EECS 16A Designing Information Devices and Systems I
Spring 2023 Homework 5

This homework is due Friday, February 24, 2023 at 23:59.
Self-grades are due Friday, March 3, 2023 at 23:59.

Submission Format
Your homework submission should consist of **one** file.

We strongly recommended that you submit your self-grades PRIOR to taking Midterm 1 on March 1, 2023, since looking at the solutions earlier will help you to study for the midterm.

1. **Reading Assignment**

   For this homework, please read Notes 8 and 9. These notes will give you an overview of matrix subspaces and eigenvalues/eigenvectors. Note that Note 10 covers change of basis and diagonalizaiton, which is not in-scope for this couse; however you are welcome to read it if interested, as these topics will be emphasized in EECS 16B. You are always welcome and encouraged to read beyond this as well.

   How do we compute eigenvalues and, subsequently, corresponding eigenvectors? What is the eigenvalue corresponding to the steady state of a system?

2. **Mechanical Determinants**

   For each of the following matrices, compute their determinant and state whether they are invertible.

   (a) $\begin{bmatrix} 6 & 9 \\ 4 & 6 \end{bmatrix}$.

   (b) $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

   (c) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

   (d) $\begin{bmatrix} -4 & 2 & 1 \\ 5 & 1 & -3 \\ 7 & 3 & 1 \end{bmatrix}$.

   (e) $\begin{bmatrix} -4 & 0 & 0 \\ 5 & 1 & -3 \\ 7 & 3 & 1 \end{bmatrix}$.

3. **Introduction to Eigenvalues and Eigenvectors**

   Learning Goal: *Practice calculating eigenvalues and eigenvectors. The importance of eigenvalues and eigenvectors will become clear in the following problems.*

   For each of the following matrices, find their eigenvalues and the corresponding eigenvectors. For simple matrices, you may do this by inspection if you prefer.

(a) $\mathbf{A} = \begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(d) Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a general square matrix. Show that the set of eigenvectors corresponding to a particular eigenvalue of $\mathbf{A}$ is a subspace of $\mathbb{R}^n$. In other words, show that

$$\{\vec{x} \in \mathbb{R}^n : \mathbf{A}\vec{x} = \lambda\vec{x}, \lambda \in \mathbb{R}\}$$

is a subspace. You have to show that all three properties of a subspace (as mentioned in Note 8) hold.

4. Properties of Pump Systems - II

Learning Objectives: *This problem builds on the pump examples we have been doing, but is meant to help you learn to do proofs in a step by step fashion. Can you generalize intuition from a simple example?*

We consider a system of reservoirs connected to each other through pumps. An example system is shown below in Figure 1, represented as a graph. Each node in the graph is marked with a letter and represents a reservoir. Each edge in the graph represents a pump which moves a fraction of the water from one reservoir to the next at every time step. The fraction of water moved is written on top of the edge.

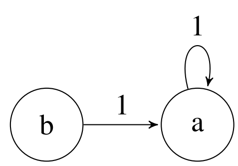


Figure 1: Pump system

We want to prove the following theorem. We will do this step by step.

Theorem: Consider a system consisting of k reservoirs such that the entries of each column in the system's state transition matrix sum to one. If s is the total amount of water in the system at timestep n, then total amount of water at timestep $n+1$ will also be s.

(a) Rewrite the theorem statement for a graph with only two reservoirs.

(b) Since the problem does not specify the transition matrix, let us consider the transition matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and the state vector $\vec{x}[n] = \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix}$. Write out what is "known" or what is given to you in the theorem statement in mathematical form.
Note: In general, it is helpful to write as much out mathematically as you can in proofs. It can also be helpful to draw the transition graph.

(c) Now write out the theorem we want to prove mathematically.

(d) Prove the statement for the case of two reservoirs. In other words, combine parts b and c to prove the theorem.

(e) Now use what you learned to generalize to the case of k reservoirs. *Hint:* Think about $\mathbf{A}$ in terms of its columns, since you have information about the columns.

5. Page Rank

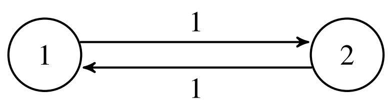
Learning Goal: *This problem highlights the use of transition matrices in modeling dynamical linear systems. Predictions about the steady state of a system can be made using the eigenvalues and eigenvectors of this matrix.*

In homework and discussion, we have discussed the behavior of water flowing in reservoirs and the people flowing in social networks. We now consider the setting of web traffic where the dynamical system can be described with a directed graph, also known as state transition diagram.

