EECS 16A Designing Information Devices and Systems I
Summer 2023 Homework 2

This homework is due June 30th, 2023, at 23:59.
Self-grades are due July 7th, 2023, at 23:59.

Submission Format

Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

1. Reading Assignment

For this homework, please read Note 1B, Note 2A, Note 2B, Note 3, Note 4, and Note 11A. Note 1B covers Gaussian Elimination. Notes 2A and 2B provide an overview of vectors, matrices, and operations among them. Note 3 covers the concepts of linear dependence and span, and Note 4 is a helpful guide for how to approach proofs. Note 11A is an introduction to circuit analysis.

Please answer the following questions:

(a) Can you name a few applications where we might use vectors? What do the components represent in each application?

(b) Are there special matrices where the order of their multiplication does *not* matter? If so come up with an example and explain why.

(c) What is the mathematical definition of span?

(d) List in your own words the steps used to construct a proof.

Solution:

(a) See Note 2A 2.2.1.

(b) Identity with any matrix, 0 matrix with any matrix, diagonal matrices with each other.

(c) From Note 3, the span of a set of vectors $\{v_1,\dots,v_n\}$ is the set of all linear combinations of $\{v_1,...v_n\}$. This can be expressed mathematically as $span(v_1,\dots,v_n) = \{\sum_{i=1}^{n}\alpha_i\vec{v}_i \mid \alpha_i \in \mathbb{R}\}$

(d) From Note 4, the steps used to construct a proof are

i. Read the entire proof statement carefully.

ii. Identify what we know from the proof statement.

iii. Identify what we want to prove.

iv. Observe the beginning (i.e., knowns) and end (i.e., final result) of the proof and try to identify connecting similarities.

v. Manipulate both sides of the claim and connect them together. Make sure to carefully justify each step.

vi. Repeatedly try different approaches until you find an idea that works.

vii. Complete the proof.

2. Image Masks

Learning Objective: *Learn to setup imaging problems with matrices.*

For these word problems, you only need to setup the problem in augmented matrix or matrix-vector notation. Of course, you may solve the system for practice (e.g., with Gaussian elimination), but no additional credit is awarded.

Solution: Full credit is awarded for setting up the augmented matrix or matrix-vector notation correctly.

After your first EECS16A lecture, you decide to try to build a single-pixel camera. You want to take a 2x2 image, i.e. 4 tiles, and based on the first lecture, you choose to take 4 measurements. Recall that each measurement is the sum of the illuminated tiles. For each measurement, you will use a different mask.

(a) Initially, you want to illuminate only one tile for each measurement. That is, you will first illuminate x_1, then you will illuminate x_2, etc. The outputs of your 4 measurements are y_1, y_2, y_3, and y_4 respectively. The 4 measurements you take are shown in Figure 1. Explicitly setup the matrix problem for this in the $A\vec{x} = \vec{b}$ form.

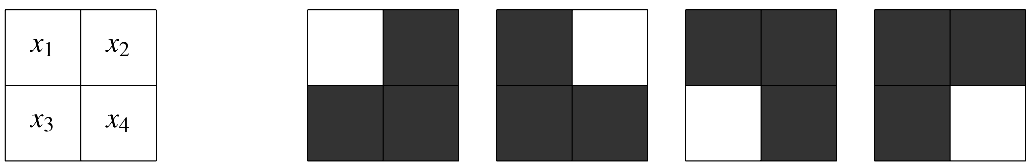


Image tile labels   Measurement 1   Measurement 2   Measurement 3   Measurement 4

Figure 1: Four image masks.

Solution: The augmented matrix setup looks like this:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & y_1 \\ 0 & 1 & 0 & 0 & y_2 \\ 0 & 0 & 1 & 0 & y_3 \\ 0 & 0 & 0 & 1 & y_4 \end{array}\right]$$

The matrix-vector setup looks like this:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

(b) While setting up your code to create the masks, you forget to turn off the illuminated tiles from the previous measurement. As a result, measurement one contains x_1, measurement two contains $x_1 + x_2$, etc. The outputs of your 4 measurements are z_1, z_2, z_3, and z_4 respectively. The 4 measurements you take are shown in Figure 2. Explicitly setup the matrix problem for this in $A\vec{x} = \vec{b}$ form.

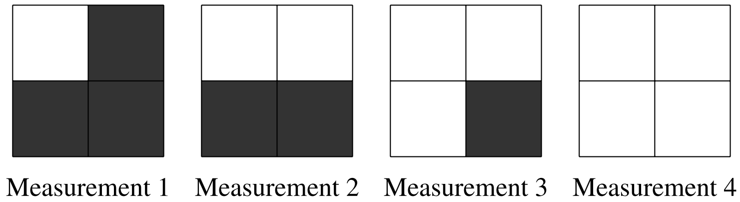


Figure 2: Four image masks.

Solution: The augmented matrix setup looks like this:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & z_1 \\ 1 & 1 & 0 & 0 & z_2 \\ 1 & 1 & 1 & 0 & z_3 \\ 1 & 1 & 1 & 1 & z_4 \end{array}\right]$$

The matrix-vector setup looks like this:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

(c) Your friend is also building their own single pixel camera. However, they make a different mistake in their code and during each measurement, instead of lighting up one tile, you light up the other 3 tiles instead. That is, instead of measuring x_1, they measure $x_2 + x_3 + x_4$. The output of the 4 measurements are w_1, w_2, w_3, and w_4. The 4 measurements from their setup are shown in Figure 3. Explicitly setup the matrix problem for this in $A\vec{x} = \vec{b}$ form.

