EECS 16A Designing Information Devices and Systems I
Summer 2023 Discussion 07C

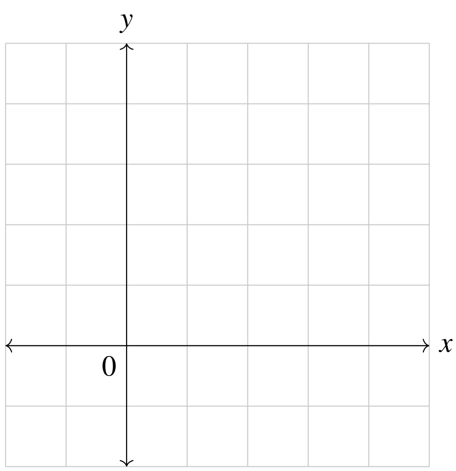
1. Mechanical Projection

In $\mathbb{R}^n$, the vector valued projection of vector $\vec{b}$ onto vector $\vec{a}$ is defined as:

$$\text{proj}_{\vec{a}}\left(\vec{b}\right) = \frac{\left\langle \vec{a}, \vec{b} \right\rangle}{\|\vec{a}\|^2}\vec{a}.$$

Recall $\|\vec{a}\|^2 = \langle \vec{a}, \vec{a} \rangle$.

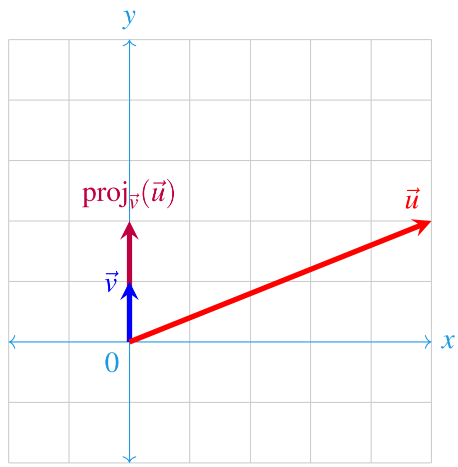
(a) Project $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ onto $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ — that is, onto the y-axis. Graph these two vectors and the projection.



Answer:

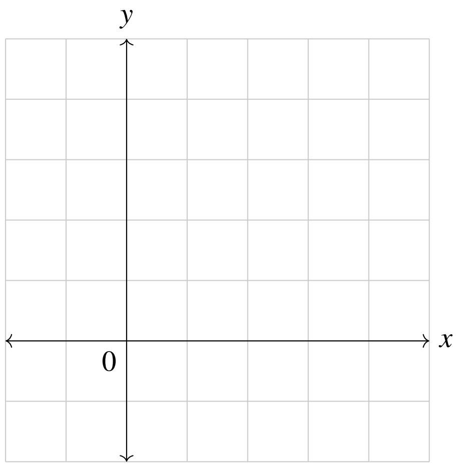
$$\vec{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \text{proj}_{\vec{v}}(\vec{u}) &= \frac{\vec{u}^\top \vec{v}}{\|\vec{v}\|^2}\vec{v} \\ &= \frac{2}{1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \end{aligned}$$



y
$\text{proj}_{\vec{v}}(\vec{u})$
$\vec{u}$
$\vec{v}$
x
0

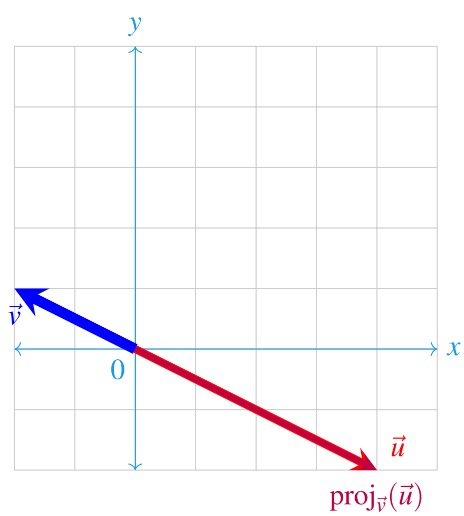
(b) Project $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Graph these two vectors and the projection.



