

Welcome to EECS 16A!

Designing Information Devices and Systems I



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Fa 2022

Lecture 3A

Matrix xForms



Announcements

- Last time:
 - Proofs
 - Span
- Today:
 - Linear (in)dependance
 - Matrix Transformations

Span / Column Space / Range

- Span of the columns of A is the set of all vectors \vec{b} such that $A\vec{x} = \vec{b}$ has a solution
 - i.e. the set of all vectors that can be reached by all possible linear combinations of the columns of A
- Definition:
If $\exists \vec{x}$ s.t. $A\vec{x} = \vec{b}$ then $\vec{b} \in \text{span}\{\text{cols}(A)\}$

Proof: Span

Theorem: $\text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} = \mathbb{R}^2$

Know:

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \Rightarrow \left\{ \vec{v} \mid \vec{v} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \alpha, \beta \in \mathbb{R} \right\} = \mathbb{S}$$

Concept: pick some specific $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in R^2$, and show that it belongs to \mathbb{S}

Need to solve:

Need to show:

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} = \mathbb{R}^2$$

Proof: Span

Theorem: $\text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} = \mathbb{R}^2$

Know:

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \Rightarrow \left\{ \vec{v} \mid \vec{v} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \alpha, \beta \in \mathbb{R} \right\} = \mathbb{S}$$

Need to show:

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} = \mathbb{R}^2$$

Concept: pick some specific $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$, and show that it belongs to \mathbb{S}

Need to solve:

$$\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Diagram illustrating the components:
- The first column of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is labeled "Known and $\in \mathbb{R}^2$ ".
- The scalar α is labeled "Unknown".
- The scalar β is labeled "Unknown".

Proof: Span

Need to solve:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Gaussian Elimination:

Proof: Span

Need to solve:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Gaussian Elimination:

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & -1 & b_2 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & -2 & b_2 - b_1 \end{array} \right]$$

$$\frac{b_1 + b_2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{b_1 - b_2}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Constructive proof

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{b_1 + b_2}{2} \\ 0 & 1 & \frac{b_1 - b_2}{2} \end{array} \right] \Rightarrow \alpha = \frac{b_1 + b_2}{2}, \beta = \frac{b_1 - b_2}{2},$$

Every $\vec{b} \in \mathbb{R}^2$ can be written as linear combinations!
So also, $\vec{b} \in \mathbb{S}$

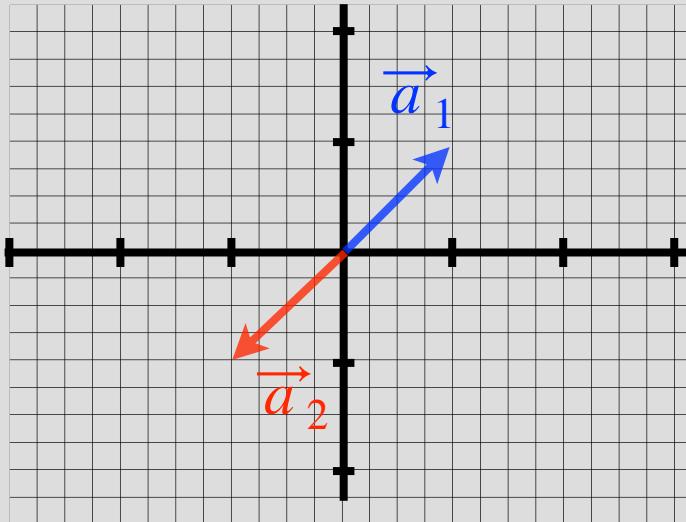


Linear Dependence

Recall:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$\downarrow \vec{a}_1$ $\downarrow \vec{a}_2$



\vec{a}_1 and \vec{a}_2 are linearly dependent

$$\vec{a}_1 = -\vec{a}_2$$

Department of
Redundancy
Department

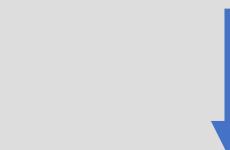
Linear Dependence

- Definition 1:

A set of vectors $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_N\}$ are linearly dependent if $\exists \{\alpha_1, \alpha_2, \dots, \alpha_N\} \in \mathbb{R}$, such that:

$$\vec{a}_i = \sum_{j \neq i} \alpha_j \vec{a}_j \quad 1 \leq i, j \leq M$$

For example: if $\vec{a}_2 = 3\vec{a}_1 - 2\vec{a}_5 + 6\vec{a}_7$



\vec{a}_i in the span of all \vec{a}_j s

Linear Dependence

Are these linearly dependent?

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Need to solve:

Linear Dependence

Are these linearly dependent?

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

Are linearly dependent

Need to solve:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

but we showed that....

$$\frac{b_1 + b_2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{b_1 - b_2}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

So....

$$\frac{3+1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3-1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Linear dependence / independence

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} \Rightarrow 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 0$$

- Definition 2:
A set of vectors $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_N\}$ are linearly dependent if
 $\exists \{\alpha_1, \alpha_2, \dots, \alpha_N\} \in \mathbb{R}$, such that:
- $$\sum_{i=1}^N \alpha_i \vec{a}_i = 0$$

As long as not all $\alpha_i = 0$
- Definition:
A set of vectors $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_N\}$ are linearly independent if they are not dependent

Linear dependence / independence

Are these linearly dependent?

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} \pi \\ \sqrt{2} \end{bmatrix} \right\}$$

span = \mathbb{R}^2

linearly dependent!

$$\in \mathbb{R}^2$$

Solutions for linear equations

- Theorem: if the columns of the matrix A are linearly dependent then, $A \vec{x} = \vec{b}$ does not have a unique solution

PROOF Consider the counter-example $\mathbf{S} \triangleq \{\circ, \bullet\}$, $\tau \triangleq \{\langle \bullet, \bullet \rangle, \langle \bullet, \circ \rangle, \langle \circ, \circ \rangle\}$ so that $\mathcal{M}_\tau = \{\langle i, \lambda \ell \cdot \bullet \rangle, \langle j, \lambda \ell \cdot \circ \rangle, \langle k, \lambda \ell \cdot (\ell < m ? \bullet \& \circ) \rangle\}$. We let $\mathcal{X} \triangleq \{\langle i, \sigma \rangle \mid \forall j < i : \sigma_j = \bullet\}$ so that $\neg FD(\mathcal{X})$. We have $\mathcal{M}_{\tau \downarrow \bullet} = \{\langle i, \lambda \ell \cdot \bullet \rangle, \langle k, \lambda \ell \cdot (\ell < m ? \bullet \& \circ) \rangle \mid k < m\}$, $\mathcal{M}_{\tau \downarrow \circ} = \{\langle j, \lambda \ell \cdot \circ \rangle, \langle k, \lambda \ell \cdot (\ell < m ? \bullet \& \circ) \rangle \mid k \geq m\}$ and $\oplus\{\mathcal{X}\} = \{\langle i, \sigma \rangle \mid \forall j \leq i : \sigma_j = \bullet\}$. We have $\alpha_{\mathcal{M}_\tau}^*(\oplus\{\mathcal{X}\}) = \{s \mid \mathcal{M}_{\tau \downarrow s} \subseteq \oplus\{\mathcal{X}\}\} = \{\bullet\}$ whereas $\widetilde{pre}[\tau](\alpha_{\mathcal{M}_\tau}^*(\mathcal{X})) = \widetilde{pre}[\tau](\{s \mid \mathcal{M}_{\tau \downarrow s} \subseteq \mathcal{X}\}) = \widetilde{pre}[\tau](\{\bullet\}) = \{s \mid \forall s' : t(s, s') \Rightarrow s' = \bullet\} = \emptyset$ since $t(s, \bullet)$ implies $s = \bullet$ and $t(\bullet, \circ)$ holds. ■

Solutions for linear equations

- Theorem: if the columns of the matrix A are linearly dependent then, $A \vec{x} = \vec{b}$ does not have a unique solution

Proof for $A \in \mathbb{R}^{3 \times 3}$

know: columns are linearly ~~independent~~

show: more than 1 solution

Concept: pick some specific solution \vec{x}^* , and show that there's another one

