

Welcome to EECS 16A!

Designing Information Devices and Systems I

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Sp 2022

Lecture 1B

Vectors, Matrices, Multiplications,
start span



Announcements

- Quest: Tuesday 02/8/22 in Class (first 30min)
 - Covers everything including today
- Last time:
 - Gaussian Elimination
- Today:
 - vectors
 - Matrix-Matrix and Matrix-vector Multiplications
 - Matrix-Vector Multiplications as linear set of equations
 - Intro to span (most likely next Tuesday)

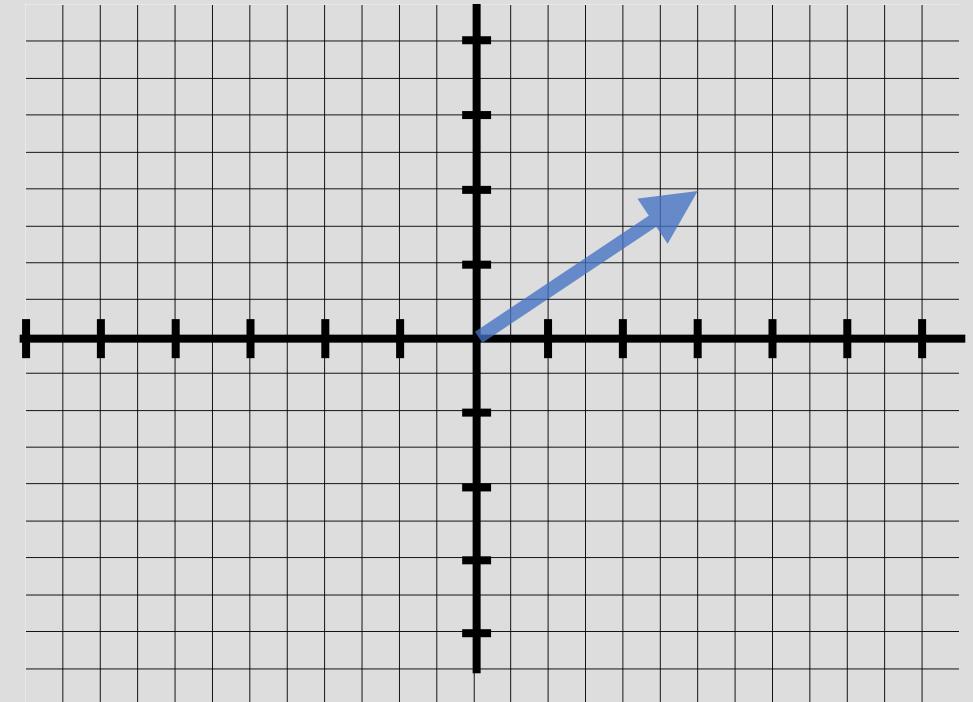
Vectors

- An array of N numbers
 - Represents coordinates in an N-dimensional space

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \quad \vec{x} \in \mathbb{R}^N$$

- For example:

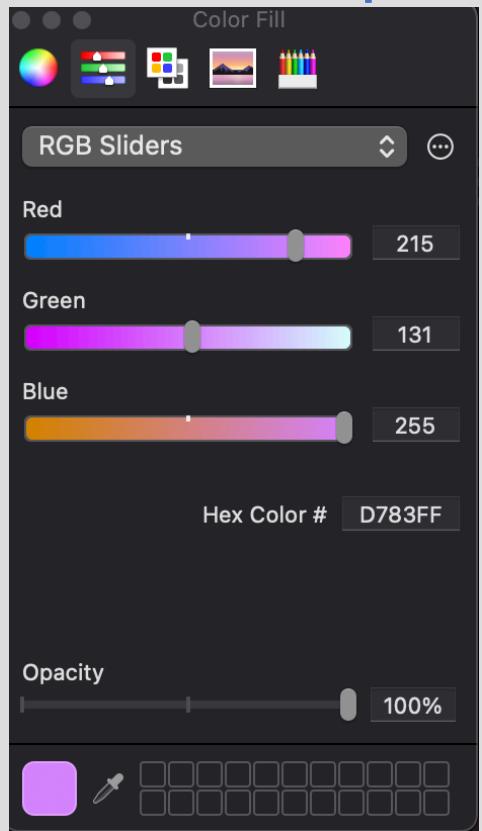
$$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \vec{x} \in \mathbb{R}^2$$



Vectors

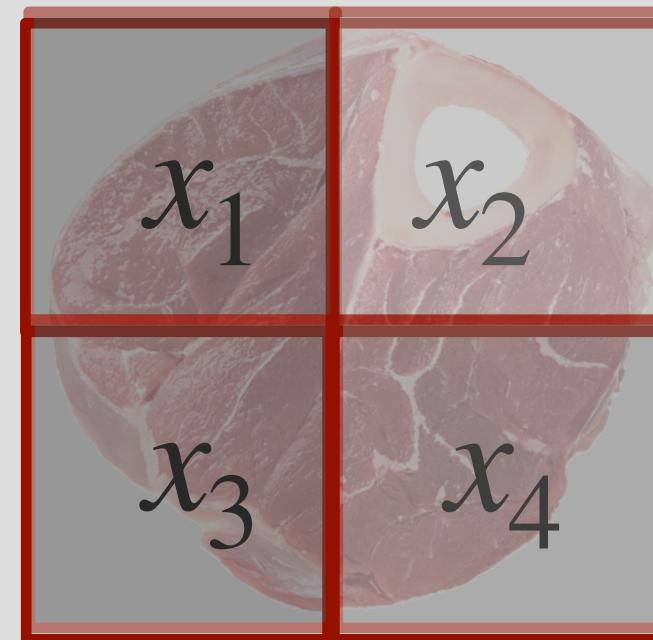
- Since it's an array of numbers, it can represent other things....

pixel color



$$\vec{x} = \begin{bmatrix} 215 \\ 131 \\ 25 \end{bmatrix}$$

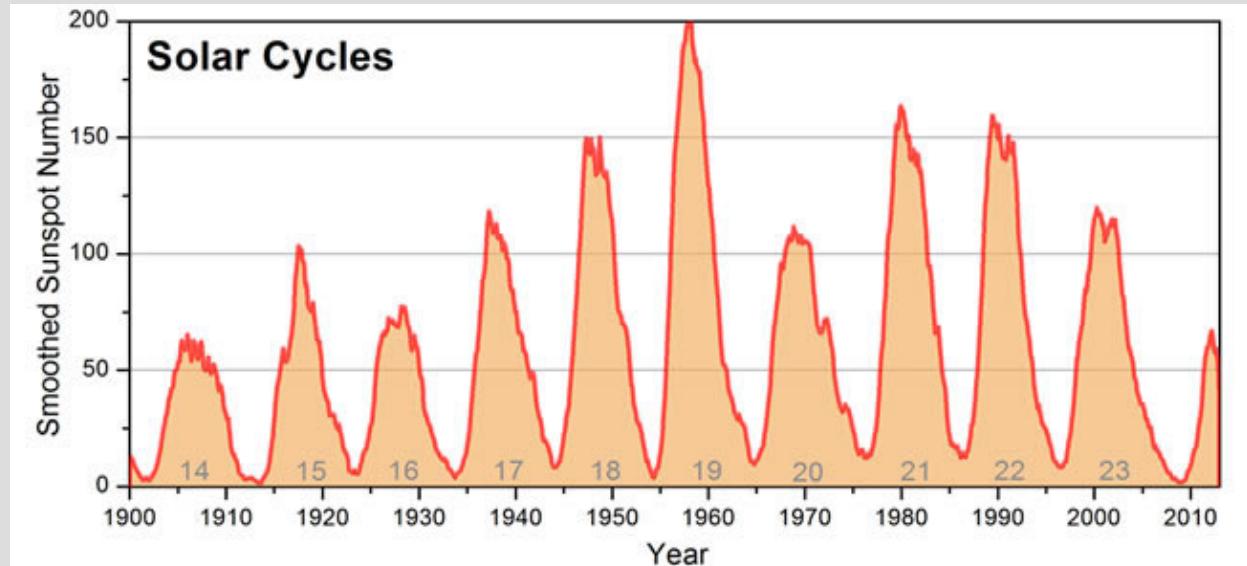
pixel values in an image



$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Vectors

Data



$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{120} \end{bmatrix}$$

Special Vectors

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_N = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Matrices

- A collection of numbers in a rectangular form

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1M} \\ x_{21} & x_{22} & \cdots & x_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{NM} \end{bmatrix}, \quad X \in \mathbb{R}^{N \times M}$$

- Or a collection of M, N-length vectors

$$X = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_M \end{bmatrix}, \quad X \in \mathbb{R}^{N \times M}$$

Vectors as Matrices

- A vector is a degenerate matrix

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \quad \vec{x} \in \mathbb{R}^{N \times 1}$$