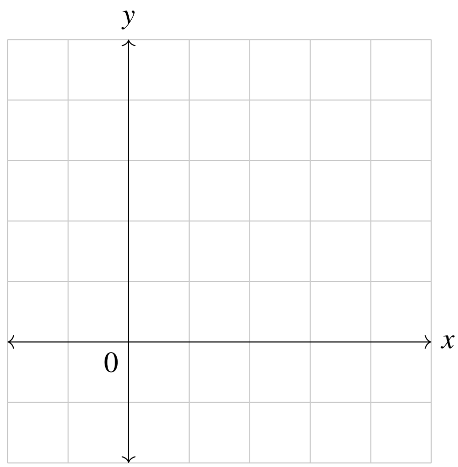
Answer:

$$\vec{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \text{proj}_{\vec{v}}(\vec{u}) &= \frac{\vec{u}^\top \vec{v}}{\|\vec{v}\|^2} \vec{v} \\ &= \frac{-10}{5} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} \end{aligned}$$



(c) Project $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Graph these two vectors and the projection.

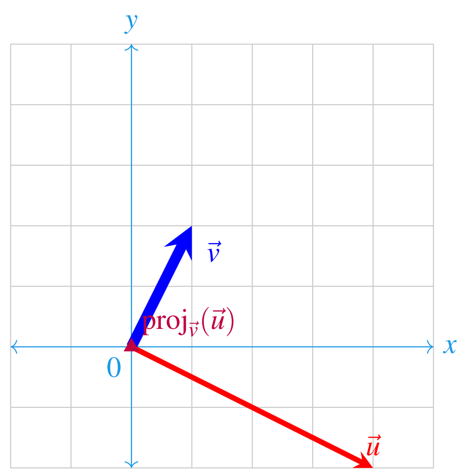


Answer:

$$\vec{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}^\top \vec{v}}{\|\vec{v}\|^2} \vec{v}$$

$$= \frac{0}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



2. Least Squares with Orthogonal Columns

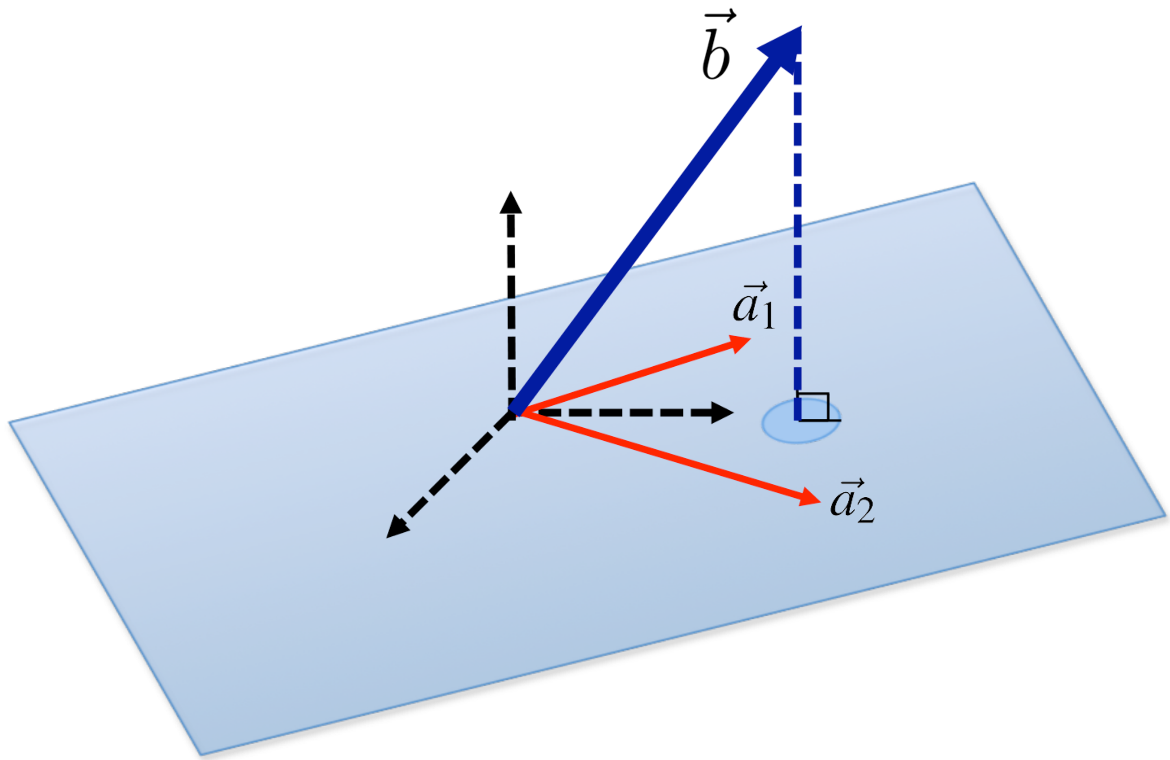
(a) Consider a least squares problem of the form

