EECS 16A Designing Information Devices and Systems I
Summer 2023 Discussion 2A

1. Matrix Multiplication

Consider the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix}$$

For each matrix multiplication problem, *if the product exists*, find the product by hand. Otherwise, explain why the product does not exist.

(a) **A B**

Answer: A is a 1×2 vector and B is a 2×1 vector, so the product exists!
$\mathbf{AB} = 1 \cdot 3 + 4 \cdot 2 = 11$

(b) **C D**

Answer:

Since both **C** and **D** are 2×2 matrices, the product exists and is a 2×2 matrix.

$$\mathbf{CD} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 4 \cdot 2 & 1 \cdot 2 + 4 \cdot 1 \\ 2 \cdot 3 + 3 \cdot 2 & 2 \cdot 2 + 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 11 & 6 \\ 12 & 7 \end{bmatrix}.$$

(c) **D C**

Answer:

Since both **C** and **D** are 2×2 matrices, the product exists and is a 2×2 matrix.

$\mathbf{DC} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 + 2 \cdot 2 & 3 \cdot 4 + 2 \cdot 3 \\ 2 \cdot 1 + 1 \cdot 2 & 2 \cdot 4 + 1 \cdot 3 \end{bmatrix} = \begin{bmatrix} 7 & 18 \\ 4 & 11 \end{bmatrix}$.

(d) **C E**

Answer:

Since **C** is a 2×2 matrix and **E** is a 2×4 matrix, the product exists and is a 2×4 matrix.

$\mathbf{CE} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 4 \cdot 4 & 1 \cdot 9 + 4 \cdot 3 & 1 \cdot 5 + 4 \cdot 2 & 1 \cdot 7 + 4 \cdot 2 \\ 2 \cdot 1 + 3 \cdot 4 & 2 \cdot 9 + 3 \cdot 3 & 2 \cdot 5 + 3 \cdot 2 & 2 \cdot 7 + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 17 & 21 & 13 & 15 \\ 14 & 27 & 16 & 20 \end{bmatrix}$.

(e) **F E** (only note whether or not the product exists and optionally compute the product if it does)

Answer:

Since **F** is a 4×3 matrix and **E** is a 2×4 matrix, the product does not exist.
This is because the number of columns in the first matrix (**F**) should match the number of rows in the second matrix (**E**) for this product to be defined.

(f) **E F** (only note whether or not the product exists and optionally compute the product if it does)

Answer:
Since $\mathbf{E}$ is a 2×4 matrix and $\mathbf{F}$ is a 4×3 matrix, the product exists and is a 2×3 matrix.

$$\begin{aligned}
\mathbf{EF} &= \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1\cdot5+9\cdot6+5\cdot4+7\cdot3 & 1\cdot5+9\cdot1+5\cdot1+7\cdot2 & 1\cdot8+9\cdot2+5\cdot7+7\cdot2 \\ 4\cdot5+3\cdot6+2\cdot4+2\cdot3 & 4\cdot5+3\cdot1+2\cdot1+2\cdot2 & 4\cdot8+3\cdot2+2\cdot7+2\cdot2 \end{bmatrix} \\
&= \begin{bmatrix} 100 & 33 & 75 \\ 52 & 29 & 56 \end{bmatrix}
\end{aligned}$$

(g) **G H** (Practice on your own)

Answer:
Since $\mathbf{G}$ and $\mathbf{H}$ are both 3×3 matrices, the product exists and is another 3×3 matrix.

$$\mathbf{GH} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 8\cdot5+1\cdot1+6\cdot2 & 8\cdot3+1\cdot8+6\cdot3 & 8\cdot4+1\cdot2+6\cdot5 \\ 3\cdot5+5\cdot1+7\cdot2 & 3\cdot3+5\cdot8+7\cdot3 & 3\cdot4+5\cdot2+7\cdot5 \\ 4\cdot5+9\cdot1+2\cdot2 & 4\cdot3+9\cdot8+2\cdot3 & 4\cdot4+9\cdot2+2\cdot5 \end{bmatrix} = \begin{bmatrix} 53 & 50 & 64 \\ 34 & 70 & 57 \\ 33 & 90 & 44 \end{bmatrix}.$$

(h) **H G** (Practice on your own)

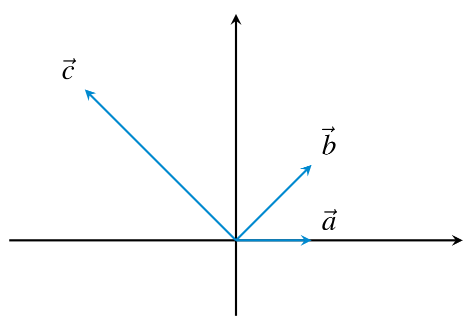
Answer:

Since **H** and **G** are both 3×3 matrices, the product exists and is another 3×3 matrix.

$$\mathbf{GH} = \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} = \begin{bmatrix} 5\cdot8+3\cdot3+4\cdot4 & 5\cdot1+3\cdot5+4\cdot9 & 5\cdot6+3\cdot7+4\cdot2 \\ 1\cdot8+8\cdot3+2\cdot4 & 1\cdot1+8\cdot5+2\cdot9 & 1\cdot6+8\cdot7+2\cdot2 \\ 2\cdot8+3\cdot3+5\cdot4 & 2\cdot1+3\cdot5+5\cdot9 & 2\cdot6+3\cdot7+5\cdot2 \end{bmatrix} = \begin{bmatrix} 65 & 56 & 59 \\ 40 & 59 & 66 \\ 45 & 62 & 43 \end{bmatrix}.$$

