$\boxed{\text{Lecture 2A}}$ Mon 6/26

Today:

- one last Gaussian elimination thing: free variables
- vectors
- matrices
- multiplications
- $A\vec{x} = \vec{b}$ as a matrix-vector multiplication

[copied from last time's lecture notes]

Ex 5. $\begin{cases} 2y + 3z = 2 \\ x + y = 1 \end{cases}$

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 2 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{\text{swap } R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 3 & 2 \end{array}\right]$$

$$\xrightarrow{\frac{R_2}{2} \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} & 1 \end{array}\right]$$

this also counts as upper triangular!

missing 3rd pivot. can't eliminate in 3rd col.

$$\xrightarrow{R_1 - R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 1 \end{array}\right]$$

counts as RREF!

doesn't matter what's happening in 3rd column because it has no pivot.

But what is the answer in this case?

- We didn't find a contradiction, so there isn't no solution.
- But we have 3 equations, 2 unknowns so there must be INFINITE solutions!

Let's see if we can write what the set of solutions is
- because infinite solutions doesn't mean ANYTHING is a solution!

The way we're going to do this is by starting with which column doesn't have a pivot: the z column.
- we're gonna call z a "free variable".
- the first row gives a relationship between x + z, and the 2nd between y + z.
- so if we just choose z to be something, x and y will be determined!

(anything! "free"!)

Let's move back to equation form.

$$\begin{aligned} x - \tfrac{3}{2}z &= 0 \\ y + \tfrac{3}{2}z &= 1 \end{aligned} \quad \rightarrow \quad \begin{cases} x = \tfrac{3}{2}z \\ y = 1 - \tfrac{3}{2}z \\ z = \text{anything} \end{cases}$$

(infinite) set of solutions to this system

Sometimes we give z another name, so we could write the set of solutions as:

$$\begin{cases} x = \tfrac{3}{2}s \\ y = 1 - \tfrac{3}{2}s \\ z = s \end{cases}, \quad s \in \mathbb{R}$$

"in" the set of all real numbers

if you choose an s, I can give you a solution!

It's possible for a system to have multiple free variables!
- which you will see in homework

Moving on.

I'm going to introduce the fundamental objects of linear algebra, which are vectors and matrices.

- I want to say that there are 2 complementary parts of learning and doing this stuff:
  1. mechanical (robot)
  2. visual / intuition (elegant! deep! requires real understanding and thinking...)
- BOTH are necessary to succeed in this class.
  - ② is much more IMPORTANT in the long term
    - ② is HARDER
  - I will try to show both to you. Dis/HW may skew to ① just because it's easier to write problems about, but don't forget about ②!

Ok. What is a vector?

- robot answer: a stack of numbers

arrow for vector

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

notation!

"in"

$\mathbb{R}$ = set of real numbers

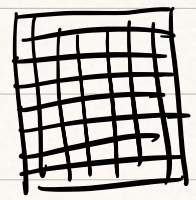
$\mathbb{R}^2$ = set of PAIRS of real numbers
$\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}, \begin{bmatrix} -0.9 \\ 0 \end{bmatrix}, \ldots \right\}$

$\mathbb{R}^n$ = n-dimensional vectors with real entries.

- a stack of numbers can represent a lot of things!

$$\begin{bmatrix} 50 \\ 52 \\ 55 \\ 60 \\ 65 \\ 70 \\ 67 \\ \vdots \end{bmatrix}$$ temperature at each hour

$$\begin{bmatrix} R \\ G \\ B \end{bmatrix}$$ color pixel



100x100 pixel image

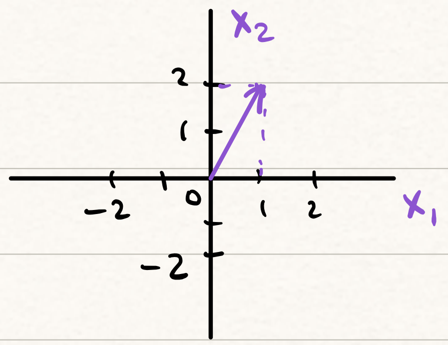
$$\rightarrow \begin{bmatrix} 0 \\ 1 \\ 2 \\ 144 \\ \vdots \end{bmatrix} \in \mathbb{R}^{10,000}$$

etc. etc.

