Lecture 4B Tues 7/11

congrats on finishing the QUEST!

Today is the last lecture of Module 1! (and of me!)

- How to find eigenstuff
- State transition steady state
- Module 1 overview

Let's start with a YouTube video.

Search "3Blue1Brown Eigenvectors and Eigenvalues" to find it.

We will watch the first 5 mins 17 seconds for a visual feel for what eigenstuff is visually.

Recall: Def. For an nxn matrix A, if

$\boxed{A\vec{v} = \lambda\vec{v}}$ for some $\vec{v} \in \mathbb{R}^n \neq \vec{0}$, $\lambda \in \mathbb{R}$,

$\vec{0}$ would always satisfy the equation

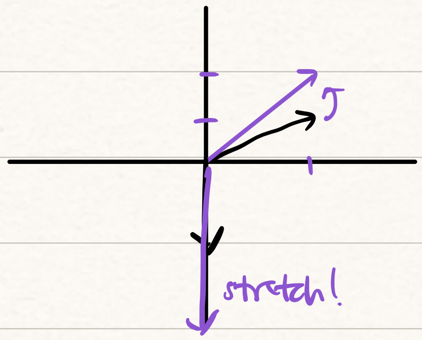
then $\vec{v}$ is called an eigenvector of A with eigenvalue λ.

- eigenvectors define "special" directions along which the matrix A only stretches/squishes vectors
- everywhere else, vectors will get rotated somewhat

Let's figure out some eigenstuff visually.

Ex 1. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ → stretches y component by 2 times

keeps x component the same



Q. What are the eigenvectors of A?
- what vectors are ONLY stretched/shrunk when A is applied?

A. Any vector on x-axis or y-axis!

Q. With what eigenvalues? How much are they stretched or shrunk?

A. $\begin{bmatrix}1\\0\end{bmatrix}$ has $\lambda = 1$

$\begin{bmatrix}0\\1\end{bmatrix}$ has $\lambda = 2$

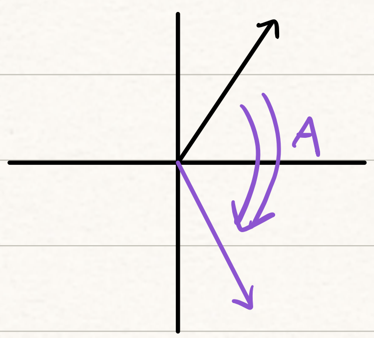
Notice: for each eigenvector, every vector along that line (in its span) is always also an eigenvector.

- we say that each eigenvalue is associated with an "eigenspace".

Also notice: diagonal matrix: can read off eigenvalues, and eigenvectors are normal basis vectors

tinyurl.com/16aeigen

Ex 2. $A = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ ← flips y component = reflection across x-axis



Q. What is an eigenvector of this transformation?
- what vector isn't "knocked off its span"?

A. any vector on the x-axis!

Q. For what eigenvalue? How much are they stretched/squished?

A. $\lambda = 1$. They stay the same.

Ok we've gonna go mechanical now.

Start with defining equation $A\vec{v} = \lambda\vec{v}$

We want to find $\lambda, \vec{v}$ that satisfy this equation.

make sure all the logic makes sense!

$$A\underline{\vec{v}} - \lambda\underline{\vec{v}} = \vec{0}$$

want to factor out $\vec{v}$, but A is a matrix and λ is a scalar

- what's the matrix that scales $\vec{v}$ by λ?

$$\begin{bmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{bmatrix} = \lambda I$$

$$A\vec{v} - \lambda I\vec{v} = \vec{0}$$

$$(A - \underline{\lambda} I)\,\underline{\vec{v}} = \vec{0}$$

Again: goal: find $\lambda, \vec{v}$ that satisfy this equation.

- Notice: it's finding the <u>nullspace</u> of matrix <u>$A - \lambda I$</u>!
- Recall: eigenvectors have to be $\neq \vec{0}$.

$\Rightarrow$ Need $A - \lambda I$ to have nontrivial nullspace!

- Recall: invertible matrix theorem said that "P has a nontrivial nullspace" is equivalent to "det (P) = 0" (or "P is not invertible", etc.)

