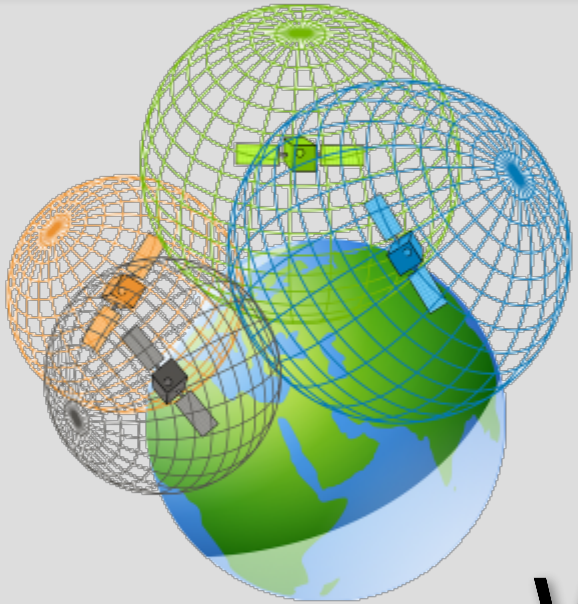
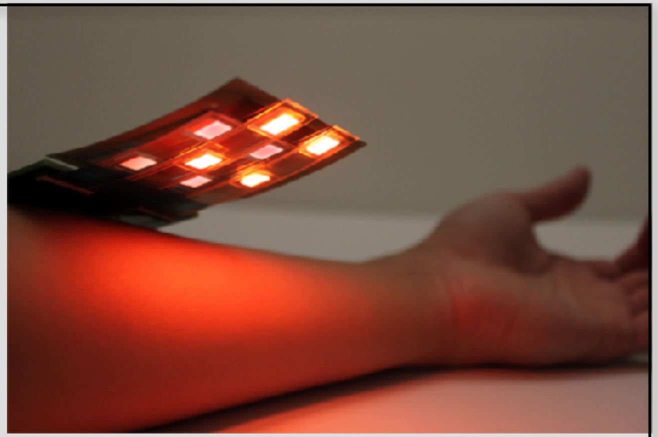




Welcome to EECS 16A!

Designing Information Devices and Systems I

Ana Arias and Miki Lustig

Fa 2022

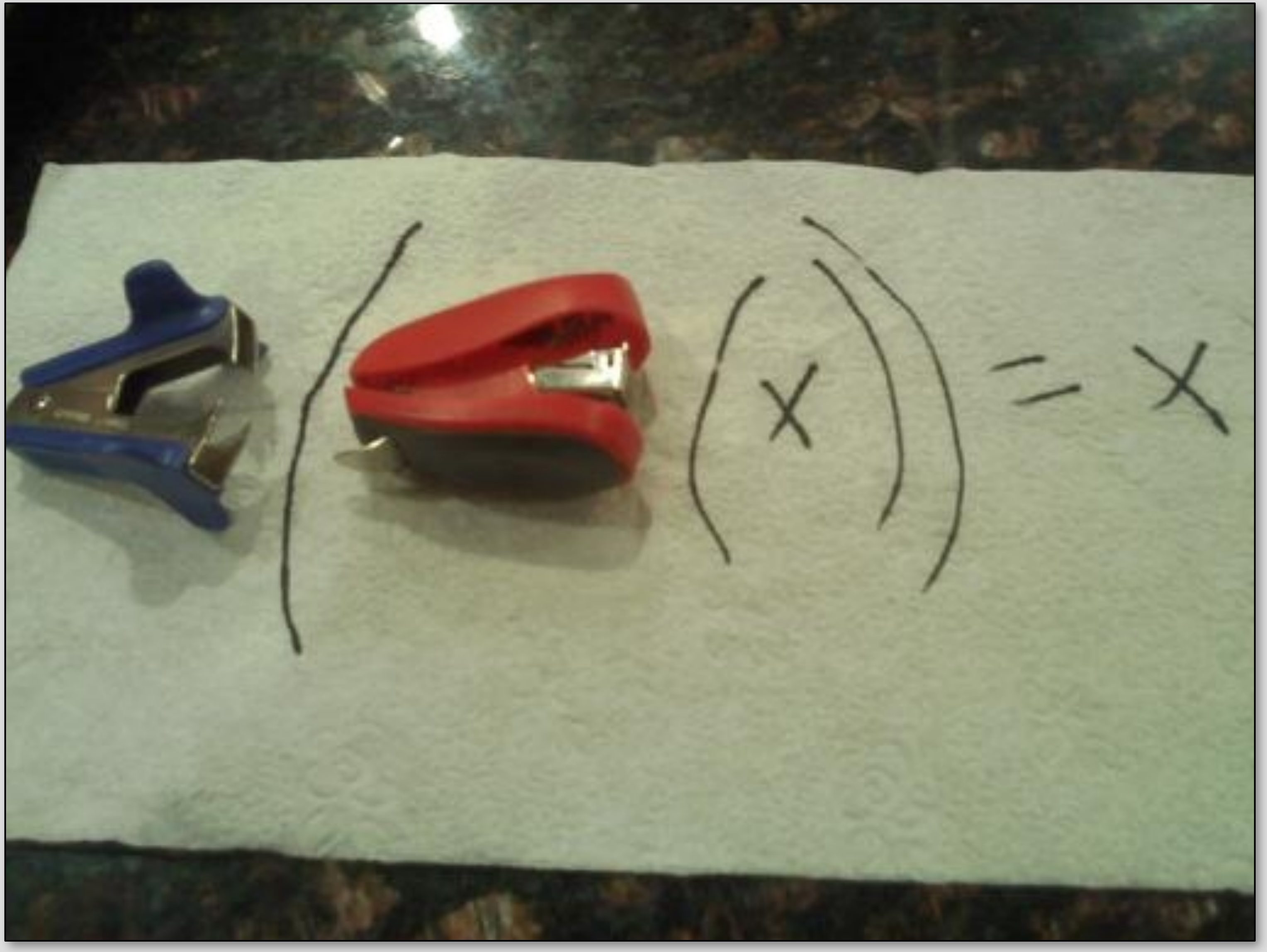
Lecture 4B

Vector Spaces

Announcements

- Last time:
  - Continue with Matrix transformations
  - Matrix Inverse
- Today:
  - Vector spaces
  - Null spaces
  - Subspaces / Row

Matrix Inversion



Invertibility of Linear Transformations

- Theorem: A is invertible, if and only if (iff) the columns of A are linearly independent.
    1. If columns of A are lin. dep. then A^{-1} does not exist
    2. If A^{-1} exists, then the cols. of A are linearly independent

Proof concept: Assume linear dependence and invertibility and show that it is a contradiction

From linear ~~in~~dependence: $\exists\, \overrightarrow{\alpha} \neq 0$ such that $A\overrightarrow{\alpha} = 0$

Assume A^{-1} exists

$$A\overrightarrow{\alpha} = 0$$

$$A^{-1}A\overrightarrow{\alpha} = A^{-1}0$$

$$I\overrightarrow{\alpha} = 0$$

But $\overrightarrow{\alpha} \neq 0$! Hence A^{-1} does not exist

Inverse of a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

1. Flip a and d
2. Negate b and c
3. Divide by $ad - bc$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Derive via Gauss Elimination!