Let: $A \vec{x}^* = \vec{b}$ and $A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3]$

From linear dependence Def 2:

$$\alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \alpha_3 \vec{a}_3 = 0$$

Solutions for linear equations

- Theorem: if the columns of the matrix A are linearly dependent then, $A \vec{x} = \vec{b}$ does not have a unique solution

Proof for $A \in \mathbb{R}^{3 \times 3}$

know: columns are linearly dependent

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Concept: pick some specific solution \vec{x}^* , and show that there's another one

Let: $A \vec{x}^* = \vec{b}$ and $A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3]$

From linear dependence Def 2:

$$\alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \alpha_3 \vec{a}_3 = 0 \rightarrow \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0 \Rightarrow A \vec{\alpha} = 0$$

Set $\vec{x}^\dagger = \vec{x}^* + \vec{\alpha}$

$$\Rightarrow A \vec{x}^\dagger = A(\vec{x}^* + \vec{\alpha}) = A \vec{x}^* + A \vec{\alpha} = \vec{b} + 0 \quad \text{So } \vec{x}^\dagger \text{ is another solution!}$$

Matrix Transformations

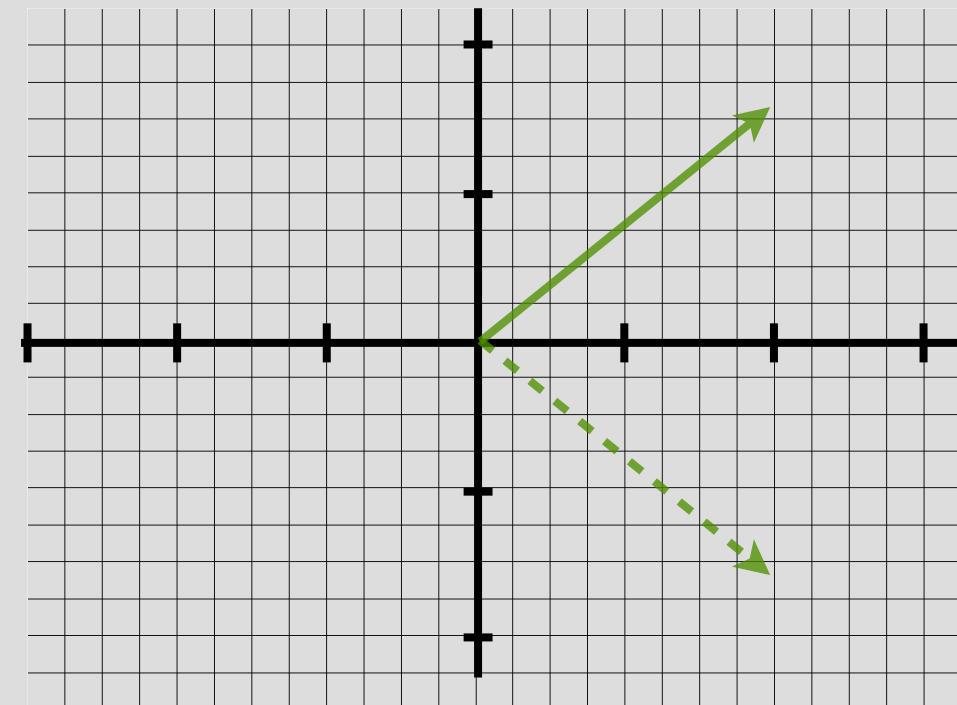
$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Matrices are operators that transform vectors

$$A \xrightarrow{\quad} \vec{x} = b$$

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$



Matrices are operators that transform vectors

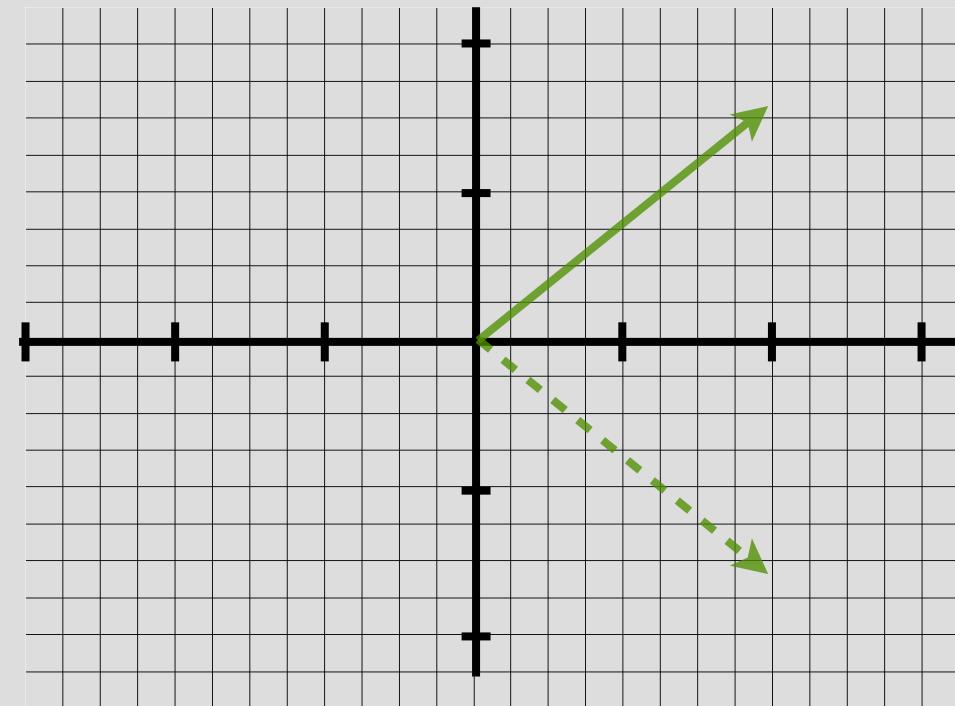
$$A \xrightarrow{\quad} \vec{x} = b$$

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$



https://www.youtube.com/watch?v=LhF_56SxrGk



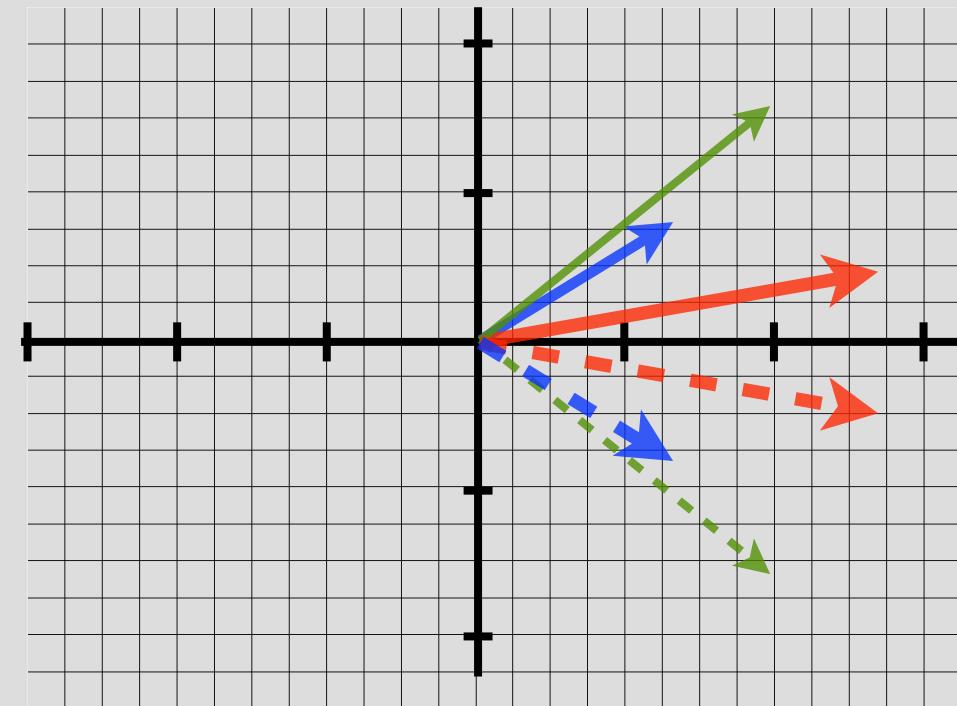
Matrices are operators that transform vectors

$$\overrightarrow{A\vec{x}} = \vec{b}$$

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

Reflection Matrix!



