EECS 16A Designing Information Devices and Systems I
Summer 2023 Discussion 2D

1. **Proofs**

Definition: A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ is **linearly dependent** if there exist constants c_1, c_2, … c_n such that

$$\sum_{i=1}^{i=n} c_i \vec{v}_i = \vec{0}$$

and at least one c_i is non-zero.

This condition intuitively states that it is possible to express any one vector in the set in terms of the others.

(a) Suppose for some non-zero vector $\vec{x}$, $\mathbf{A}\vec{x} = \vec{0}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

(b) (Optional) For a matrix $\mathbf{A}$, suppose there exist two unique vectors $\vec{x}_1$ and $\vec{x}_2$ that both satisfy $\mathbf{A}\vec{x} = \vec{b}$, that is, $\mathbf{A}\vec{x}_1 = \vec{b}$ and $\mathbf{A}\vec{x}_2 = \vec{b}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

(c) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix for which there exists a non-zero $\vec{y} \in \mathbb{R}^n$ such that $\mathbf{A}\vec{y} = \vec{0}$. Let $\vec{b} \in \mathbb{R}^m$ be some non-zero vector. Show that if there is one solution to the system of equations $\mathbf{A}\vec{x} = \vec{b}$, then there are infinitely many solutions.

2. Visualizing Matrices as Operations

This problem is going to help you visualize matrices as operations. For example, when we multiply a vector by a "rotation matrix," we will see it "rotate" in the true sense here. Similarly, when we multiply a vector by a "reflection matrix," we will see it be "reflected." The way we will see this is by applying the operation to all the vertices of a polygon and seeing how the polygon changes.

Your TA will now show you how a unit square can be rotated, scaled, or reflected using matrices! Note that in this exercise we are applying a matrix transformation on each of the vertices of the unit square separately.

(a) First, we will look at reflections. The transformation matrix that reflects a vector about the y-axis is:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

since any vector of the form $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is transformed to $\begin{bmatrix} -x_0 \\ y_0 \end{bmatrix}$.
What are the matrices that reflect a vector about the (i) x-axis and (ii) line $x = y$?

(b) We are given matrices $\mathbf{T}_1$ and $\mathbf{T}_2$, and we are told that they will rotate the unit square by $15°$ and $30°$ respectively. Suggest some methods to rotate the unit square by $45°$ using only $\mathbf{T}_1$ and $\mathbf{T}_2$. How would you rotate the square by $60°$? Your TA will show you the result in the iPython notebook.

(c) Find a single matrix $\mathbf{T}_3$ to rotate the unit square by $60°$. Your TA will show you the result in the iPython notebook.

(d) $\mathbf{T}_1$, $\mathbf{T}_2$, and the matrix you used in part (b) are called "rotation matrices." They rotate any vector by an angle θ. Show that a rotation matrix has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where θ is the angle of rotation. To do this, consider rotating the unit vector $\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ by θ degrees using the matrix $\mathbf{R}$.

(**Definition:** A vector, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix}$, is a unit vector if $\sqrt{v_1^2 + v_2^2 + \ldots} = 1$.)

(*Hint: Use your trigonometric identities:* $\cos(a)\cos(b) - \sin(a)\sin(b) = \cos(a+b)$, $\cos(a)\sin(b) + \sin(a)\cos(b) = \sin(a+b)$.)

(e) Now, we want to get back the original unit square from the rotated square in part (c). What matrix should we use to do this? (**Note:** Don't use inverses! Answer this question using your intuition; we will visit inverses very soon in lecture!)

(f) Use part (e) to obtain the rotation matrix that reverses the operation of a matrix that rotates a vector by θ. Multiply the reverse rotation matrix with the rotation matrix and vice-versa. What do you get?

(g) A natural question to ask is the following: does the *order* in which you apply transformations matter? Let's see what happens to a vector when we rotate it by $60°$ and then reflect it along the y-axis (matrix given in part (a)). Next, let's see what happens when we first reflect the vector along the y-axis and then rotate it by $60°$. You will need to multiply the corresponding rotation and reflection matrices in the correct order. Are the results the same?

(h) Now, let's perform the operations in part (g) on the unit square in our iPython notebook. Are the results the same?