EECS 16A Designing Information Devices and Systems I
Spring 2023 Discussion 6B

1. Matrix Multiplication Proof

(a) Given that matrix A is square and has linearly independent columns, which of the following are true?

i. A is full rank
ii. A has a trivial nullspace
iii. A$\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$
iv. A is invertible
v. The determinant of A is non-zero

Solution/Answer: They are all true. Below are some informal explanations of why.

i.: If a square matrix has linearly independent columns then the dimension of its columnspace is equal to the number of columns in the matrix, which by definition means it is full rank.

ii.: Since A is square and has linearly independent columns, the dimension of it's columnspace is equal to the number of columns in the matrix. Thus, by the rank-nullity theorem, the dimension of its nullspace is 0 meaning that it is a trivial nullspace.

iii. : Let's say $A \in \mathbb{R}^{n\times n}$. Since A has linearly independent columns, then its columns form a basis for $\mathbb{R}^n$. This means any vector for any vector $\vec{b} \in \mathbb{R}^n$ there exists a unique $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{b}$.

iv. : From iii. we saw that $A\vec{x} = \vec{b}$ has a unique solution for any $\vec{b}$. Consequently $AB = I$ must have a unique solution B where I is the identity matrix. By definition, $B = A^{-1}$ and so A is invertible.

v. : Recall the geometric interpretation of the determinant of A as the volume of the parallelepiped formed by its columns. Since A has linearly independent columns, none of the sidelengths of the parallelepiped will be 0 and so the determinant will always be nonzero.

(b) Let two square matrices $\mathrm{M}_1, \mathrm{M}_2 \in \mathbb{R}^{2x2}$ each have linearly independent columns. Prove that $\mathrm{G} = \mathrm{M}_1\mathrm{M}_2$ also has linearly independent columns.

Solution/Answer: If $\mathrm{M_i}$ is square and has linearly independent columns, then it is also invertible.

Now, let's consider the toy case of $\mathrm{G} = \mathrm{M}_1$. Since we've established that $\mathrm{M_i}$ is invertible, we get ${\mathrm{M}_1}^{-1}\mathrm{G} = \mathrm{I}$. By definition, we've found that G has an inverse, namely ${\mathrm{M}_1}^{-1}$. Since G has an inverse, it must have linearly independent columns.

We're now in a position to extend this argument:

$$\begin{aligned} \mathrm{G} &= \mathrm{M}_1\mathrm{M}_2 \\ \implies {\mathrm{M}_1}^{-1}\mathrm{G} &= I\mathrm{M}_2 \\ \implies {\mathrm{M}_2}^{-1}{\mathrm{M}_1}^{-1}\mathrm{G} &= I \\ \implies \mathrm{G}^{-1} &= {\mathrm{M}_2}^{-1}{\mathrm{M}_1}^{-1} \end{aligned}$$

Thus, we've shown that G has an inverse composed of two invertible matrices, which implies that it has linearly independent columns.

Note that there are many other ways to prove this. For example, consider a proof by contradiction in which you assume G has a nontrivial nullspace. What does this imply about M_1 and/or M_2?

2. The Romulan Ruse

While scanning parts of the galaxy for alien civilization, the starship USS Enterprise NC-1701D encounters a Romulan starship that is known for advanced cloaking devices.

(a) The Romulan illusion technology causes a point (x_0, y_0) to transform or *map* to (u_0, v_0). Similarly, (x_1, y_1) is mapped to (u_1, v_1). Figure 1 and Table 1 show these points.