Now the *VISUAL* way to think about vectors:

- let's do it in 2D.

$$\begin{array}{c} x_1 \\ x_2 \end{array} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

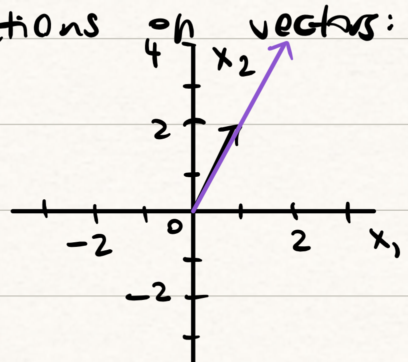


We find the point (1, 2) and draw an arrow to it from the origin.

Let's do some simple operations on vectors:

- scale a vector

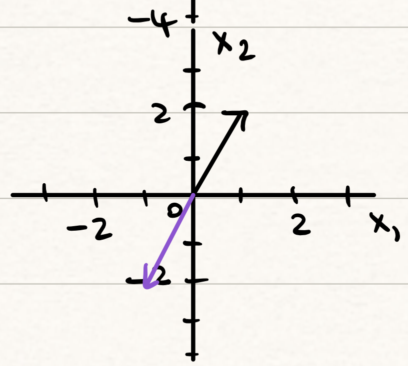
$$2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$



make it 2x longer! but point in same direction.

scale all entries

$$-1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

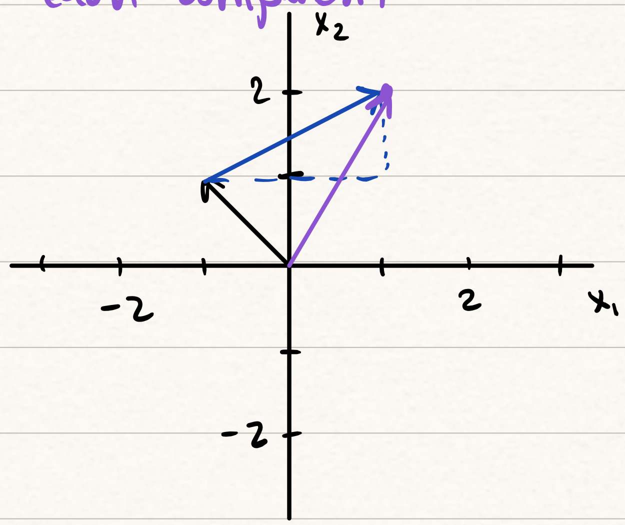


negative flips it around! (but stays on same line)

- adding vectors

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

add each component



1. draw first vector
2. draw 2^{nd} vector starting from end of 1^{st}.
3. where you end up = sum

- notice: when you scaled a vector, you could never move off the line the vector lived on
  - but when you add vectors, you can reach new places!

Some special vectors (with some notation)

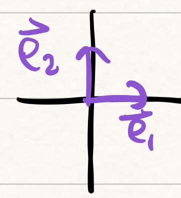
$$\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \vec{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

zero vector   ones vector

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

"basis vectors"

in 2D: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



axes!

sometimes called: $\hat{i}$ "i hat" $\hat{j}$ (in 3D, the 3^{rd} is called $\hat{k}$)

What is a matrix? A rectangular grid of numbers

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 & 5 \\ 1 & -1 & 0 \end{bmatrix}$$

- note : $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix} \neq \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 4 & 6 \end{array}\right]$

special notation for Gaussian elim

- matrix with m rows, n columns

$$\begin{array}{c} m=2 \\ \text{rows} \end{array} \begin{bmatrix} 0 & 3 & 5 \\ 1 & -1 & 0 \end{bmatrix}$$

"$m \times n$" matrix

rows × cols

$n = 3$ columns

- notation : $A \in \mathbb{R}^{m \times n}$ ← set of all $m \times n$ matrices with real entries

- can think of them as collections of vectors

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Let $\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Then can write $A = \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix}$

lines remind you that these are vectors.

