EECS 16A Designing Information Devices and Systems I
Summer 2023 Discussion 3A

1. Mechanical Inverses

For each sub-part below, compute the inverse of $\mathbf{A}$ using the Gauss-Jordan method.

(a) $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Answer: We can use the Gauss-Jordan method:

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{a}R_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - cR_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d-\frac{c}{a}b & -\frac{c}{a} & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{d-\frac{c}{a}b}R_2} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-\frac{c}{a}}{d-\frac{c}{a}b} & \frac{1}{d-\frac{c}{a}b} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - \frac{b}{a}R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a}+\frac{b}{a}\frac{c}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right].$$

Therefore, we get that $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

This is a known formula which, if you find useful, you can use for any general 2x2 matrix. The expression $ad-bc$ in a matrix has a special name called the determinant, which will be more useful later in the class when we talk about eigenvalues and eigenvectors.

Note that the matrix does not have an inverse if $ad-bc=0$.

(b) Use the answer in part (a) to find the inverse of the rotation matrix $\mathbf{A} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$

Answer: Using the formula from part a, we get:

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{cos^2(\theta)-(-sin^2(\theta))}\begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} \\ &= \frac{1}{cos^2(\theta)+sin^2(\theta)}\begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} \\ &= \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} \end{aligned}$$

We can quickly confirm that this is correct since the inverse of rotating by θ counterclockwise is rotating by $-\theta$ degrees counterclockwise. So $\begin{bmatrix} cos(-\theta) & -sin(-\theta) \\ sin(-\theta) & cos(-\theta) \end{bmatrix} = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix}$ since $sin(-\theta) = -sin(\theta)$ and $cos(-\theta) = cos(\theta)$ by trigonometry identities.

(c) $\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

Answer:

We use Gaussian elimination:

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{3} & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow 3R_3} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{3} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - \frac{1}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - \frac{1}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & 1 & 0 & -1 \\ 0 & \frac{1}{2} & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow 2R_2} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right]$$

Therefore, we get $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}$.

(d) $\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix}$

Answer: We use Gaussian elimination:

$$\left[\begin{array}{ccc|ccc} 5 & 5 & 15 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{5}R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 1 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{2}{5} & \frac{1}{2} & 1 \end{array}\right].$$

While row-reducing, we notice that the second column doesn't have a pivot (and that there is also a row of zeros). Therefore, no inverse exists.

(e) (PRACTICE)

$$\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 0 & 4 \end{bmatrix}$$

Answer:

We use Gaussian elimination:

$$\left[\begin{array}{ccc|ccc} 5 & 5 & 15 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{5}R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 1 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & -\frac{1}{5} & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & -1 & 1 & -\frac{1}{5} & 0 & 1 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow -R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{5} & 0 & -1 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \end{array}\right] \xrightarrow{R_3 \leftarrow -R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{5} & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right]$$

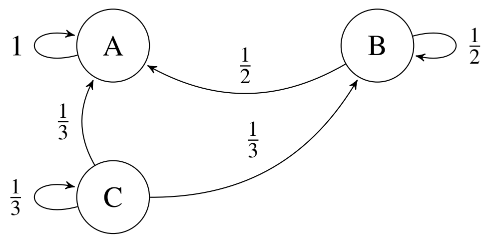
$$\xrightarrow{R_2 \leftarrow R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -\frac{2}{5} & \frac{3}{2} & 0 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{4}{5} & 2 & 1 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right].$$

Therefore, we get $\mathbf{A}^{-1} = \begin{bmatrix} -\frac{4}{5} & 2 & 1 \\ \frac{2}{5} & -\frac{1}{2} & -1 \\ \frac{1}{5} & -\frac{1}{2} & 0 \end{bmatrix}$.

2. Transition Matrix

Suppose we have a network of pumps as shown in the diagram below. Let us describe the state of A, B, and C using a state vector $\vec{x}[n] = \begin{bmatrix} x_A[n] \\ x_B[n] \\ x_C[n] \end{bmatrix}$ where $x_A[n]$, $x_B[n]$, and $x_C[n]$ are the states at time-step n.



(a) Find the state transition matrix S, such that $\vec{x}[n+1] = S\,\vec{x}[n]$.
Separately, find the sum of the terms for each column vector in S. Do you notice a pattern?

Answer:
We can write the following equations by examining the state transition diagram:

$$\begin{aligned} x_A[n+1] &= x_A[n] + (1/2)\,x_B[n] + (1/3)x_C[n] \\ x_B[n+1] &= (1/2)\,x_B[n] + (1/3)\,x_C[n] \\ x_C[n+1] &= (1/3)\,x_C[n] \end{aligned}$$

From here we can directly write down the state transition matrix as:

$$S = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Note the columns of S each sum to 1, which ensures that nothing enters or exits the system at any timestep.

(b) Let us now find the matrix S^{-1} such that we can recover the previous state $\vec{x}[n-1]$ from $\vec{x}[n]$.
Specifically, solve for S^{-1} such that $\vec{x}[n-1] = S^{-1}\,\vec{x}[n]$.