Equivalent Statements

- Matrix A is **invertible**
- $A\overrightarrow{x} = \overrightarrow{b}$ has a unique solution
- A has linearly independent columns (A is **full rank**)
- A has a **trivial nullspace**
- The **determinant** of A is not zero

Today (and next time's) Jargon

- **Rank** a matrix A is the number of linearly independent columns
- **Nullspace** of a matrix A is the set of solutions to $A\overrightarrow{x} = 0$
- A **vector space** is a set of vectors connected by two operators (+,x)
- A vector **subspace** is a subset of vectors that have "nice properties"
- A **basis** for a vector space is a minimum set of vectors needed to represent all vectors in the space
- **Dimension** of a vector space is the number of basis vectors
- **Column space** is the span (range) of the columns of a matrix
- **Row space** is the span of the rows of a matrix

ESPIRiT—an eigenvalue approach to autocalibrating parallel MRI: where SENSE meets GRAPPA 1147 2014
M Uecker, P Lai, MJ Murphy, P Virtue, M Elad, JM Pauly, SS Vasanawala, ...
Magnetic resonance in medicine 71 (3), 990-1001

https://www.ncbi.nlm.nih.gov/pmc/articles/PMC4142121/

- Basis - 3 times
- Rank - 4 times
- Row space - 4 times
- Columns (of a matrix) - 6 times
- Subspace - 17 times
- Null Space - 29 times
- Eigen - 87 times

Vector Space

- From Merriam Webster:

Definition of *vector space*

a set of vectors along with operations of addition and multiplication such that the set is a commutative group under addition, it includes a multiplicative inverse, and multiplication by scalars is both associative and distributive

Vector Space

- A vector space, is a set of vectors and scalars ($\mathbb{V} \in \mathbb{R}^N, \mathbb{F} \in \mathbb{R}$) and two operators $\cdot$, $+$ that satisfy the following:

Axioms of closure

1. $\alpha \overrightarrow{x} \in \mathbb{V}$
2. $\overrightarrow{x} + \overrightarrow{y} \in \mathbb{V}$

Axioms of addition (+)

3. $\overrightarrow{x} + (\overrightarrow{y} + \overrightarrow{z}) = (\overrightarrow{x} + \overrightarrow{y}) + \overrightarrow{z}$ (associativity)
4. $\overrightarrow{x} + \overrightarrow{y} = \overrightarrow{y} + \overrightarrow{x}$ (commutativity)
5. $\exists \overrightarrow{0} \in \mathbb{V}$ s.t. $\overrightarrow{x} + \overrightarrow{0} = \overrightarrow{x}$ (additive identity)
6. $\exists (-\overrightarrow{x}) \in \mathbb{V}$ s.t. $\overrightarrow{x} + (-\overrightarrow{x}) = \overrightarrow{0}$ (additive inverse)

Axioms of scaling (·)

7. $\alpha(\overrightarrow{x} + \overrightarrow{y}) = \alpha \overrightarrow{x} + \alpha \overrightarrow{y}$ (distributivity)
8. $\alpha \cdot (\beta \overrightarrow{x}) = (\alpha\beta) \cdot \overrightarrow{x}$
9. $(\alpha + \beta)\overrightarrow{x} = \alpha \overrightarrow{x} + \beta \overrightarrow{x}$
10. $1 \cdot \overrightarrow{x} = \overrightarrow{x}$

Vector Space

- A vector space $\mathbb{V}$ is a set of vectors and two operators $\cdot$, $+$ that satisfy the following:
  1. $\alpha \vec{x} \in \mathbb{V}$
  2. $\vec{x} + \vec{y} \in \mathbb{V}$
  3. $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$ (associativity)
  4. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (commutativity)
  5. $\exists \vec{0} \in \mathbb{V}$ s.t. $\vec{x} + \vec{0} = \vec{x}$ (additive identity)
  6. $\exists (-\vec{x}) \in \mathbb{V}$ s.t. $\vec{x} + (-\vec{x}) = \vec{0}$
  7. $\alpha(\vec{x} + \vec{y}) = \alpha \vec{x} + \alpha \vec{y}$ (distributivity)
  8. $\alpha \cdot (\beta \vec{x}) = (\alpha\beta) \cdot \vec{x}$
  9. $(\alpha + \beta)\vec{x} = \alpha \vec{x} + \beta \vec{x}$
  10. $1 \cdot \vec{x} = \vec{x}$

Is $\mathbb{R}^2$ a vector space?

Is $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$?

Is $\alpha \in \mathbb{R}, \alpha \geq 0$?

Is $\text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$?

Is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

Is 0?

Subspaces

- A subspace $\mathbb{U}$ consists of a subset of $\mathbb{V}$ in vector space $(\mathbb{V}, \mathbb{F}, +, \cdot)$
    - $\mathbb{U} \subset \mathbb{V}$ and have 3 properties

1. Contains $\overrightarrow{0}$, i.e., $\overrightarrow{0} \in \mathbb{U}$
2. Closed under vector addition: $\overrightarrow{v}_1, \overrightarrow{v}_2 \in \mathbb{U}, \ \Rightarrow \overrightarrow{v}_1 + \overrightarrow{v}_2 \in \mathbb{U}$
3. Closed under scalar multiplication: $\overrightarrow{v}_1 \in \mathbb{U}, \alpha \in \mathbb{F}, \ \Rightarrow \alpha \overrightarrow{v} \in \mathbb{U}$

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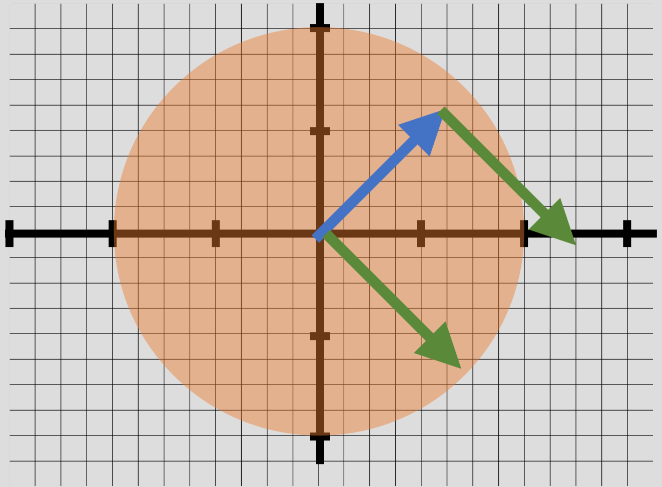
  3. Closed under scalar multiplication: $\overrightarrow{v}_1 \in \mathbb{U}, \alpha \in \mathbb{F}, \ \Rightarrow \alpha\overrightarrow{v} \in \mathbb{U}$

Q: Consider all vectors $\overrightarrow{v}$ who's length < 1. Is this a subspace?