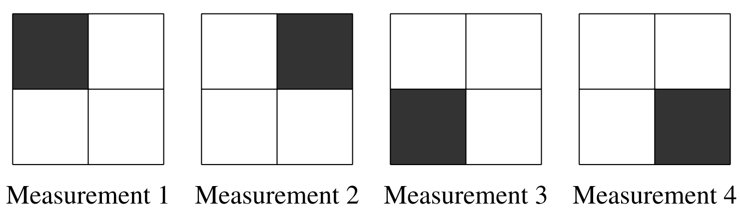


Figure 3: Four image masks.

Solution: The augmented matrix setup looks like this:

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & w_1 \\ 1 & 0 & 1 & 1 & w_2 \\ 1 & 1 & 0 & 1 & w_3 \\ 1 & 1 & 1 & 0 & w_4 \end{array}\right]$$

The matrix-vector setup looks like this:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

3. Gaussian Elimination

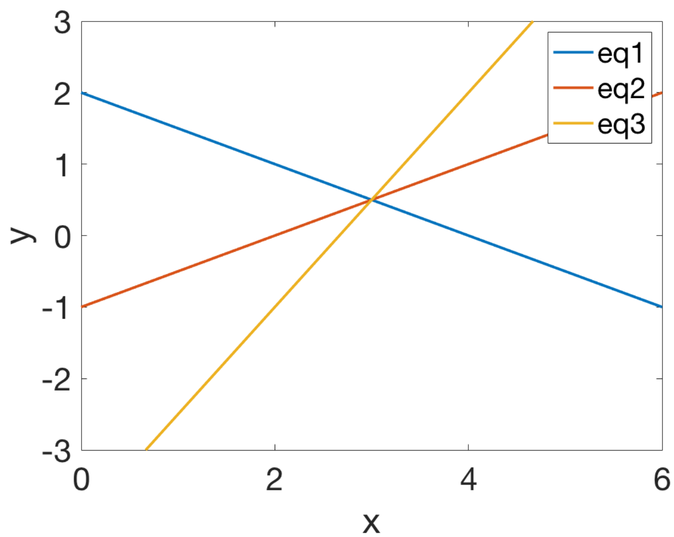
Learning Goal: *Understand the relationship between Gaussian elimination and the graphical representation of linear equations, and explore different types of solutions to a system of equations. You will also practice determining the parametric solutions when there are infinitely many solutions.*

(a) In this problem we will investigate the relationship between Gaussian elimination and the geometric interpretation of linear equations. You are welcome to draw plots by hand or using software. Please be sure to label your equations with a legend on the plot.

i. Plot the following set of linear equations in the x-y plane. If the lines intersect, write down the point or points of intersection.

$$x+2y=4 \tag{1}$$
$$2x-4y=4 \tag{2}$$
$$3x-2y=8 \tag{3}$$

Solution:
The three lines intersect at the point (3,0.5).



ii. Write the above set of linear equations in augmented matrix form and do the first step of Gaussian elimination to eliminate the x variable from equation 2. Now, the second row of the augmented matrix has changed. Plot the corresponding new equation created in this step on the same graph as above. What do you notice about the new line you draw?

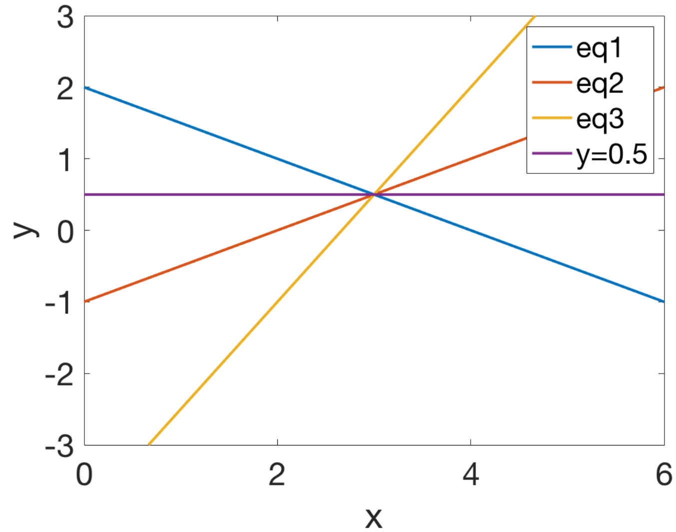
Solution: We start with the following augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & -4 & 4 \\ 3 & -2 & 8 \end{array}\right]$$

We then eliminate x from the second equation by subtracting $2\times$Row 1 from Row 2:

$$\text{Row 2: subtract } 2\times\text{Row 1} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -8 & -4 \\ 3 & -2 & 8 \end{array}\right]$$

So equation 2 becomes $-8y = -4$, which is equivalent to $y = 0.5$. You will notice that the line $y = 0.5$ intersects with the three lines you drew previously.



iii. Complete all of the steps of Gaussian elimination including back substitution. Now plot the new equations represented by the rows of the augmented matrix in the last step (after completing back substitution) on the same graph as above. What do you notice about the new line you draw?