$$\min_{\vec{x}} \left\| \vec{b} - \mathbf{A}\vec{x} \right\|^2 = \min_{\vec{x}} \left\| \mathbf{A}\vec{x} - \vec{b} \right\|^2 = \min_{\vec{x}} \left\| \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} - \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|^2$$

Let the solution be $\hat{\vec{x}} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$.

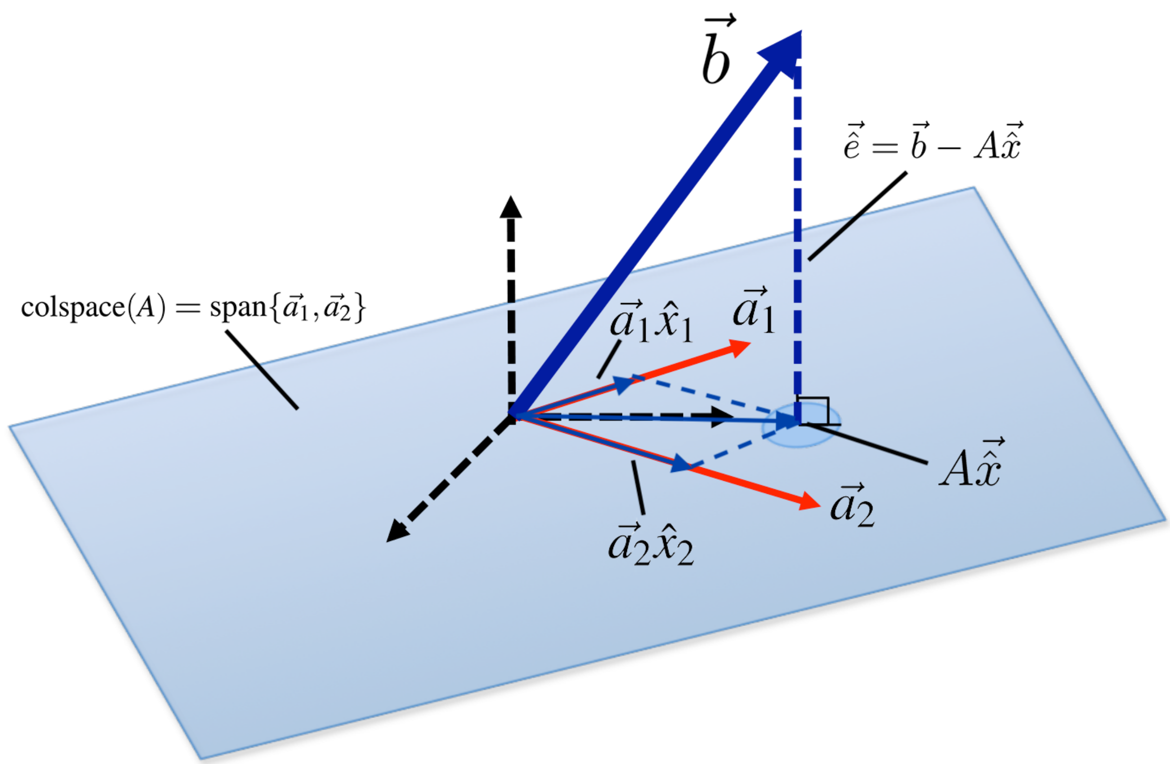
Label the following elements in the diagram below.

$$\text{span}\{\vec{a}_1, \vec{a}_2\}, \qquad \vec{e} = \vec{b} - \mathbf{A}\hat{\vec{x}}, \qquad \mathbf{A}\hat{\vec{x}}, \qquad \vec{a}_1\hat{x}_1,\ \vec{a}_2\hat{x}_2, \qquad \text{colspace}(\mathbf{A})$$



