EECS 16A Designing Information Devices and Systems I
Summer 2023 Discussion 06A

1. Dynamical Systems (Spring 2020 Midterm 1 Question 7)

Define matrices $Q, R \in \mathbb{R}^{2\times 2}$ according to

$$Q = \begin{bmatrix} 0 & 3/4 \\ 1 & 1/4 \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(a) Find the eigenvalues for the matrix Q.

Answer: The eigenvalues are $\lambda_1 = 1, \lambda_2 = -3/4$.

(b) Consider a system with state vector $\vec{x}[n] \in \mathbb{R}^2$ at time $n \geq 1$ given by

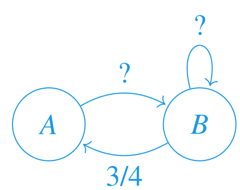
$$\vec{x}[n] = Q\vec{x}[n-1].$$

Is there a non-zero vector $\vec{x}$ satisfying $\vec{x} = Q\vec{x}$? If yes, give one such vector.

Answer: Yes; $\vec{x} = [3/4,\ 1]^\top$.

(c) Draw the state-transition diagram for the system in part (b). Label your nodes "A" and "B".

Answer:



(d) Now, consider a system with state vector $\vec{w}[n] \in \mathbb{R}^2$ at time $n \geq 1$ given by:

$$\vec{w}[n] = \begin{cases} Q\,\vec{w}[n-1] & \text{if } n \text{ is odd} \\ R\,\vec{w}[n-1] & \text{if } n \text{ is even.} \end{cases}$$

Write expressions for $\vec{w}[1]$, $\vec{w}[2]$, $\vec{w}[3]$ and $\vec{w}[4]$ in terms of $\vec{w}[0]$ and Q and R. Write each answer in the form of a matrix-vector product.

Answer:

$$\vec{w}[1] = Q\vec{w}[0], \quad \vec{w}[2] = RQ\vec{w}[0], \quad \vec{w}[3] = Q(RQ)\vec{w}[0], \quad \vec{w}[4] = (RQ)^2\vec{w}[0].$$

(e) Suppose we start the system of part (d) with state $\vec{w}[0] = \begin{bmatrix} 11/14 & 3/14 \end{bmatrix}^\top$. Find expressions for $\vec{w}_{\text{even}}$ and $\vec{w}_{\text{odd}}$, which are defined according to

$$\vec{w}_{\text{even}} = \lim_{k\to\infty} \vec{w}[2k], \qquad \vec{w}_{\text{odd}} = \lim_{k\to\infty} \vec{w}[2k+1].$$

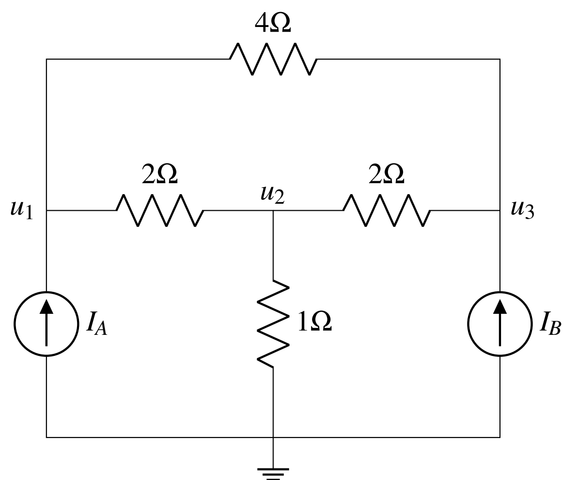
In words, $\vec{w}_{\text{even}}$ and $\vec{w}_{\text{odd}}$ describe the long-term behavior of the system at even and odd time-instants, respectively. (Hint: you can avoid computation by thinking about the system at even time-instants in terms of a state-transition diagram.)