$\Rightarrow$ Eigenvectors only exist for the eigenvalues λ that make $\boxed{\det(A - \lambda I) = 0}$

We will use this to solve for λ.

Ex. $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \rightarrow$

$$A - \lambda I = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 0$$

$(3-\lambda)(2-\lambda) = 0$ "characteristic equation" of A

$\Rightarrow$ $\lambda = 2, 3$

These are the λ's for which eigenvectors exist!

Now how do we find those eigenvectors?

Well they satisfy $(A - \lambda I)\vec{v} = \vec{0}$

for each λ

$\rightarrow$ Plug in each value of λ and find the null space/ solve the system of equations!

$\lambda_1 = 2$

$$A - \lambda_1 I = \begin{bmatrix} 3-2 & 1 \\ 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$(A - \lambda_1 I)\vec{v_1} = \vec{0}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

How many solutions? infinite!

$$v_{11} + v_{12} = 0$$

$$v_{11} = -v_{12}$$

So any $\vec{v}_1 \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ is an eigenvector with eigenvalue $\lambda_1 = 2$!

Notice: $\lambda_1 = 2$ was the perfect value to subtract from diagonals to actually get a matrix with a nontrivial null space/ infinite solutions to $(A - \lambda I)\vec{v} = \vec{0}$.

↳ Think about: why does $A\vec{x} = \vec{0}$ always either have infinite solutions or the unique solution $\vec{x} = \vec{0}$?

Now repeat for $\lambda_2 = 3$ (try it on your own)

$$A - 3I = \begin{bmatrix} 3-3 & 1 \\ 0 & 2-3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$(A - 3I)\vec{v}_2 = \vec{0}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_{22} = 0$$

$$v_{21} = \text{free}$$

$$\Rightarrow \vec{v}_2 \in \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

Here are the steps to find eigenstuff:

1. First find eigenvalues: solutions to equation $\det(A-\lambda I)=0$
2. Plug in each eigenvalue that you find to

   $(A-\lambda I)\vec{v}=\vec{0}$

   Solve the system of equations for $\vec{v}$.

   There will be infinite solutions = your eigenspace for that eigenvalue
3. Repeat for all eigenvalues.

But WHY should I care about eigenstuff?

- it's a very valid question.
- I won't be able to answer it fully today.
- I will say that eigenstuff is THE linear algebra thing that comes up EVERYWHERE in science/engineering. (it's a bit of a meme because of that)
- We're going to see one example today for dynamic systems
- The real answer is in the magic of "diagonalization", which you will learn in 16B. You can get a glimpse if you watch the rest of the YouTube video.

Back to the question of steady state convergence in state transition systems.

- if I run my pumps for ∞ time steps, will I eventually converge to some steady state?

Recall: We said steady state is when

$$\vec{x}[t+1] = \boxed{Q\,\vec{x}[t] = \vec{x}[t]}$$

Q ← system's state transition matrix

- advance one time step (with Q) but stay the same – every tank loses same water it gains

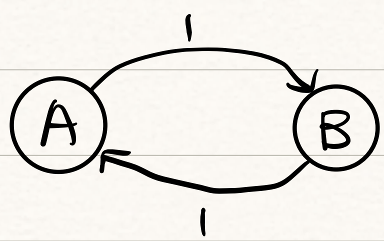
- in other words, steady state is the eigenvector of Q with eigenvalue 1.

So if we start at $\vec{x}[0] = \vec{x}_{ss}$, we will never change.

But how do we know we will actually converge to $\vec{x}_{ss}$ from an arbitrary start $\vec{x}[0]$?

- turns out we won't always converge.

Ex.

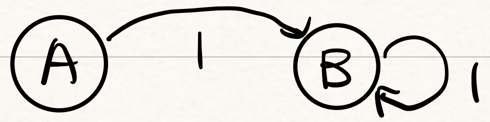


switch water every time step

If $\vec{x}[0] = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow \vec{x}[1] = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{x}[2] = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ldots$

never converges!

But steady state exists: $\vec{x}[0] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ switching doesn't change anything

Ex. 

Steady state? $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ all water in B

Can we converge?

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \vec{x}[1] = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$ and stays same forever after.