- A scalar is a degenerate vector or matrix

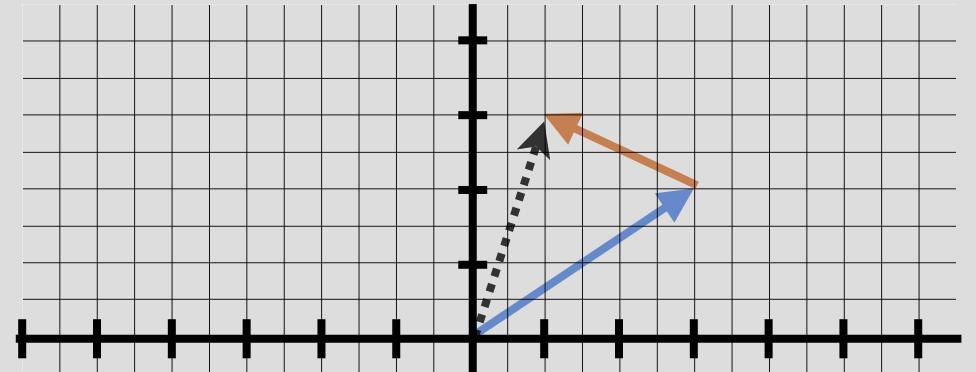
$$a \in \mathbb{R}^{1 \times 1}$$

Vector addition

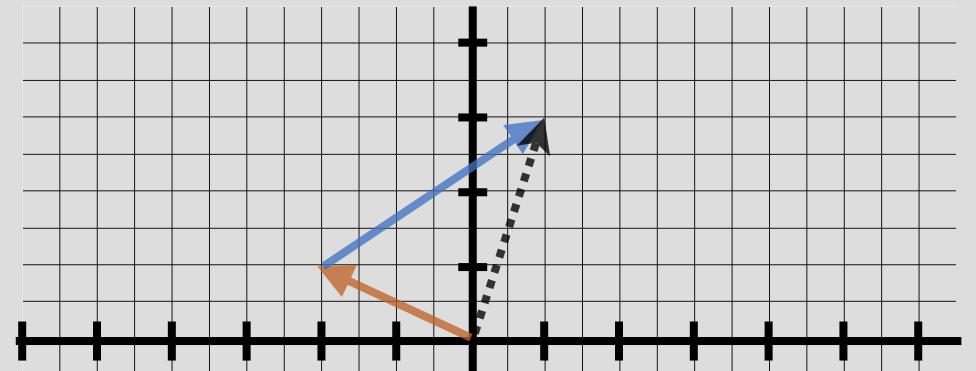
- Two vectors of the same length can be added
 - Addition is element-wise

$$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\vec{x} + \vec{y} =$$



$$\vec{y} + \vec{x} =$$

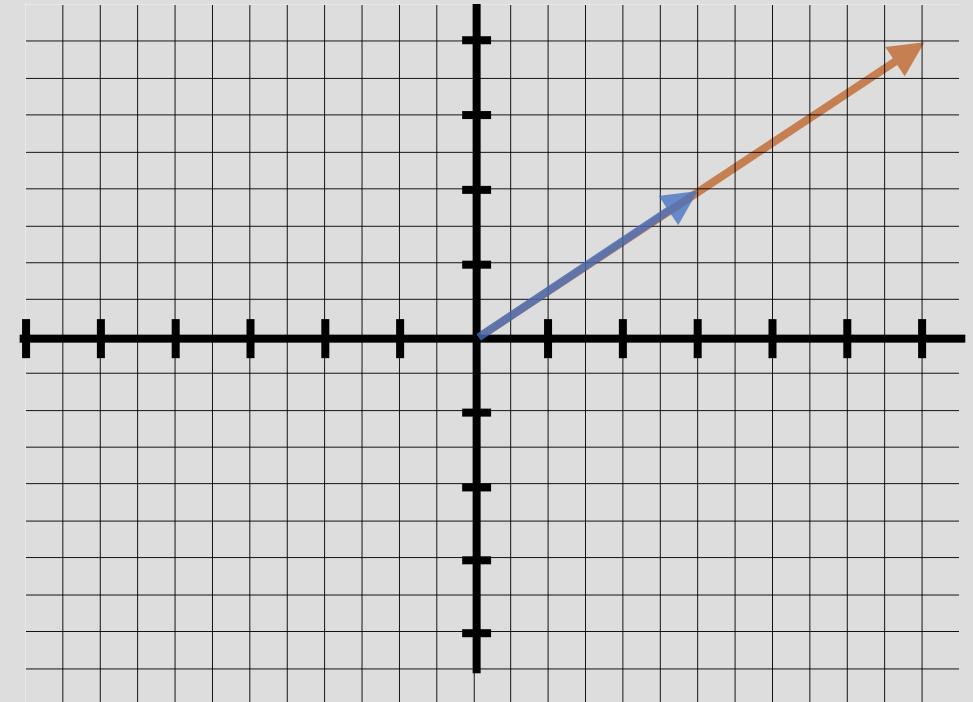


Scalar Vector Multiplication

- Multiplying with a scalar result in multiplying each element.

$$a\vec{x} = \begin{bmatrix} ax_1 \\ ax_1 \\ \vdots \\ ax_{N \times 1} \end{bmatrix}$$

$$2 \cdot \vec{x} = 2 \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$



Properties of vector addition

- Commutativity: $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
- Associativity: $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
- Additive negative: $\vec{x} + (-\vec{x}) = \vec{0}$
- Additive identity: $\vec{x} + \vec{0} = \vec{x}$

Matrix Addition

- When matrices are the same size, they can be added element-wise

$$X + Y = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

- Scalar multiplication – by all elements

$$2X = 2 \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Matrix Addition

- When matrices are the same size, they can be added element-wise

$$X + Y = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Scalar multiplication – by all elements

$$2X = 2 \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & -4 \end{bmatrix}$$

diagonal matrix

Vector Transpose

- \vec{x}^T is the transpose of \vec{x}

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \quad \vec{x} \in \mathbb{R}^{N \times 1}$$

$$\vec{x}^T = [\ x_1 \ x_2 \ \cdots \ x_N \], \quad \vec{x}^T \in \mathbb{R}^{1 \times N}$$

- \vec{x} is always a column vector
- To represent a row vector, write: \vec{x}^T

Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N \times M}$ are a_{ij}

The elements of $A^T \in \mathbb{R}^{M \times N}$ are a_{ji}

Matrix transpose is not (generally) an inverse!

$$A \in \mathbb{R}^{N \times M} \quad \left[\begin{array}{c} \vec{a}_1 \vec{a}_2 \cdots \vec{a}_m \\ \vdots \end{array} \right] \quad A^T \in \mathbb{R}^{M \times N} \quad \left[\begin{array}{c} \vec{a}_1^T \\ \vec{a}_2^T \\ \vdots \\ \vec{a}_m^T \end{array} \right]$$

Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N \times M}$ are a_{ij}

The elements of $A^T \in \mathbb{R}^{M \times N}$ are a_{ji}

Matrix transpose is not (generally) an inverse!

$$A \in \mathbb{R}^{N \times M}$$

$$\begin{bmatrix} A \rightarrow A \\ B \rightarrow A \\ C \rightarrow B \\ D \rightarrow C \\ E \rightarrow D \end{bmatrix}$$

$$A^T \in \mathbb{R}^{M \times N}$$

$$\begin{bmatrix} A \\ B \end{bmatrix}$$

Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N \times M}$ are a_{ij}

The elements of $A^T \in \mathbb{R}^{M \times N}$ are a_{ji}

Matrix transpose is not (generally) an inverse!

$$A \in \mathbb{R}^{N \times M}$$

$$\begin{bmatrix} A \rightarrow A \\ B \rightarrow A \\ A \rightarrow B \\ B \rightarrow B \\ A \rightarrow C \\ B \rightarrow C \end{bmatrix}$$

$$A^T \in \mathbb{R}^{M \times N}$$

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N \times M}$ are a_{ij}

The elements of $A^T \in \mathbb{R}^{M \times N}$ are a_{ji}

Matrix transpose is not (generally) an inverse!

$$A \in \mathbb{R}^{N \times M}$$

$$\begin{bmatrix} A \rightarrow A \\ \frac{1}{2}B \\ \frac{1}{2}C \end{bmatrix}$$

$$\left[\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_m \right]$$

$$A^T \in \mathbb{R}^{M \times N}$$

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

$$\left[\vec{a}_1^T \ \vec{a}_2^T \ \dots \ \vec{a}_m^T \right]$$

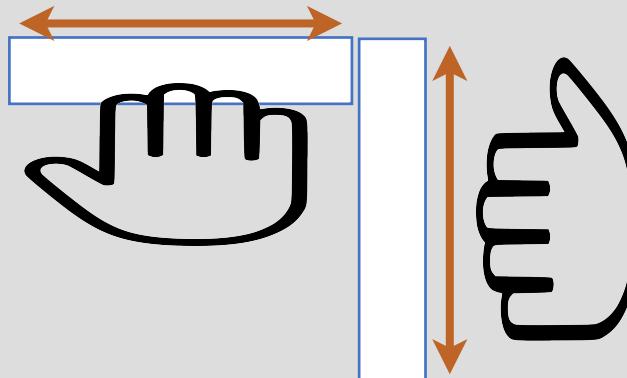
Vector-Vector Multiplication

- Multiplication is valid only for specific matching dimensions!
 - Width of the 1st, matches length of the second

Like this....



and like that!

