Welcome to EECS 16A!
Designing Information Devices and Systems I

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Sp 2022

Lecture 1A
Gaussian Elimination
Vectors

What is linear algebra?

- The study of linear functions and linear equations, typically using vectors and matrices

- Linearity is not always applicable, but can be a good first-order approximation

- There exist good fast algorithms to solve these problems

Linear Equations

- Definition:

Consider: $f(x_1, x_2, \cdots, x_N) : \mathbb{R}^n \to \mathbb{R}$

f is *linear if the following identity holds:*

(1) Homogeneity:

$$f(\alpha x_1, \cdots, \alpha x_N) = \alpha f(x_1, \ldots, x_N)$$

(2) Super Position (distributivity): if $x_i = y_i + z_i$, then

$$f(y_1 + z_1, \cdots, y_N + z_N) = f(y_1, \cdots, y_N) + f(z_1, \cdots, z_N)$$

Claim: linear functions can always be expressed as:

$$f(x_1, x_2, \cdots, x_N) = c_1 x_1 + c_2 x_2 + \cdots + c_N x_N$$

Proof for $\mathbb{R}^2$

- $f(x_1, x_2) : \mathbb{R}^2 \Rightarrow \mathbb{R}$ is linear. Need to prove: $f(x_1, x_2) = c_1 x_1 + c_2 x_2$

Trick:

$$x_1 = 1 \cdot x_1 + 0 \cdot x_2 \qquad \Rightarrow x_1 = x_1 y_1 + x_2 z_1$$

$$x_2 = 0 \cdot x_1 + 1 \cdot x_2 \qquad \Rightarrow x_2 = x_1 y_2 + x_2 z_2$$

So,

$$f(x_1, x_2) = f(x_1 y_1 + x_2 z_1, \ x_1 y_2 + x_2 z_2)$$

$$= f(x_1 y_1, x_1 y_2) + f(x_2 z_1, x_2 z_2)$$

$$= x_1 f(y_1, y_2) + x_2 f(z_1, z_2)$$

$$= x_1 f(1,0) + x_2 f(0,1)$$

$$= c_1 x_1 + c_2 x_2$$

Linear Set of Equations

- Consider the set of M linear equations with N variables:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N = b_1$$

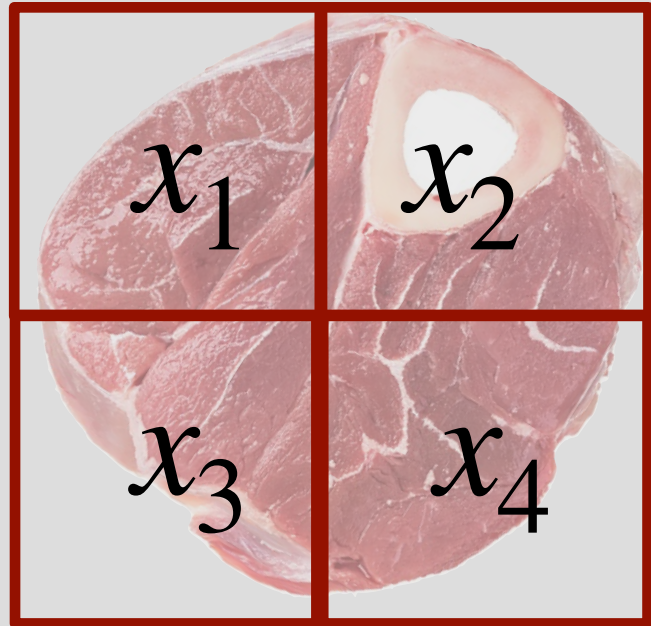$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N = b_2$$

$$\vdots \qquad\qquad \vdots$$

$$a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N = b_M$$

- Can be written compactly using augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1N} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2N} & b_2 \\ \vdots & & \vdots & & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} & b_M \end{array} \right]$$