Q. Can we predict if we will converge given an arbitrary starting state $\vec{x}[0]$?

A. Yes! And the way is EIGENSTUFF

Say we start at $\vec{x}[0] = \vec{v}_1$, an eigenvector of Q with eigenvalue λ_1.

$$\rightarrow \vec{x}[1] = Q\vec{x}[0] = Q\vec{v}_1 = \lambda_1 \vec{v}_1$$

$$\vec{x}[2] = Q\vec{x}[1] = Q(\lambda_1 \vec{v}_1) = \lambda_1 (Q\vec{v}_1) = \lambda_1^2 \vec{v}_1$$

$$\vdots$$

$$\vec{x}[t] = \lambda_1^t \vec{v}_1$$

many matrix multiplications are so easy with eigenvectors!

$\vec{x}[\infty] = ?$

$$\lim_{t \to \infty} \vec{x}[t] = \lim_{t \to \infty} \lambda_1^t \vec{v}_1 = \vec{v}_1 \left(\lim_{t \to \infty} \lambda_1^t \right)$$

what's the limit?

depends on λ_1!

- $\lambda_1 = 1 \rightarrow \vec{x}[\infty] = \vec{x}[0]$

steady state

- $|\lambda_1| < 1 \rightarrow \vec{x}[\infty] = \vec{0}$

Converges

- $|\lambda_1| > 1 \rightarrow \vec{x}[\infty] = \infty$
- $\lambda_1 = -1 \rightarrow \vec{x}[\infty]$ flips sign every step

doesn't converge

Ok but what about an arbitrary start $\vec{x}[0]$?

- Consider our example $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

  We found $\lambda_1 = 2$, $\vec{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\lambda_2 = 3$, $\vec{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- Consider the set $\{\vec{v_1}, \vec{v_2}\} = \left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

  They're linearly indep so form a basis for $\mathbb{R}^2$!
- An nxn matrix A can have up to n eigenvectors (spaces) that can span $\mathbb{R}^n$.
  - they won't always... when it doesn't is outside the scope of this class.

We will live in a world only with eigenvectors that form a basis for $\mathbb{R}^n$.

- If they're a basis, any vector $\vec{x}[0]$ can be written as a linear combination of them!

Let's be concrete.

$$Q = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$$

turns out has eigenvalue - eigenvector pairs:

$$\lambda_1 = \frac{1}{2}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = 1, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\{\vec{v}_1, \vec{v}_2\}$ are a basis for $\mathbb{R}^2$.

Now let's say we start with $\vec{x}[0] = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

- there must exist $\alpha, \beta \in \mathbb{R}$ such that

$$\vec{x}[0] = \alpha \vec{v}_1 + \beta \vec{v}_2$$ because $\vec{v}_1, \vec{v}_2$ form a basis!

Find α, β (you know how to do this!)

$$\vec{x}[0] = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha + 0 \\ -\alpha + \beta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ -1 & 1 & 2 \end{array}\right] \xrightarrow{R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 4 \end{array}\right] \begin{array}{l} \to \alpha = 2 \\ \to \beta = 4 \end{array}$$

$$\Rightarrow \vec{x}[0] = 2\vec{v}_1 + 4\vec{v}_2$$

Great. why did we do that?

- well we know what happens with each eigenvector after many time steps
- so maybe we can use that to figure out what's going on with $\vec{x}[0]$.

$$\vec{x}[0] = 2\vec{v_1} + 4\vec{v_2}$$

$$\vec{x}[1] = Q\vec{x}[0] = Q(2\vec{v_1} + 4\vec{v_2})$$

$$= 2(Q\vec{v_1}) + 4(Q\vec{v_2})$$ whoa we skipped the matrix

$$= 2(\lambda_1\vec{v_1}) + 4(\lambda_2\vec{v_2})$$ ← multiplication

$$\vec{x}[2] = Q\vec{x}[1]$$

$$= Q(2\lambda_1\vec{v_1} + 4\lambda_2\vec{v_2})$$

$$= 2\lambda_1(Q\vec{v_1}) + 4\lambda_2(Q\vec{v_2})$$

$$= 2\lambda_1^2\vec{v_1} + 4\lambda_2^2\vec{v_2}$$

$$\vdots$$

$$\vec{x}[t] = 2\lambda_1^t\vec{v_1} + 4\lambda_2^t\vec{v_2}$$

should've had to do $\underbrace{Q^t}_{\text{terrible}}\vec{x}[n]$, but we didn't!!