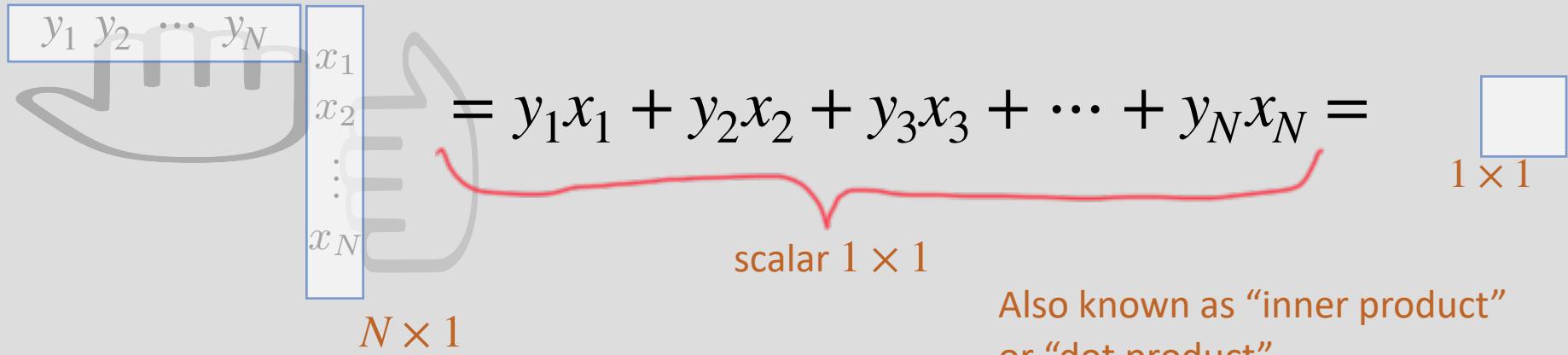


Vector Vector Multiplication

$$\vec{x}, \vec{y} \in \mathbb{R}^{N \times 1}$$

$1 \times N$

$$\vec{y}^T \vec{x} =$$



Like this....



and like that!



Matrix-Vector Multiplication

$$\vec{x}, \vec{y} \in \mathbb{R}^{N \times 1}$$

$$\vec{y}^T \vec{x} =$$

$1 \times N$

$$\begin{matrix} y_1 & y_2 & \cdots & y_N \end{matrix} \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{matrix} = y_1x_1 + y_2x_2 + y_3x_3 + \cdots + y_Nx_N =$$

1×1

scalar 1×1

What about this case....

$A \in \mathbb{R}^{M \times N}, \vec{x} \in \mathbb{R}^{N \times 1}$

$$A \vec{x} =$$

M N

x_1 x_2 \vdots x_N $N \times 1$

= ?

Like this....



and like that!



Also known as “inner product”
or “dot product”

Matrix-Vector Multiplication

$$A \in R^{M \times N}, \vec{x} \in \mathbb{R}^{N \times 1}$$

$$A \vec{x} = \begin{matrix} & N \\ \begin{matrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{matrix} & \times \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_N \\ N \times 1 \end{matrix} \end{matrix} = \left[a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \right]$$

$$[a_{11} \ a_{12} \ \cdots \ a_{1N}] \vec{x} = \vec{y}_1^T \vec{x}$$

Like this....



and like that!



Matrix-Vector Multiplication

$$A \in R^{M \times N}, \vec{x} \in \mathbb{R}^{N \times 1}$$

$$A \vec{x} = \begin{matrix} & N \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix} & \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{matrix} \end{matrix} = \boxed{\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N \end{bmatrix}}$$

$$[a_{11} \ a_{12} \ \cdots \ a_{1N}] \vec{x} = \vec{y}_1^T \vec{x}$$

$$[a_{21} \ a_{22} \ \cdots \ a_{2N}] \vec{x} = \vec{y}_2^T \vec{x}$$

Like this....



and like that!



Matrix-Vector Multiplication

$$A \in R^{M \times N}, \vec{x} \in \mathbb{R}^{N \times 1}$$

$$A \vec{x} = \begin{matrix} & N \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix} & \end{matrix} \begin{matrix} M \\ \vec{x} \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{matrix} \\ N \times 1 \end{matrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N \\ \vdots \\ a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N \end{bmatrix} = \begin{matrix} 1 \\ M \end{matrix}$$

$$\begin{aligned} [a_{11} & a_{12} & \cdots & a_{1N}] \vec{x} &= \vec{y}_1^T \vec{x} \\ [a_{21} & a_{22} & \cdots & a_{2N}] \vec{x} &= \vec{y}_2^T \vec{x} \\ \vdots & & & \\ [a_{M1} & a_{M2} & \cdots & a_{MN}] \vec{x} &= \vec{y}_M^T \vec{x} \end{aligned}$$

Like this....



and like that!



Matrix-Vector Multiplication

$$A \in R^{M \times N}, \vec{x} \in \mathbb{R}^{N \times 1}$$

$$A \vec{x} = \begin{matrix} & N \\ \begin{matrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{matrix} & \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{matrix} & N \times 1 \end{matrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N \\ \vdots \\ a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N \end{bmatrix} = \begin{matrix} & 1 \\ M & \end{matrix}$$

Like this....



and like that!



What about this case....

$$A \in R^{M \times N}, B \in \mathbb{R}^{N \times L}$$

$$AB = \begin{matrix} & N \\ \begin{matrix} A & \end{matrix} & \begin{matrix} B & \end{matrix} & N \times L \end{matrix} = ?$$

Matrix-Matrix Multiplication

$$A \in \mathbb{R}^{M \times N}, B \in \mathbb{R}^{N \times L}$$

Like this....



and like that!



$$AB = \begin{matrix} & N \\ \text{Matrix } A & \end{matrix} \cdot \begin{matrix} & L \\ \text{Matrix } B & \end{matrix} = ?$$

The diagram illustrates matrix multiplication AB . Matrix A is shown as a vertical stack of N horizontal rows, with its width labeled M . Matrix B is shown as a vertical stack of L horizontal columns, with its height labeled N . The result of the multiplication is represented by a large bracket on the right side of the equation, containing the expression $a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1N}b_{N1}$.

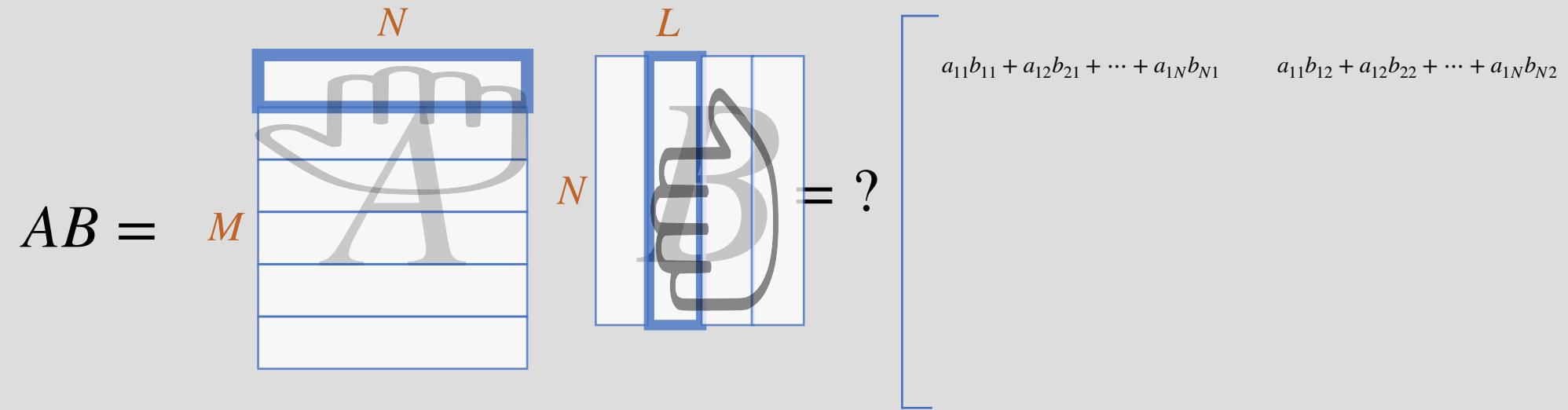
Matrix-Matrix Multiplication

$$A \in \mathbb{R}^{M \times N}, B \in \mathbb{R}^{N \times L}$$

Like this....



and like that!