A: not closed under addition,
nor scalar mult.

Subspaces

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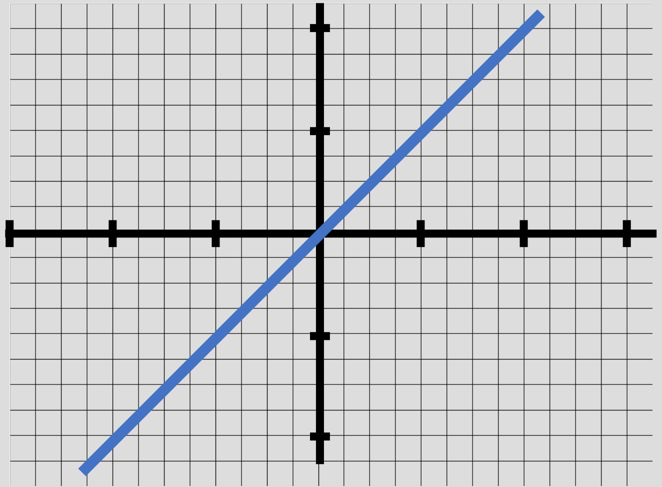
  3. Closed under scalar multiplication: $\overrightarrow{v}_1 \in \mathbb{U}, \alpha \in \mathbb{F}, \;\Rightarrow \alpha \overrightarrow{v} \in \mathbb{U}$

Q: Is $\mathrm{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ a subspace?



A: Yes!

Subspaces

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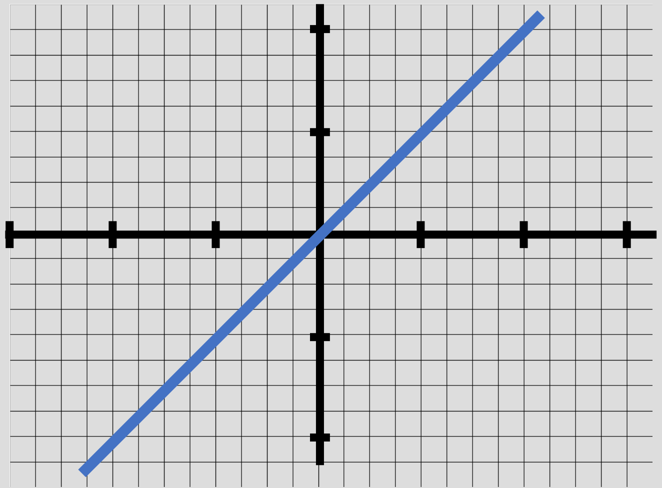
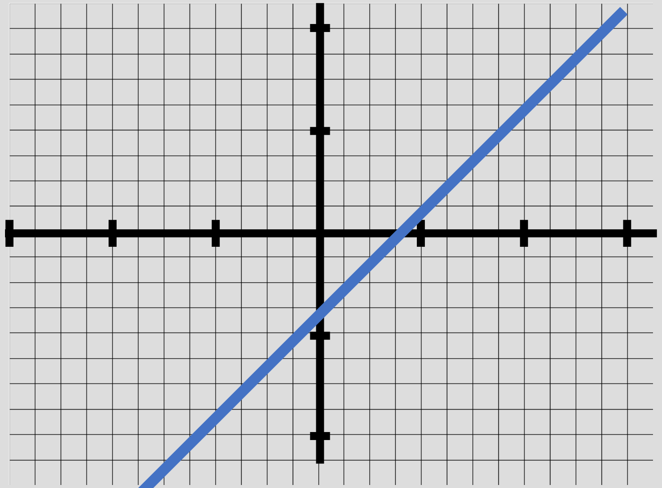


A: Yes!

Q: What about this?



A: $\vec{0} \notin \mathbb{U}$

No!

Subspaces

- A subspace $\mathbb{U}$ consists of a subset of $\mathbb{V}$ in vector space $(\mathbb{V}, \mathbb{F}, +, \cdot)$
    - $\mathbb{U} \subset \mathbb{V}$ and have 3 properties

    1. Contains $\overrightarrow{0}$, i.e., $\overrightarrow{0} \in \mathbb{U}$
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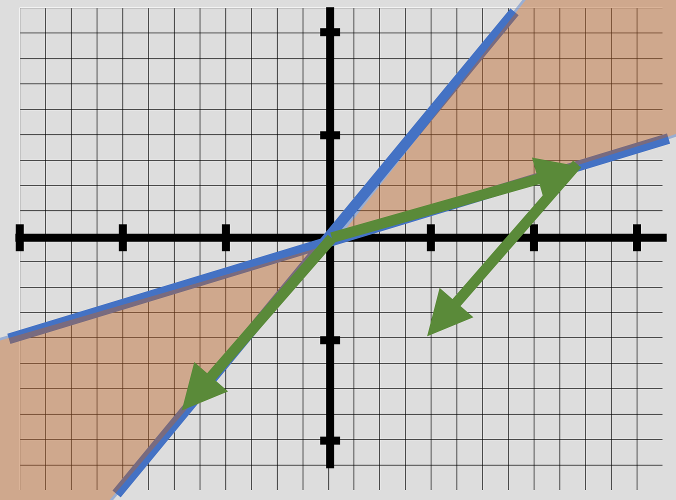
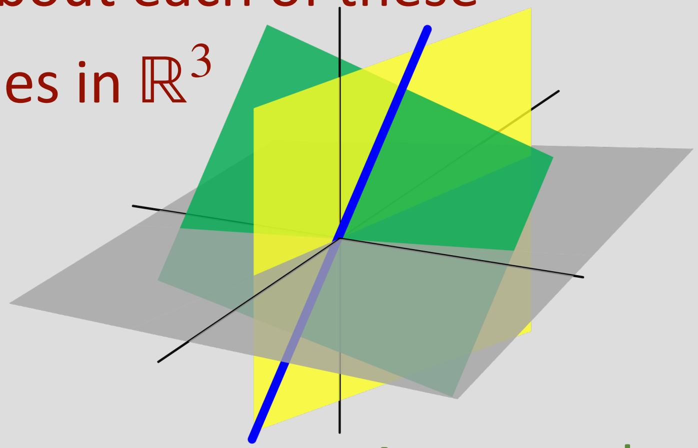
Q: What about this?



A: Not closed under addition!

