EECS 16A   Designing Information Devices and Systems I
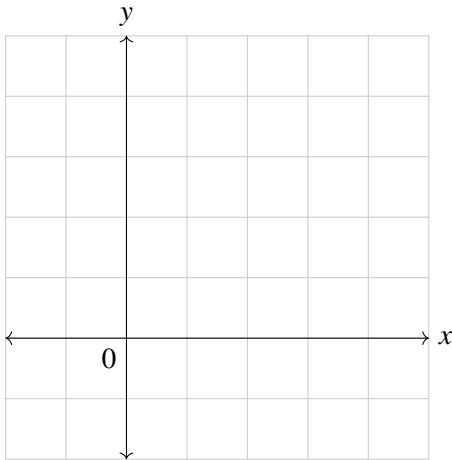## Summer 2023
Discussion 07C

1. Mechanical Projection

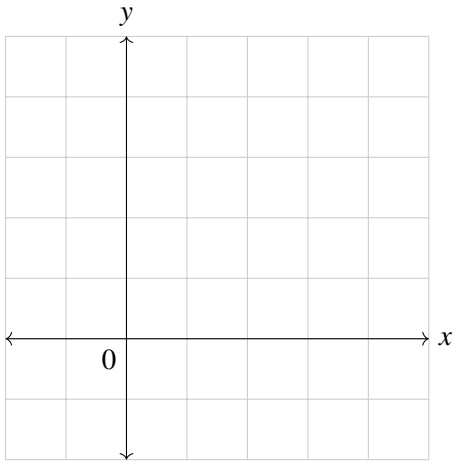In $\mathbb{R}^n$, the vector valued projection of vector $\vec{b}$ onto vector $\vec{a}$ is defined as:

$$\text{proj}_{\vec{a}}\left(\vec{b}\right) = \frac{\left\langle \vec{a}, \vec{b} \right\rangle}{\|\vec{a}\|^2}\vec{a}.$$

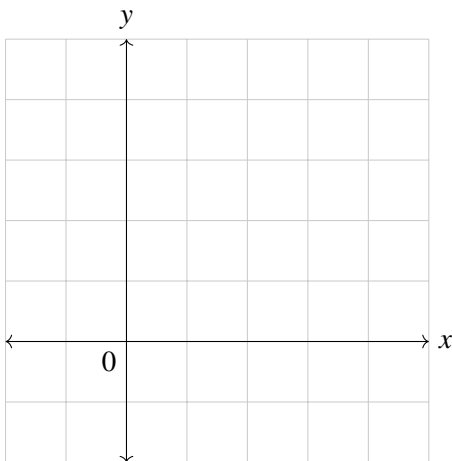Recall $\|\vec{a}\|^2 = \langle \vec{a}, \vec{a} \rangle$.

(a) Project $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ onto $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ — that is, onto the y-axis. Graph these two vectors and the projection.

(b) Project $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Graph these two vectors and the projection.

(c) Project $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Graph these two vectors and the projection.
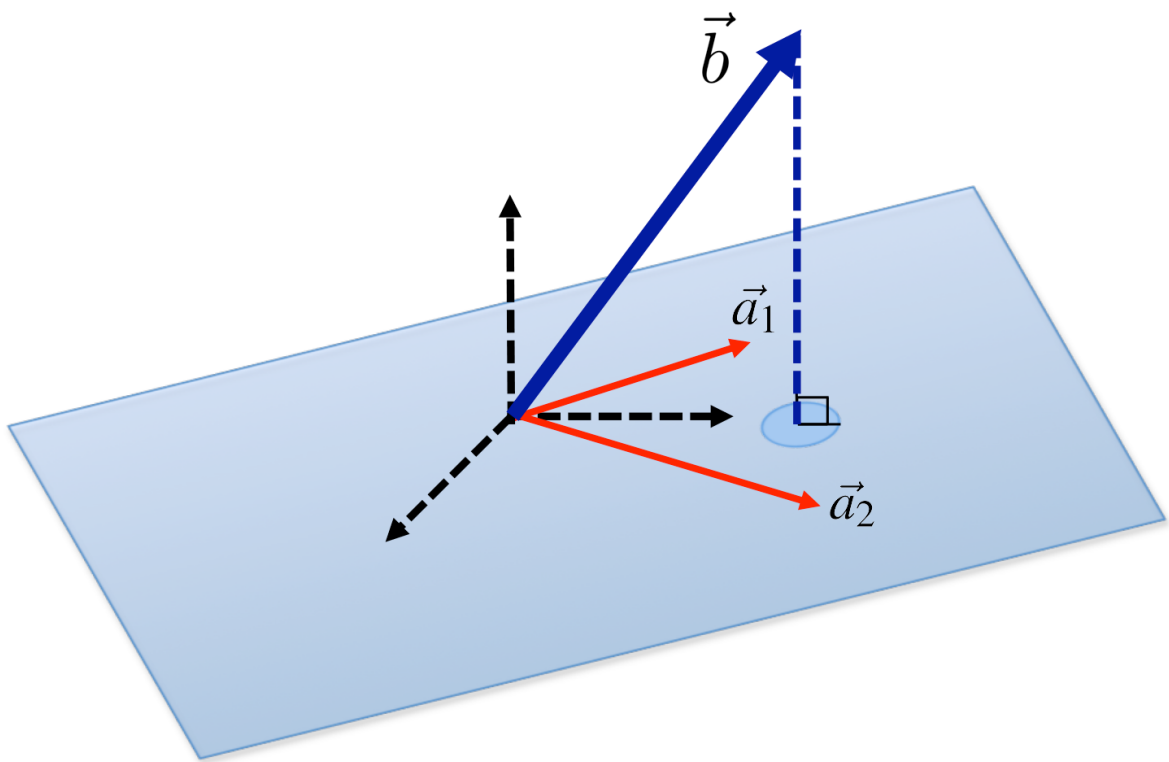
2. Least Squares with Orthogonal Columns

(a) Consider a least squares problem of the form

$$\min_{\vec{x}} \ \left\| \vec{b} - \mathbf{A}\vec{x} \right\|^2 \ = \ \min_{\vec{x}} \ \left\| \mathbf{A}\vec{x} - \vec{b} \right\|^2 \ = \ \min_{\vec{x}} \ \left\| \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} - \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|^2$$

Let the solution be $\vec{\hat{x}} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$.

Label the following elements in the diagram below.

$$\text{span}\{\vec{a}_1, \vec{a}_2\}, \qquad \vec{e} = \vec{b} - \mathbf{A}\vec{\hat{x}}, \qquad \mathbf{A}\vec{\hat{x}}, \qquad \vec{a}_1 \hat{x}_1, \ \vec{a}_2 \hat{x}_2, \qquad \text{colspace}(\mathbf{A})$$

(b) We now consider the special case of least squares where the columns of $\mathbf{A}$ are orthogonal. Given that $\vec{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b}$ and $A\vec{x} = \text{proj}_{\mathbf{A}}(\vec{b}) = \hat{x}_1\vec{a}_1 + \hat{x}_2\vec{a}_2$, show that

$$\text{proj}_{\vec{a}_1}(\vec{b}) = \hat{x}_1\vec{a}_1$$
$$\text{proj}_{\vec{a}_2}(\vec{b}) = \hat{x}_2\vec{a}_2$$

(c) Compute the least squares solution to

$$\min_{\vec{x}} \quad \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|^2 .$$