As we have seen in lecture and discussion, the "transition matrix", $\mathbf{T}$, can be constructed using the state transition diagram as follows: entries t_{ji} represent the *proportion* of the people who are at website i that click the link for website j.

The **steady-state frequency** (i.e. fraction of visitors in steady-state) for a graph of websites is related to the eigenspace associated with eigenvalue 1 for the "transition matrix" of the graph. Once computed, an eigenvector with eigenvalue 1 will have values which correspond to the steady-state frequency for the fraction of people for each webpage. When the elements of this eigenvector are made to **sum to one** (to conserve population), the i^{th} element of the eigenvector will correspond to the fraction of people on the i^{th} website.

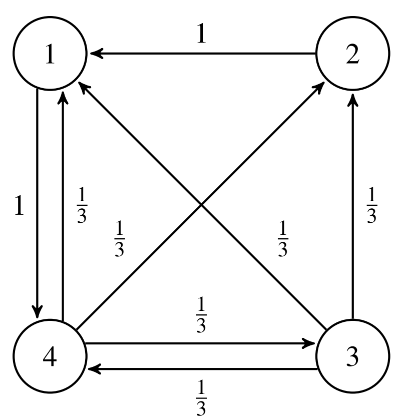
(a) For graph A shown below, what are the steady-state frequencies (i.e. fraction of visitors in steady-state) for the two webpages? Graph A has weights in place to help you construct the transition matrix. Remember to ensure that your steady state-frequencies sum to 1 to maintain conservation.



Graph A

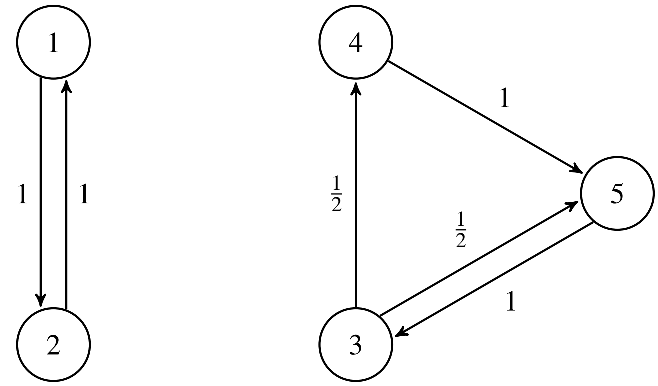
(b) For graph B, what are the steady-state frequencies for the webpages? You may use IPython and the Numpy command `numpy.linalg.eig` for this. Graph B is shown below, with weights in place to help you construct the transition matrix.
Hint: `numpy.linalg.eig` returns eigenvectors and eigenvalues. The eigenvectors are arranged in a matrix in *column-major* order. In other words, given eigenvectors
$\vec{v}_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}$
NumPy will return:

$$\begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} \tag{1}$$



Graph B

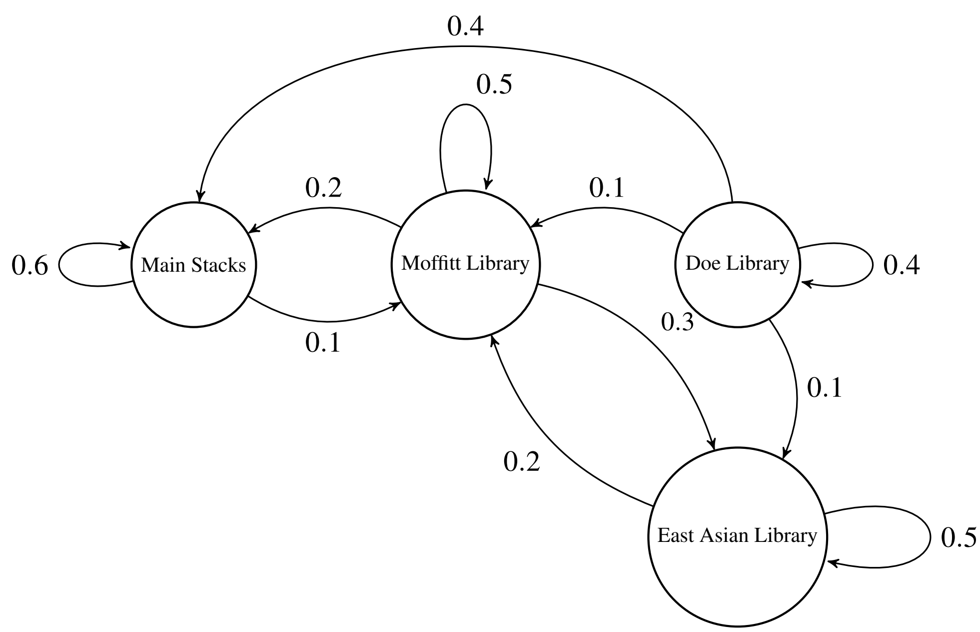
(c) Graph C with weights in place is shown below. Find the eigenspace that corresponds to the steady-state for graph C. How many independent systems (disjoint sets of webpages) are there in graph C versus in graph B? What is the dimension of the eigenspace corresponding to the steady-state for graph C? You may use IPython to compute the eigenvalues and eigenvectors again.



