try and do this yourself):

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 3 & 16 \\ 0 & 0 & 1 & 0 & -3 & -17 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

How many variables are free variables? Which ones? Find the general form of the solutions in terms of arbitrary real numbers.

Solution:

We first note that the given augmented matrix is in reduced row echelon form, which makes sense as it is the final output of the Gaussian elimination algorithm. We observe that the second and fifth columns do not have 1s in pivot position so there are two free variables corresponding to y and v.

Let $y = s$ and let $v = t$, where $s \in \mathbb{R}$ and $t \in \mathbb{R}$.

Using back substitution, we can solve for x, y, z, w, and v in terms of s and t:

$$\begin{aligned}
&\text{Row 1:} \quad x + y + 3v = 16 \quad \implies x = 16 - 3t - s \\
&\text{Row 2:} \quad z - 3v = -17 \quad \implies z = -17 + 3t \\
&\text{Row 3:} \quad w + v = 5 \quad \implies w = 5 - t
\end{aligned}$$

The solutions to the system of equations are therefore:

$$\begin{aligned}
x &= 16 - 3t - s \\
y &= s \\
z &= -17 + 3t \\
w &= 5 - t \\
v &= t
\end{aligned}$$

The solutions can also be represented by a set as:

$$S = \left\{ \vec{u} \;\middle|\; \vec{u} = \begin{bmatrix} 16 \\ 0 \\ -17 \\ 5 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3 \\ 0 \\ 3 \\ -1 \\ 1 \end{bmatrix} t \quad , \quad s \in \mathbb{R}, \quad t \in \mathbb{R} \right\}.$$

4. Linearity

Learning Goal: *Review the definition of linearity and further develop the ability to recognize and identify linear functions.*

In this question, we will explore in further detail what exactly it means for a function to be linear. For each of the following, please specify the value(s) of $a \in \mathbb{R}$, if any, for which the function is linear. Here, x and y are variables.

(a)

$$f(x,y) = (3-a)x + 2ay$$

Solution: Recall the definition of linearity, from Note 1:

A real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is *linear* if for all real-valued $\alpha, \beta, y_1, \ldots, y_n, z_1, \ldots, z_n$, the following identity holds:

$$f(\alpha y_1 + \beta z_1, \alpha y_2 + \beta z_2, \ldots, \alpha y_n + \beta z_n) = \alpha f(y_1, \ldots, y_n) + \beta f(z_1, \ldots, z_n). \tag{4}$$

For this question, the equivalent theorem will be more useful: If $f : \mathbb{R}^n \to \mathbb{R}$ is linear, then there exist coefficients $c_1, c_2, \ldots, c_n$ (i.e., real constants, not depending on the input to the function) such that

$$f(x_1, \ldots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \quad \text{for all } x_1 \in \mathbb{R}, \ldots, x_n \in \mathbb{R}. \tag{5}$$

We can see that our variables are x and y (analogs of $x_1 \ldots, x_n$), and their coefficients are $(3-a)$ and $2a$, respectively. Any real value of a will make these coefficients real numbers, and the function linear.

(b)

$$f(x,y) = a^2 x + 8y$$

Solution: Following a similar line of logic as in part a, the coefficients of our variables are a^2 and 8. Any real value of a results in a real value of a^2, and hence real values for all the c_i.

(c)

$$f(x,y) = y + axy - 3x$$

Solution: Notice that this equation is the first where we have a term that multiplies two of our variables. Referencing our theorem, we see that this is not allowed–the function may only be a summation of variables multiplied by a scalar. In other words, we need to eliminate the axy term; the only way to do so is if a is zero.

(d)

$$f(x,y) = (x + ay)^2$$

Solution: If we expand this function, we would get:

$$f(x,y) = x^2 + 2axy + a^2 y^2 \tag{6}$$

Once again, we have nonlinear terms; however, the x^2 term cannot be eliminated, so there are no values of a for which this equation is linear.

5. Vector-Vector, Matrix-Vector, and Matrix-Matrix Multiplication

Learning Objective: *Practice evaluating vector-vector, matrix-vector, and matrix-matrix multiplication.*

(a) For the following multiplications, state the dimensions of the result. If the product is not defined and thus has no solution, state this and justify your reasoning. For this problem $\vec{x} \in \mathbb{R}^N, \vec{y} \in \mathbb{R}^N, \vec{z} \in \mathbb{R}^M$, with $N \neq M$.

i. $\vec{x}^T \cdot \vec{z}$

Solution: This is invalid. $\vec{x}^T$ is an $1 \times N$ vector meaning that it has 1 rows and N column but $\vec{z}$ is an $M \times 1$ vector meaning that is has M rows and 1 column. Since $\vec{x}^T$ does not have the same number of columns as $\vec{z}$ has rows there is no solution.