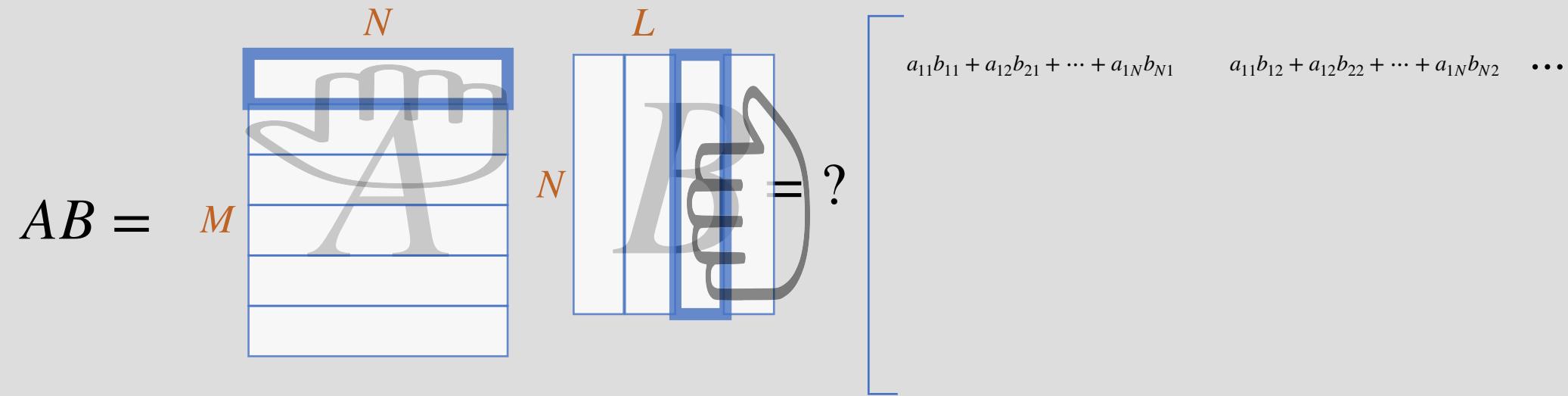
Matrix-Matrix Multiplication

$$A \in R^{M \times N}, B \in R^{N \times L}$$

Like this....



and like that!



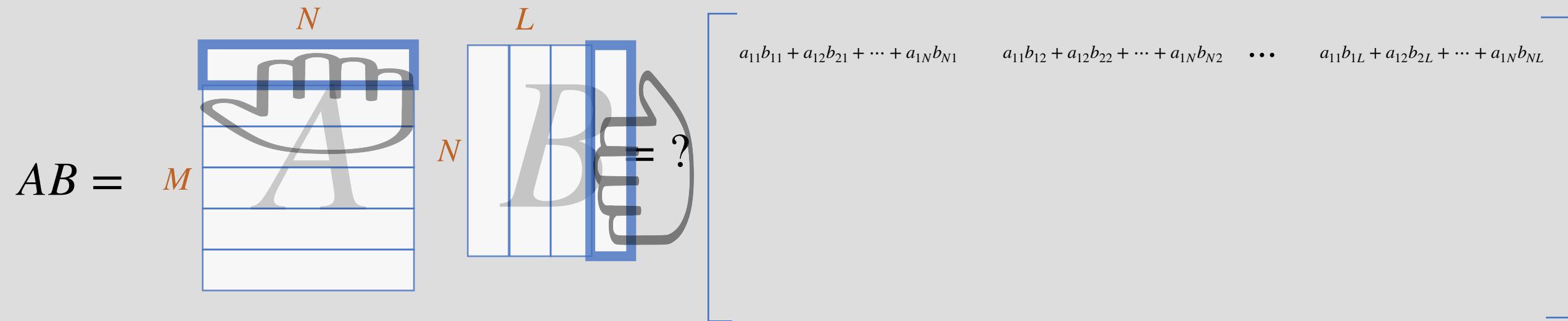
Matrix-Matrix Multiplication

$$A \in \mathbb{R}^{M \times N}, B \in \mathbb{R}^{N \times L}$$

Like this....



and like that!



Matrix-Matrix Multiplication

$$A \in R^{M \times N}, B \in R^{N \times L}$$

Like this....



and like that!



$$AB = \begin{matrix} & N \\ \begin{matrix} M \\ \text{Matrix } A \end{matrix} & \end{matrix} \begin{matrix} L \\ \text{Matrix } B \end{matrix} = ?$$

The diagram illustrates matrix multiplication AB . Matrix A (labeled M) has dimensions $M \times N$, represented by a grid of M rows and N columns. Matrix B (labeled L) has dimensions $N \times L$, represented by a grid of N rows and L columns. The result of the multiplication is a matrix with dimensions $M \times L$, shown as a large bracketed area. The entries in the first row of the result matrix are calculated as follows:

$a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1N}b_{N1}$	$a_{11}b_{12} + a_{12}b_{22} + \dots + a_{1N}b_{N2}$	\dots	$a_{11}b_{1L} + a_{12}b_{2L} + \dots + a_{1N}b_{NL}$
$a_{21}b_{11} + a_{22}b_{21} + \dots + a_{2N}b_{N1}$	$a_{21}b_{12} + a_{22}b_{22} + \dots + a_{2N}b_{N2}$	\dots	$a_{21}b_{1L} + a_{22}b_{2L} + \dots + a_{2N}b_{NL}$

Matrix-Matrix Multiplication

$$A \in R^{M \times N}, B \in R^{N \times L}$$

Like this....



and like that!



$$AB = \begin{matrix} & N \\ \begin{matrix} M \\ \text{Matrix A} \end{matrix} & \end{matrix} \begin{matrix} L \\ \text{Matrix B} \end{matrix} = ?$$

The diagram illustrates matrix multiplication AB . Matrix A (labeled M) has dimensions $M \times N$, represented by a grid of M rows and N columns. Matrix B (labeled L) has dimensions $N \times L$, represented by a grid of N rows and L columns. The result of the multiplication is a matrix with dimensions $M \times L$, shown as a grid of M rows and L columns. The entries in the resulting matrix are calculated as dot products of corresponding rows from A and columns from B .

$a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1N}b_{N1}$	$a_{11}b_{12} + a_{12}b_{22} + \dots + a_{1N}b_{N2}$	\dots	$a_{11}b_{1L} + a_{12}b_{2L} + \dots + a_{1N}b_{NL}$
$a_{21}b_{11} + a_{22}b_{21} + \dots + a_{2N}b_{N1}$	$a_{21}b_{12} + a_{22}b_{22} + \dots + a_{2N}b_{N2}$	\dots	$a_{21}b_{1L} + a_{22}b_{2L} + \dots + a_{2N}b_{NL}$
\vdots	\vdots	\vdots	\vdots
$a_{M1}b_{11} + a_{M2}b_{21} + \dots + a_{MN}b_{N1}$	$a_{M1}b_{12} + a_{M2}b_{22} + \dots + a_{MN}b_{N2}$	\dots	$a_{M1}b_{1L} + a_{M2}b_{2L} + \dots + a_{MN}b_{NL}$

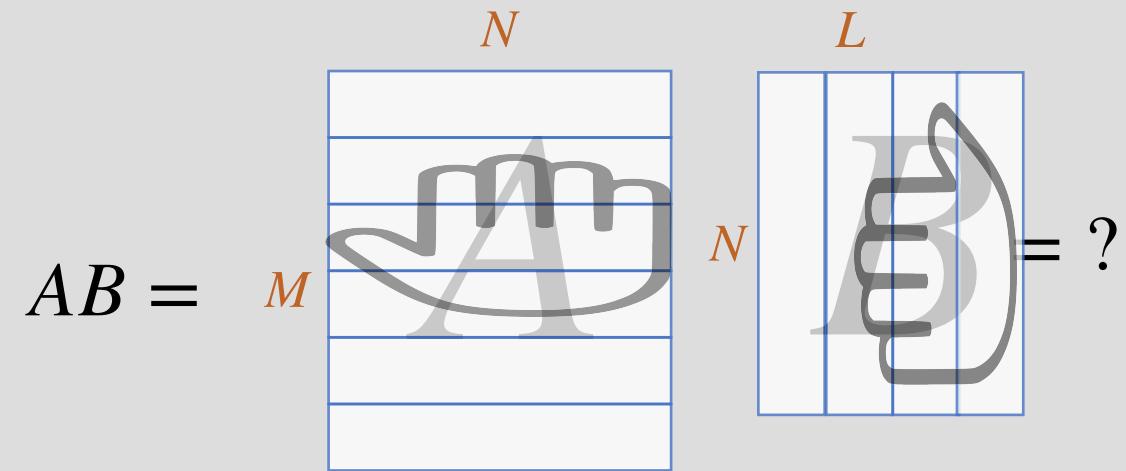
Matrix-Matrix Multiplication

$$A \in R^{M \times N}, B \in R^{N \times L}$$

Like this....



and like that!