Graph C

6. Favorite Study Spots in Berkeley

Berkeley students are some of the most studious in the nation! Thus, it is not uncommon to find them studying in various spots on campus. Due to class schedules, students often move to different libraries based on their proximity to different classes. The flow of students across the four most popular libraries is as follows:



Let the number of students at the libraries be represented in the following way:

$$A \begin{bmatrix} x_{ML}[t] \\ x_{DL}[t] \\ x_{MS}[t] \\ x_{EAL}[t] \end{bmatrix} = \begin{bmatrix} x_{ML}[t+1] \\ x_{DL}[t+1] \\ x_{MS}[t+1] \\ x_{EAL}[t+1] \end{bmatrix}$$

(a) Write the transition matrix A corresponding to the diagram above.

(b) Determine if the transition matrix is conservative or not. Explain why or why not either conceptually or mathematically.

(c) For a research project, your friend wants to predict the number of students studying in these libraries in the future. Ignoring your answer from the previous part, use the following transition matrix A and the current number of students given by $\vec{x}[t]$:

$$A = \begin{bmatrix} 0.5 & 0.4 & 0 & 0.3 \\ 0.3 & 0.4 & 0.5 & 0 \\ 0 & 0 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.3 & 0.5 \end{bmatrix} \quad \vec{x}[t] = \begin{bmatrix} 140 \\ 400 \\ 210 \\ 90 \end{bmatrix}$$

Help your friend predict the number of students in each libraries in the next time step.

(d) You want to expand upon your friend's research. Thus, you tracked the number of students at 2 lesser known libraries (Hangrove Library and Mathematics/Statistics Library) and calculated their corresponding state-transition matrix to form the following model:

$$\begin{bmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{bmatrix} \begin{bmatrix} x_H[t] \\ x_{MS}[t] \end{bmatrix} = \begin{bmatrix} x_H[t+1] \\ x_{MS}[t+1] \end{bmatrix}$$

Given that there are in total 1500 students, determine the number of students in these two libraries after infinite time steps ($\vec{x}[\infty]$). If the answer can not be determined, give a brief explanation why.

7. Is There A Steady State?

So far, we've seen that for a conservative state transition matrix $\mathbf{A}$, we can find the eigenvector, $\vec{v}$, corresponding to the eigenvalue $\lambda = 1$. This vector is the steady state since $\mathbf{A}\vec{v} = \vec{v}$. However, we've so far taken for granted that the state transition matrix even has the eigenvalue $\lambda = 1$. Let's try to prove this fact.

(a) Show that if λ is an eigenvalue of a matrix $\mathbf{A}$, then it is also an eigenvalue of the matrix $\mathbf{A}^T$.
Hint: The determinants of $\mathbf{A}$ and $\mathbf{A}^T$ are the same. This is because the volumes which these matrices represent are the same.

(b) Let a square matrix $\mathbf{A}$ have, for each row, entries that sum to one. Show that $\vec{1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$ is an eigenvector of $\mathbf{A}$. What is the corresponding eigenvalue?

(c) Let's put it together now. From the previous two parts, show that any conservative state transition matrix will have the eigenvalue $\lambda = 1$. Recall that conservative state transition matrices have, for each column, entries that sum to 1.

8. Traffic Flows (OPTIONAL Practice for Midterm 1)

Learning Objective: *The learning objective of this problem is to see how the concept of nullspaces can be applied to flow problems.*

Your goal is to measure the flow rates of vehicles along roads in a town. It is prohibitively (too) expensive to place a traffic sensor along every road. You realize, however, that the number of cars flowing into an intersection must equal the number of cars flowing out. You can use this "flow conservation" to determine the traffic along all roads in a network by measuring the flow along only some roads. In this problem, we will explore this concept.