Q: What about each of these 2D planes in $\mathbb{R}^3$



A: yes, as long as passing through 0

By Alksentrs at en.wikipedia -

Subspaces

Example:

$$\mathbb{W} = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| \, a, b, d \in \mathbb{R} \right\}, \quad \mathbb{V} = \mathbb{R}^{2\times 2}$$

Is $\mathbb{W} \subset \mathbb{V}$?

 1. Zero vector?

 2. Closed under addition?

 3. Closed under scalar multiplication?

Bases

- In words: Minimum set of vectors that spans a vector space

- Definition: given $\mathbb{V}$, a set of vectors $\{\overrightarrow{v}_1, \overrightarrow{v}_2, \cdots, \overrightarrow{v}_N\}$ is a basis of the vector space, if it satisfies:
  - $\{\overrightarrow{v}_1, \overrightarrow{v}_2, \cdots, \overrightarrow{v}_N\}$ are linearly independent
  - $\forall \overrightarrow{v} \in \mathbb{V}, \;\; \exists \;\; \alpha_1, \alpha_2, \cdots, \alpha_N \in \mathbb{R}^{\cancel{N}}$ such that $\overrightarrow{v} = \alpha_1 \overrightarrow{v}_1 + \alpha_2 \overrightarrow{v}_2 + \cdots + \alpha_N \overrightarrow{v}_N$

Bases examples

Q: Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{V} = \mathbb{R}^3$?

Q: Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{V} = \mathbb{R}^3$?

Q: Is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{V} = \mathbb{R}^3$?

Bases examples

$$\text{Q: Is } \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix} \right\} \text{ a basis for } \mathbb{V} = \mathbb{R}^3\text{?}$$

$$\text{Q: Is } \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\10\\0\end{bmatrix} \right\} \text{ a basis for } \mathbb{V} = \mathbb{R}^3\text{?}$$

Column Space

- The range/span/columnspace of a set of vectors is a set of all possible linear combinations:

$$\text{span}\left\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_M\right\} = \triangleq \left\{\sum_{m=1}^{M} \alpha_m \vec{a}_m \,\middle|\, \alpha_1, \alpha_2, \cdots, \alpha_M \in \mathbb{R}\right\}$$

Consider:

$$A = \begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix}$$

Q: Are the columns of A, a basis? 

Q: Is the column space of A, a subspace?

Column Space

1. Zero vector?
2. Closed under addition?
3. Closed under scalar multiplication?

Consider:

$$A = \begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix} \qquad \vec{v}_1 = A\vec{u}_1, \quad \vec{v}_2 = A\vec{u}_2$$

Q: Is the column space of A, a subspace?

$$A\vec{0} = \vec{0}$$

$$\vec{v}_1 + \vec{v}_2 = A\vec{u}_1 + A\vec{u}_2 = A(\vec{u}_1 + \vec{u}_2)$$

$$\alpha\vec{v}_1 = \alpha A\vec{u}_1 = A(\alpha\vec{u}_1)$$

Rank

- USA Today University Ranking for Cal:
  - #1 (joint) in Computer Science
  - #3 in Electrical Engineering
  - #3 in Computer Engineering

Rank

- $A \in \mathbb{R}^{N \times M}, \ \text{Rank}\{A\} = \dim\left\{\text{Span}\left\{\overset{\text{cols}(A)}{\cancel{A}}\right\}\right\}$

$$A = \begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{bmatrix}$$

2 2 1

- $\text{Rank}\{A\} = \dim\left\{\text{Span}\{A\}\right\} \leq \min(M, N)$

Null Space

- Definition: The null-space of $A \in \mathbb{R}^{N \times M}$ is the set of all vectors $\overrightarrow{x} \in \mathbb{R}^{M}$ such that: $A\overrightarrow{x} = 0$

$$A\overrightarrow{x} = 0$$

How many solutions for $\overrightarrow{x}$ satisfy the above?

Examples

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linearly independent!

$$\overrightarrow{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\overrightarrow{0}$ is always in the null space — trivial Null space

Examples

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linearly dependent!

$$\vec{x} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Gaussian elimination:

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 = 2x_2$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix}$$

A has a non-trivial null-space, span $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

Example

$$A\overrightarrow{x} = \overrightarrow{b}$$

We know that $\overrightarrow{v}_0 \in \mathrm{Null}(A)$

$$\rightarrow A\overrightarrow{v}_0 = \overrightarrow{0}$$

We know 1 solution: $\overrightarrow{x}_0$

$$\rightarrow A\overrightarrow{x}_0 = b$$

Example

$$A\vec{x} = \vec{b}$$

We know that $\vec{v}_0 \in \mathrm{Null}(A)$

$$\rightarrow A\vec{v}_0 = \vec{0}$$

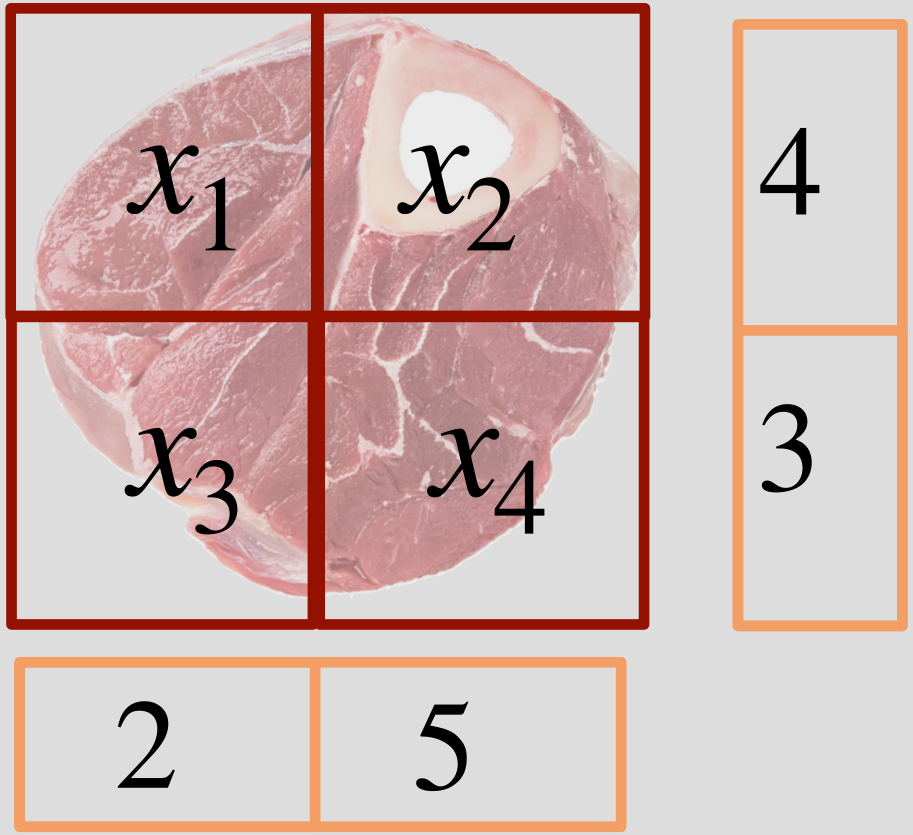
We know 1 solution: $\vec{x}_0$

$$\rightarrow A\vec{x}_0 = b$$

Then: $\vec{x}_0 + \alpha\vec{v}_0$ is also a solution

$$\begin{aligned}\rightarrow A(\vec{x}_0 + \alpha\vec{v}_0) &= A\vec{x}_0 + A(\alpha\vec{v}_0)\\ &= \vec{b} + \alpha A\vec{v}_0\\ &= \vec{b}\end{aligned}$$

Back to Tomography



$$1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4$$
$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 = 3$$
$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2$$
$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$

$$A$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]$$

Null Space of the Tomography System (4 measur.)