$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1N}b_{N1} & a_{11}b_{12} + a_{12}b_{22} + \dots + a_{1N}b_{N2} & \dots & a_{11}b_{1L} + a_{12}b_{2L} + \dots + a_{1N}b_{NL} \\ a_{21}b_{11} + a_{22}b_{21} + \dots + a_{2N}b_{N1} & \textcolor{blue}{a_{21}b_{12} + a_{22}b_{22} + \dots + a_{2N}b_{N2}} & \dots & a_{21}b_{1L} + a_{22}b_{2L} + \dots + a_{2N}b_{NL} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1}b_{11} + a_{M2}b_{21} + \dots + a_{MN}b_{N1} & a_{M1}b_{12} + a_{M2}b_{22} + \dots + a_{MN}b_{N2} & \dots & a_{M1}b_{1L} + a_{M2}b_{2L} + \dots + a_{MN}b_{NL} \end{bmatrix}$$

Result at location $2 \times 2 = a_{21}b_{12} + a_{22}b_{22} + \dots + a_{2N}b_{N2}$

Matrix-Vector Multiplication

Like this....

and like that!



$$A$$

$$B$$

=

$$C$$

$M \times N$

$N \times L$

$M \times L$

Vector Vector Multiplication

$$\vec{x}, \vec{y} \in \mathbb{R}^{N \times 1}$$

$$\vec{x} \vec{y}^T = \begin{matrix} & & & 1 \times N \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{matrix} & \boxed{\begin{matrix} y_1 & y_2 & \cdots & y_N \end{matrix}} & = & \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_N \\ x_2y_1 & x_2y_2 & \cdots & x_2y_N \\ \vdots & \vdots & \ddots & \vdots \\ x_Ny_1 & x_Ny_2 & \cdots & x_Ny_N \end{bmatrix} & = & \boxed{\quad} \\ N \times 1 & & & & & N \times N \end{matrix}$$

Like this....



and like that!



Vector Vector Multiplication

$$\vec{x}, \vec{y} \in \mathbb{R}^{N \times 1}$$

$$\vec{y}^T \vec{x} =$$

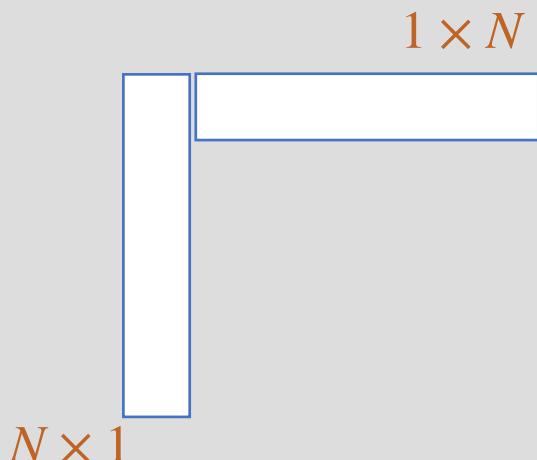


$$= y_1x_1 + y_2x_2 + y_3x_3 + \cdots + y_Nx_N$$

scalar 1×1

Also known as “inner product”
or “dot product”

$$\vec{x} \vec{y}^T =$$



$$= \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_N \\ x_2y_1 & x_2y_2 & \cdots & x_2y_N \\ \vdots & \vdots & \ddots & \vdots \\ x_Ny_1 & x_Ny_2 & \cdots & x_Ny_N \end{bmatrix}_{N \times N}$$

Do not commute!
Also known as “outer product”

Like this....



and like that!



Matrix Matrix Multiplication

$$\begin{matrix} N \times M \\ A \end{matrix} \quad \begin{matrix} M \times N \\ B \end{matrix} = \begin{matrix} N \times N \end{matrix}$$


$$\begin{matrix} 1 \times M \\ I \times 1 \end{matrix} \quad \begin{matrix} 1 \times N \end{matrix} = \begin{matrix} M \times N \end{matrix}$$


$$\begin{matrix} M \times N \\ B \end{matrix} \quad \begin{matrix} N \times M \\ A \end{matrix} = \begin{matrix} M \times M \end{matrix}$$


$$\begin{matrix} 1 \times N \\ 1 \times N \end{matrix} \quad \begin{matrix} N \times M \\ N \times M \end{matrix} = \begin{matrix} 1 \times M \\ 1 \times M \end{matrix}$$


Matrix multiplication does not commute!

Matrix-Vector Form of Systems of Linear Equations

- Consider the matrix equation: $A \vec{x} = \vec{b}$

$$A \vec{x} = \begin{array}{c|c} \begin{matrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{matrix} & \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{matrix} \end{array} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N \\ \vdots \\ a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N \end{bmatrix}_{M \times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}$$

Same as the Augmented Matrix!

$A \vec{x} = \vec{b}$ is another way to write
A linear set of equations!

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1N} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2N} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} & b_M \end{array} \right]$$

Row vs Column Perspective

- Row / Measurement Perspective of $A \vec{x} = \vec{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Row vs Column Perspective

- Row / Measurement Perspective of $A \vec{x} = \vec{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Q: What does a row mean?

A: How each variable affect a particular measurement

Row vs Column Perspective

- Column Perspective of $A \vec{x} = \vec{b}$

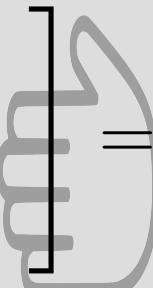
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

Row vs Column Perspective

- Column Perspective of $A \vec{x} = \vec{b}$

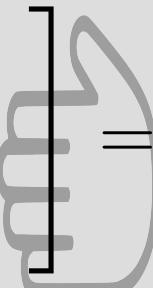
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$


Row vs Column Perspective

- Column Perspective of $A \vec{x} = \vec{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 =$$


Row vs Column Perspective

- Column Perspective of $A \vec{x} = \vec{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 =$$

$$= \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \end{bmatrix} + \begin{bmatrix} a_{13}x_3 \\ a_{23}x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Q: What does a column mean?

A: How a particular variable affects all measurements.

Linear combination of vectors

- Given set of vectors $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_M\} \in \mathbb{R}^N$, and coefficients $\{\alpha_1, \alpha_2, \dots, \alpha_M\} \in \mathbb{R}$
- A linear combination of vectors is defined as: $\vec{b} \triangleq \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \dots + \alpha_M \vec{a}_M$

Recall: $A \vec{x}$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3$$

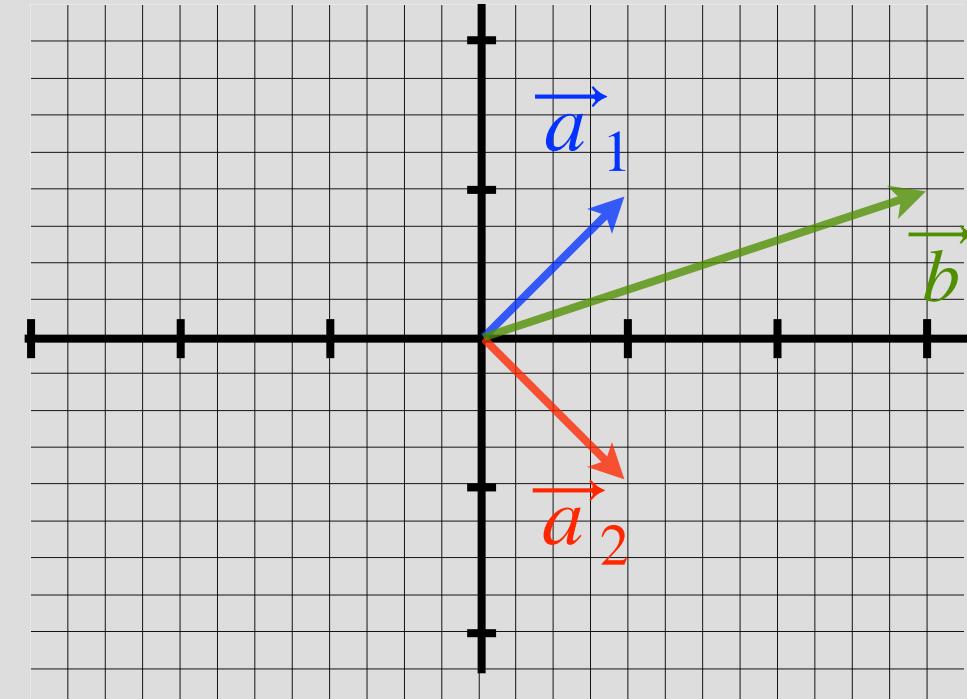
Matrix-vector multiplication is a linear combination of the columns of A!