Solution:

We continue from the previous part, where we had the following augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -8 & -4 \\ 3 & -2 & 8 \end{array}\right]$$

and take the following steps to complete Gaussian elimination:

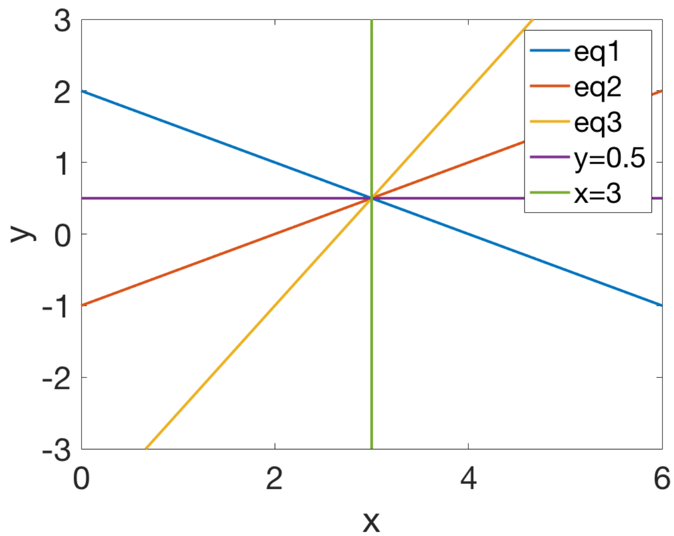
$$\text{Row 3: subtract } 3\times\text{Row 1} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -8 & -4 \\ 0 & -8 & -4 \end{array}\right]$$

$$\text{Row 2: divide by } -8 \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 0.5 \\ 0 & -8 & -4 \end{array}\right]$$

$$\text{Row 3: subtract } -8\times\text{Row 2} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{array}\right]$$

$$\text{Row 1: subtract } 2\times\text{Row 2} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, we end up with the solution $x = 3$ and $y = 0.5$.
Plotting the new equation $x = 3$ on the same graph as before, we see that all five lines intersect at the same point (3,0.5).



(b) Write the following set of linear equations in augmented matrix form and use Gaussian elimination to determine if there are no solutions, infinite solutions, or a unique solution. If any solutions exist, determine what they are. You may do this problem by hand or use a computer. We encourage you to try it by hand to ensure you understand Gaussian elimination. Remember that it is possible to end up with fractions during Gaussian elimination.

$$\begin{aligned} x + 2y + 5z &= 3 \\ x + 12y + 6z &= 1 \\ 2y + z &= 4 \\ 3x + 16y + 16z &= 7 \end{aligned}$$

Solution:
Writing the system in augmented matrix form we get the following:

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 1 & 12 & 6 & 1 \\ 0 & 2 & 1 & 4 \\ 3 & 16 & 16 & 7 \end{array}\right]$$

We eliminate the x variables from the second and fourth equations:

$$\begin{array}{c} \text{Row 2: subtract Row 1} \\ \text{Row 4: subtract } 3\times\text{Row 1} \end{array} \implies \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 10 & 1 & -2 \\ 0 & 2 & 1 & 4 \\ 0 & 10 & 1 & -2 \end{array}\right]$$

We then divide Row 2 by 10 to get a 1 in the pivot position:

$$\text{Row 2: divide by 10} \implies \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 1 & 0.1 & -0.2 \\ 0 & 2 & 1 & 4 \\ 0 & 10 & 1 & -2 \end{array}\right]$$

Next, we eliminate the y variables from the third and fourth equations:

$$\begin{array}{c} \text{Row 3: subtract } 2\times\text{Row 2} \\ \text{Row 4: subtract } 10\times\text{Row 2} \end{array} \implies \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 1 & 0.1 & -0.2 \\ 0 & 0 & 0.8 & 4.4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We divide Row 3 by 0.8 to get a 1 in the pivot position:

$$\text{Row 3: divide by 0.8} \implies \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 1 & 0.1 & -0.2 \\ 0 & 0 & 1 & 5.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We then proceed with back-substitution:

$$\begin{array}{c} \text{Row 2: subtract } 0.1\times\text{Row 3} \\ \text{Row 1: subtract } 5\times\text{Row 3} \end{array} \implies \left[\begin{array}{ccc|c} 1 & 2 & 0 & -24.5 \\ 0 & 1 & 0 & -0.75 \\ 0 & 0 & 1 & 5.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{Row 1: subtract } 2\times\text{Row 2} \implies \left[\begin{array}{ccc|c} 1 & 0 & 0 & -23 \\ 0 & 1 & 0 & -0.75 \\ 0 & 0 & 1 & 5.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This final matrix is in reduced row echelon form. The first three rows of the matrix have non-zero elements in pivot position, for a system with three unknowns, and the fourth row is a row of zeros, so we can conclude there is a unique solution: $x = -23$, $y = -0.75$, and $z = 5.5$.

(c) Consider the following system:

$$\begin{aligned} 4x+4y+4z+w+v &= 1 \\ x+y+2z+4w+v &= 2 \\ 5x+5y+5z+w+v &= 0 \end{aligned}$$

If you were to write the above equations in augmented matrix form and use Gaussian elimination to solve the system, you would get the following (for extra practice, you can try and do this yourself):

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 3 & 16 \\ 0 & 0 & 1 & 0 & -3 & -17 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

How many variables are free variables? Which ones? Find the general form of the solutions in terms of real constants (e.g., $s \in \mathbb{R}$).