Step I

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

Step II

$$\begin{array}{r} \\ \\ (3)-(1) \\ \\ \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

Step III

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Step IV

$$\begin{array}{r} \\ \\ (3)+(2) \\ \\ \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Step V

$$\begin{array}{r} \\ \\ \\ (4)-(3) \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Step VI

$$\begin{array}{r} (1)-(2) \\ \\ \\ \\ \end{array}\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

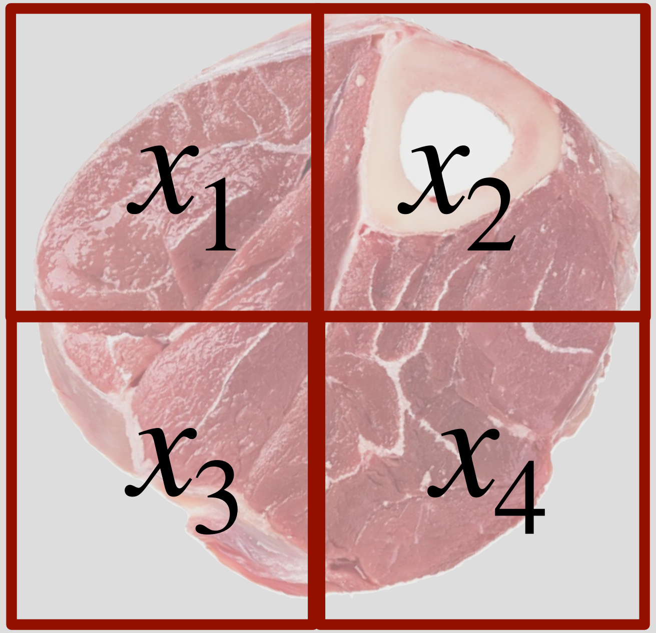
Null Space of the Tomography System (4 measur.)

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_4 is the free variable:

$$\Rightarrow \vec{x} = \alpha \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Possible reconstruction



$$= \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} + \alpha \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Rank

- $A \in \mathbb{R}^{N \times M}$, $\text{Rank}\{A\} = \dim\left\{\text{Span}\{A\}\right\}$

- $\text{Rank}\{A\} = \dim\left\{\text{Span}\{A\}\right\} \leq \min(M, N)$

- $\text{Rank} = L$, mean the matrix $A \in \mathbb{R}^{N \times M}$ has L independent rows&columns

- $\text{Rank}\{A\} + \dim\left\{\text{Null}\{A\}\right\} = \min(M, N)$

Equivalent Statements

- Matrix A is **invertible**
- $A\overrightarrow{x} = \overrightarrow{b}$ has a unique solution
- A has linearly independent columns (A is **full rank**)
- A has a **trivial nullspace**
- The **determinant** of A is not zero

The Determinant

- For $A \in \mathbb{R}^{2\times 2}$

$$\det(A) = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

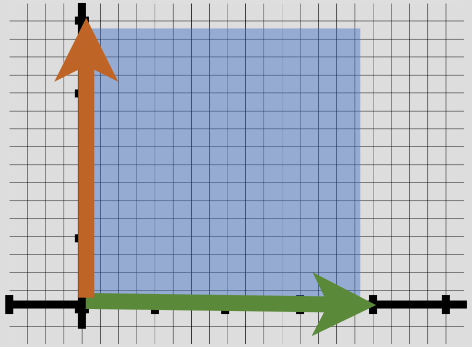
When $\det(A) \neq 0$, A is invertible

Recall:

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Interpretation of Determinant of a Matrix in $\mathbb{R}^{2\times 2}$

- Area of a parallelogram



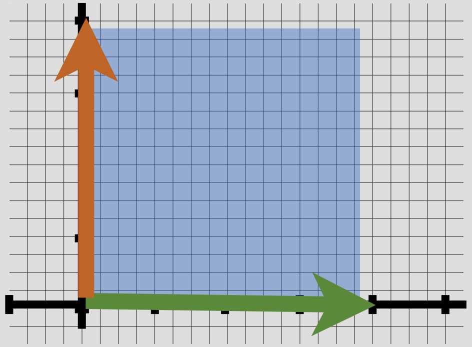
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Area$\neq 0$

$$\det(A) = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Interpretation of Determinant of a Matrix in $\mathbb{R}^{2\times 2}$

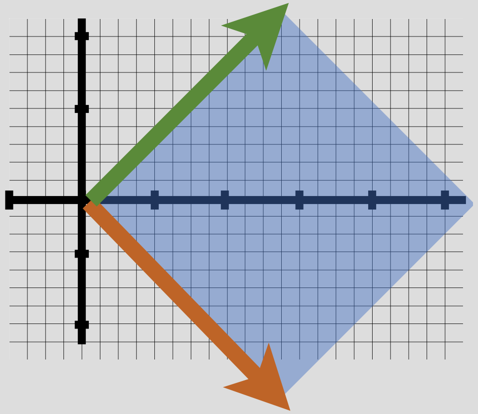
- Area of a parallelogram

$$\det(A) = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$



$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Area $\neq 0$



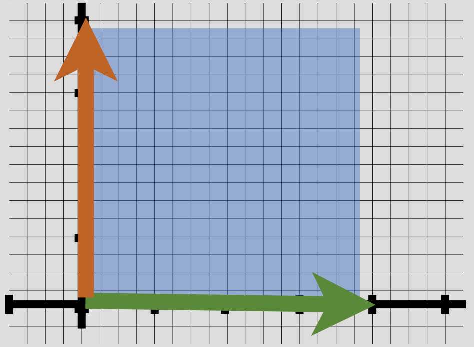
$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Area $\neq 0$

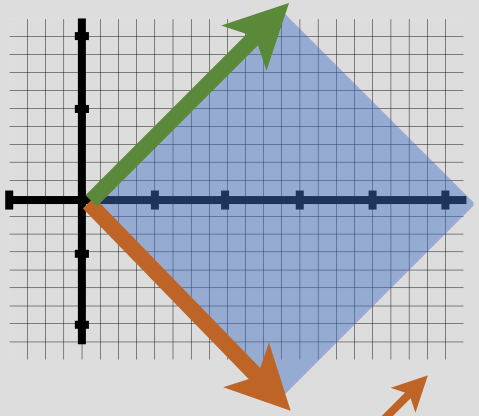
Interpretation of Determinant of a Matrix in $\mathbb{R}^{2\times2}$