ii. $\vec{x} \cdot \vec{x}^T$

Solution:

$$N \times N$$

$\vec{x}$ has N row and 1 columns and $\vec{x}^T$ has 1 row and N columns. Since the number of columns of $\vec{x}$ is the same as the number of rows of $\vec{x}^T$, there is a solution. The solution would have the dimensions of the number of rows of $\vec{x}$ times the number of columns of $\vec{x}^T$.

iii. $\vec{x} \cdot \vec{y}^T$

Solution:

$$N \times N$$

$\vec{x}$ has N row and 1 columns and $\vec{y}^T$ has 1 row and N columns. Since the number of columns of $\vec{x}$ is the same as the number of rows of $\vec{y}^T$, there is a solution. The solution would have the dimensions of the number of rows of $\vec{x}$ times the number of columns of $\vec{y}^T$.

iv. $\vec{x} \cdot \vec{z}^T$

Solution:

$$N \times M$$

$\vec{x}$ has N row and 1 columns and $\vec{z}^T$ has 1 row and M columns. Since the number of columns of $\vec{x}$ is the same as the number of rows of $\vec{z}^T$, there is a solution. The solution would have the dimensions of the number of rows of $\vec{x}$ times the number of columns of $\vec{z}^T$.

For questions (b) through (d), complete the matrix-vector multiplication. If the product is not defined and thus has no solution, state this and justify your reasoning:

(b)

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution: No solution. The number of columns in the matrix does not match the number of elements (aka the number of rows) in the column vector.

(c)

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 * \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 * \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 1 * \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2 * \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

(e) Compute **AB** by hand, where **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -3 & 0 & 2 & -1 \end{bmatrix}$$

What are the dimensions of **AB**? Compute **BA** too if the operation is valid. If it is invalid, explain why. Make sure you show the work for your calculations.

Solution:

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 0 \\ -3 & 0 & 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1\times 1+0\times -3 & 1\times 2+0\times 0 & 1\times -1+0\times 2 & 1\times 0+0\times -1 \\ 2\times 1+1\times -3 & 2\times 2+1\times 0 & 2\times -1+1\times 2 & 2\times 0+1\times -1 \\ 0\times 1+1\times -3 & 0\times 2+1\times 0 & 0\times -1+1\times 2 & 0\times 0+1\times -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -3 & 0 & 2 & -1 \end{bmatrix} \end{aligned}$$

BA does not exist since the number of columns in **B** is not equal to the number of rows in **A**.

(f) Compute **AB** by hand, where **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} 3 & 21 & 9 \\ -1 & 14 & 4 \\ 7 & -8 & 2 \end{bmatrix}, \text{ and } \qquad \mathbf{B} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \\ 3 & -6 \end{bmatrix}$$

Compute **BA** too if the operation is valid. If it is invalid, explain why. Make sure you show the work for your calculations.

Solution:

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 3 & 21 & 9 \\ -1 & 14 & 4 \\ 7 & -8 & 2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ -1 & 2 \\ 3 & -6 \end{bmatrix} \\ &= \begin{bmatrix} 3\times -2+21\times -1+9\times 3 & 3\times 4+21\times 2+9\times -6 \\ -1\times -2+14\times -1+4\times 3 & -1\times 4+14\times 2+4\times -6 \\ 7\times -2+-8\times -1+2\times 3 & 7\times 4+-8\times 2+2\times -6 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

BA does not exist since the number of columns in **B** is not equal to the number of rows in **A**.

6. Vectors in the Span

Learning Goal: *Practice determining whether a vector is in the span of a set of vectors.*

Determine whether a vector $\vec{v}$ is in the span of the given set of vectors. If it is in the span of given set, write $\vec{v}$ as a linear combination of given set of vectors (you will need to find the scalar coefficients in the linear combination).

(a) $\vec{v} = \begin{bmatrix} -10 \\ 4 \end{bmatrix}$ and $\left\{ \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$

Solution: We see that $\vec{v}$ is a scaled multiple of one of the vectors. $\vec{v} = 2\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ thus $\vec{v}$ is in the span of the set of vectors.

(b) $\vec{v} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ and $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix} \right\}$

Solution:

We want to write the following:

$$\vec{v} = a\begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix}$$

We can use Gaussian Elimination to find a and b:

$$\left[\begin{array}{cc|c} -1 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 3 & -1 \\ -1 & -2 & 0 \\ 1 & -1 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow -R_1} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 0 & 3 & -1 \\ -1 & -2 & 0 \\ 1 & -1 & 1 \end{array}\right] \xrightarrow[R_4 \leftarrow R_4 + R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 3 & -1 \\ 0 & -3 & 1 \\ 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \leftarrow R_2/3]{R_5 \leftarrow R_5 - R_1} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 3 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow[R_4 \leftarrow R_4 + 3R_2]{R_3 \leftarrow R_3 - 3R_2} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

This gives us a = $\frac{2}{3}$ and b = -$\frac{1}{3}$ and we can write: $\vec{v} = \frac{2}{3}\begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix}$.