Solution:

We first note that the given augmented matrix is in reduced row echelon form, which makes sense as it is the final output of the Gaussian elimination algorithm. We observe that the second and fifth columns do not have 1s in pivot position so there are two free variables corresponding to y and v.

Let $y = s$ and let $v = t$, where $s \in \mathbb{R}$ and $t \in \mathbb{R}$.

Using back substitution, we can solve for x, y, z, w, and v in terms of s and t:

$$\begin{aligned} &\text{Row 1:} \quad x + y + 3v = 16 \quad \implies x = 16 - 3t - s \\ &\text{Row 2:} \quad z - 3v = -17 \quad \implies z = -17 + 3t \\ &\text{Row 3:} \quad w + v = 5 \quad \implies w = 5 - t \end{aligned}$$

The solutions to the system of equations are therefore:

$$\begin{aligned} x &= 16 - 3t - s \\ y &= s \\ z &= -17 + 3t \\ w &= 5 - t \\ v &= t \end{aligned}$$

The solutions can also be represented by a set as:

$$S = \left\{ \vec{u} \;\middle|\; \vec{u} = \begin{bmatrix} 16 \\ 0 \\ -17 \\ 5 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3 \\ 0 \\ 3 \\ -1 \\ 1 \end{bmatrix} t \quad , \quad s \in \mathbb{R}, \quad t \in \mathbb{R} \right\}.$$

4. Vector-Vector, Matrix-Vector, and Matrix-Matrix Multiplication

Learning Objective: *Practice evaluating vector-vector, matrix-vector, and matrix-matrix multiplication.*

(a) For the following multiplications, state the dimensions of the result. If the product is not defined and thus has no solution, state this and justify your reasoning. For this problem $\vec{x} \in \mathbb{R}^N, \vec{y} \in \mathbb{R}^N, \vec{z} \in \mathbb{R}^M$, with $N \neq M$.

i. $\vec{x}^T \cdot \vec{z}$

Solution: This is invalid. $\vec{x}^T$ is an $1 \times N$ vector meaning that it has 1 rows and N column but $\vec{z}$ is an $M \times 1$ vector meaning that is has M rows and 1 column. Since $\vec{x}^T$ does not have the same number of columns as $\vec{z}$ has rows there is no solution.

ii. $\vec{x} \cdot \vec{x}^T$

Solution:

$$N \times N$$

$\vec{x}$ has N row and 1 columns and $\vec{x}^T$ has 1 row and N columns. Since the number of columns of $\vec{x}$ is the same as the number of rows of $\vec{x}^T$, there is a solution. The solution would have the dimensions of the number of rows of $\vec{x}$ times the number of columns of $\vec{x}^T$.

iii. $\vec{x} \cdot \vec{y}^T$

Solution:

$$N \times N$$

$\vec{x}$ has N row and 1 columns and $\vec{y}^T$ has 1 row and N columns. Since the number of columns of $\vec{x}$ is the same as the number of rows of $\vec{y}^T$, there is a solution. The solution would have the dimensions of the number of rows of $\vec{x}$ times the number of columns of $\vec{y}^T$.

iv. $\vec{x} \cdot \vec{z}^T$

Solution:

$$N \times M$$

$\vec{x}$ has N row and 1 columns and $\vec{z}^T$ has 1 row and M columns. Since the number of columns of $\vec{x}$ is the same as the number of rows of $\vec{z}^T$, there is a solution. The solution would have the dimensions of the number of rows of $\vec{x}$ times the number of columns of $\vec{z}^T$.

For questions (b) through (d), complete the matrix-vector multiplication. If the product is not defined and thus has no solution, state this and justify your reasoning:

(b)

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution: No solution. The number of columns in the matrix does not match the number of elements (aka the number of rows) in the column vector.

(c)

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 * \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 * \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 1 * \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2 * \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

(e) Compute $\mathbf{AB}$ by hand, where $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -3 & 0 & 2 & -1 \end{bmatrix}$$

What are the dimensions of $\mathbf{AB}$? Compute $\mathbf{BA}$ too if the operation is valid. If it is invalid, explain why. Make sure you show the work for your calculations.

Solution:

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 0 \\ -3 & 0 & 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1\times 1 + 0\times -3 & 1\times 2 + 0\times 0 & 1\times -1 + 0\times 2 & 1\times 0 + 0\times -1 \\ 2\times 1 + 1\times -3 & 2\times 2 + 1\times 0 & 2\times -1 + 1\times 2 & 2\times 0 + 1\times -1 \\ 0\times 1 + 1\times -3 & 0\times 2 + 1\times 0 & 0\times -1 + 1\times 2 & 0\times 0 + 1\times -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -3 & 0 & 2 & -1 \end{bmatrix} \end{aligned}$$

$\mathbf{BA}$ does not exist since the number of columns in $\mathbf{B}$ is not equal to the number of rows in $\mathbf{A}$.

(f) Compute $\mathbf{AB}$ by hand, where $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{bmatrix} 3 & 21 & 9 \\ -1 & 14 & 4 \\ 7 & -8 & 2 \end{bmatrix}, \text{ and } \qquad \mathbf{B} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \\ 3 & -6 \end{bmatrix}$$

Compute $\mathbf{BA}$ too if the operation is valid. If it is invalid, explain why. Make sure you show the work for your calculations.