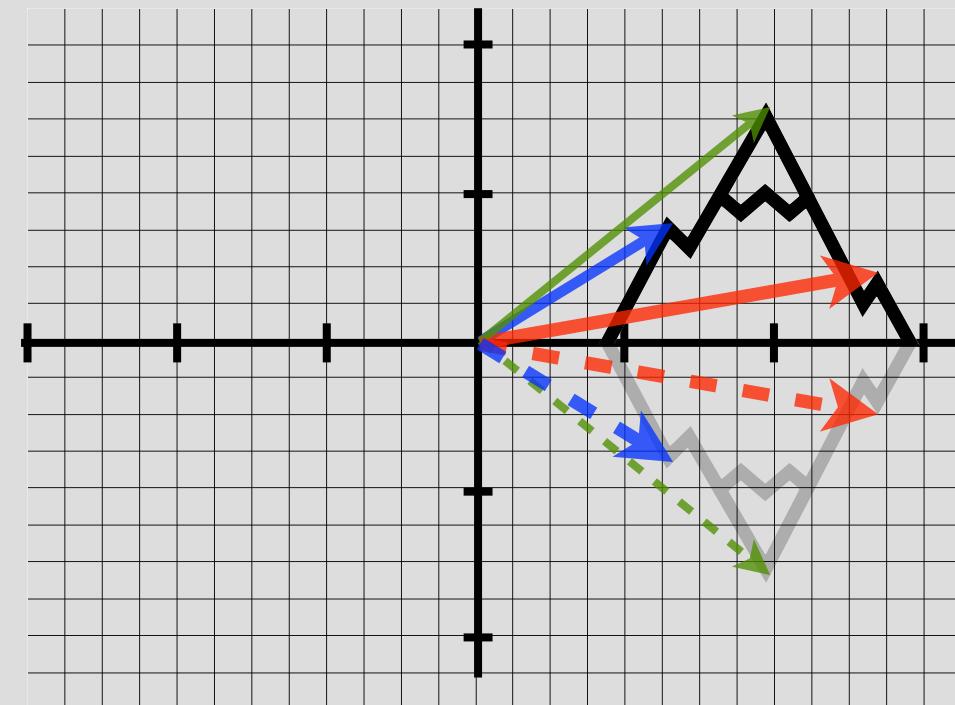
Matrices are operators that transform vectors

$$A \vec{x} = \vec{b}$$

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

Reflection Matrix!



Matrices are operators that transform vectors

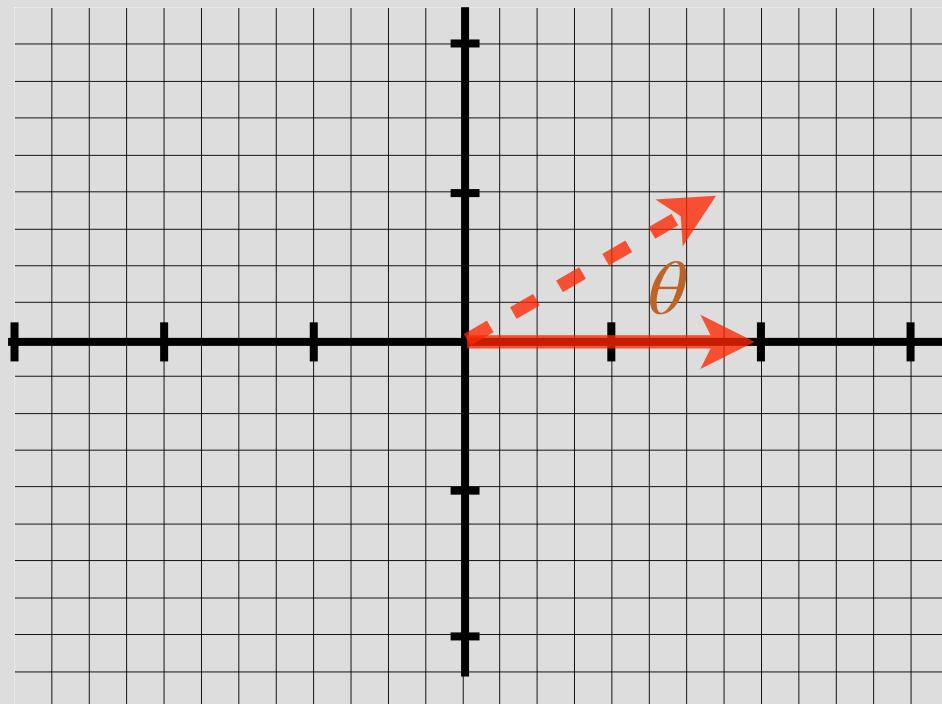
Example 2:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta)x_1 - \sin(\theta)x_2 \\ \sin(\theta)x_1 + \cos(\theta)x_2 \end{bmatrix}$$

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

Rotation Matrix!

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$



Linear Transformation of vectors

f : is a linear transformation if:

$$f(\alpha \vec{x}) = \alpha f(\vec{x}) \quad \alpha \in \mathbb{R}$$

$$f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$$

Claim: Matrix-vector multiplications satisfy linear transformation

$$A \cdot (\alpha \vec{x}) = \alpha A \vec{x}$$

Proof via explicitly writing the elements

$$A \cdot (\vec{x} + \vec{y}) = A \vec{x} + A \vec{y}$$

Vectors as states, Matrices as state transition

Vectors can represent states of a system

Example: The state of a car at time = t

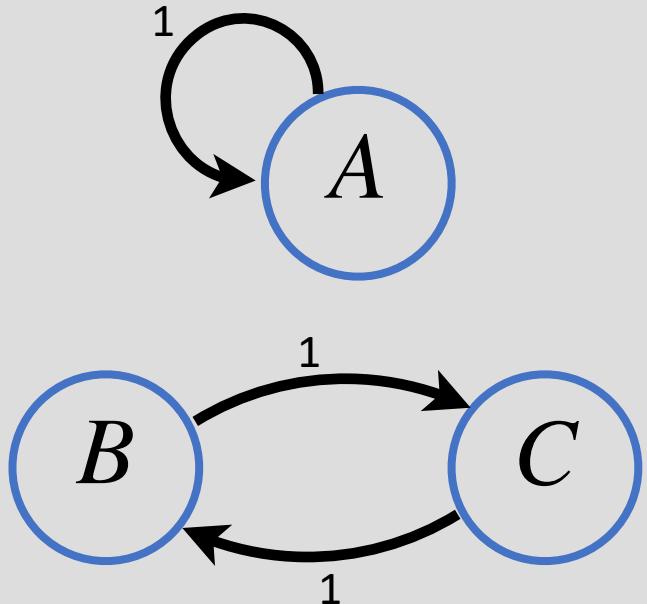
$$\vec{s}(t) = \begin{bmatrix} x(t) \\ y(t) \\ v(t) \\ \theta(t) \end{bmatrix} \quad \left. \begin{array}{l} \text{position} \\ \text{velocity} \end{array} \right\}$$

Q: Is that enough?