Answer:

$$\vec{w}_{\text{even}} = \lim_{k\to\infty} \vec{w}[2k] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \text{and, } \vec{w}_{\text{odd}} = Q\vec{w}_{\text{even}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

2. Superposition (Fall 2020 Midterm 2 Question 7)

For this question, we will analyze the circuit shown below with the two current sources of strength I_A and I_B as inputs. It may be observed that the network of resistors shown in the circuit is symmetric. We will first solve this circuit for symmetric inputs $I_A = I_B$, and then for anti-symmetric inputs $I_A = -I_B$. Using these two results, we we will solve the circuit for arbitrary inputs I_A, I_B.

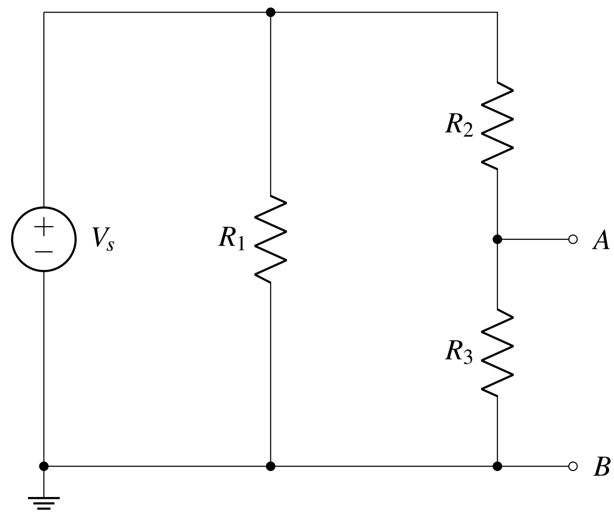


(a) Consider the circuit above with symmetric inputs, $I_A = I_B = 1\text{A}$. Using superposition, solve for the node voltages at the nodes marked u_1, u_2 and u_3.

(b) Consider the same circuit as before with anti-symmetric inputs, $I_A = 1\,\mathrm{A}$ and $I_B = -1\,\mathrm{A}$. Using superposition solve for the node voltages at the nodes marked u_1, u_2 and u_3.

3. Thévenin/Norton Equivalence

(a) Find the Thévenin resistance R_{th} of the circuit shown below, with respect to its terminals A and B.

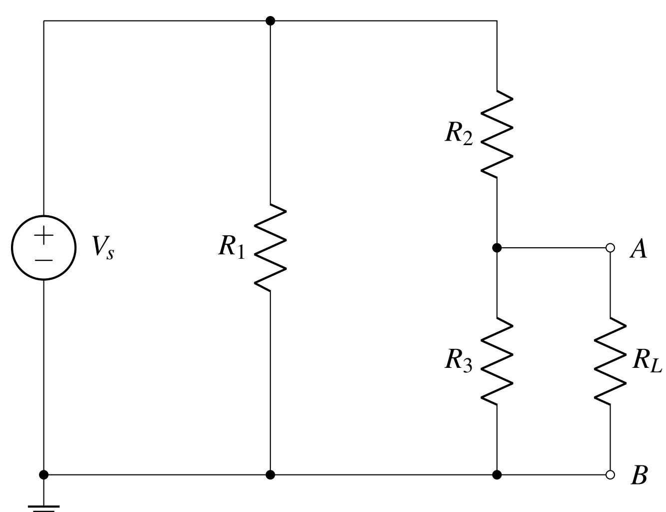


Answer: To find the Thévenin resistance, we null out the voltage source (which shorts out R_1) and find the equivalent resistance, which is:

$$R_{th} = R_2 \parallel R_3$$

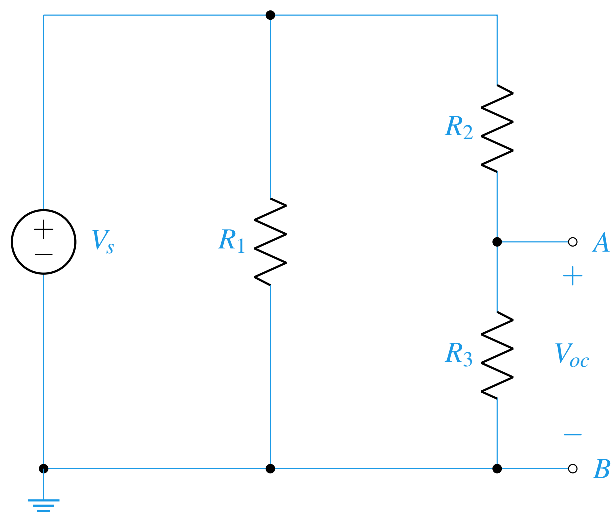
since resistors R_2 and R_3 are in parallel.

(b) Now a load resistor, R_L, is connected across terminals A and B, as shown in the circuit below. Using Thévenin equivalence, find the power dissipated in the load resistor in terms of the given variables.



Answer:

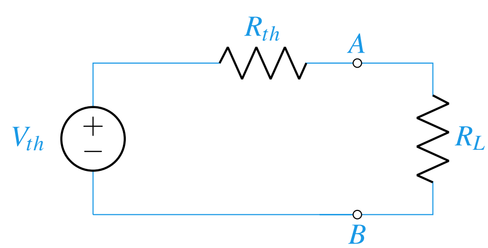
To help simplify the analysis, we replace the circuit with its Thévenin equivalent circuit. In order to do so, we first need to find the Thévenin voltage, V_{th}. One way to determine V_{th} is to find the open circuit voltage, $V_{AB} = V_{oc}$, in the original circuit when an open circuit is connected externally to terminals A and B.



The open circuit can be derived from a voltage divider:

$$V_{th} = V_{oc} = \frac{R_3}{R_2 + R_3} V_s$$

We also already know the Thévenin resistance $R_{th} = R_2 \parallel R_3$ from part (a).
Thus, the circuit can be simplified to:

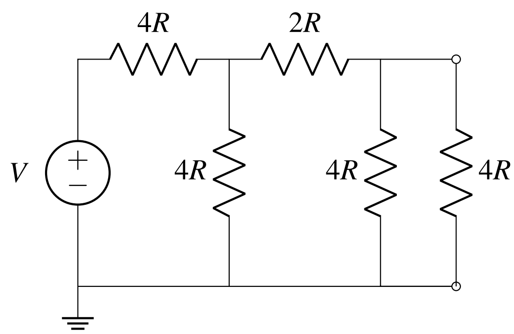


The power through the load resistor is then given by:

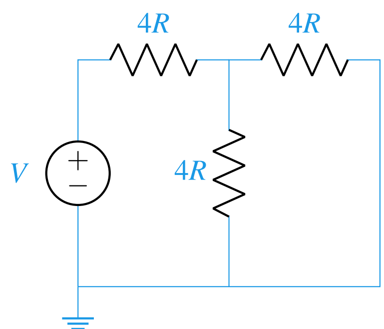
$$P_{R_L} = V_{R_L} \cdot I_{R_L} = I_{R_L}^2 \cdot R_L = \left(\frac{V_{th}}{R_L + R_{th}}\right)^2 R_L = \left(\frac{R_3}{R_2 + R_3} V_s \cdot \frac{1}{R_L + R_2 \parallel R_3}\right)^2 R_L$$

4. OPTIONAL: Power to Resist (from Spring 2018 midterm 2)

Find the power dissipated by the voltage source in the circuit below. Be sure to use passive sign convention.



Answer: We want to find the equivalent resistance across the voltage source in Figure 6.2. Start by reducing the two resistors on the right to $4R \parallel 4R = 2R$. Then combine the other $2R$ resistor with this to get a new resistor of value $4R$ as in the circuit below.



Once again we have $4R \parallel 4R = 2R$. This is finally in series with $4R$ giving us a total resistance of $4R + 2R = 6R$

$$P = VI = V\frac{-V}{6R} = -\frac{V^2}{6R}$$

The negative sign is present because the voltage source actually provides power, which can also be seen by using passive sign convention.