Linear Set of Equations as a Linear Combination

- Consider the problem: $A \vec{x} = \vec{b}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\downarrow \overrightarrow{a}_1 \quad \downarrow \overrightarrow{a}_2 \quad \downarrow \overrightarrow{b}$



Linear Set of Equations as a Linear Combination

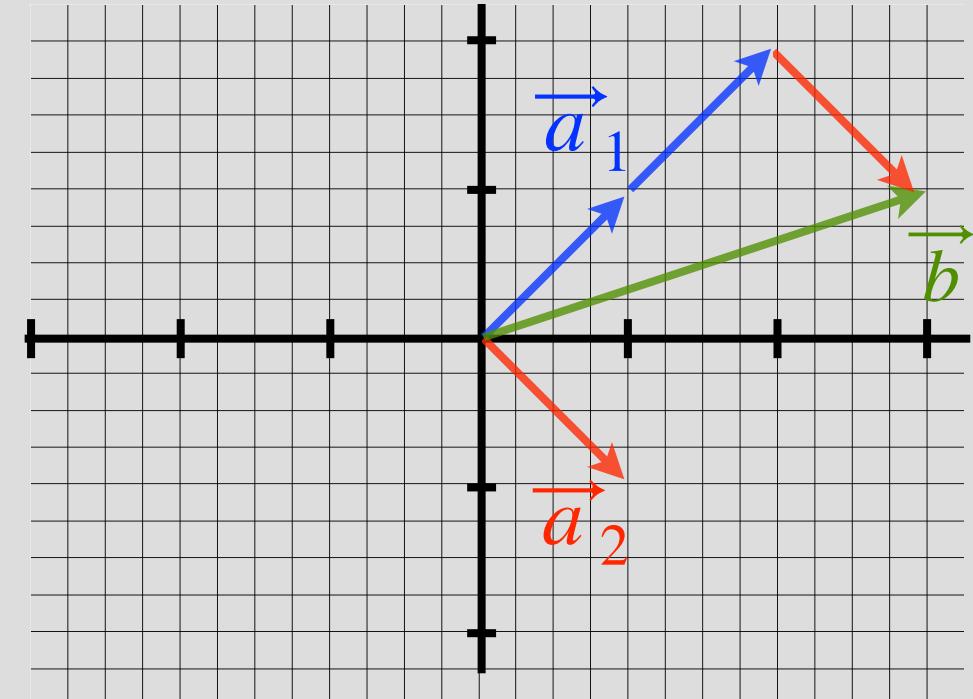
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$\downarrow \quad \downarrow \quad \downarrow$

$\vec{a}_1 \quad \vec{a}_2 \quad \vec{b}$

Q: What linear combination of \vec{a}_1, \vec{a}_2 will give \vec{b} ?



Linear Set of Equations as a Linear Combination

- Consider the problem: $A \vec{x} = \vec{b}$:

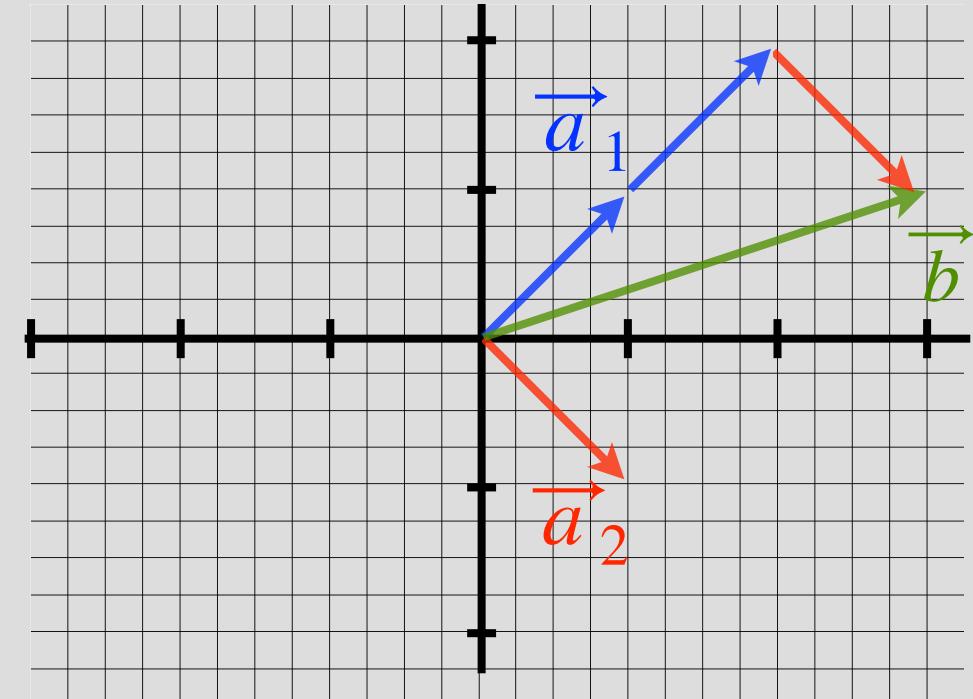
$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\downarrow \quad \downarrow \quad \downarrow$

$\vec{a}_1 \quad \vec{a}_2 \quad \vec{b}$

Q: What linear combination of \vec{a}_1 , \vec{a}_2 will give \vec{b} ?

A: $2\vec{a}_1 + 1\vec{a}_2$



Linear Set of Equations as a Linear Combination

- Consider the problem: $A \vec{x} = \vec{b}$:

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$\downarrow \vec{a}_1 \quad \downarrow \vec{a}_2 \quad \downarrow \vec{b}$

Q: What linear combination of \vec{a}_1, \vec{a}_2 will give \vec{b} ?

A: $2\vec{a}_1 + 1\vec{a}_2$

Gaussian Elimination:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array} \right]$$

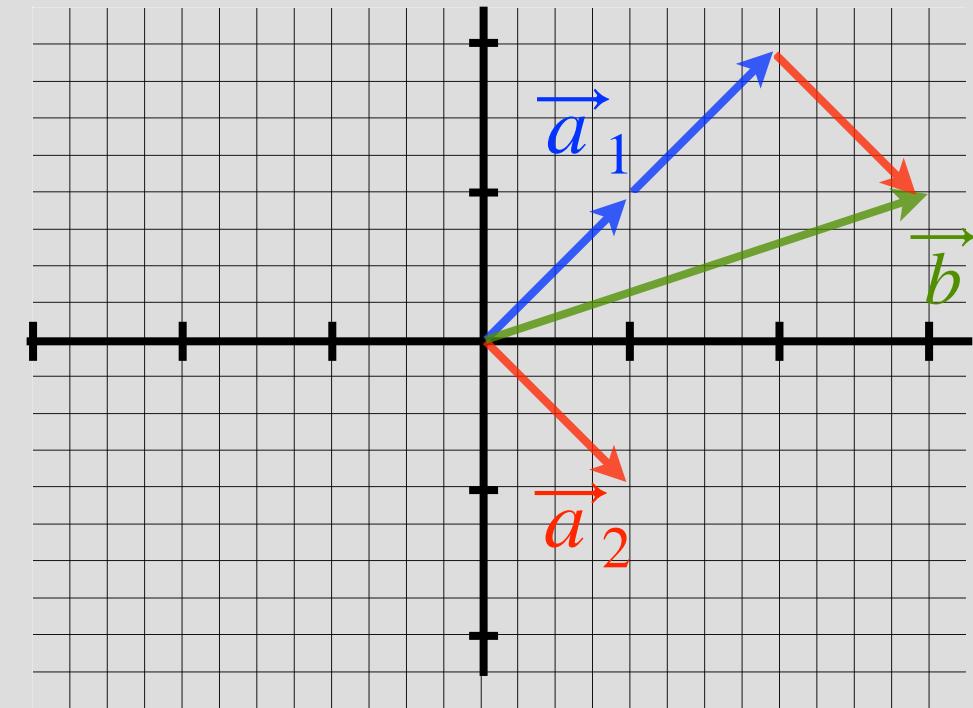
$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 2 & 1 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -2 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array} \right]$$

$$x_1 = 2$$

$$x_2 = 1$$



same as

$$\vec{b} = \underline{\underline{2}}\vec{a}_1 + \underline{\underline{1}}\cdot\vec{a}_2$$



Linear Set of Equations as a Linear Combination

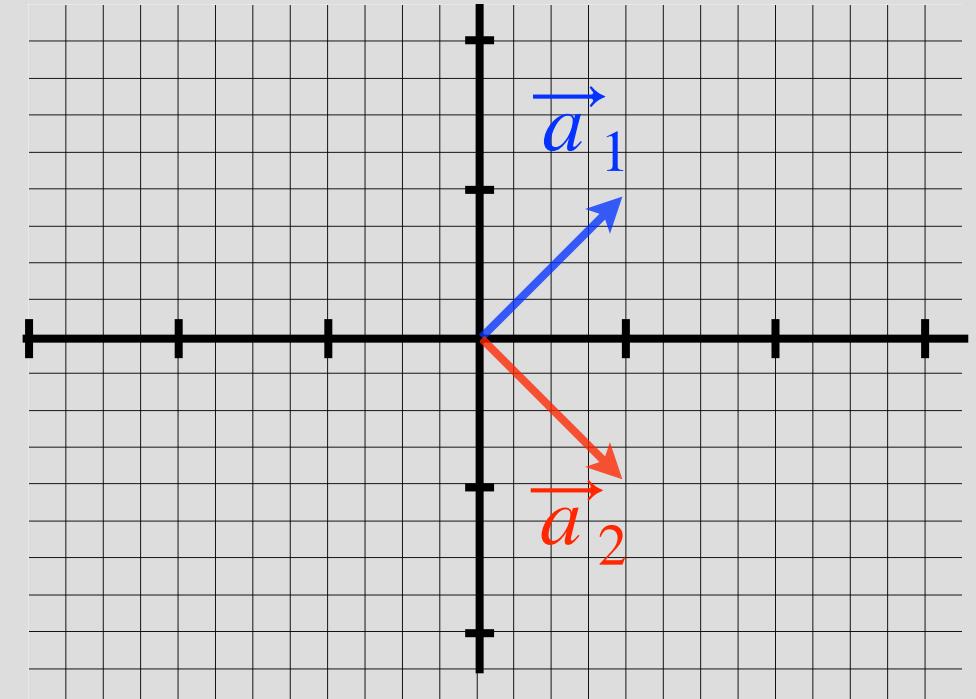
- Consider the problem: $A \vec{x} = \vec{b}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}$$

\downarrow \downarrow
 \vec{a}_1 \vec{a}_2

Q: Can linear combination of \vec{a}_1, \vec{a}_2 give any \vec{b} ?