- Area of a parallelogram

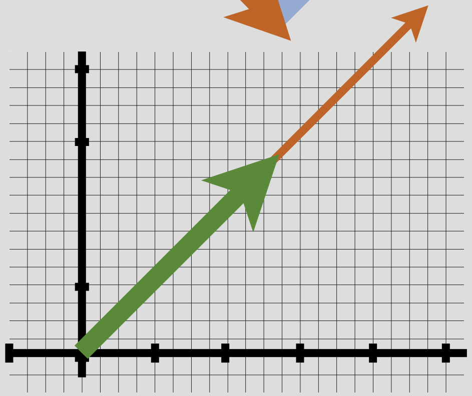
$$\det(A) = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$



$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Area≠0



$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$ Area≠0



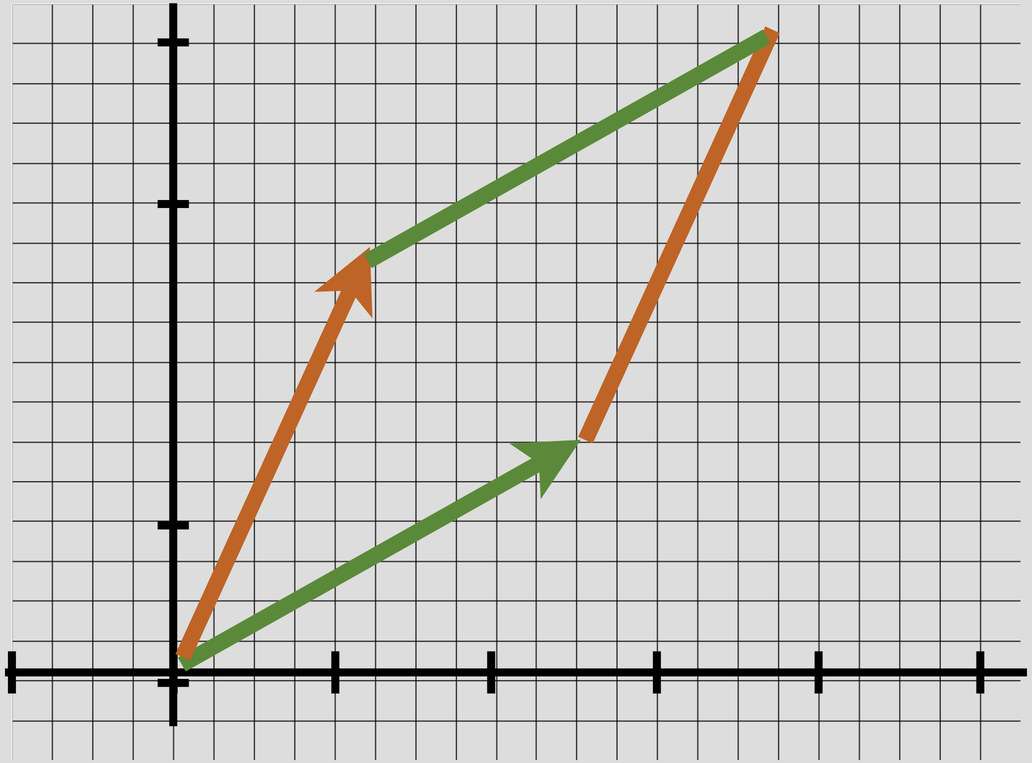
$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$ Area=0

$$\det(A) = 1 \cdot 2 - 1 \cdot 2 = 0$$

Interpretation of Determinant of a Matrix in $\mathbb{R}^{2\times 2}$

- Area of a parallelogram

$$\det(A) = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = ad - bc$$



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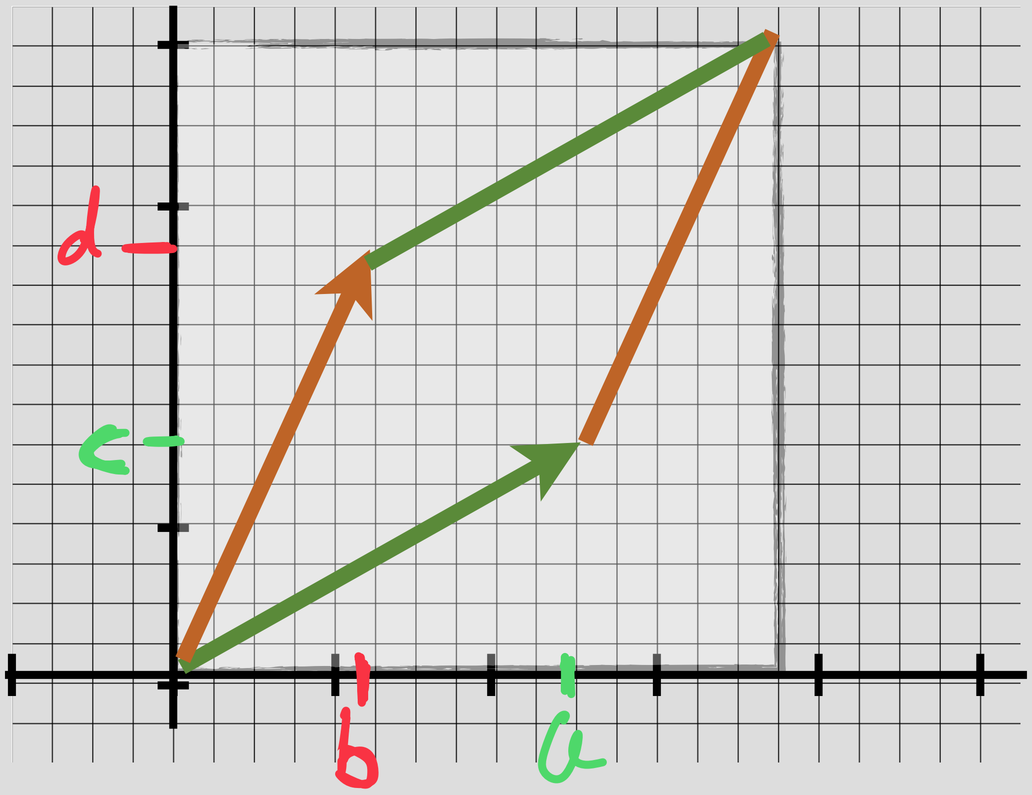