A: need orientation or $v_x(t), v_y(t)$

Graph Transition Matrices

Example: Reservoirs and Pumps



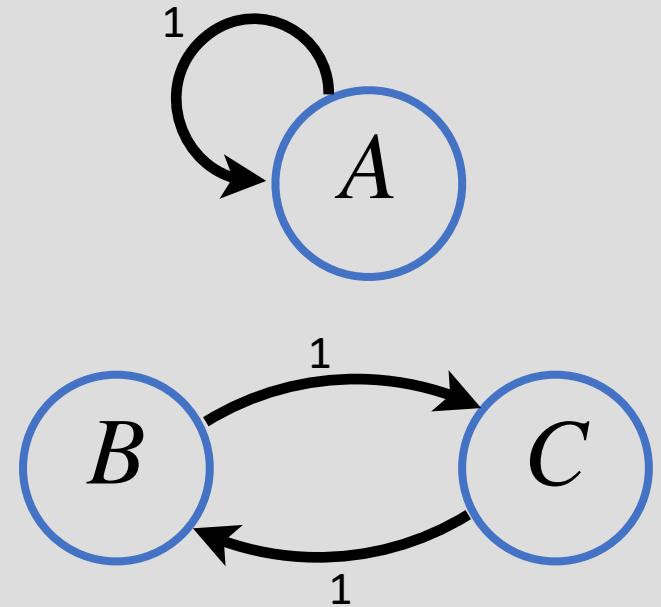
Q: What is the state?

A: Water in each reservoir

$$\vec{x}(t) = \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

Pumps move water...
What would the state be tomorrow?

State Transition Matrices



State Transition Matrices

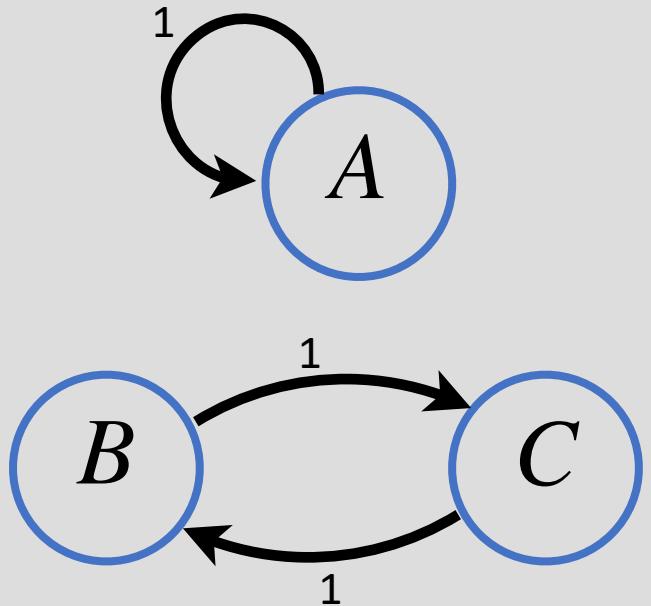
$$x_A(t+1) = x_A(t)$$

$$x_B(t+1) = x_C(t)$$

$$x_C(t+1) = x_B(t)$$

Write as a matrix-vector multiplication:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \quad \quad \quad \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



State Transition Matrices

$$x_A(t+1) = x_A(t)$$

$$x_B(t+1) = x_C(t)$$

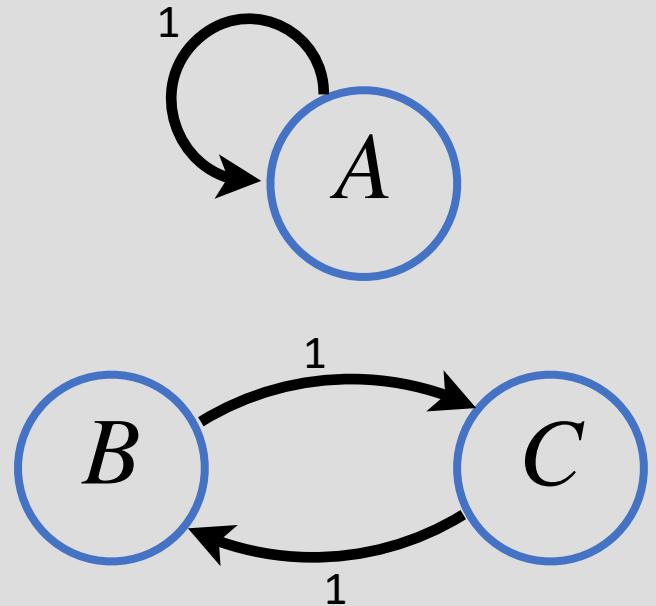
$$x_C(t+1) = x_B(t)$$

Write as a matrix-vector multiplication:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix} \quad \text{or } \vec{x}(t+1) = Q \vec{x}(t)$$

What is the state after 2 times?

$$\vec{x}(t+2) = Q \vec{x}(t+1) = QQ \vec{x}(t) = Q^2 \vec{x}(t)$$

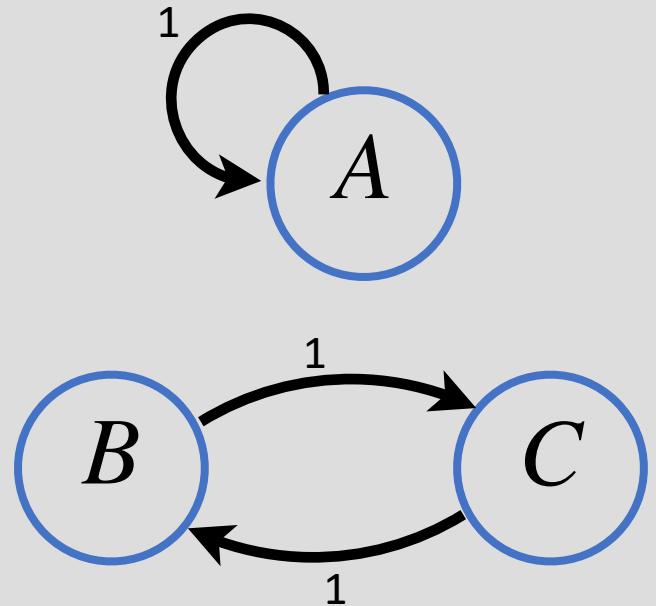


State Transition Matrices

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

What is the state after at t=1, 2?

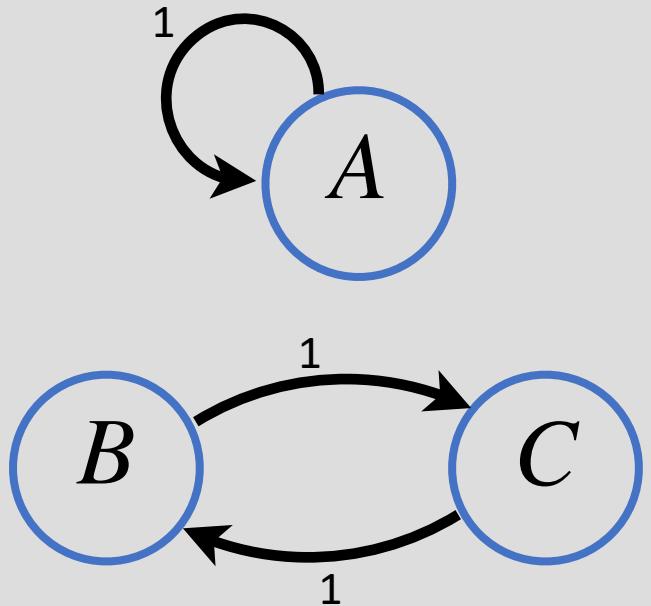


State Transition Matrices

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

What is the state after at t=1, 2?