$$\lim_{t\to\infty}\vec{x}[t] = 2\vec{v_1}\Big(\underbrace{\lim_{t\to\infty}\lambda_1^t}_{(\frac{1}{2})^t \to 0}\Big) + 4\vec{v_2}\Big(\underbrace{\lim_{t\to\infty}\lambda_2^t}_{(1)^t \to 1}\Big) \quad !$$

$$= 4\vec{v_2}$$

We just predicted the infinitely far future!!
- using eigenstuff!

1. Write $\vec{x}[0]$ as linear combo of eigenvectors of Q.

   $\vec{x}[0] = \alpha_1\vec{v_1} + \alpha_2\vec{v_2} + \dots + \alpha_n\vec{v_n}$ Find all α_i.

2. Terms corresponding to
   $\lambda_i = 1 \to \alpha_i\vec{v_i}$
   $|\lambda_i| < 1 \to 0$
   $|\lambda_i| > 1 \to \infty$ ← entire sum can't converge
   $\lambda_i = -1 \to$ limit doesn't exist

So... which starting $\vec{x}[0]$ are the ones that will converge to the steady state?

A. the ones that only contain components in the directions of eigenvectors with eigenvalues $\lambda = 1$ or $|\lambda| < 1$.

- in other words, the α_i on all the diverging terms has to be zero for the limit to exist.

My attempt to explain why eigenstuff shows up everywhere:

- The EIGENBASIS is the world where every matrix

basis formed by eigenvectors

multiplication is a set of scalar multiplications.

- Scalar multiplication is WAY easier than matrix multiplication
- but more fundamentally, "changing to the eigenbasis"

↳ write vector as lin combo of eigen vectors

brings you to the ~true coordinate system~ of a matrix

(instead of the normal $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ coord system)

- where a matrix can show you their true colors
- ("diagonalization" is the name of this concept)

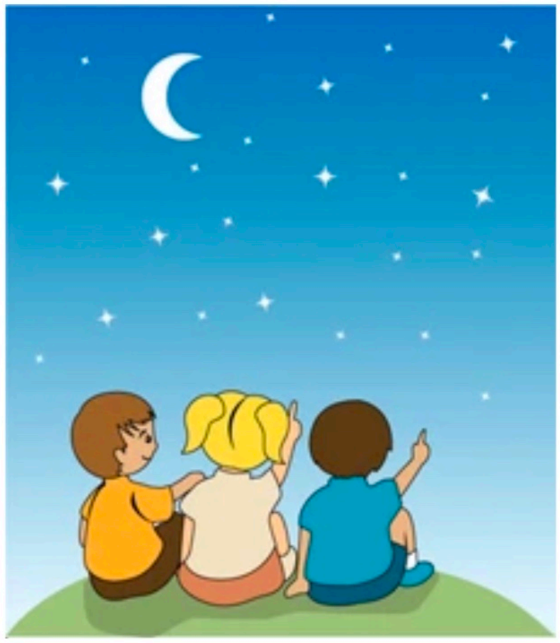
↳ where every matrix becomes a diagonal matrix = scalar multiplication

Enough. I will let your futures convince you that eigenstuff is cool and important.

Let's take a look at everything we learned in Module 1!

You've come so far so quickly

- Tomography
- Linear systems of equations
- Gaussian elimination algorithm
- Vectors and matrices, "multiplying" them
- Span, linear independence, basis
- Proofs!
- State transition systems
- Matrix inversion
- Vector spaces
- Null spaces, column spaces
- Eigenvalues and eigenvectors
- Steady state convergence behavior



I'm proud of you 💙

Linear algebra is one of the rare things that's both beautiful and useful.

Don't worry if you didn't get everything this time - I learned it at least 3 times before I felt like I knew it.

And thank you all for listening to me.

It's been a lot but also it's been a good time ❤️