Solution:

$$\mathbf{AB} = \begin{bmatrix} 3 & 21 & 9 \\ -1 & 14 & 4 \\ 7 & -8 & 2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ -1 & 2 \\ 3 & -6 \end{bmatrix}$$
$$= \begin{bmatrix} 3\times -2 + 21\times -1 + 9\times 3 & 3\times 4 + 21\times 2 + 9\times -6 \\ -1\times -2 + 14\times -1 + 4\times 3 & -1\times 4 + 14\times 2 + 4\times -6 \\ 7\times -2 + -8\times -1 + 2\times 3 & 7\times 4 + -8\times 2 + 2\times -6 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

BA does not exist since the number of columns in **B** is not equal to the number of rows in **A**.

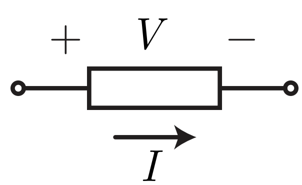
5. Basic Circuit Components

Learning Objectives: *Review basics of cucuit components and current-voltage relationships*

In the laboratory, you are tasked with identifying a single unknown component within a circuit. You can use a piece of electrical equipment called a multimeter to measure either voltage (voltmeter) across or the current (ammeter) through the component (you can measure both quantities simultaneously using two multimeters).

For each part of the problem, deduce the most likely type of circuit component based on the provided voltage and current measurements. Also draw the component circuit symbol and sketch the IV curve.

*Hint: You are told the possible choices are **short circuit (wire)**, **open circuit**, **resistor**, **voltage source**, and **current source**. Moreover, each part in this problem has a unique component, there are no repeats.*



(a) First, to familiarize yourself with common quantities of *voltage*, *current*, and *resistance*, fill in the unit name and unit symbol for each:

Quantity	Symbol	Unit Name	Unit Symbol
Voltage	V		
Current	I		
Resistance	R		

Solution:

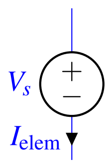
Quantity	Symbol	Unit Name	Unit Symbol
Voltage	V	Volts	V
Current	I	Amperes	A
Resistance	R	Ohms	Ω

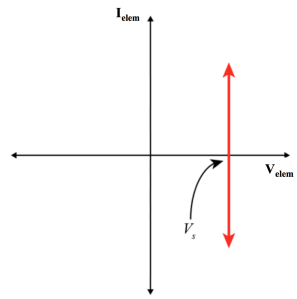
(b) You take one measurement and find the voltage is $V = 10$ V and current is $I = 1$ A. After changing a part of the circuit (not the part you are measuring), you take another measurement and find $V = 10$ V and $I = 2$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

Solution: The component is most likely a **voltage source** with $V_s = 10$ V. The voltage across the voltage source is always equal to the source value, V_s. The current through a voltage source is determined by the rest of the circuit.

IV Relationship

Symbol



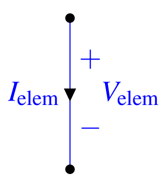


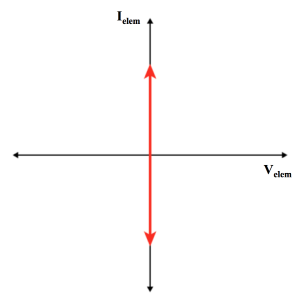
(c) You take one measurement and find the voltage is $V = 0$ V and current is $I = 1$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

Solution: The component is most likely a **short circuit (wire)**. A wire is an ideal connection with zero voltage across it. The current through the wire is determined by the rest of the circuit.

IV Relationship

Symbol



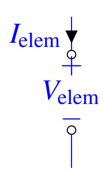


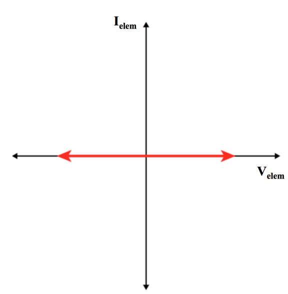
(d) You take one measurement and find the voltage is $V = 5$ V and current is $I = 0$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

Solution: The component is most likely an **open circuit**. There is no current going through an open circuit. The voltage potential across an open circuit is determined by the rest of the circuit.

IV Relationship

Symbol

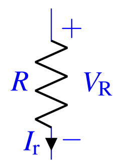




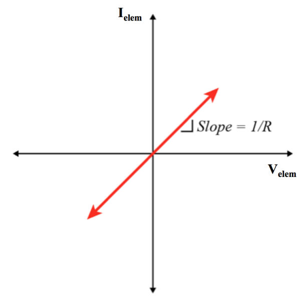
(e) You take one measurement and find the voltage is $V = 10$ V and current is $I = 2$ A. After changing a part of the circuit (not the part you are measuring), you take another measurement and find $V = -5$ V and $I = -1$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

Solution: The component is most likely an **resistor**. The relationship is described by Ohm's Law: $V_R = I_R R$

Symbol



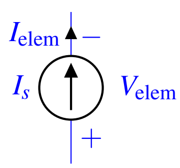
IV Relationship



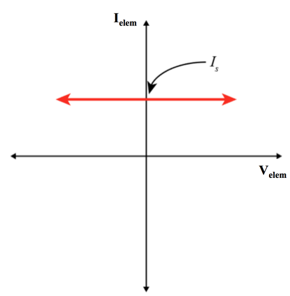
(f) You take one measurement and find the voltage is $V = 10$ V and current is $I = 2$ A. After changing a part of the circuit (not the part you are measuring), you take another measurement and find $V = -3$ V and $I = 2$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

Solution: The component is most likely a **current source** with $I_s = 2$ A. The current through a current source is always equal to the source value I_s. The voltage across a current source is determined by the rest of the circuit.