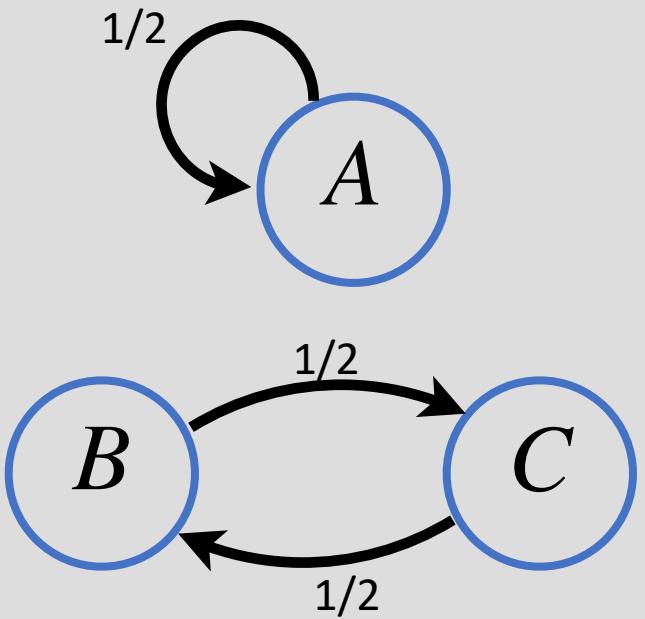


① $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

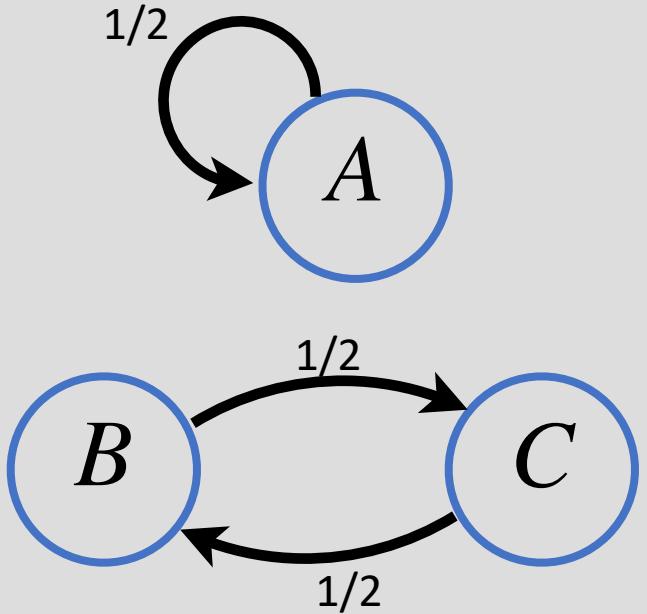
② $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$Q \cdot Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

State Transition Matrices



State Transition Matrices



$$\mathcal{X}[t+1] = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \mathcal{X}(t)$$

Non-conservative!

$$Q^2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

Q) What will happen if we keep going?

A) Numbers will diminish to zero

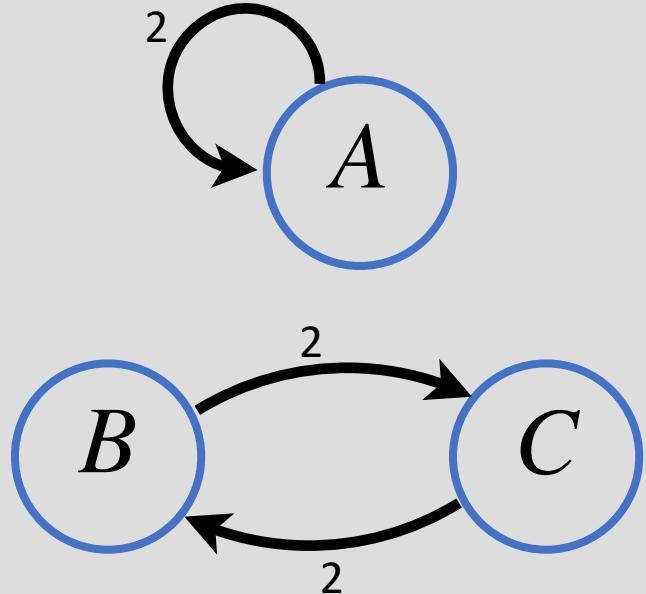
Google

DEAD SEA SOUTH
1984





State Transition Matrices



$$x(t+1) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} x(t)$$

$$\tilde{x} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

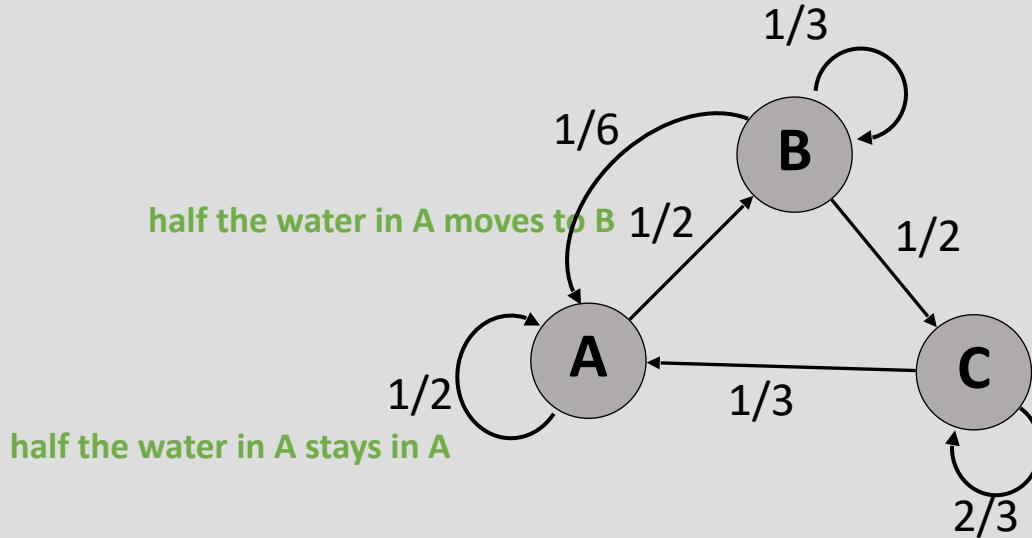
Q) What will happen if we keep going?

A) Numbers will explode to infinity



Graph Representation

Ex: Reservoirs and Pumps



Nodes

I have 3 reservoirs: A,B,C
and I want to keep track of how
much water is in each

When I turn on some pumps, water
moves between the reservoirs.

Where the water moves and what
fraction is represented by arrows.
Edge weights

Edges

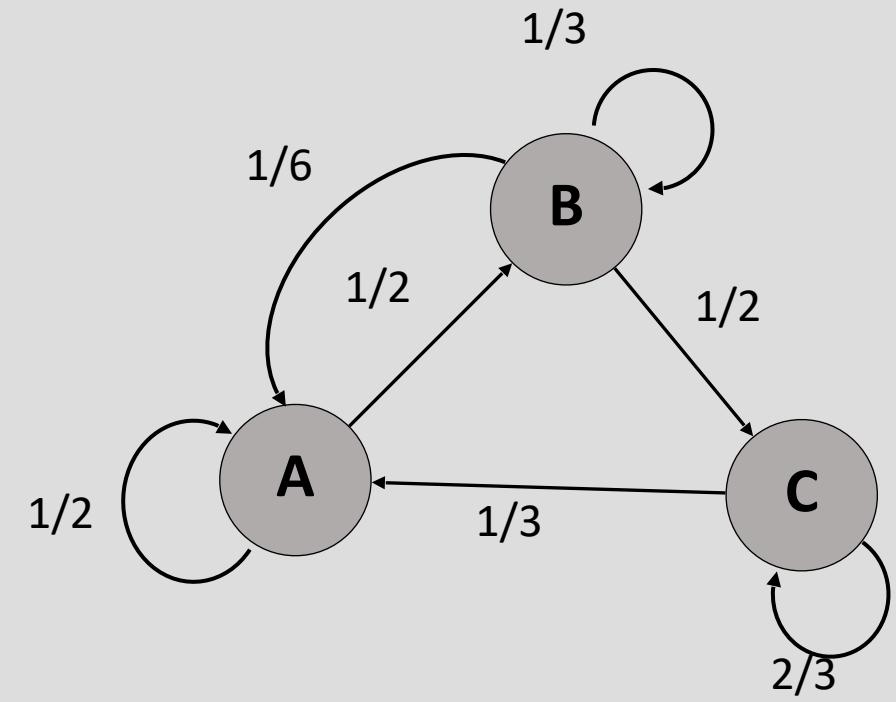
“directed” graph because
arrows have a direction

Where does the rest of the water in A go? Need to label that too...