Back to Tomography

$$1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4$$

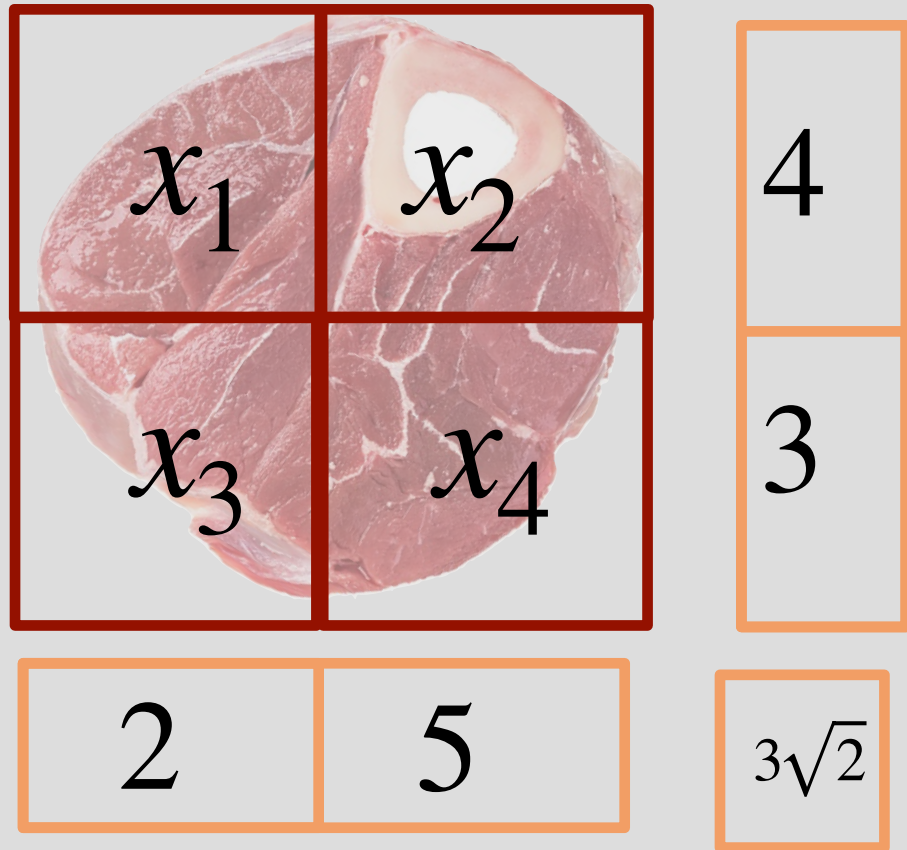$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 = 3$$

$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2$$

$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$

$$\sqrt{2}x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \sqrt{2}x_4 = 3\sqrt{2}$$

x_1	x_2	4
x_3	x_4	3

| 2 | 5 |

| $3\sqrt{2}$ |

Back to Tomography

$$1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4$$

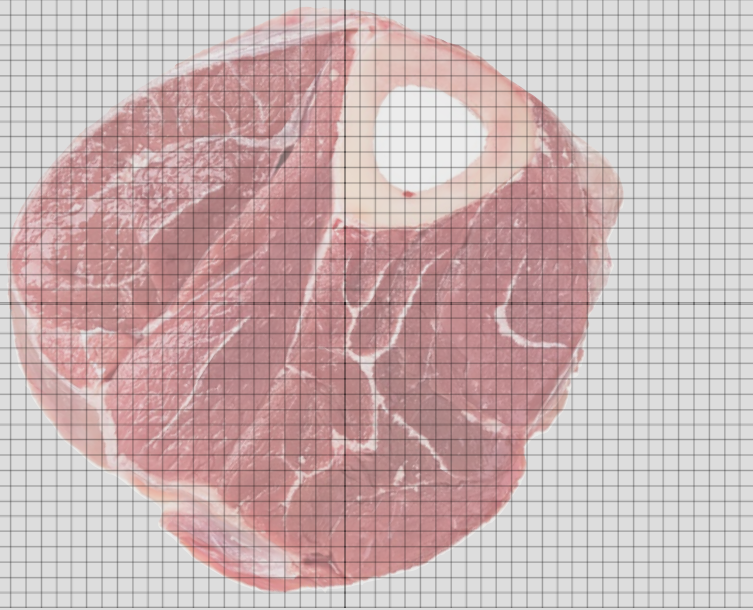$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 = 3$$