Symbol



IV Relationship



6. Easing into Proofs

Learning Objectives: *This is an opportunity to practice your proof development skills.*

Proof: Show that if the system of linear equations, $\mathbf{A}\vec{x} = \vec{b}$, has infinitely many solutions, then columns of $\mathbf{A}$ are linearly dependent.

To approach this proof, let us use a simplified version of the methodology delineated in Note 4. Although the final proof would read sequentially as in the *Proof Steps* indicated in Table 1, each part of this question will tackle each proof step but in a non-sequential order.

(a) **Proof Step 1: Write what is known**

Think about the information we already know from the problem statement. Every detail could be important and some details could be implicit.

We know that system of equations, $\mathbf{A}\vec{x} = \vec{b}$, has infinitely many solutions which can be difficult to work with, but perhaps we can simplify to a case that we can work with. It turns out that if a linear system has at least **two** distinct solutions, then it must also have an infinite number of solutions.

Table 1: Fundamental steps to a proof

Proof Steps	Description	Question Part
1	Identify what is known	(a)
2	Manipulate what is known	(c)
3	Connecting it up	(d)
4	Identify what is to be proved	(b)

We can also construct arbitrary vectors $\vec{u}$ and $\vec{v}$ which, in this case, are each a solution to the system of equations $\mathbf{A}\vec{x} = \vec{b}$ but not the same vector. Express the previous sentence in a mathematical form (just writing the equations will suffice; no need to do further mathematical manipulation).

Solution: $\vec{u}$ and $\vec{v}$ must satisfy:

$$\mathbf{A}\vec{u} = \vec{b}, \quad \mathbf{A}\vec{v} = \vec{b}. \tag{4}$$

$$\vec{u} \neq \vec{v}. \tag{5}$$

(b) **Proof Step 4: Identify what is to be proved**

We have to show that the columns of $\mathbf{A}$ are linearly dependent. The matrix $\mathbf{A}$ can always be deconstructed and the columns explicitly denoted as vectors $\vec{c}_1$, $\vec{c}_2$, ..., and $\vec{c}_n$, i.e. $\mathbf{A} = \begin{bmatrix} | & | & \cdots & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & \cdots & | \end{bmatrix}$.

Using the Definition 3.1 of linear dependence from Note 3.1, write a mathematical equation that conveys linear dependence of $\vec{c}_1$, $\vec{c}_2$, ..., and $\vec{c}_n$.

Solution: According to the definition of linear dependence:

$$\alpha_1\vec{c}_1 + \alpha_2\vec{c}_2 + \ldots + \alpha_n\vec{c}_n = \vec{0}. \tag{6}$$

where at least one α_i is not equal to zero.

(c) **Proof Step 2: Manipulating what is known**

Now let us try to start from the givens in part (a) and make mathematically logical steps to reach the final result in part (b). Since your answer to (b) is expressed in terms of the column vectors of $\mathbf{A}$, try to express the mathematical equations from part (a) in terms of the column vectors too. Using the column interpretation of matrix-vector multiplication, we can write

$$\mathbf{A}\vec{x} = \vec{b} \quad \Longrightarrow \quad \begin{bmatrix} | & | & \cdots & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \vec{b} \quad \Longrightarrow \quad x_1\vec{c}_1 + x_2\vec{c}_2 + \ldots + x_n\vec{c}_n = \vec{b}$$

Now rewrite your answer to part (a) using the above formulation to get equations for distinct solutions $\vec{u}$ and $\vec{v}$ in terms of the column vectors of $\mathbf{A}$.

Solution:

$$\mathbf{A}\vec{u} = \vec{b} \quad \Longrightarrow \quad \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \end{bmatrix} = \vec{b} \quad \Longrightarrow \quad u_1\vec{c}_1 + u_2\vec{c}_2 + \ldots + u_n\vec{c}_n = \vec{b}$$

$$\mathbf{A}\vec{v} = \vec{b} \implies \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} = \vec{b} \implies v_1\vec{c}_1 + v_2\vec{c}_2 + \dots + v_n\vec{c}_n = \vec{b}$$

(d) **Proof Step 3: Connecting it up**
Now think about how you can mathematically manipulate your answer from part (c) to match the pattern of your desired final proof statement in part (b).