Can you tell me how much water in each after pumps start? Need to know initial amounts

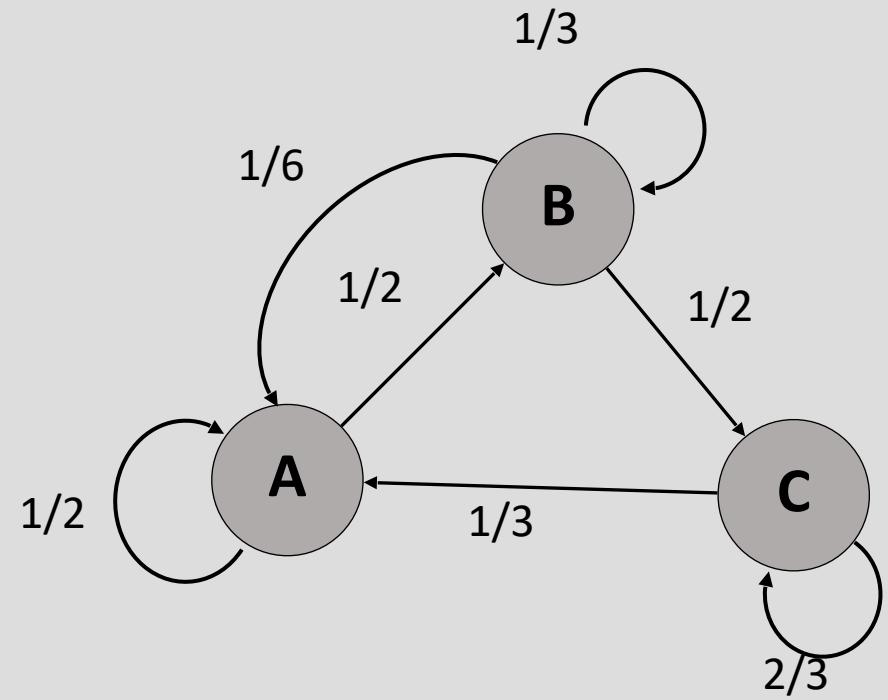
Exercise:

$$\begin{bmatrix} \lambda_{C_A}(t+1) \\ \lambda_{C_B}(t+1) \\ \lambda_{C_C}(t+1) \end{bmatrix} = \begin{bmatrix} A \rightarrow A & B \rightarrow A & C \rightarrow A \\ A \rightarrow B & B \rightarrow B & C \rightarrow B \\ A \rightarrow C & B \rightarrow C & C \rightarrow C \end{bmatrix} \begin{bmatrix} \lambda_A(t) \\ \lambda_B(t) \\ \lambda_C(t) \end{bmatrix}$$



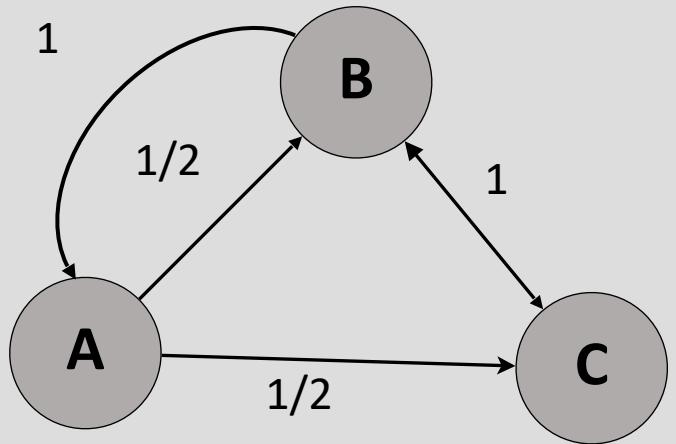
Exercise:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



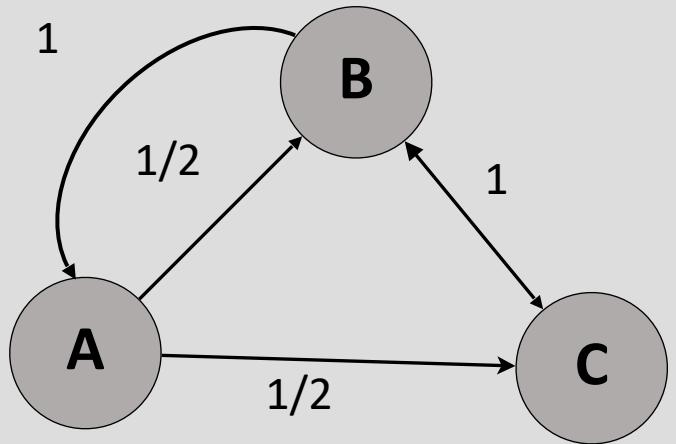
Example 2:

$$\begin{bmatrix} \chi_A(t+1) \\ \chi_B(t+1) \\ \chi_C(t+1) \end{bmatrix} = \begin{bmatrix} A \rightarrow A & B \rightarrow A & C \rightarrow A \\ A \rightarrow B & B \rightarrow B & C \rightarrow B \\ A \rightarrow C & B \rightarrow C & C \rightarrow C \end{bmatrix} \begin{bmatrix} \chi_A(t) \\ \chi_B(t) \\ \chi_C(t) \end{bmatrix}$$



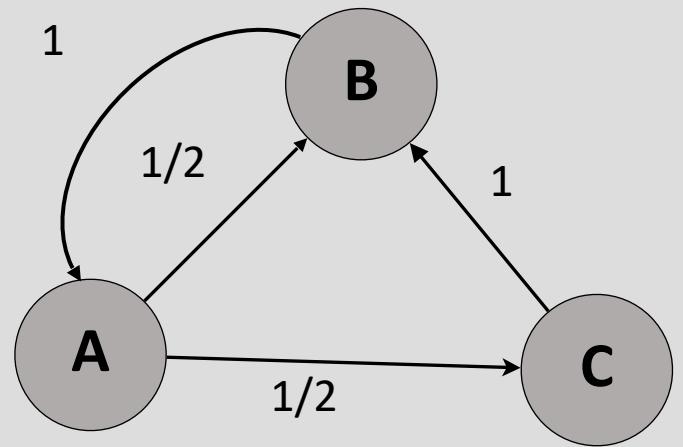
Example 2:

$$\begin{bmatrix} x_{c_A}(t+1) \\ x_{c_B}(t+1) \\ x_{c_C}(t+1) \end{bmatrix} = \begin{bmatrix} A \xrightarrow{1} A & B \xrightarrow{1} A & C \xrightarrow{1} A \\ A \xrightarrow{1/2} B & B \xrightarrow{1} B & C \xrightarrow{1} B \\ A \xrightarrow{1/2} C & B \xrightarrow{1} C & C \xrightarrow{1} C \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



What about the reverse?

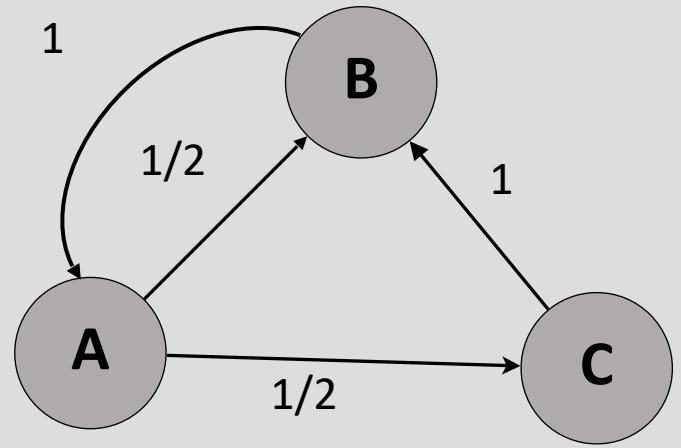
$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} A \rightarrow A & B \rightarrow A & C \rightarrow A \\ A \rightarrow B & B \rightarrow B & C \rightarrow B \\ A \rightarrow C & B \rightarrow C & C \rightarrow C \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



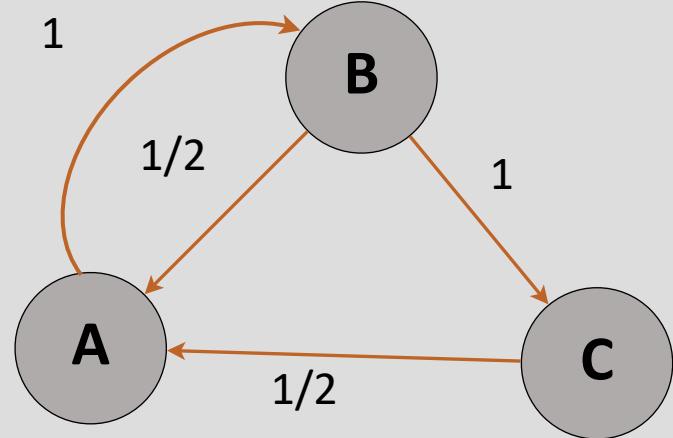
Q) Will flipping the arrows make us go back in time?