$\vec{b}$
$\vec{a}_1$
$\vec{a}_2$

Answer:



$\vec{b}$
$\vec{\hat{e}} = \vec{b} - A\vec{\hat{x}}$
$\text{colspace}(A) = \text{span}\{\vec{a}_1, \vec{a}_2\}$
$\vec{a}_1\hat{x}_1$
$\vec{a}_1$
$A\vec{\hat{x}}$
$\vec{a}_2$
$\vec{a}_2\hat{x}_2$

(b) We now consider the special case of least squares where the columns of $\mathbf{A}$ are orthogonal. Given that $\vec{\hat{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b}$ and $A\vec{\hat{x}} = \text{proj}_{\mathbf{A}}(\vec{b}) = \hat{x}_1\vec{a}_1 + \hat{x}_2\vec{a}_2$, show that

$$\text{proj}_{\vec{a}_1}(\vec{b}) = \hat{x}_1\vec{a}_1$$
$$\text{proj}_{\vec{a}_2}(\vec{b}) = \hat{x}_2\vec{a}_2$$

Answer: The projection of $\vec{b}$ onto $\vec{a}_1$ and $\vec{a}_2$ are given by:

$$\text{proj}_{\vec{a}_1}(\vec{b}) = \frac{\langle \vec{a}_1, \vec{b}\rangle}{\|\vec{a}_1\|^2}\vec{a}_1 \qquad \text{proj}_{\vec{a}_2}(\vec{b}) = \frac{\langle \vec{a}_2, \vec{b}\rangle}{\|\vec{a}_2\|^2}\vec{a}_2$$

$$\text{Length:} \quad \frac{\langle \vec{a}_1, \vec{b}\rangle}{\|\vec{a}_1\|} \qquad \frac{\langle \vec{a}_2, \vec{b}\rangle}{\|\vec{a}_2\|}$$

The least squares solution is given by:

$$\begin{aligned}
\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} &= \left(\begin{bmatrix} - & \vec{a}_1{}^T & - \\ - & \vec{a}_2{}^T & - \end{bmatrix}\begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix}\right)^{-1}\begin{bmatrix} - & \vec{a}_1{}^T & - \\ - & \vec{a}_2{}^T & - \end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix} \\
&= \begin{bmatrix}\frac{1}{\|\vec{a}_1\|^2} & 0 \\ 0 & \frac{1}{\|\vec{a}_2\|^2}\end{bmatrix}\begin{bmatrix} - & \vec{a}_1{}^T & - \\ - & \vec{a}_2{}^T & - \end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix} \\
&= \begin{bmatrix}\frac{\vec{a}_1{}^T\vec{b}}{\|\vec{a}_1\|^2} \\ \frac{\vec{a}_2{}^T\vec{b}}{\|\vec{a}_2\|^2}\end{bmatrix}
\end{aligned}$$

Thus,

$$\text{proj}_{\vec{a}_1}(\vec{b}) = \frac{\langle \vec{a}_1, \vec{b}\rangle}{\|\vec{a}_1\|^2}\vec{a}_1 = \frac{\vec{a}_1{}^T\vec{b}}{\|\vec{a}_1\|^2}\vec{a}_1 = \hat{x}_1\vec{a}_1$$
$$\text{proj}_{\vec{a}_2}(\vec{b}) = \frac{\langle \vec{a}_2, \vec{b}\rangle}{\|\vec{a}_2\|^2}\vec{a}_2 = \frac{\vec{a}_2{}^T\vec{b}}{\|\vec{a}_2\|^2}\vec{a}_2 = \hat{x}_2\vec{a}_2$$

(c) Compute the least squares solution to

$$\min_{\vec{x}} \left\| \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\right\|^2.$$

Answer: Noticing that the columns of A are orthogonal, we can use the result we proved in the previous part to solve for the least squares solution without explicitly evaluating the formula.

$$\text{proj}_{\vec{a}_1}(\vec{b}) = \frac{1}{1}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\text{proj}_{\vec{a}_2}(\vec{b}) = \frac{3}{1}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\rightarrow \vec{\hat{x}} = \begin{bmatrix}1\\3\end{bmatrix}$$