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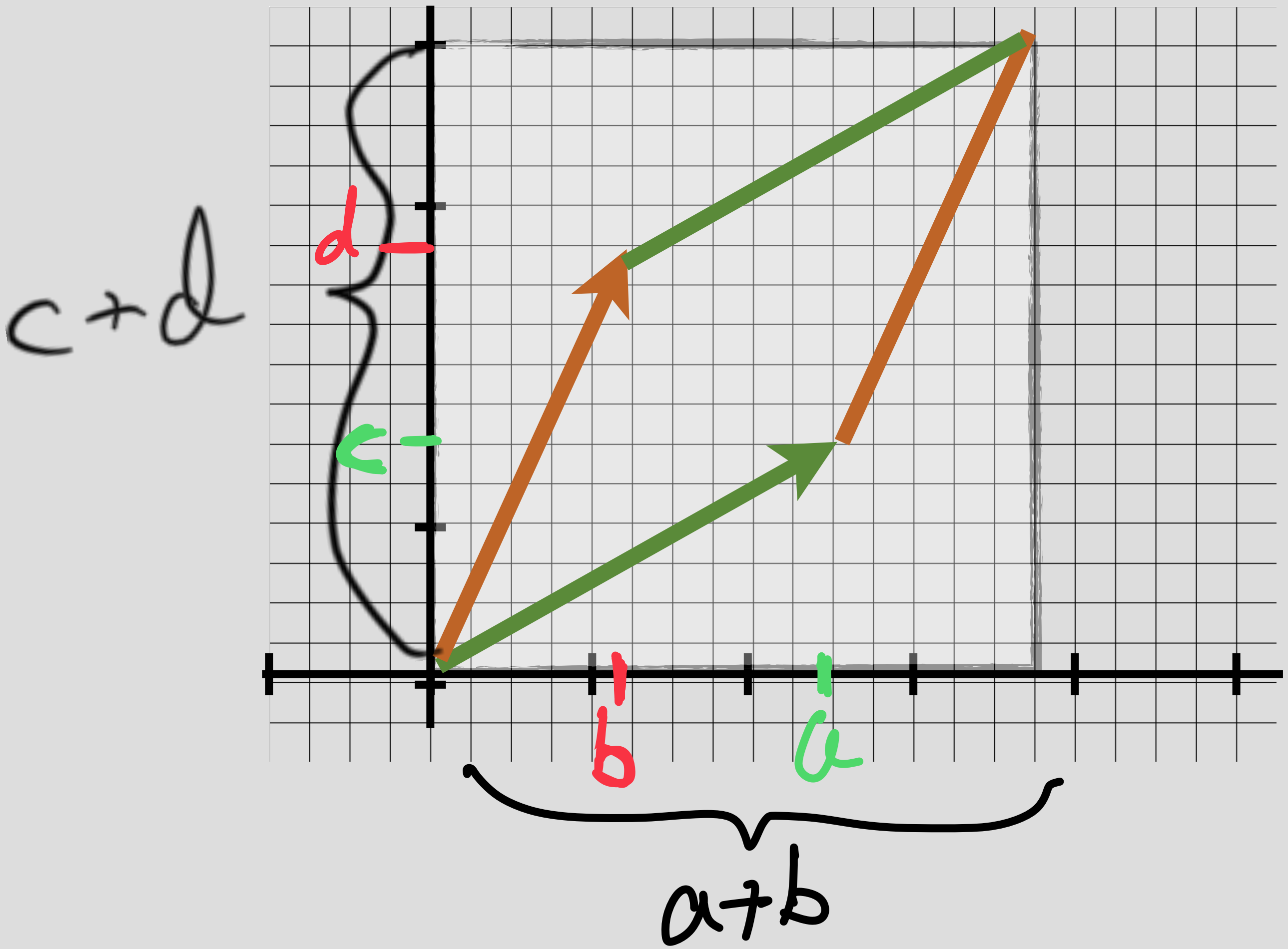

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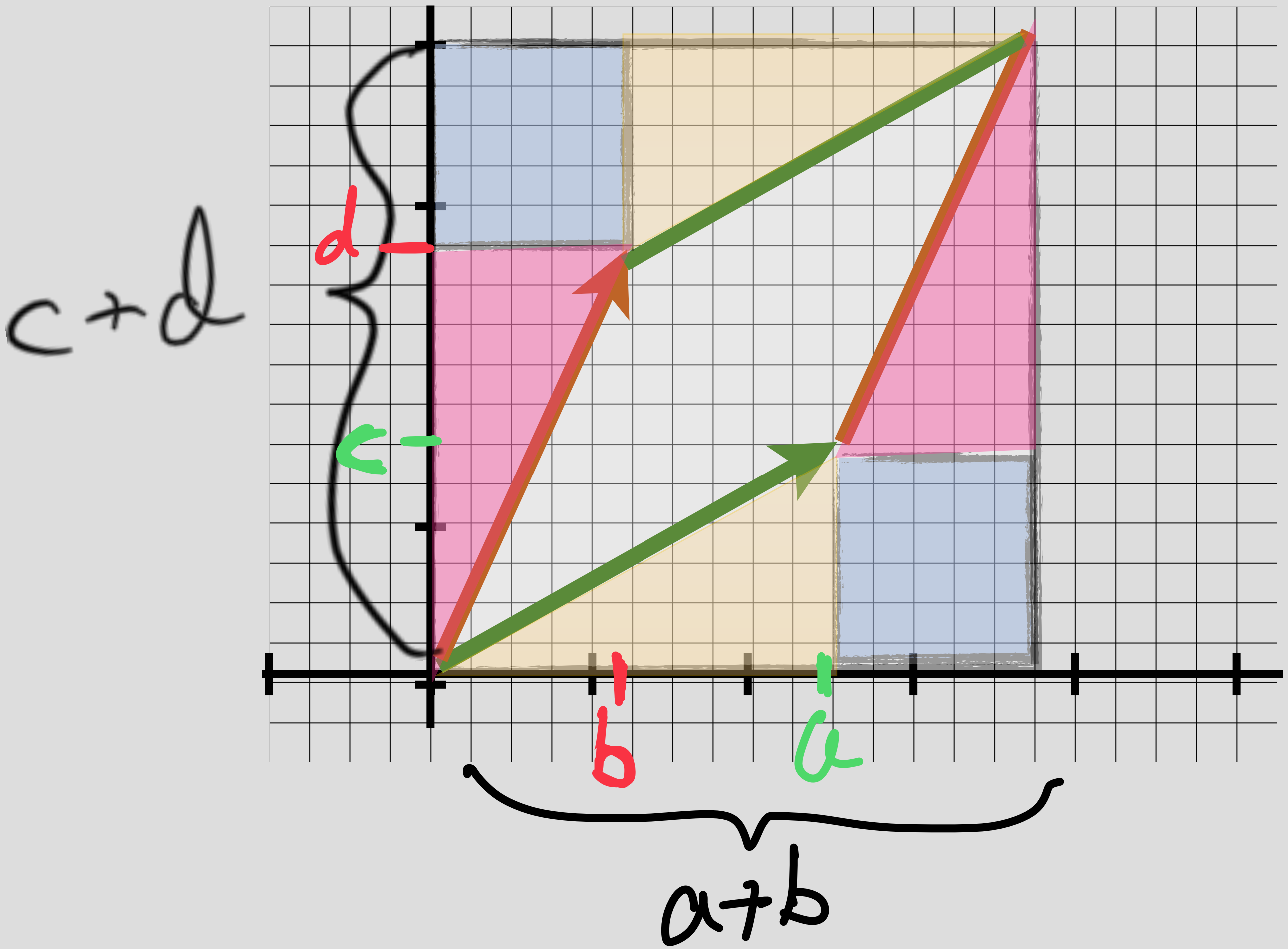
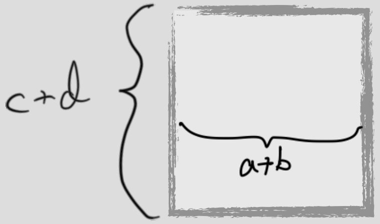
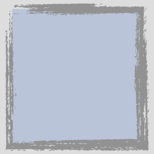