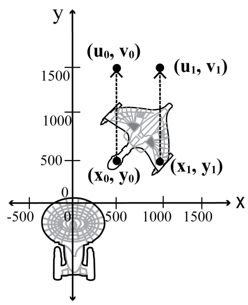

Figure 1: Figure for part (a)

Original Point	Mapped Point
$(x_0, y_0) = (500, 500)$	$(u_0, v_0) = (500, 1500)$

Original Point	Mapped Point
$(x_1, y_1) = (1000, 500)$	$(u_1, v_1) = (1000, 1500)$

Table 1: Original and Mapped Points

Find a transformation matrix $\mathbf{A}_0$ such that

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \mathbf{A}_0 \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \text{ and } \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \mathbf{A}_0 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Solution/Answer: *Concept: Matrix Transformations*

Let us assume $\mathbf{A_0} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Hence for point (x_0, y_0), we have:

$$\begin{bmatrix} 500 \\ 1500 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 500 \\ 500 \end{bmatrix} \implies \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

i.e.

$$a + b = 1; \tag{1}$$

$$c + d = 3. \tag{2}$$

Similarly, for point (x_1, y_1), we have

$$\begin{bmatrix} 1000 \\ 1500 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1000 \\ 500 \end{bmatrix} \implies \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

i.e.

$$2a + b = 2; \tag{3}$$

$$2c + d = 3. \tag{4}$$

Solving Equations (1) and (3) for a and b, we have:

$$a = 1, \text{ and } b = 0.$$

Solving Equations (2) and (4) for c and d, we have:

$$c = 0, \text{ and } d = 3.$$

Substituting values of a, b, c, and d, we have

$$\mathbf{A}_0 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

Additionally, it can be observed from Figure 1 that the mapped vectors are derived by scaling the original vectors by 3 in the y-direction and by unity in the x-direction. Using Figure 1 and Table 1, we can write

$$u_0 = x_0, \text{ and } v_0 = 3y_0, \tag{5}$$

and

$$u_1 = x_1, \text{ and } v_1 = 3y_1. \tag{6}$$

Writing equations 5 and 6 in matrix-vector product form, we have

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix};$$
$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Hence

$$\mathbf{A}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}. \tag{7}$$

(b) In this scenario, every point on the Romulan ship (x_m, y_m) is mapped to (u_m, v_m), such that vector $\begin{bmatrix} x_m \\ y_m \end{bmatrix}$ is rotated counterclockwise by 30° and then scaled by 2 in the x- and y-directions. This transformation is shown in Figure 2.

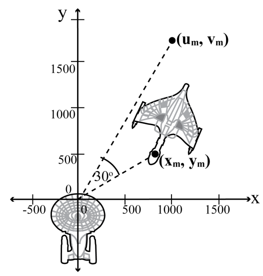

Figure 2: Figure for part (b)

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	∞

Table 2: Trigononometric Table

Find a transformation matrix R such that $\begin{bmatrix} u_m \\ v_m \end{bmatrix} = \mathbf{R} \begin{bmatrix} x_m \\ y_m \end{bmatrix}$.

Solution/Answer: *Concept: Matrix Transformations* Transformation matrix that rotates a vector counterclockwise by 30° is:

$$\mathbf{R}_\theta = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Transformation matrix that rotates a vector counterclockwise by 30° and scales by 2 is:

$$\mathbf{R} = 2\mathbf{R}_\theta = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}.$$

Alternatively, the transformation matrix can be written as:

$$\mathbf{R} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}.$$

The Romulan ship has launched a probe into space and the Enterprise is trying to destroy the probe by firing a photon torpedo along a straight line from point $(0,0)$ towards the probe.

(c) The Romulan generals found a clever way to hide the probe by transforming (mapping) its position with a *cloaking* (transformation) matrix $\mathbf{A}_p$:

$$\mathbf{A}_p = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

They positioned the probe at (x_p, y_p) so that it maps to $(u_p, v_p) = (0,0)$, where $\begin{bmatrix} u_p \\ v_p \end{bmatrix} = \mathbf{A}_p \begin{bmatrix} x_p \\ y_p \end{bmatrix}$. This scenario is shown in Figure 3. The initial position of the torpedo is $(0,0)$ and the torpedo cannot be fired on its initial position! Impressive trick indeed!

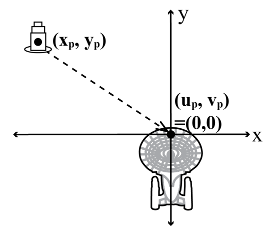

Figure 3: Figure for part (c)

Find the possible positions of the probe (x_p, y_p) so that $(u_p, v_p) = (0,0)$.