(a) Let's begin with a network with three intersections, A, B and C. Define the flow t_1 as the rate of cars (cars/hour) on the road between B and A, flow t_2 as the rate on the road between C and B, and flow t_3 as the rate on the road between C and A.

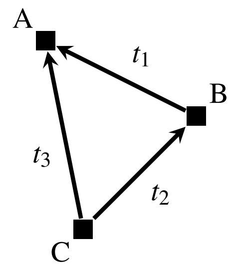


Figure 2: A simple road network.

(Note: The directions of the arrows in the figure are the way that we define positive flow by convention. For example, if there were 100 cars per hour traveling from A to C, then $t_3 = -100$. The flows now are not fractions of water in reservoirs as in the pumps setting, but numbers of cars.)

We assume the "flow conservation" constraints: the net number of cars per hour flowing into each intersection is zero. For example at intersection B, we have the constraint $t_2 - t_1 = 0$. The full set of constraints (one per intersection) is:

$$\begin{cases} t_1 + t_3 = 0 \\ t_2 - t_1 = 0 \\ -t_3 - t_2 = 0 \end{cases}$$

As mentioned earlier, we can place sensors on a road to measure the flow through it, but we have a limited budget, and we would like to determine all of the flows with the smallest possible number of sensors.

Suppose for the network above we have one sensor reading, $t_1 = 10$. Can we figure out the flows along the other roads? (That is, the values of t_2 and t_3). If we can, find the values of t_2 and t_3.

(b) Now suppose we have a larger network, as shown in Figure 3.

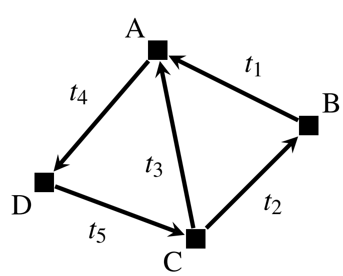


Figure 3: A larger road network.

We would again like to determine the traffic flows on all roads, using measurements from some sensors. A Berkeley student claims that we need two sensors placed on the roads CA (measuring t_3) and DC (measuring t_5). A Stanford student claims that we need two sensors placed on the roads CB (measuring t_2) and BA (measuring t_1). Write out the system of linear equations that represents this flow graph. Is it possible to determine all traffic flows, $\left[t_1, t_2, t_3, t_4, t_5\right]^T$, with the Berkeley student's suggestion? How about the Stanford student's suggestion? *Hint: This can be solved just writing out the relevant equations and reasoning about them.*

(c) We would like a more general way of determining the possible traffic flows in a network. Suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. As a first step, let us try to write all the flow conservation constraints (one per intersection) i.e. the system of equations from part (b) as a matrix equation.

Construct a 4×5 matrix $\mathbf{B}$ such that the equation $\mathbf{B}\vec{t} = \vec{0}$:

$$\begin{bmatrix} & & \mathbf{B} & & \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

represents the flow conservation constraints for the network in Figure 3.

*Hint: You can construct **B** using only $0, 1,$ and -1 entries. Each row represents the inflow/outflow of an intersection. This matrix is called the **incidence matrix**.*

(d) Again, suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. Then, determine the subspace of all valid traffic flows for the network of Figure 3. Notice that the set of all vectors $\vec{t}$ that satisfy $\mathbf{B}\vec{t} = \vec{0}$ is exactly the null space of the matrix $\mathbf{B}$. That is, we can find all valid traffic flows by computing the null space of $\mathbf{B}$. What is the dimension of the nullspace?

(e) **[Challenge]** Now let us analyze more general road networks. Say there is a road network graph G, with incidence matrix $\mathbf{B}_G$. If $\mathbf{B}_G$ has a k-dimensional null space, does this mean measuring the flows along ***any*** k **roads** is always sufficient to recover all of the true flows? In other words, is there ever a possibility of being unable to recover the true flows depending on which k roads you choose?

If you think measuring the flows along any k roads will always work, then prove it showing various possible scenarios. Otherwise give an example showing a scenario where it does not work (such an example is called a counter example).

Hint: Consider the Stanford student's measurement from part (b).

(f) **[Challenge]** If the incidence matrix $\mathbf{B}_G$ has a k-dimensional null space, does this mean we can **always pick a set of** k **roads** such that measuring the flows along these roads is sufficient to recover the exact flows? If this is true, explain how you would pick these k roads to guarantee that you could recover the missing information. Otherwise, give a counterexample.

9. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.