What about the reverse?

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} A \rightarrow A & B \rightarrow A & C \rightarrow A \\ A \rightarrow B & B \rightarrow B & C \rightarrow B \\ A \rightarrow C & B \rightarrow C & C \rightarrow C \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

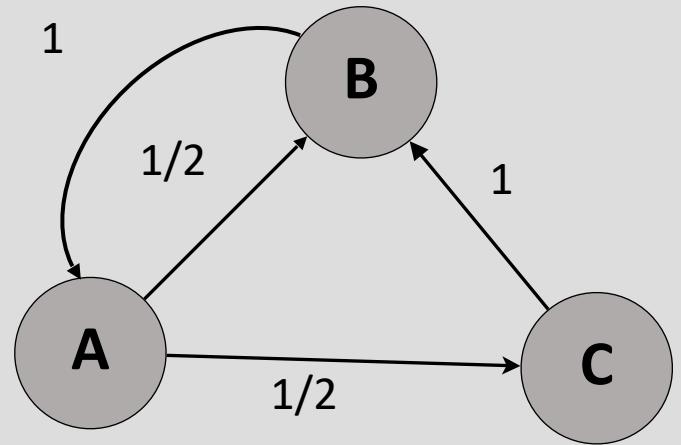


Q) Will flipping the arrows make us go back in time?



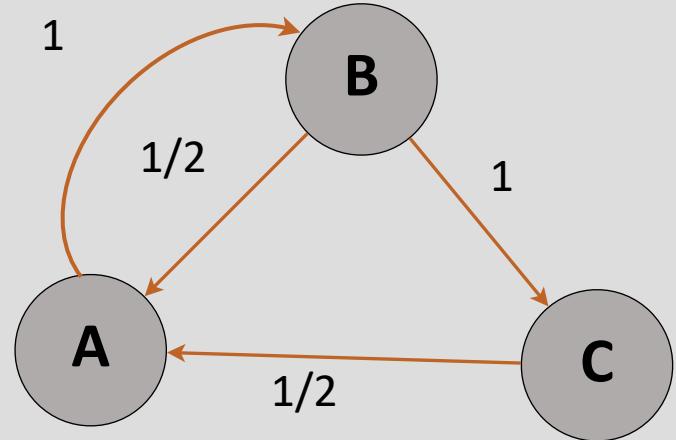
What about the reverse?

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} A \rightarrow A & B \rightarrow A & C \rightarrow A \\ A \rightarrow B & B \rightarrow B & C \rightarrow B \\ A \rightarrow C & B \rightarrow C & C \rightarrow C \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



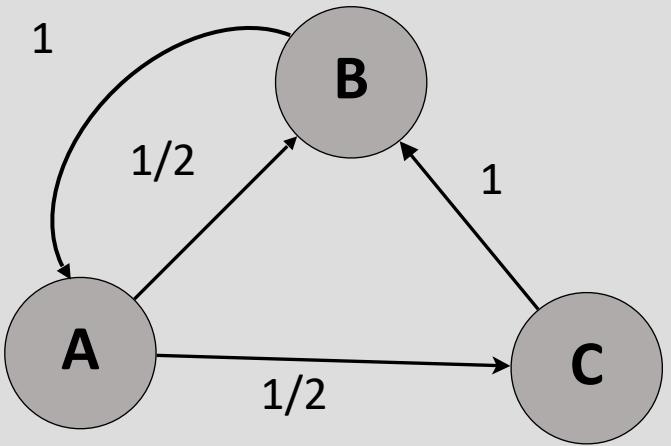
Q) Will flipping the arrows make us go back in time?

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



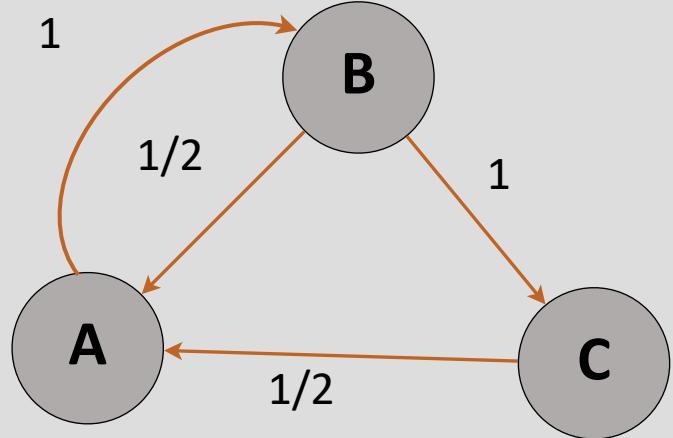
What about the reverse?

$$\begin{bmatrix} 6 & x_A(t+1) \\ 10 & x_B(t+1) \\ 2 & x_C(t+1) \end{bmatrix} = \begin{bmatrix} A \rightarrow A & B \rightarrow A & C \rightarrow A \\ A \rightarrow B & B \rightarrow B & C \rightarrow B \\ A \rightarrow C & B \rightarrow C & C \rightarrow C \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



Q) Will flipping the arrows make us go back in time?

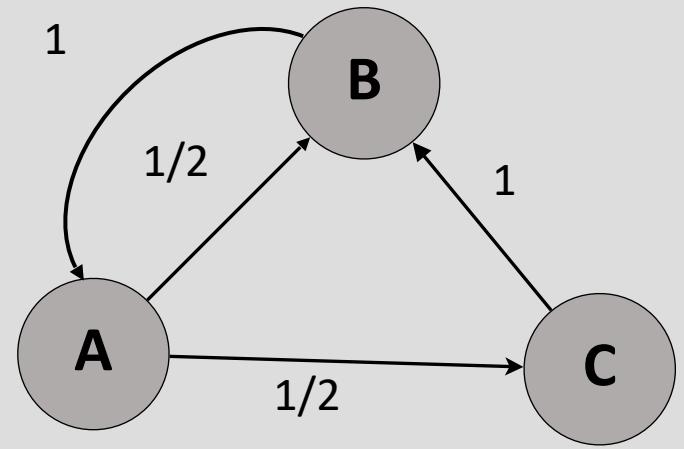
$$\begin{bmatrix}] \\] \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix}] \\] \end{bmatrix}$$



What about the reverse?

$$\begin{bmatrix} 6 \\ 10 \\ 2 \end{bmatrix} \begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} A \rightarrow A & B \rightarrow A & C \rightarrow A \\ A \rightarrow B & B \rightarrow B & C \rightarrow B \\ A \rightarrow C & B \rightarrow C & C \rightarrow C \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

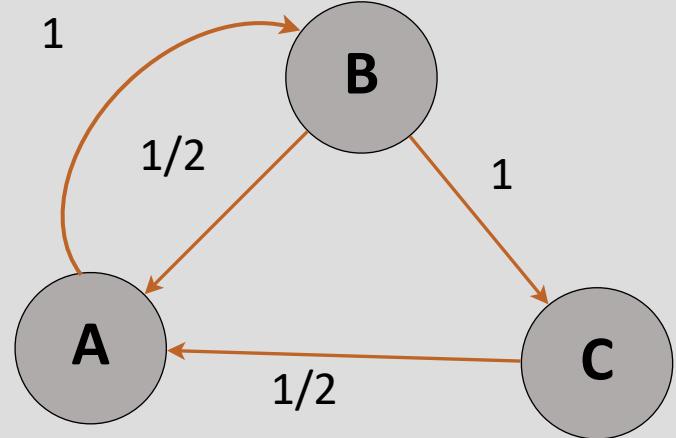
4
6
8



Q) Will flipping the arrows make us go back in time?

$$\begin{bmatrix} 7 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 10 \\ 2 \end{bmatrix}$$

A) In general, no!



Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N \times M}$ are a_{ij}

The elements of $A^T \in \mathbb{R}^{M \times N}$ are a_{ji}

Matrix transpose is not (generally) an inverse!

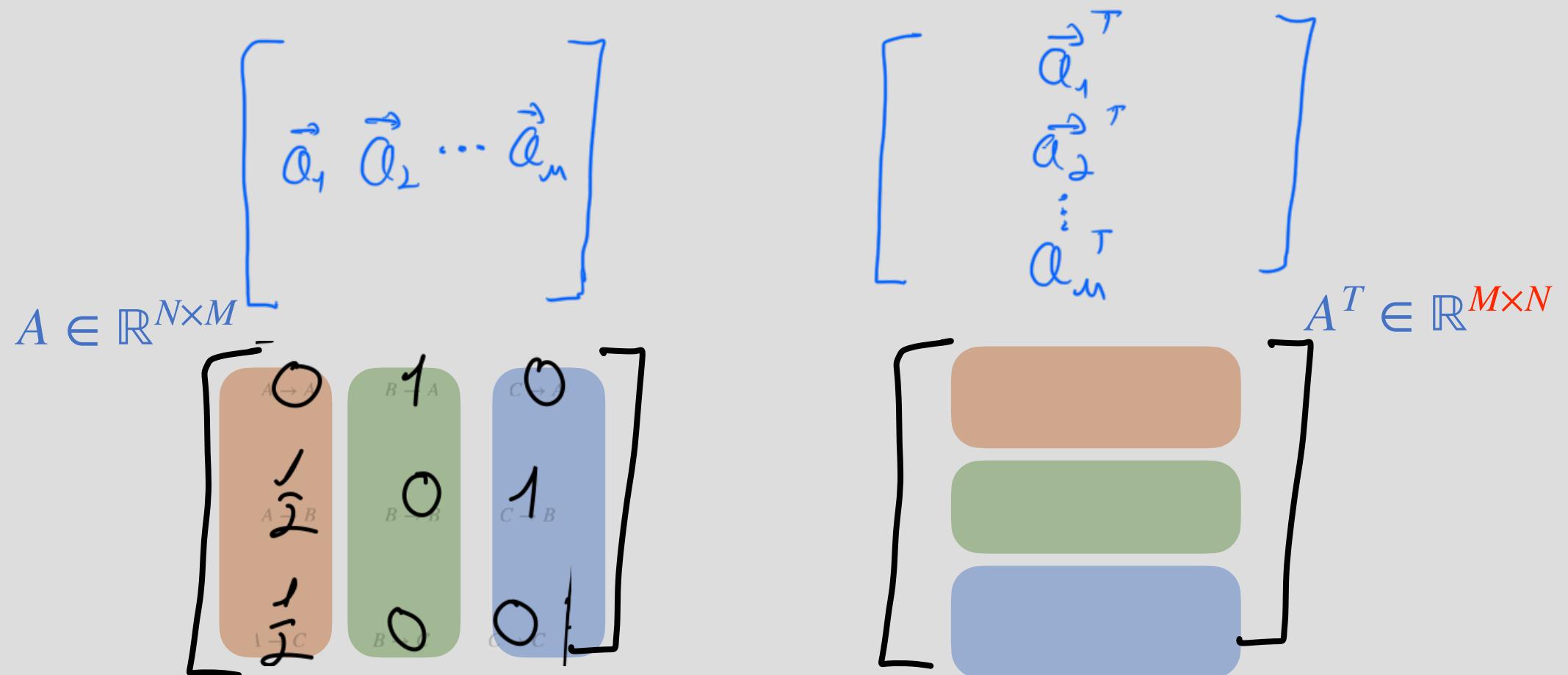
$$A \in \mathbb{R}^{N \times M} \quad \left[\begin{array}{c} \vec{a}_1 \vec{a}_2 \cdots \vec{a}_m \\ \vdots \end{array} \right] \quad A^T \in \mathbb{R}^{M \times N} \quad \left[\begin{array}{c} \vec{a}_1^T \\ \vec{a}_2^T \\ \vdots \\ \vec{a}_m^T \end{array} \right]$$

Matrix Transpose

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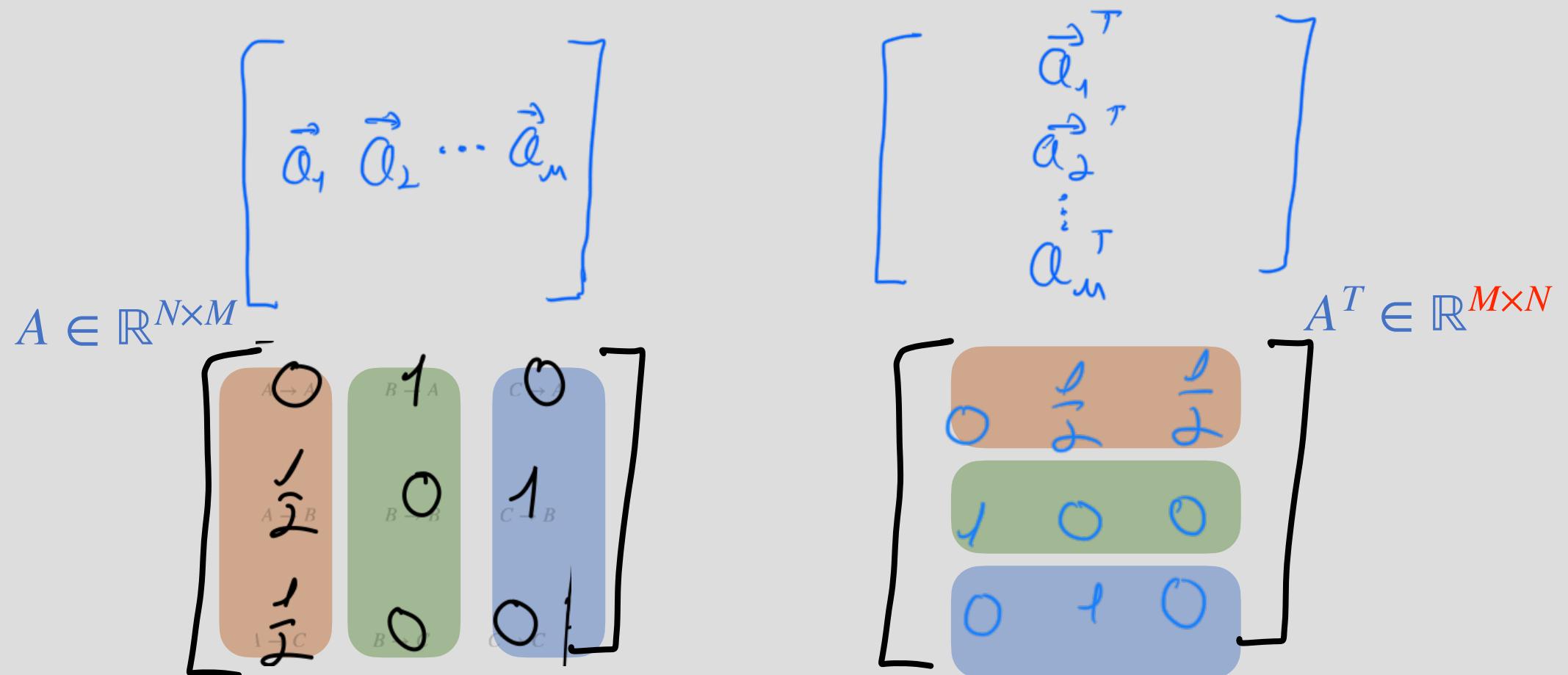


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Matrix Inversion

