Welcome to Module 3!

Key Question: How do we "learn" models from data, and make predictions?

Agenda

- Introduction to Global Positioning Systems
- Inner Products
- Norms
- Orthogonality
- Cauchy-Schwarz Inequality
- Trilateration

Global Positioning Systems (GPS)

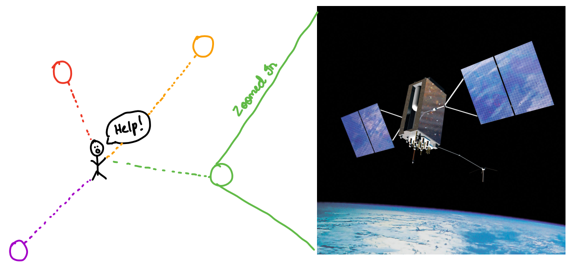
How does it work?

We're designers, so let's think about how we'd design this system.

Goal: Find out my location. Where am I?

Ideas:

- Find out how far away I am from known locations
  - But, how does knowing these distances tell me my position?
  - How can I measure those distances to known locations?
  - How many known locations do I need?
  - How accurate is my distance data? What if there is noise?

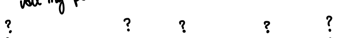


GPS uses satellites as the known locations with which we can find our position on Earth.

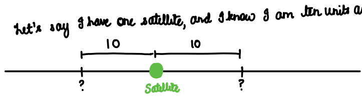
How do they work?

Let's consider the one-dimensional case.

All my possible locations lie on a straight line:

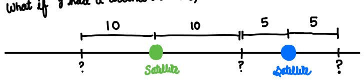


Let's say I have one satellite, and I know I am ten units away from it. Do I know where I am?

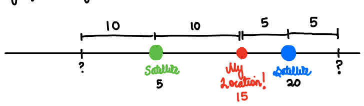


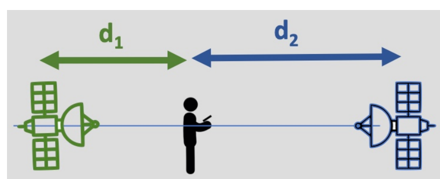
No! In which direction am I 10 units away?

What if I had a second satellite, that I knew I was 5 units away from? Now, do I know where I am?



Yes! As long as I know where each satellite is, which is which, and the satellites aren't in the same place!



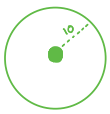


What happens in two dimensions?

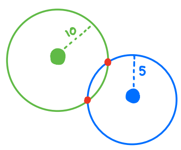
There is more ambiguity in where I could be, so I need more measurements. Thus, this problem becomes finding the intersections of circles.

Instead of two different directions from a satellite, I could be anywhere in the 360° around the satellite — a circle!

If I know I am 10 units away from satellite 1... Where am I?

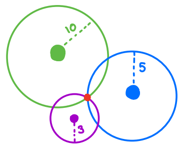


What if I also know I am 5 units away from satellite 2?

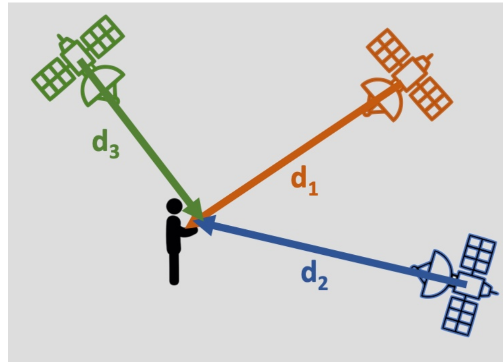


It is possible I could know my location if the circles intersected like , but I could also possibly have two solutions.

I need a third satellite! What if I know I am 3 units away from satellite 3?



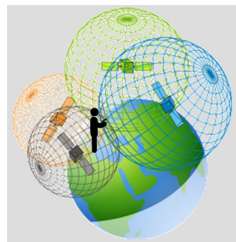
Now do I know where I am? Yes!



What about the three-dimensional case?

An intersection of spheres!

Do I need 3D GPS every day? Probably not, since I'll likely stay on the surface of the Earth.



The United States has, at the minimum, 24 satellites in orbit, each of which circles the Earth twice a day and are time synchronized. Currently, there are 31 satellites in orbit, for redundancy.

OK, even if we know where each satellite is, how do we know which distances correspond to which satellites? And how do we even know how far we are from each satellite? What if there are external factors that impact communication, like different behavior at different levels of the atmosphere? How do we deal with noise?