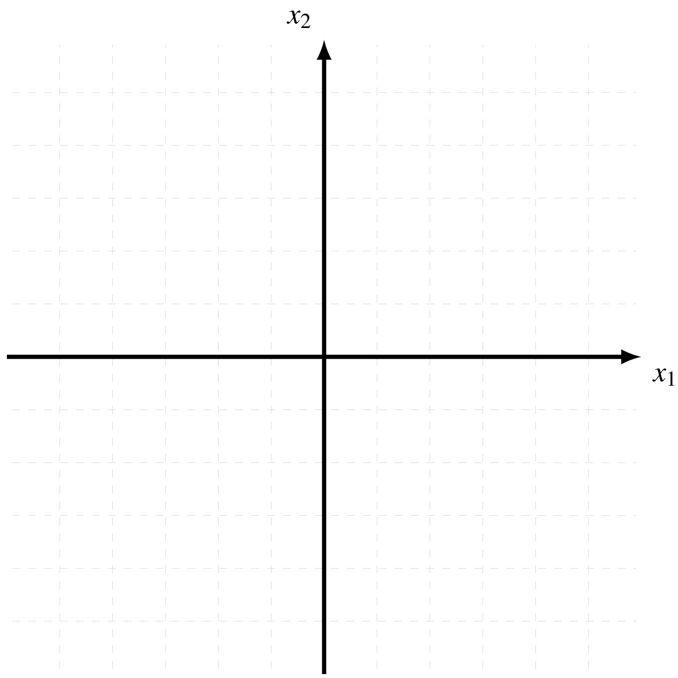
2. Visualizing Linear Combinations of Vectors

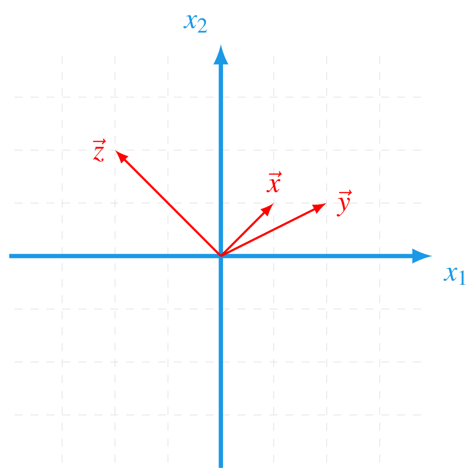
We are given a point $\vec{c}$ that we want to get to, but we can only move in two directions: $\vec{a}$ and $\vec{b}$. We know that to get to $\vec{c}$, we can travel along $\vec{a}$ for some amount α, then change direction and travel along $\vec{b}$ for some amount β. We want to find these two scalars α and β, such that we reach point $\vec{c}$. That is, $\alpha\vec{a}+\beta\vec{b}=\vec{c}$.



(a) First, consider the case where $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\vec{z} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Draw these vectors.

Answer:





(b) We want to find the two scalars α and β, such that by moving α along $\vec{x}$ and β along $\vec{y}$, we can reach $\vec{z}$. Write a system of equations to find α and β in matrix form.

Answer:

$$\alpha\vec{x} + \beta\vec{y} = \vec{z}$$

$$\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\begin{cases} \alpha + \beta \cdot 2 = -2 \\ \alpha + \beta = 2 \end{cases}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

(c) Solve for α, β.

Answer:

We start by writing the system in the augmented matrix form

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 1 & 1 & 2 \end{array}\right]$$

Then we solve the system using Gaussian Elimination. First, we subtract the second row by the first row:

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & -1 & 4 \end{array}\right]$$

Next, we multiply the second row by -1 to solve for β.

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & -4 \end{array}\right]$$

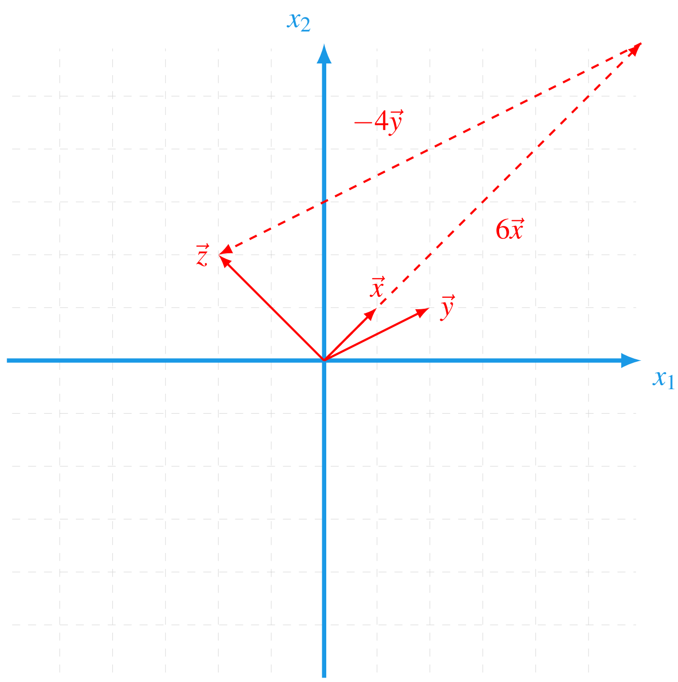
We get $\beta = -4$. Then, we take the first row and subtract it by two times the second row.

$$\left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & -4 \end{array}\right]$$

So the solution is $\alpha = 6$ and $\beta = -4$.

(d) Superimpose the scaled vectors $\alpha\vec{x}$ and $\beta\vec{y}$ on your graph in part (a) and confirm $\alpha\vec{x} + \beta\vec{y} = \vec{z}$.

Answer:


x_2
$-4\vec{y}$
$6\vec{x}$
$\vec{z}$
$\vec{x}$
$\vec{y}$
x_1