A: Hmm....I think so....



Linear Set of Equations as a Linear Combination

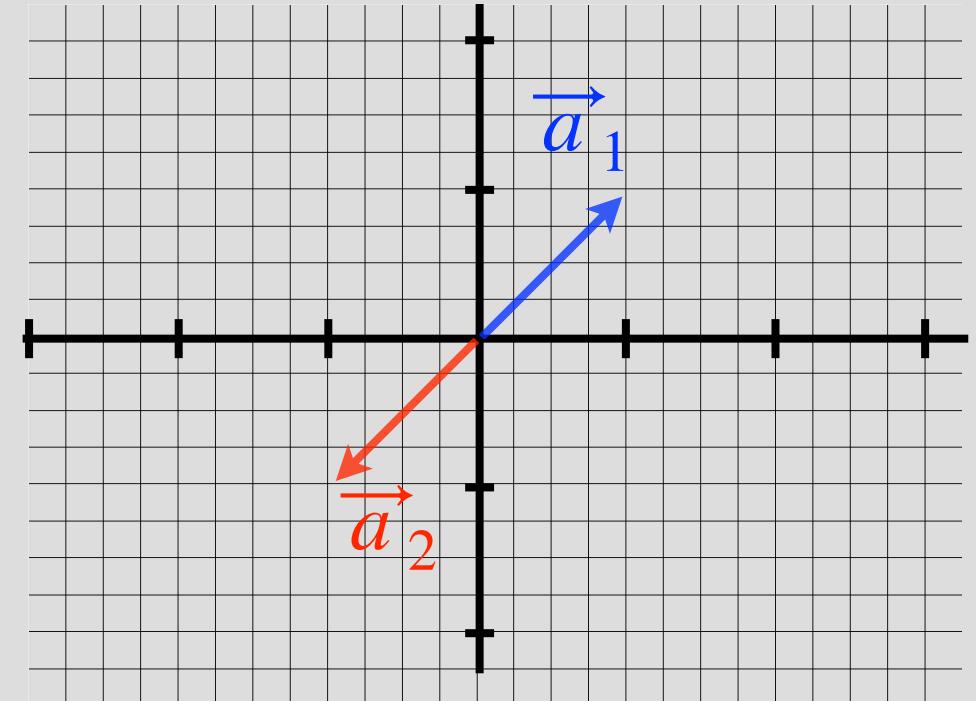
- Consider the problem: $A \vec{x} = \vec{b}$:

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}$$

\downarrow \downarrow
 \vec{a}_1 \vec{a}_2

Q: Can linear combination of \vec{a}_1, \vec{a}_2 give any \vec{b} ?

A: Hmm....I don't think so.... Unless its along the line \vec{a}_1



Linear Set of Equations as a Linear Combination

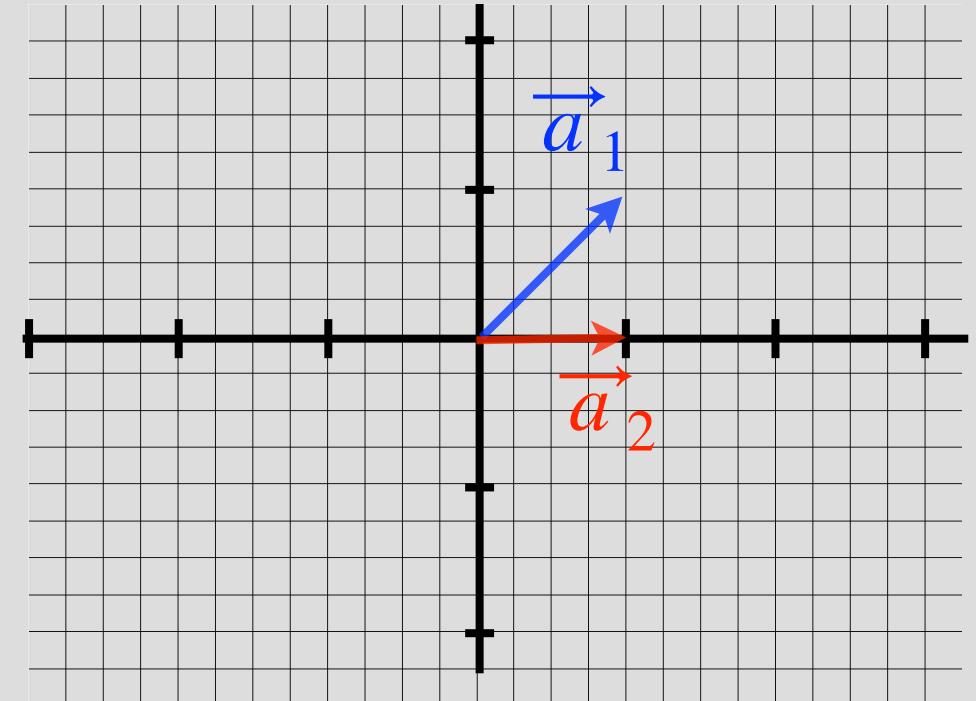
- Consider the problem: $A \vec{x} = \vec{b}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}$$

\downarrow \downarrow
 \vec{a}_1 \vec{a}_2

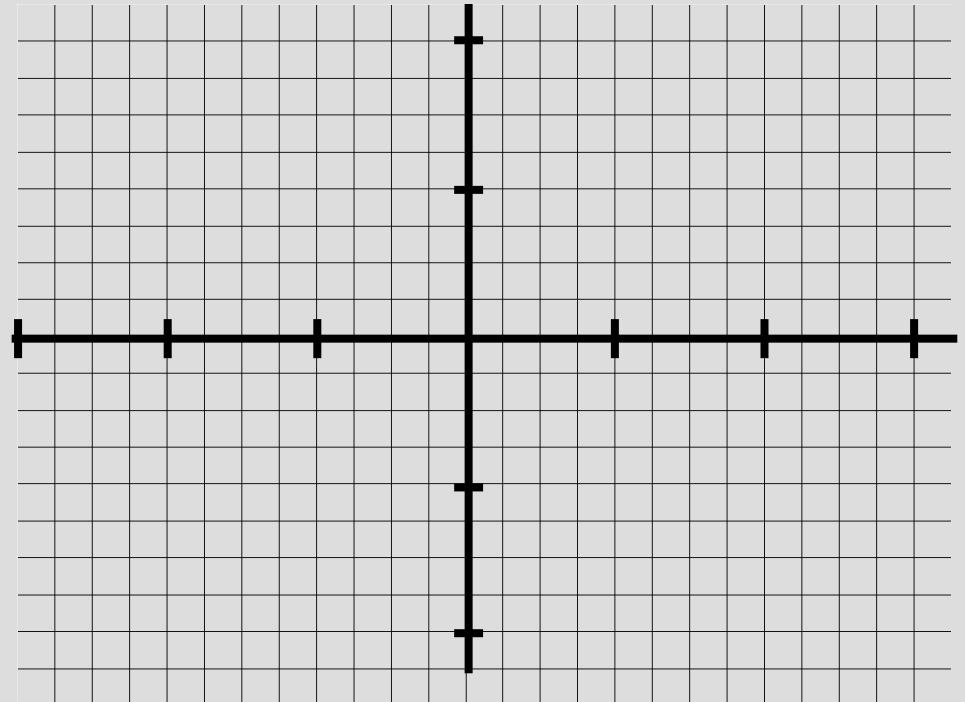
Q: Can linear combination of \vec{a}_1, \vec{a}_2 give any \vec{b} ?

A: Hmm....yes!