Solution:
Subtracting the second equation from the first equation in part (iii), we have

$$u_1\vec{c}_1 + u_2\vec{c}_2 + \dots + u_n - v_1\vec{c}_1 - v_2\vec{c}_2 - \dots - v_n\vec{c}_n = \vec{b} - \vec{b} \tag{7}$$

$$\implies (u_1 - v_1)\vec{c}_1 + (u_2 - v_2)\vec{c}_2 + \dots + (u_n - v_n)\vec{c}_n = \vec{0} \tag{8}$$

Let $\alpha_1 = u_1 - v_1$, ..., and $\alpha_n = u_n - v_n$, i.e. $\vec{\alpha} = \vec{u} - \vec{v}$. Here, at least one α_i is not equal to zero since $\vec{u} \neq \vec{v}$. In other words, we find that because $\vec{u} \neq \vec{v}$, there exists a $(u_i - v_i)\vec{c}_i = \vec{0}$. Thus, $\mathbf{A}$ must have linearly dependent columns. Hence the final mathematical expression from part (b) is satisfied, i.e. the proof is complete!

7. Span Proofs

Learning Objectives: *This is an opportunity to practice your proof development skills. Keep in mind the method and tips given in Note 4!*

(a) Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, show the following:

$$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \dots, \vec{v}_n\}$$

In other words, we can replace one vector with the sum of itself and another vector and not change their span.

In order to show this, you have to prove the two following statements:

- If a vector $\vec{q}$ belongs in $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, then it must also belong in $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \dots, \vec{v}_n\}$.
- If a vector $\vec{r}$ belongs in $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \dots, \vec{v}_n\}$, then it must also belong in $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$.

In summary, you have to prove the problem statement from both directions.

Solution:
Suppose $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. For some scalars a_i:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = a_1(\vec{v}_1 + \vec{v}_2) + (-a_1 + a_2)\vec{v}_2 + \cdots + a_n\vec{v}_n$$

We can change the scalar values to adjust for the combined vectors. Thus, we have shown that $\vec{q} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \dots, \vec{v}_n\}$.
Now, we must show the other direction. Suppose we have some arbitrary $\vec{r} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \dots, \vec{v}_n\}$. For some scalars b_i:

$$\vec{r} = b_1(\vec{v}_1 + \vec{v}_2) + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n = b_1\vec{v}_1 + (b_1 + b_2)\vec{v}_2 + \cdots + b_n\vec{v}_n.$$

Thus, we have shown that $\vec{r} \in \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. Combining this with the earlier result, the spans are the same.

(b) Consider the span of the set $(\vec{v}_1,\ldots,\vec{v}_n,\vec{u})$. Suppose $\vec{u}$ is in the span of $\{\vec{v}_1,\ldots,\vec{v}_n\}$. Then, show that any vector $\vec{r}$ in $span\{\vec{v}_1,\ldots,\vec{v}_n,\vec{u}\}$ is in $span\{\vec{v}_1,\ldots,\vec{v}_n\}$.

Solution: From the first sentence of the question, by definition of span, we know that any vector $\vec{r}$ in $span\{\vec{v}_1,\ldots,\vec{v}_n,\vec{u}\}$ can be written $\vec{r} = k\vec{u} + a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_n\vec{v}_n$. Using the summation symbol, we can also write $\vec{r} = k\vec{u} + \sum_{i=1}^{n} a_i\vec{v}_i$.

From the second sentence of the question, since $\vec{u}$ is in the span of $\{\vec{v}_1,\ldots,\vec{v}_n\}$, we can write $\vec{u} = b_1\vec{v}_1 + b_2\vec{v}_2 + \ldots b_n\vec{v}_n$ or $\sum_{i=1}^{n} b_i\vec{v}_i$. Now that we have an expression for $\vec{u}$, let's substitute it into the previous expression.

$$\vec{r} = k\vec{u} + \sum_{i=1}^{n} a_i\vec{v}_i$$

$$\vec{r} = k(\sum_{i=1}^{n} b_i\vec{v}_i) + \sum_{i=1}^{n} a_i\vec{v}_i$$

Finally, gathering up coefficients, we get:

$$\vec{r} = \sum_{i=1}^{n} (k * b_i + a_i)\vec{v}_i,$$

so this arbitrary vector $\vec{r}$ is also in $span\{\vec{v}_1,\ldots,\vec{v}_n)\}$.

Intuitively, $\vec{u}$ is redundant, so we can safely remove it without reducing our span.

8. Filtering Out The Troll

Learning Objectives: *The goal of this problem is to explore the problem of sound reconstruction by solving a system of linear equations.*

You were attending the 16A lecture the day before the first exam, and decided to record it using two directional microphones (one microphone receives sound from the 'x' direction and the other from the 'y' direction). However, someone in the audience was trolling around loudly, adding interference to the recording! The troll's interference dominates both of your microphones' recordings, so you cannot hear the recorded speech. Fortunately, since your recording device contained two microphones, you can combine the two individual microphone recordings to remove the troll's interference.

The diagram shown in Figure 4 shows the locations of the speaker, the troll, and you and your two microphones (at the origin).

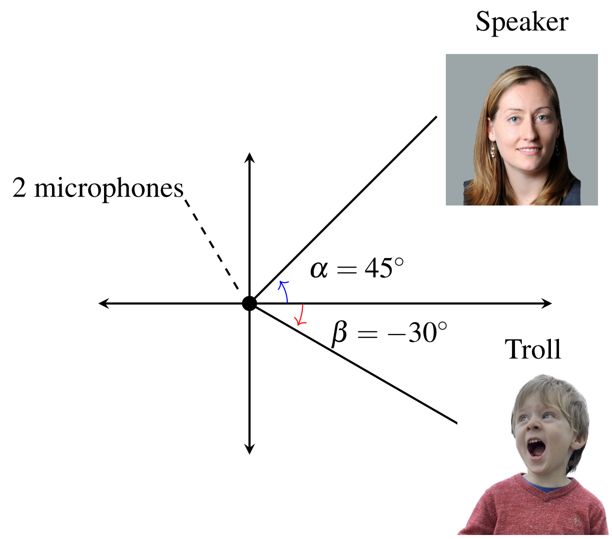


Figure 4: Locations of the speaker and the troll.