This week, we'll learn tools to help us answer these questions!

But first, some fundamental concepts that we'll use a lot, both in this class and in the future!

Inner Product

- Provides a measure of 'similarity' between vectors

Definition: For a real-valued vector space, $\mathbb{V}$, the mapping

$$\vec{u}, \vec{v} \in \mathbb{V} \longrightarrow \langle \vec{u}, \vec{v} \rangle \in \mathbb{R}$$

is called an inner product if it satisfies:

1. Symmetry: $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
2. Linearity: $\langle \vec{u} + \vec{w}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle$
   $\langle \alpha \vec{u}, \vec{v} \rangle = \alpha \langle \vec{u}, \vec{v} \rangle$ for $\alpha \in \mathbb{R}$
3. Positive Definiteness: $\langle \vec{v}, \vec{v} \rangle \geq 0$ ($\langle \vec{v}, \vec{v} \rangle = 0$ iff $\vec{v} = 0$)

In other words, an inner product is a function that takes in two vectors and outputs a real value.

In this class, we utilize the Euclidean inner product, also called a dot product.

For two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, the inner product, or dot product, is:

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} = \underset{1 \times n}{\vec{x}^T} \underset{n \times 1}{\vec{y}} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = \underset{\substack{1 \times 1 \\ \text{scalar}}}{\square}$$

Let's check the above properties for the Euclidean inner product.

1. Symmetry: $\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + \ldots + x_n y_n = y_1 x_1 + \ldots + y_n x_n = \langle \vec{y}, \vec{x} \rangle$. ✓
2. Linearity: $\langle \vec{x} + \vec{z}, \vec{y} \rangle = (\vec{x} + \vec{z})^T \vec{y} = \vec{x}^T \vec{y} + \vec{z}^T \vec{y} = \langle \vec{x}, \vec{y} \rangle + \langle \vec{z}, \vec{y} \rangle$.
   $\langle \alpha \vec{x}, \vec{y} \rangle = (\alpha \vec{x})^T \vec{y} = \alpha \vec{x}^T \vec{y} = \alpha \langle \vec{x}, \vec{y} \rangle$ ✓
3. Positive Definiteness: $\langle \vec{x}, \vec{x} \rangle = \vec{x}^T \vec{x} = \underbrace{x_1^2 + x_2^2 + \ldots + x_n^2}_{x_i^2 \geq 0}$ ✓

Example: Weighted Inner Product

Let $Q \in \mathbb{R}^{n\times n}$ symmetric matrix with positive eigenvalues. $(A = A^T)$

Define $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T Q \vec{y}$

$\vec{x}, \vec{y} \in \mathbb{R}^2$

$Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

Is this a valid inner product?

① Symmetry: $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T Q \vec{y} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & 3x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 y_1 + 3x_2 y_2$

$\langle \vec{y}, \vec{x} \rangle = \vec{y}^T Q \vec{x} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 & 3y_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = y_1 x_1 + 3y_2 x_2$

$\Big\} \; \langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle \; \checkmark$

② Linearity: $\langle \alpha\vec{x} + \vec{y}, \vec{z} \rangle = (\alpha\vec{x} + \vec{y})^T \vec{z} = \alpha\vec{x}^T\vec{z} + \vec{y}^T\vec{z} = \alpha\langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle \; \checkmark$

③ Positive Definiteness: $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T Q \vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 3x_2^2 \geq 0 \; \checkmark$

Yes!

Example: Calculate the following Euclidean inner products:

$\left\langle \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\rangle = 1\cdot 3 + 2\cdot 4 = 9$

$\langle [1\ 1\ 1 \ldots 1]^T, [x_0\ x_1 \ldots x_n]^T \rangle = x_0 + x_1 + \ldots + x_n$ Sum of Components

$\langle [\tfrac{1}{n}\ \tfrac{1}{n} \cdots \tfrac{1}{n}]^T, [x_0\ x_1 \ldots x_n]^T \rangle = \dfrac{x_0 + x_1 + \ldots + x_n}{n}$ Average

$\langle [x_0\ x_1 \ldots x_n]^T, [x_0\ x_1 \ldots x_n]^T \rangle = x_0^2 + x_1^2 + \ldots + x_n^2$ Sum of Squares

$\langle [0\ 1\ 0\ 0\ 1]^T, [x_0\ x_1\ x_2\ x_3\ x_4]^T \rangle = x_1 + x_4$ Selective Sum

Norm

- Provides a measure of 'length' of elements in the vector space.