- One more notation : "transpose"
  - for a vector

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \rightarrow \vec{x}^T = [x_1 \dots x_n]$$

flip it on its side

- for a matrix, flip every col:

$$A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \dots & \vec{a}_n \\ | & & | \end{bmatrix} \rightarrow A^T = \begin{bmatrix} - & \vec{a}_1^T & - \\ & \vdots & \\ - & \vec{a}_n^T & - \end{bmatrix}$$

$m \times n \longrightarrow n \times m$

But how can we think of matrices visually?

- let's watch a pretty video together.

Search on Youtube: "3Blue1Brown linear algebra Chapter 3: Linear Transformations and Matrices"

To recap the video:

- We think of matrices as functions that take in vectors and spit out vectors.

$$\begin{bmatrix} & A & \end{bmatrix} \underbrace{\begin{bmatrix} \vec{v_1} \end{bmatrix}}_{\text{input}} = \underbrace{\begin{bmatrix} \vec{v_2} \end{bmatrix}}_{\text{output}}$$

We write it like this.

- Visually, A can act on all vectors in space, so you can think of it as a certain warping of space: every point in space ends up somewhere else.
- Each column of a matrix tells you where that basis vector is going to end up after the warping.

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

- To figure out where some other vector will go, remember that every vector is a weighted sum of basis vectors.

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then $A\begin{bmatrix}5\\3\end{bmatrix} = A\left(5\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}\right)$

$$= 5 \cdot \underbrace{A\begin{bmatrix}1\\0\end{bmatrix}}_{\text{where } \vec{e}_1 \text{ goes}} + 3 \cdot \underbrace{A\begin{bmatrix}0\\1\end{bmatrix}}_{\text{where } \vec{e}_2 \text{ goes}}$$

And we just said that's written in the cols of A.

$$= 5\begin{bmatrix}1\\2\end{bmatrix} + 3\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}5\\10\end{bmatrix} + \begin{bmatrix}9\\12\end{bmatrix} = \begin{bmatrix}45\\120\end{bmatrix}$$

- symbolically,

$$\begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2$$

$$\begin{bmatrix} | & & | \\ \vec{a}_1 & \dots & \vec{a}_n \\ | & & | \end{bmatrix}\begin{bmatrix}x_1\\ \vdots \\x_n\end{bmatrix} = x_1\vec{a}_1 + \dots + x_n\vec{a}_n$$

this is what this means!

"column view of matrix - vector multiplication"

Ok. isn't that cool? I will now show you the robot way.

- it's going to be like a game we play with matrices and vectors that doesn't mean anything
- but now maybe you can remember that actually it does!

Let's actually start with vector-vector multiplication

$$\underbrace{[x_1 \ \ldots \ x_n]}_{\vec{x}_1^T} \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}_{\vec{y}_1} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

- the rules of the game:
  1. Use your left hand for the 1st vector, right for the 2nd.
  2. Your left hand moves horizontally $\longleftrightarrow$
     Your right hand moves vertically.
  3. Start at x_1 and y_1.
     Move both hands so you're at $x_2 + y_2$
     $\vdots$
  4. Each place you stop, multiply the 2 numbers.
  5. Add up all the products you get (all the way to $x_n y_n$)

Let's try.

$$[1 \ \ 2] \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$ move them arms!

$= 1 \cdot 3 + 2 \cdot 4 = 3 + 8 = 11$ vector · vector = scalar!