$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2$$

$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$

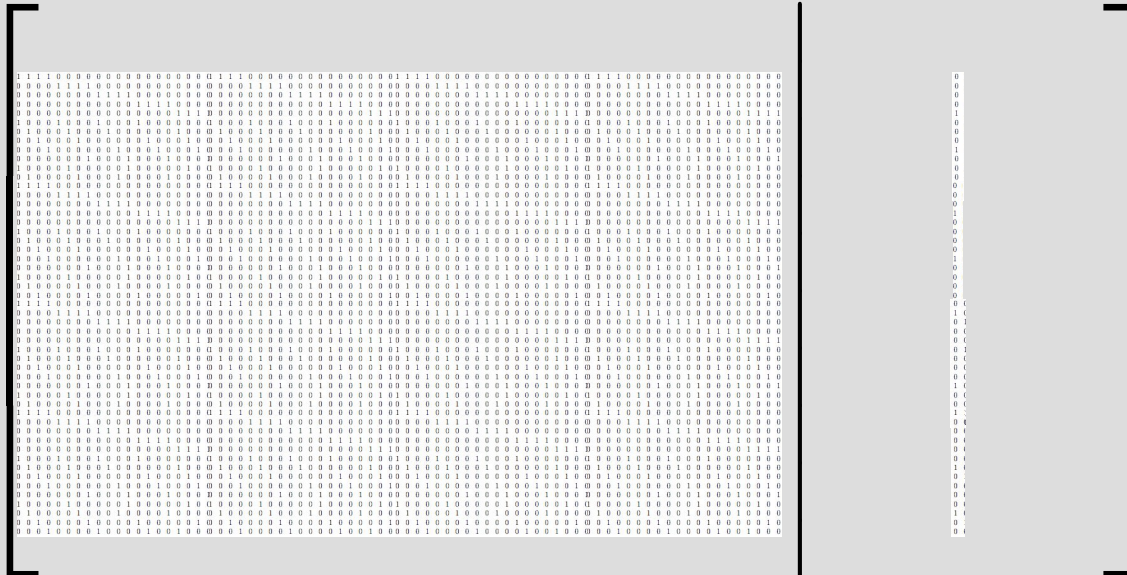$$\sqrt{2}x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \sqrt{2}x_4 = 3\sqrt{2}$$

$$
\left[
\begin{array}{cccc|c}
1 & 1 & 0 & 0 & 4 \\
0 & 0 & 1 & 1 & 3 \\
1 & 0 & 1 & 0 & 2 \\
0 & 1 & 0 & 1 & 5 \\
\sqrt{2} & 0 & 0 & \sqrt{2} & 3\sqrt{2}
\end{array}
\right]
$$

x_1	x_2	4
x_3	x_4	3

| 2 | 5 | $3\sqrt{2}$ |

How do we solve it?

Back to Tomography

$$x_1 + x_2 + \cdots + x_N = b_1$$
$$= b_2$$
$$\vdots$$
$$= b_M$$

How do we systematically solve it?

Algorithm for solving linear equations

- Three basic operations that don't change a solution:

    1. Multiply an equation with *nonzero* scalar

    2. Adding a scalar constant multiple of one equation to another

    3. Swapping equations

Algorithm for solving linear equations

- Three basic operations that don't change a solution:

  1. Multiply an equation with *nonzero* scalar

  2. Adding a scalar constant multiple of one equation to another

  3. Swapping equations

$$\begin{array}{llllll}
(1) & x & + & y & = & 2 \\
(2) & 3x & + & 2y & = & 5
\end{array}$$

and

$$\begin{array}{llllll}
(1) & 3x & + & 2y & = & 5 \\
(2) & x & + & y & = & 2
\end{array}$$

Have the same solution

Proof: Pretty obvious!

Algorithm for solving linear equations

- Three basic operations that don't change a solution:

  1. Multiply an equation with *nonzero* scalar

$$2x + 3y = 4 \quad \text{has the same solution as:} \quad 4x + 6y = 8$$

Proof for N=2:

Let $ax + by = c$, with solution x_0, y_0
$\Rightarrow ax_0 + by_0 = c$

Show that $\beta ax + \beta by = \beta c$,
has the same solution.

Substitute x_0, y_0 for x, y:

$$\beta ax_0 + \beta by_0 = \beta c$$

$$\beta(ax_0 + by_0) = \beta c$$

$$\beta c = \beta c \quad \text{But is it the only solution?}$$

$\beta ax + \beta by = \beta c$, with solution: x_1, y_1
$\Rightarrow \beta ax_1 + \beta by_1 = \beta c$

Show that $ax + by = c$,
has the same solution.....

Since $\beta \neq 0$....

$$\beta ax_1 + \beta by_1 = \beta c \Rightarrow ax_1 + by_1 = c$$

SOLUTION OF ONE, IMPLIES THE OTHER
AND VICE-VERSA!