(c) $\vec{v} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} \right\}$

Solution:

Using the same method as part (b):

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 2 & 1 & 4 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1/2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 4 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -3 & 2 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3/-3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -\frac{2}{3} \end{array}\right] \xrightarrow[R_1 \leftarrow R_1 - R_3]{R_2 \leftarrow R_2 - 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{2}{3} \end{array}\right]$$

We can write $\vec{v} = \frac{2}{3}\begin{bmatrix}2\\2\\0\end{bmatrix} + \frac{1}{3}\begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}2\\4\\-1\end{bmatrix}$

(d) $\vec{v} = \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$ and $\left\{\begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-2\\2\\0\end{bmatrix}\right\}$

Solution: Note that $\vec{v}$ is $\vec{0}$, the zero vector since all its values are zeros. We can write a linear combination using zeros as scalars (this is also called "trivial linear combination"):

$$0\begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix} + 0\begin{bmatrix}0\\1\\-2\\2\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$$

There will always exist a zero vector that exists in the span of a set of vectors.

7. Span Proofs

Learning Objectives: *This is an opportunity to practice your proof development skills.*

(a) Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, show the following:

$$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \dots, \vec{v}_n\}$$

In other words, we can replace one vector with the sum of itself and another vector and not change their span.

In order to show this, you have to prove the two following statements:

- If a vector $\vec{q}$ belongs in $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, then it must also belong in $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \dots, \vec{v}_n\}$.
- If a vector $\vec{r}$ belongs in $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \dots, \vec{v}_n\}$, then it must also belong in $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$.

In summary, you have to prove the problem statement from both directions.

Solution:

Suppose $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. For some scalars a_i:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = a_1(\vec{v}_1 + \vec{v}_2) + (-a_1 + a_2)\vec{v}_2 + \cdots + a_n\vec{v}_n$$

We can change the scalar values to adjust for the combined vectors. Thus, we have shown that $\vec{q} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \dots, \vec{v}_n\}$.

Now, we must show the other direction. Suppose we have some arbitrary $\vec{r} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \dots, \vec{v}_n\}$. For some scalars b_i:

$$\vec{r} = b_1(\vec{v}_1 + \vec{v}_2) + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n = b_1\vec{v}_1 + (b_1 + b_2)\vec{v}_2 + \cdots + b_n\vec{v}_n.$$

Thus, we have shown that $\vec{r} \in \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. Combining this with the earlier result, the spans are the same.

(b) Consider the span of the set $(\vec{v}_1,\ldots,\vec{v}_n,\vec{u})$. Suppose $\vec{u}$ is in the span of $\{\vec{v}_1,\ldots,\vec{v}_n\}$. Then, show that any vector $\vec{r}$ in $span\{\vec{v}_1,\ldots,\vec{v}_n,\vec{u}\}$ is in $span\{\vec{v}_1,\ldots,\vec{v}_n\}$.

Solution: From the first sentence of the question, by definition of span, we know that any vector $\vec{r}$ in $span\{\vec{v}_1,\ldots,\vec{v}_n,\vec{u}\}$ can be written $\vec{r} = k\vec{u} + a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_n\vec{v}_n$. Using the summation symbol, we can also write $\vec{r} = k\vec{u} + \sum_{i=1}^{n} a_i\vec{v}_i$.

From the second sentence of the question, since $\vec{u}$ is in the span of $\{\vec{v}_1,\ldots,\vec{v}_n\}$, we can write $\vec{u} = b_1\vec{v}_1 + b_2\vec{v}_2 + \ldots b_n\vec{v}_n$ or $\sum_{i=1}^{n} b_i\vec{v}_i$. Now that we have an expression for $\vec{u}$, let's substitute it into the previous expression.

$$\vec{r} = k\vec{u} + \sum_{i=1}^{n} a_i\vec{v}_i$$

$$\vec{r} = k(\sum_{i=1}^{n} b_i\vec{v}_i) + \sum_{i=1}^{n} a_i\vec{v}_i$$

Finally, gathering up coefficients, we get:

$$\vec{r} = \sum_{i=1}^{n} (k * b_i + a_i)\vec{v}_i,$$

so this arbitrary vector $\vec{r}$ is also in $span\{\vec{v}_1,\ldots,\vec{v}_n)\}$.

Intuitively, $\vec{u}$ is redundant, so we can safely remove it without reducing our span.

8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.