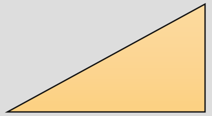


$$(c+d)(a+b)$$



x2

$$bc \times 2$$

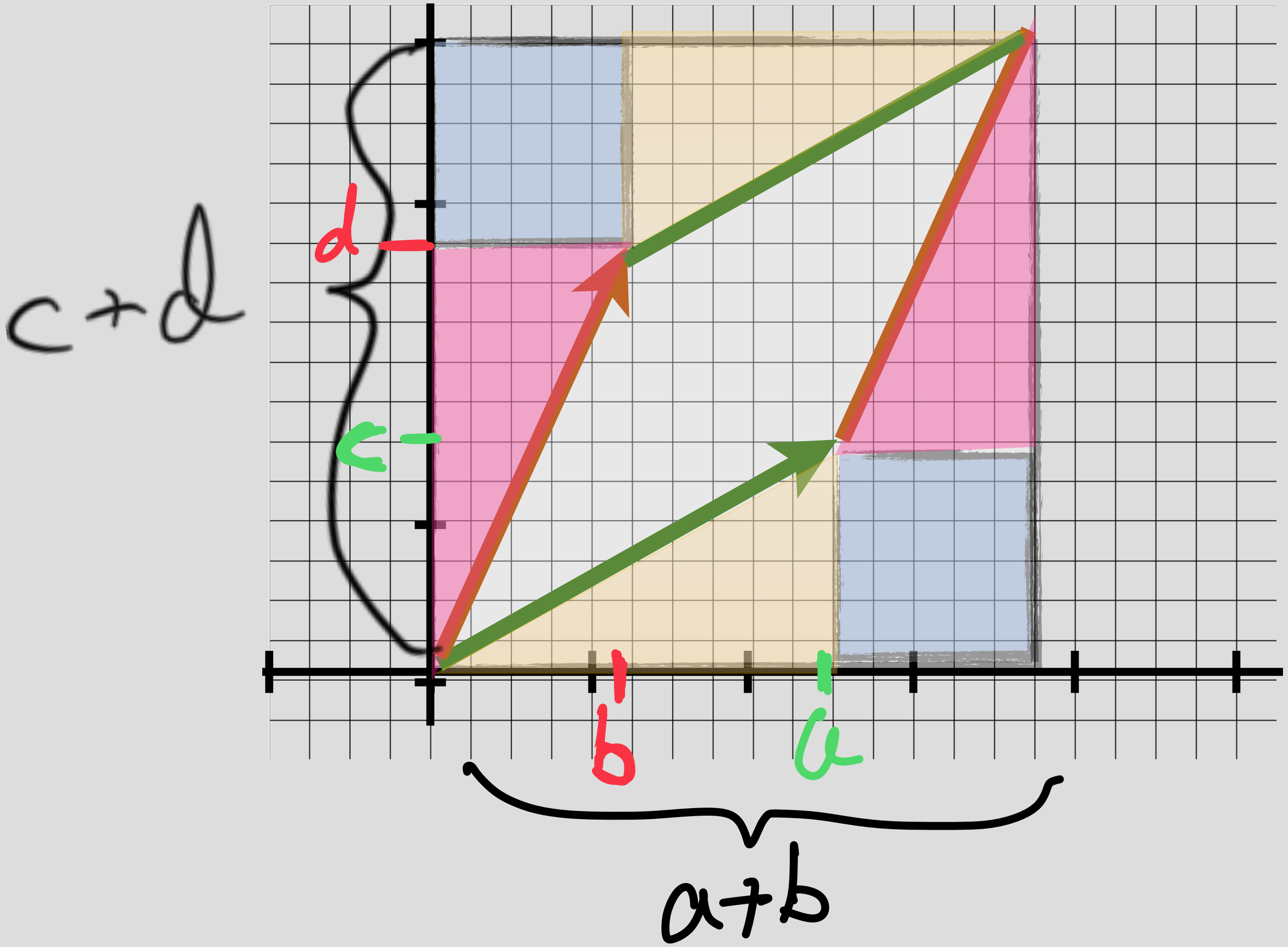


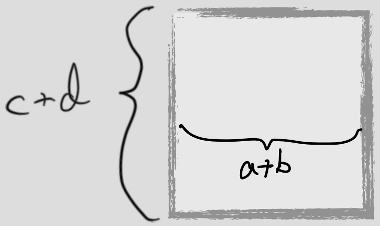
x2

$$\cancel{\frac{1}{2}}ac \times \cancel{2}$$

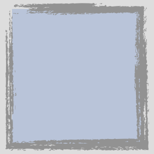
x2

$$\cancel{\frac{1}{2}}bd \times \cancel{2}$$

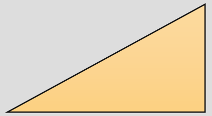




$(c+d)(a+b)$

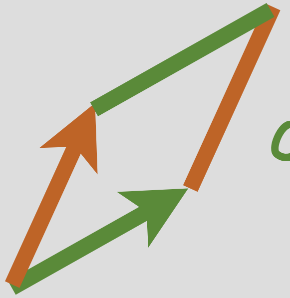


x2 $2bc$



x2 ac

x2 bd



area= $(c+d)(a+b) \; -2bc \; -ac \; -bd$

$= \cancel{c}a + \cancel{c}b + da + \cancel{d}b - \cancel{2}bc - \cancel{a}c - \cancel{b}d = ad - bc$

Determinant in $\mathbb{R}^3$

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = \left[a \times \begin{vmatrix} e & f \\ h & i \end{vmatrix} \right] - \left[b \times \begin{vmatrix} d & f \\ g & i \end{vmatrix} \right] + \left[\begin{vmatrix} d & e \\ g & h \end{vmatrix} \times c \right]$$