Algorithm for solving linear equations

- Three basic operations that don't change a solution:

  1. Multiply an equation with *nonzero* scalar

  2. Adding a scalar constant multiple of one equation to another

$$
\begin{array}{rrrrr}
(1) & x & + & y & = & 2 \\
(2) & 3x & + & 2y & = & 5
\end{array}
$$

and

$$
\begin{array}{rrrrr}
(1) & x & + & y & = & 2 \\
3 \times (1) + (2) & 6x & + & 5y & = & 11
\end{array}
$$

Have the same solution

Concept of proof: look at explicit solution, show they are the same

Also show the reverse — by applying the reverse operations

Upper Diagonal Systems

- Consider the following equations:

$$
\begin{array}{rcrcrcr}
x & - & y & + & 2z & = & 1 \\
  &   & y & - & z & = & 2 \\
  &   &   &   & z & = & 1
\end{array}
$$

$$
\left[\quad \middle| \quad \right]
$$

Upper Diagonal Systems

- Consider the following equations:

$$\begin{array}{rcrcrcr} x & - & y & + & 2z & = & 1 \\ & & y & - & z & = & 2 \\ & & & & z & = & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

- Why are they easy to solve?

More general Row Echelon in the notes!

Upper Triangular matrix \ Row Echelon

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$$

Pivots

Gaussian Elimination

$$\begin{array}{rcrcrcr}
x & - & y & + & 2z & = & 1 \\
2x & + & y & + & z & = & 8 \\
-4x & + & 5y & & & = & 7
\end{array}$$

- Row-reduction to upper triang (Row echelon):

Step I

$$\begin{bmatrix} & & & | & \\ & & & | & \\ & & & | & \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

Gaussian Elimination

$$\begin{array}{rcrcrcr} x & - & y & + & 2z & = & 1 \\ 2x & + & y & + & z & = & 8 \\ -4x & + & 5y & & & = & 7 \end{array}$$

- Row-reduction to upper triang (Row echelon):

Step I

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ -4 & 5 & 0 & 7 \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Step II

$$\begin{array}{l} \\ (2) - 2 \times (1) \\ (3) + 4 \times (1) \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ & & & \\ & & & \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Gaussian Elimination

$$\begin{array}{rcrcrcl}
x & - & y & + & 2z & = & 1 \\
2x & + & y & + & z & = & 8 \\
-4x & + & 5y & & & = & 7
\end{array}$$

- Row-reduction to upper triang (Row echelon):

Step I

$$\left[\begin{array}{ccc|c}
1 & -1 & 2 & 1 \\
2 & 1 & 1 & 8 \\
-4 & 5 & 0 & 7
\end{array}\right] \begin{array}{l}(1)\\(2)\\(3)\end{array}$$

Step II

$$\begin{array}{l} \\ (2)-2\times(1) \\ (3)+4\times(1) \end{array} \left[\begin{array}{ccc|c}
1 & -1 & 2 & 1 \\
0 & 3 & -3 & 6 \\
0 & 1 & 8 & 11
\end{array}\right] \begin{array}{l}(1)\\(2)\\(3)\end{array}$$

Step III

$$(2)/3 \left[\begin{array}{ccc|c}
1 & -1 & 2 & 1 \\
& & & \\
& & &
\end{array}\right] \begin{array}{l}(1)\\(2)\\(3)\end{array}$$

Gaussian Elimination

$$\begin{aligned} x &- y &+ 2z &= 1 \\ 2x &+ y &+ z &= 8 \\ -4x &+ 5y & &= 7 \end{aligned}$$

- Row-reduction to upper triang (Row echelon):

Step I

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ -4 & 5 & 0 & 7 \end{array}\right] \begin{array}{l}(1)\\(2)\\(3)\end{array}$$

Step II

$$\begin{array}{l} \\ (2)-2\times(1) \\ (3)+4\times(1) \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & 6 \\ 0 & 1 & 8 & 11 \end{array}\right] \begin{array}{l}(1)\\(2)\\(3)\end{array}$$

Step III

$$(2)/3 \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 8 & 11 \end{array}\right] \begin{array}{l}(1)\\(2)\\(3)\end{array}$$

Step IV

$$(3)-(2) \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ & & & \end{array}\right] \begin{array}{l}(1)\\(2)\\(3)\end{array}$$