Span / Column Space / Range

- Span of the columns of A is the set of all vectors \vec{b} such that $A\vec{x} = \vec{b}$ has a solution
 - i.e. the set of all vectors that can be reached by all possible linear combinations of the columns of A

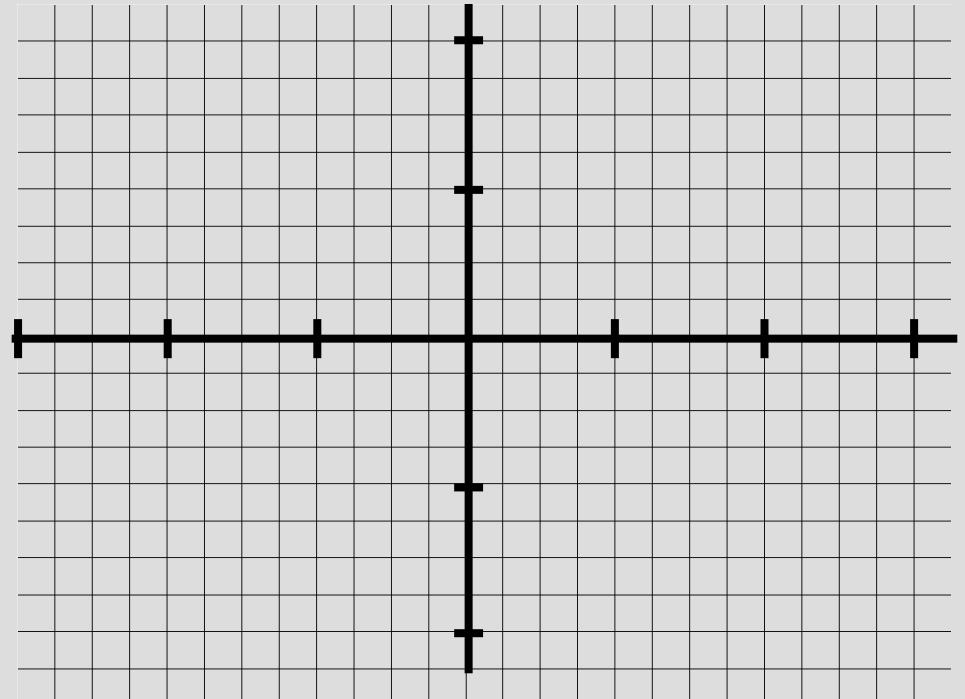


Span / Column Space / Range

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Example: What is the span of the cols of A?

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



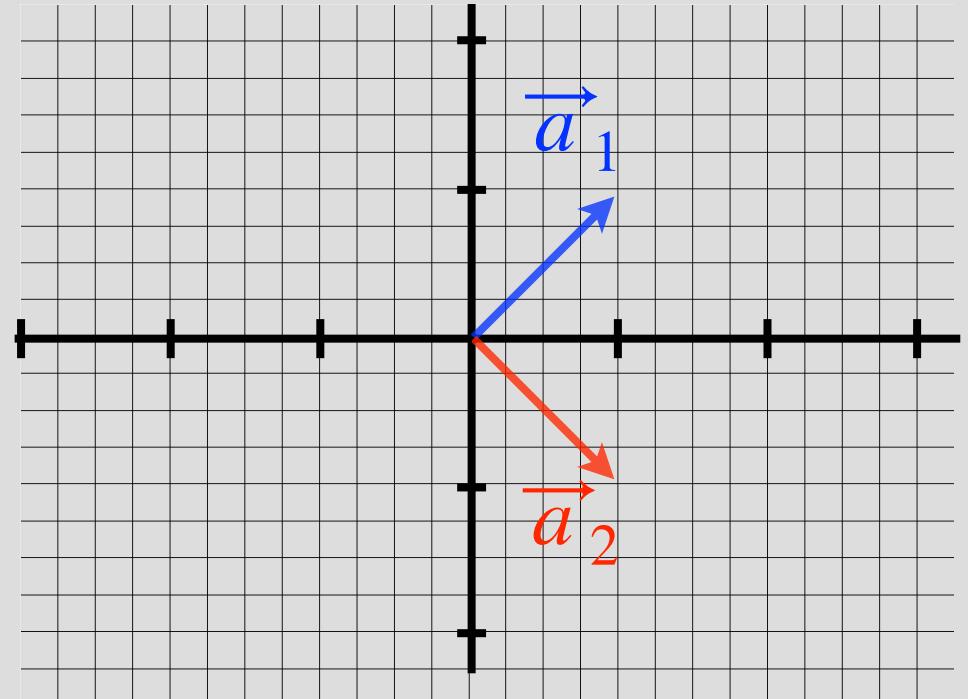
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Example: What is the span of the cols of A?

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

A: \mathbb{R}^2 !



Span / Column Space / Range

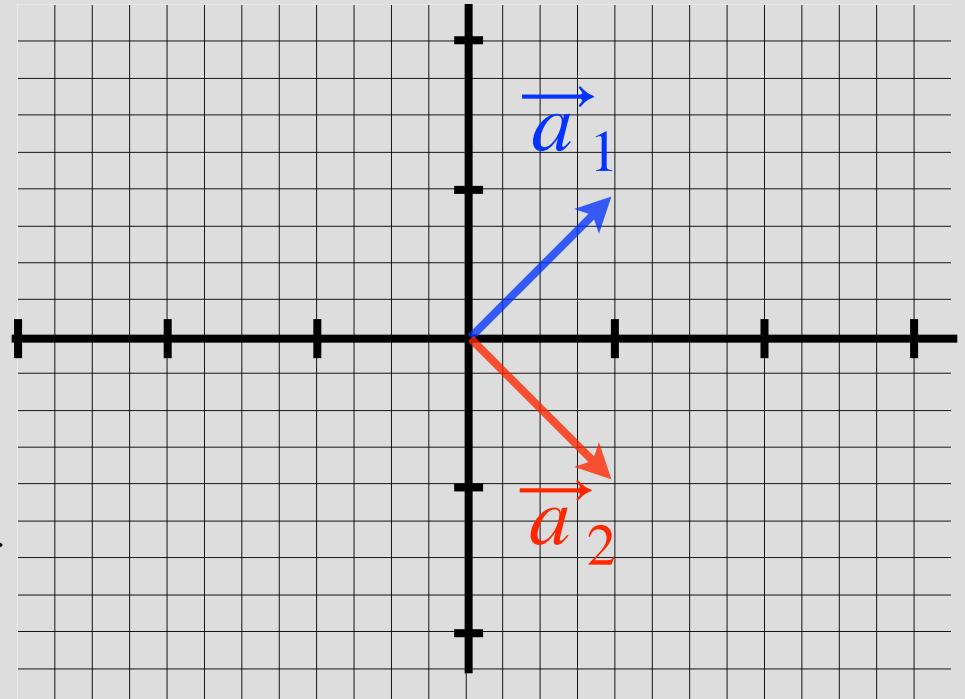
- Span of the columns of A is the set of all vectors \vec{b} such that $A\vec{x} = \vec{b}$ has a solution
 - i.e. the set of all vectors that can be reached by all possible linear combinations of the columns of A

Example: What is the span of the cols of A?

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

A: \mathbb{R}^2 !

$$\text{span}(\text{cols of } A) = \left\{ \vec{v} \mid \vec{v} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \alpha, \beta \in \mathbb{R} \right\}$$



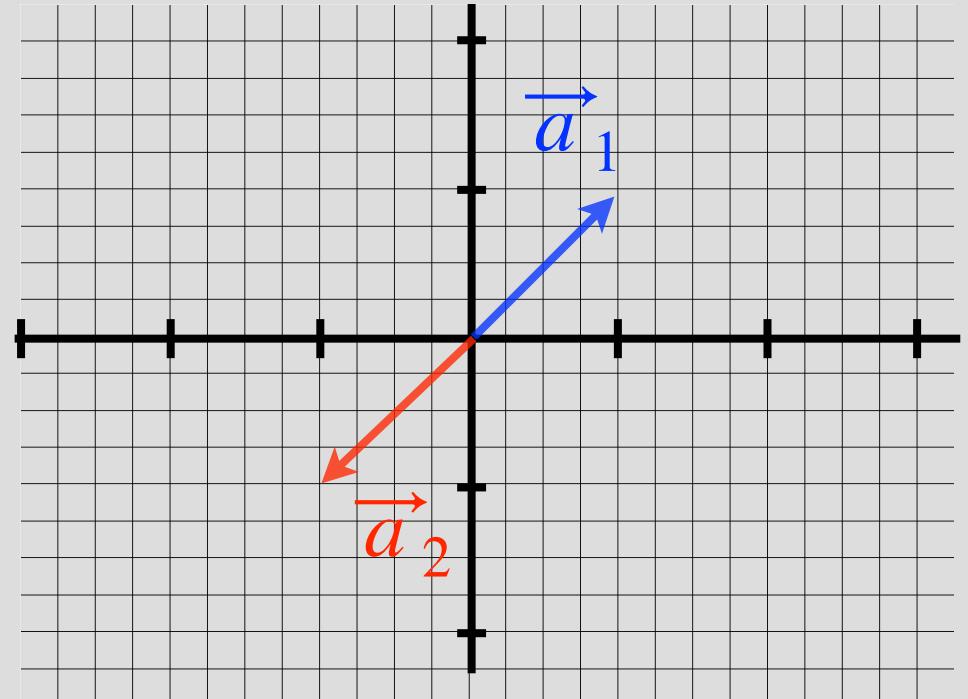
Span / Column Space / Range

Example 2: What is the span of the cols of A?

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

A: The line $x_1 = x_2$

$$\text{span}(\text{cols of } A) = \left\{ \vec{v} \mid \vec{v} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \alpha \in \mathbb{R} \right\}$$



Span / Column Space / Range

- Definition:

If $\exists \vec{x}$ s.t. $A\vec{x} = \vec{b}$ then $\vec{b} \in \text{span}\{A\}$

Converse: $\vec{b} \in \text{span}\{\text{cols}(A)\}$, there is a solution for $A\vec{x} = \vec{b}$

Q: What if $\vec{b} \notin \text{span}\{\text{cols}(A)\}$?

A: There is no solution for $A\vec{x} = \vec{b}$