Solution/Answer: *Concept: Gaussian Elimination, Systems of Equations*

We need to solve for

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So essentially we need to find the nullspace of the matrix $\mathbf{A}_p$. Using Gaussian Elimination on the augmented matrix, we have:

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 1 & 3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_p + 3y_p = 0 \Rightarrow x_p = -3y_p.$$

The solution is $\alpha \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, where α is $\{\alpha \in \mathbb{R}\}$. So $\begin{bmatrix} x_p \\ y_p \end{bmatrix}$ should be in the span of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.
Alternatively, any point (x_p, y_p) that is on the line: $x = -3y$, would represent all possible positions of the probe.

(d) It turns out the Romulan engineers were not as smart as the Enterprise engineers. Their calculations did not work out and they positioned the probe at (x_q, y_q) such that the *cloaking* (transformation) matrix, $\mathbf{A}_p$, mapped it to (u_q, v_q), where

$$\begin{bmatrix} u_q \\ v_q \end{bmatrix} = \mathbf{A}_p \begin{bmatrix} x_q \\ y_q \end{bmatrix}, \text{ and } \mathbf{A}_p = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

As a result, the torpedo, while traveling straight from $(0,0)$ to (u_q, v_q), hit the probe at (x_q, y_q) on the way! The scenario is shown in Figure 4. For the torpedo to hit the probe, we must have $\begin{bmatrix} u_q \\ v_q \end{bmatrix} = \lambda \begin{bmatrix} x_q \\ y_q \end{bmatrix}$, where λ is a real number.

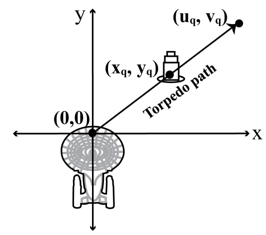

Figure 4: Figure for part (d)

Find the possible positions of the probe (x_q, y_q) so that $(u_q, v_q) = (\lambda x_q, \lambda y_q)$. Remember that the torpedo cannot be fired on $(0,0)$. This means that $(u_q, v_q) = (\lambda x_q, \lambda y_q)$ cannot be $(0,0)$.

Solution/Answer: *Concept: Eigenspaces/Eigenvectors/Eigenvalues* We need to solve for $\mathbf{A}_p \begin{bmatrix} x_q \\ y_q \end{bmatrix} = \lambda \begin{bmatrix} x_q \\ y_q \end{bmatrix}$, i.e. we need to find the eigenvectors of $\mathbf{A}_p$. Let's start by finding the eigenvalues:

$$\det\left\{\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right\} = 0$$
$$\det\left\{\begin{bmatrix} 1-\lambda & 3 \\ 2 & 6-\lambda \end{bmatrix}\right\} = 0$$

So we have the characteristic polynomial:

$$(1-\lambda)(6-\lambda) - (3)(2) = 0$$
$$\Rightarrow \lambda = 0, 7$$

Using $\lambda = 0$, we have: $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_q \\ y_q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ which will map (x_q, y_q) to the original position of the torpedo. The torpedo cannot be fired on its original position. So $\lambda = 0$ will not provide a valid solution.

Using $\lambda = 7$, we have:

$$(\mathbf{A_p} - 7\mathbf{I}) \begin{bmatrix} x_q \\ y_q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow = \left(\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} x_q \\ y_q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -6 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_q \\ y_q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using Gaussian Elimination on the augmented matrix form, we have

$$\left[\begin{array}{cc|c} -6 & 3 & 0 \\ 2 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow 2x_q - y_q = 0 \Rightarrow y_q = 2x_q$$

The solution is $\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, where α is $\{\alpha \in \mathbb{R} : \alpha \neq 0\}$. So $\begin{bmatrix} x_q \\ y_q \end{bmatrix}$ should be in the span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Alternatively, any point (x_q, y_q) that is on the line: $y = 2x$, excluding $(0,0)$, would represent all possible positions of the probe.