Definition: A function mapping vectors to scalars

$\mathbb{R}^n \longrightarrow \|\ \|_i \longrightarrow \mathbb{R}$

is a norm if it satisfies:

① Non-negativity: $\|\vec{v}\| \geq 0 \quad (\|\vec{v}\| = 0 \text{ iff } \vec{v} = 0)$

② Homogeneity: $\|\alpha\vec{v}\| = |\alpha|\,\|\vec{v}\| \quad \alpha \in \mathbb{R}$

③ Triangle Inequality: $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

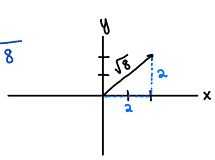
In this class, we utilize the Euclidean norm, or the 2-norm, defined as:

$\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \sqrt{\langle \vec{x}, \vec{x} \rangle}$ for $\vec{x} \in \mathbb{R}^n$

In general, the p-norm is defined as:

$$\|\vec{x}\|_p = \left(\sum_{i=1}^{n} (|x_i|)^p \right)^{\frac{1}{p}}$$

Example: $\left\| \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\| = \sqrt{\left\langle \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\rangle} = \sqrt{2^2 + 2^2} = \sqrt{8}$



Normalization and Unit Vectors

Let $\vec{x} \in \mathbb{R}^n$, $\vec{z} \in \mathbb{R}^n$

$$\vec{z} = \frac{\vec{x}}{\|\vec{x}\|}$$

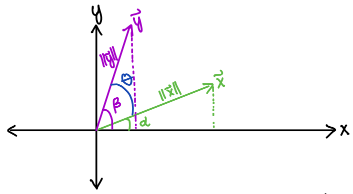
What is $\|\vec{z}\|$?

$$\|\vec{z}\| = \left\|\frac{\vec{x}}{\|\vec{x}\|}\right\| = \frac{1}{\|\vec{x}\|}\|\vec{x}\| = 1$$

$\vec{z}$ is a unit vector!

We can utilize the norm for a geometric interpretation of the inner product!

Let $\vec{x} = \begin{bmatrix} \|\vec{x}\| \cos\alpha \\ \|\vec{x}\| \sin\alpha \end{bmatrix}$, $\vec{y} = \begin{bmatrix} \|\vec{y}\| \cos\beta \\ \|\vec{y}\| \sin\beta \end{bmatrix}$



The Euclidean inner product of these vectors is:

$$\begin{aligned} \langle \vec{x}, \vec{y} \rangle &= \|\vec{x}\| \cos\alpha \, \|\vec{y}\| \cos\beta + \|\vec{x}\| \sin\alpha \, \|\vec{y}\| \sin\beta \\ &= \|\vec{x}\|\|\vec{y}\| \left(\cos\alpha\cos\beta + \sin\alpha\sin\beta\right) \\ &= \|\vec{x}\|\|\vec{y}\| \cos(\alpha-\beta) \\ &= \|\vec{x}\|\|\vec{y}\| \cos(\theta) \end{aligned}$$

where θ is the angle between $\vec{x}$ and $\vec{y}$.

Remember: $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

If $\vec{x}$ and $\vec{y}$ were pointing in the same direction, then $\theta = 0$, and thus $\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\|\|\vec{y}\|$, since $\cos(0) = 1$

The inner product of two vectors tells us how similar or different their directions are!

Orthogonality

Two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ are orthogonal if $\langle \vec{x}, \vec{y} \rangle = 0$

If $\vec{x} \perp \vec{y}$, $\theta = \frac{\pi}{2}$, so $\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\|\|\vec{y}\| \cos\frac{\pi}{2} = 0$.