Since the microphones are directional, the strength of the recorded signal depends on the angle from which the sound arrives. Suppose that the sound arrives from an angle θ relative to the x-axis (in our case, these angles are 45°and $-30°$, labeled as α and β, respectively). The first microphone scales the signal by $\cos(\theta)$, while the second microphone scales the signal by $\sin(\theta)$. Each microphone records the weighted sum (or linear combination) of all received signals.

The speech signal can be represented as a vector, $\vec{s}$, and the troll's interference as vector $\vec{r}$, with each entry representing an audio sample at a given time. The recordings of the two microphones are given by $\vec{m}_1$ and $\vec{m}_2$:

$$\vec{m}_1 = \cos(\alpha)\cdot\vec{s} + \cos(\beta)\cdot\vec{r} \tag{9}$$

$$\vec{m}_2 = \sin(\alpha)\cdot\vec{s} + \sin(\beta)\cdot\vec{r} \tag{10}$$

where α and β are the angles at which the professor and the troll respectively are located with respect to the x-axis, and variables $\vec{s}$ and $\vec{r}$ are the audio signals produced by the professor and the troll respectively.

(a) Plug in $\alpha = 45° = \frac{\pi}{4}$ and $\beta = -30° = -\frac{\pi}{6}$ to Equations 9 and 10 to write the recordings of the two microphones $\vec{m}_1$ and $\vec{m}_2$ as a linear combination (i.e. a weighted sum) of $\vec{s}$ and $\vec{r}$.

Solution:

$$\begin{aligned}
\vec{m}_1 &= \cos\left(\frac{\pi}{4}\right)\cdot\vec{s} + \cos\left(-\frac{\pi}{6}\right)\cdot\vec{r} \\
&= \frac{1}{\sqrt{2}}\cdot\vec{s} + \frac{\sqrt{3}}{2}\cdot\vec{r} \\
\vec{m}_2 &= \sin\left(\frac{\pi}{4}\right)\cdot\vec{s} + \sin\left(-\frac{\pi}{6}\right)\cdot\vec{r} \\
&= \frac{1}{\sqrt{2}}\cdot\vec{s} - \frac{1}{2}\cdot\vec{r}
\end{aligned}$$

(b) Solve the system from part (a) using any convenient method you prefer to recover the important speech $\vec{s}$ as a weighted combination of $\vec{m}_1$ and $\vec{m}_2$. In other words, write $\vec{s} = c\cdot\vec{m}_1 + k\cdot\vec{m}_2$ (where c and k are scalars). What are the values of c and k?

Solution:

Solving the system of linear equations yields

$$\vec{s} = \frac{\sqrt{2}}{1+\sqrt{3}} \cdot \left(\vec{m}_1 + \sqrt{3}\vec{m}_2\right).$$

Therefore, the values are $c = \frac{\sqrt{2}}{1+\sqrt{3}}$ and $k = \frac{\sqrt{6}}{1+\sqrt{3}}$.

It is fine if you solved this either using IPython or by hand using any valid technique. The easiest approach is to subtract either of the two equations from the other and immediately see that $\vec{r} = \frac{2}{\sqrt{3}+1}(\vec{m}_1 - \vec{m}_2)$. Substituting b back into the second equation and multiplying through by $\sqrt{2}$ gives that $\vec{s} = \sqrt{2}(\vec{m}_2 + \frac{1}{\sqrt{3}+1}(\vec{m}_1 - \vec{m}_2))$, which simplifies to the expression given above.

Notice that subtracting one equation from the other is natural given the symmetry of the microphone patterns and the fact that the patterns intersect at the 45 degree line where the important speech is happening, and the fact that $\sin(45°) = \cos(45°)$. So we know that the result of subtracting one microphone recording from the other results in only the troll's contribution. Once we have the troll contribution, we can remove it and obtain the professor's sole content.

(c) Partial IPython code can be found in `prob2.ipynb`, which you can access through the Datahub link associated with this assignment on the course website. Complete the code to get the signal of the important speech. Write out what the speaker says. (Optional: Where is the speech taken from?)

Note: You may have noticed that the recordings of the two microphones sound remarkably similar. This means that you could recover the real speech from two "trolled" recordings that sound almost identical! Leave out the fact that the recordings are actually different, and have some fun with your friends who aren't lucky enough to be taking EECS16A.

Solution:

The solution code can be found in `sol2.ipynb`. The speaker (Professor Waller) is giving a lecture on the nebulous *turboencabulator* (you can find the full lecture link here: `https://youtu.be/Ac7G7xOG2Ag`), and the audio is recorded while her son was being particularly boisterous singing a song.

The idea of using multiple microphones to isolate speech is interesting and is increasingly used in practice. Furthermore, similar techniques are used in wireless communication both by cellular systems like LTE and increasingly by WiFi hotspots. (This is why they often have multiple antennas).

9. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.