- Notice: # entries in each vector has to match
  otherwise impossible! one arm will be left hanging.

$$[1 \ \ 2] \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$! NO!

Now let's go to matrix-vector multiplication.

Level 2 of the game.

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 2 \cdot 1 \\ -1 \cdot 3 + 0 \cdot 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

2 vector-vector multiplications!

we've already seen that the output should be 2D.

Check: column-view from before

$$3 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

is this allowed?

- check if any arm is left hanging!
- looks like we're good.

$$= \begin{bmatrix} 1 \cdot 2 + -1 \cdot -1 \\ 0 \cdot 2 + 3 \cdot -1 \\ 2 \cdot 2 + 2 \cdot -1 \end{bmatrix} = \begin{bmatrix} 2+1 \\ 0-3 \\ 4-2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix}$$

We can think about what a matrix that takes in 2D and outputs 3D means in the future.

Ready to level up? Level 3: Matrix-matrix multiplication

$$\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1\cdot 4 + 0\cdot 0 & 1\cdot -2 + 0\cdot 2 \\ -3\cdot 4 + 2\cdot 0 & -3\cdot -2 + 2\cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 \\ -10 & 10 \end{bmatrix}$$

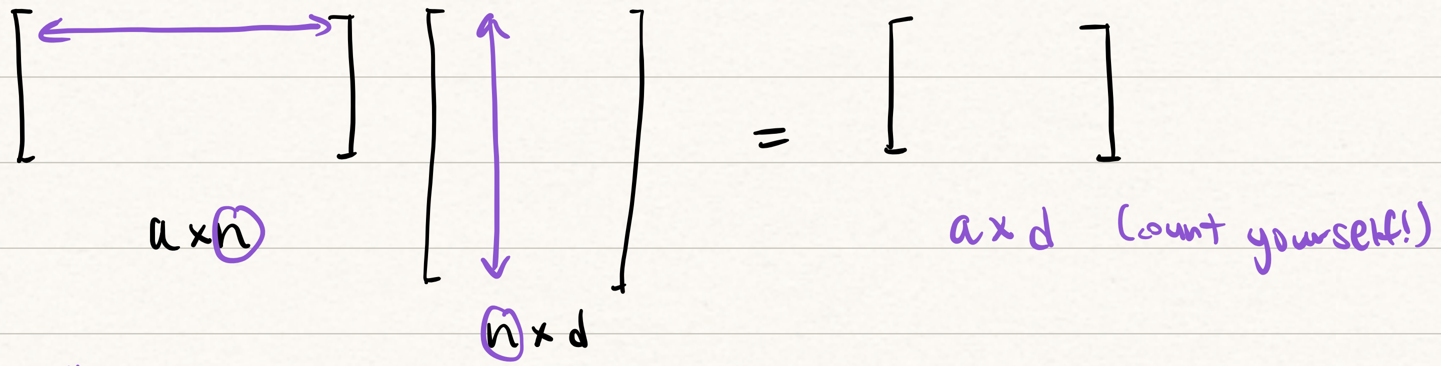
Q. If A is $a \times b$ and B is $c \times d$, when is AB a valid multiplication?

A. the vector-vector multiplication we do is

rows of A · cols of B

b elements c elements

⇒ the condition is that $b = c$



"inner dimensions match"

outer dimensions tell you output dims

Congratulations! You beat the game.

Here's the last boss battle: tinyurl.com/16afinalboss

$A \in \mathbb{R}^{M \times L}$ $B \in \mathbb{R}^{N \times L}$

- AB: $(M \times L) \cdot (N \times L)$ nope
- $A^T B$: $(L \times M) \cdot (N \times L)$ still no
- AB^T: $(M \times L) \cdot (L \times N)$ yay!

What else would work? BA^T

What is the visual intuition of matrix-matrix multiplication?

We will see soon :) on Thursday, probably.