Cauchy-Schwarz Inequality

$$|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\|\|\vec{y}\|$$

Proof:

$$\begin{aligned} |\langle \vec{x}, \vec{y} \rangle| &= \left| \|\vec{x}\|\|\vec{y}\| \cos\theta \right| \\ &= \|\vec{x}\|\|\vec{y}\| \, |\cos\theta| \\ &\leq \|\vec{x}\|\|\vec{y}\| \end{aligned}$$

since $-1 \leq \cos\theta \leq 1$, so $0 \leq |\cos\theta| \leq 1$

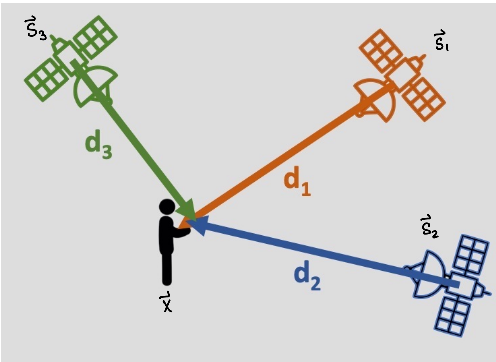
Applications: Triangle Inequality, Probability Theory, etc.

Now, we can use these fundamental ideas for our first tool: trilateration!

Trilateration

- Finding position from distances
  - Known: positions of satellites, distances to satellites
  - Unknown: your position

This is the last step in our GPS system!



Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{s}_1 = \begin{bmatrix} s_{1x} \\ s_{1y} \end{bmatrix}$, $\vec{s}_2 = \begin{bmatrix} s_{2x} \\ s_{2y} \end{bmatrix}$, $\vec{s}_3 = \begin{bmatrix} s_{3x} \\ s_{3y} \end{bmatrix}$ represent positions.

How do we solve for unknown values? Write out a system of equations!

$$\|\vec{x} - \vec{s}_1\|^2 = d_1^2 \quad ①$$

$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$, so let's square both sides to simplify the math!

$$\|\vec{x} - \vec{s}_2\|^2 = d_2^2 \quad ②$$

$$\|\vec{x} - \vec{s}_3\|^2 = d_3^2 \quad ③$$

Expanding these equations:

$$① \quad (\vec{x} - \vec{s}_1)^T (\vec{x} - \vec{s}_1) = d_1^2$$

$$\vec{x}^T\vec{x} - \vec{s}_1^T\vec{x} - \vec{x}^T\vec{s}_1 + \vec{s}_1^T\vec{s}_1 = d_1^2$$

$$\|\vec{x}\|^2 - 2\vec{s}_1^T\vec{x} + \|\vec{s}_1\|^2 = d_1^2$$

$$② \quad \|\vec{x}\|^2 - 2\vec{s}_2^T\vec{x} + \|\vec{s}_2\|^2 = d_2^2$$

$$③ \quad \|\vec{x}\|^2 - 2\vec{s}_3^T\vec{x} + \|\vec{s}_3\|^2 = d_3^2$$

We have three equations and two unknowns, but we have a problem!
These equations are not linear! We have $\|\vec{x}\|^2$ terms.
What do we do? A trick! What if we just... get rid of those terms?

$$② - ① \quad \cancel{\|\vec{x}\|^2} - 2\vec{s}_2^T\vec{x} + \|\vec{s}_2\|^2 - \cancel{\|\vec{x}\|^2} + 2\vec{s}_1^T\vec{x} - \|\vec{s}_1\|^2 = d_2^2 - d_1^2$$

$$③ - ① \quad \cancel{\|\vec{x}\|^2} - 2\vec{s}_3^T\vec{x} + \|\vec{s}_3\|^2 - \cancel{\|\vec{x}\|^2} + 2\vec{s}_1^T\vec{x} - \|\vec{s}_1\|^2 = d_3^2 - d_1^2$$

Rearranging these equations:

$$2(\vec{s}_1 - \vec{s}_2)^T\vec{x} = \|\vec{s}_1\|^2 - \|\vec{s}_2\|^2 + d_2^2 - d_1^2$$

$$2(\vec{s}_1 - \vec{s}_3)^T\vec{x} = \|\vec{s}_1\|^2 - \|\vec{s}_3\|^2 + d_3^2 - d_1^2$$

$$(\vec{s}_1 - \vec{s}_2)^T = \left(\begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix} - \begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix} \right)^T = \begin{bmatrix} s_{11} - s_{21} & s_{12} - s_{22} \end{bmatrix}$$

These are linear!!!

$$2 \begin{bmatrix} s_{11} - s_{21} & s_{12} - s_{22} \\ s_{11} - s_{31} & s_{12} - s_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \|\vec{s}_1\|^2 - \|\vec{s}_2\|^2 + d_2^2 - d_1^2 \\ \|\vec{s}_1\|^2 - \|\vec{s}_3\|^2 + d_3^2 - d_1^2 \end{bmatrix}$$

Solve using Gaussian Elimination!