Answer: We can use Gauss-Jordan method to solve for matrix S^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{3} & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{5} R_1} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{3} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{2} R_2} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & 1 & 0 & -1 \\ 0 & \frac{1}{2} & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right]$$

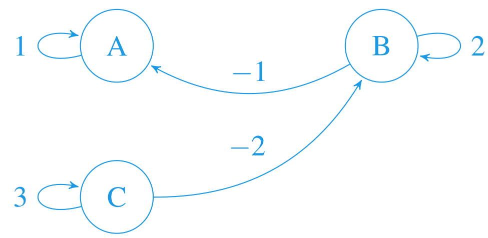
Therefore:

$$S^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}$$

You can also use your answer from the previous question as well. Note that the columns of S^{-1} still sum to 1 despite matrix elements lying outside $[0,1]$.
So while this is not physical, the inverse process obeys conservation as expected.

(c) Now, draw the state transition diagram that corresponds to the S^{-1} that you just found. Again, find the sum of the terms for each column vector in S^{-1}. Do you notice a pattern?

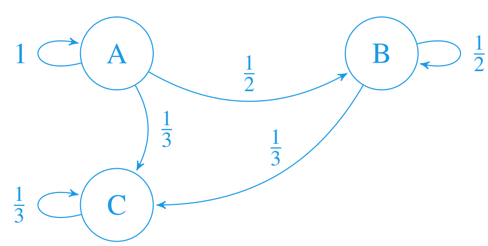
Answer:



The matrix S^{-1} can be thought of as the transition that *turns back time* for the pump system. **Although it is non-physical**, the weights that have an absolute value greater than 1 can be thought of as "generating" water, and the weights that have negative weight can be thought of as "destroying" water. However, note (as seen above) that the outflow weights of each node still sum to 1 (i.e. the columns of S^{-1} still sum to 1). This means that in total all of the water is being conserved during the transition between time steps, even when time is reversed.

(d) Redraw the diagram from the first part of the problem, but now with the directions of the arrows reversed. Let us call the state transition matrix of this "reversed" state transition diagram T. Does $T = S^{-1}$?

Answer:



After drawing the "reversed" state transition diagram, we can write the following equations:

$$\begin{aligned} x_A[n+1] &= x_A[n] \\ x_B[n+1] &= (1/2)x_A[n] + (1/2)\,x_B[n] \\ x_C[n+1] &= (1/3)\,x_A[n] + (1/3)\,x_B[n] + (1/3)\,x_C[n] \end{aligned}$$

From here, we can directly write down the state transition matrix as:

$$T = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Note that $T \neq S^{-1}$. What we have actually found is that T is equal to the *transpose* of S, denoted by $S^\top$ (the superscript $\top$ denotes the transpose of a matrix). The transpose of a matrix is when its rows become its columns. In general, a matrix's inverse and its transpose are not equal to each other.

(e) Suppose we start in the state $\vec{x}[1] = \begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix}$. Compute the state vector after two time-steps, $\vec{x}[3]$.

Answer: There are two ways to approach this problem:

i. Compute states successively: $\vec{x}[2] = S\,\vec{x}[1]$, then $\vec{x}[3] = S\,\vec{x}[2]$

ii. Compute directly: $\vec{x}[3] = S \cdot S\,\vec{x}[1]$

They are equivalent because of the fact that matrix multiplication is associative:

$$\vec{x}[3] = S(S\,\vec{x}[1]) = (S \cdot S)\,\vec{x}[1]$$

We start with method i.

$$\vec{x}[2] = S\,\vec{x}[1] = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix} = \begin{bmatrix} 12+6+4 \\ 6+4 \\ 4 \end{bmatrix} = \begin{bmatrix} 22 \\ 10 \\ 4 \end{bmatrix}$$

$$\vec{x}[3] = S\,\vec{x}[2] = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 22 \\ 10 \\ 4 \end{bmatrix} = \begin{bmatrix} 22+5+4/3 \\ 5+4/34/3 \end{bmatrix} = \begin{bmatrix} 85/3 \\ 19/3 \\ 4/3 \end{bmatrix} \qquad \square$$

Alternatively we can use method ii.

$$\vec{x}[3] = S\,S\,\vec{x}[1] \quad \rightarrow \quad S^2 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 3/4 & 11/18 \\ 0 & 1/4 & 5/18| \\ 0 & 0 & 1/9 \end{bmatrix}$$

$$\vec{x}[3] = S^2\,\vec{x}[1] = \begin{bmatrix} 1 & 3/4 & 11/18 \\ 0 & 1/4 & 5/18| \\ 0 & 0 & 1/9 \end{bmatrix} \begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix} = \begin{bmatrix} 12+12\cdot 3/4+12\cdot 11/18 \\ 12\cdot 1/4+12\cdot 5/8 \\ 12\cdot 1/9 \end{bmatrix} = \begin{bmatrix} 85/3 \\ 19/3 \\ 4/3 \end{bmatrix} \square$$