Gaussian Elimination

$$\begin{array}{rcrcrcr} x & - & y & + & 2z & = & 1 \\ 2x & + & y & + & z & = & 8 \\ -4x & + & 5y & & & = & 7 \end{array}$$

- Row-reduction to upper triang (Row echelon):

Step I

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ -4 & 5 & 0 & 7 \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Step II

$$\begin{array}{l} \\ (2)-2\times(1) \\ (3)+4\times(1) \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & 6 \\ 0 & 1 & 8 & 11 \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Step III

$$\begin{array}{l} \\ (2)/3 \\ \\ \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 8 & 11 \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Step IV

$$\begin{array}{l} \\ \\ (3)-(2) \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 9 & 9 \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Step V

$$\begin{array}{l} \\ \\ (3)/9 \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ & & & \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Gaussian Elimination

$$
\begin{array}{rrrrr}
x & - & y & + & 2z & = & 1 \\
2x & + & y & + & z & = & 8 \\
-4x & + & 5y & & & = & 7
\end{array}
$$

- Row-reduction to upper triang (Row echelon):

Step I

$$
\left[\begin{array}{rrr|r}
1 & -1 & 2 & 1 \\
2 & 1 & 1 & 8 \\
-4 & 5 & 0 & 7
\end{array}\right]
\begin{array}{l}(1)\\(2)\\(3)\end{array}
$$

Step II

$$
\begin{array}{l}\\(2)-2\times(1)\\(3)+4\times(1)\end{array}
\left[\begin{array}{rrr|r}
1 & -1 & 2 & 1 \\
0 & 3 & -3 & 6 \\
0 & 1 & 8 & 11
\end{array}\right]
\begin{array}{l}(1)\\(2)\\(3)\end{array}
$$

Step III

$$
(2)/3
\left[\begin{array}{rrr|r}
1 & -1 & 2 & 1 \\
0 & 1 & -1 & 2 \\
0 & 1 & 8 & 11
\end{array}\right]
\begin{array}{l}(1)\\(2)\\(3)\end{array}
$$

Step IV

$$
\begin{array}{l}\\\\(3)-(2)\end{array}
\left[\begin{array}{rrr|r}
1 & -1 & 2 & 1 \\
0 & 1 & -1 & 2 \\
0 & 0 & 9 & 9
\end{array}\right]
\begin{array}{l}(1)\\(2)\\(3)\end{array}
$$

Step V

$$
(3)/9
\left[\begin{array}{rrr|r}
1 & -1 & 2 & 1 \\
0 & 1 & -1 & 2 \\
0 & 0 & 1 & 1
\end{array}\right]
\begin{array}{l}(1)\\(2)\\(3)\end{array}
$$

Pivots

Gaussian Elimination Cont.

$$
\begin{aligned}
x &- y + 2z = 1 \\
2x &+ y + z = 8 \\
-4x &+ 5y \quad\quad = 7
\end{aligned}
$$

- Back substitution:

Step V

$$
\left[
\begin{array}{ccc|c}
1 & -1 & 2 & 1 \\
0 & 1 & -1 & 2 \\
0 & 0 & 1 & 1
\end{array}
\right]
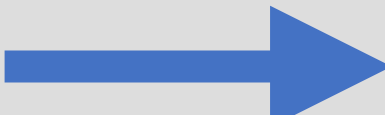\begin{array}{l}
(1) \\
(2) \\
(3)
\end{array}
$$

Step VI

$$
\begin{array}{l}
(1) - 2 \times (3) \\
\\
(2) + (3) \\
\\
\end{array}
\left[
\begin{array}{ccc|c}
 & & & \\
 & & & \\
0 & 0 & 1 & 1
\end{array}
\right]
\begin{array}{l}
(1) \\
(2) \\
(3)
\end{array}
$$

Gaussian Elimination Cont.

$$\begin{array}{ccccccc} x & - & y & + & 2z & = & 1 \\ 2x & + & y & + & z & = & 8 \\ -4x & + & 5y & & & = & 7 \end{array}$$

- Back substitution:

Step V

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Step VI

$$\begin{array}{c} (1) - 2 \times (3) \\ \\ (2) + (3) \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Step VII

$$\begin{array}{c} (1) + (2) \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} & & & \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Gaussian Elimination Cont.

$$\begin{array}{rcrcrcr} x & - & y & + & 2z & = & 1 \\ 2x & + & y & + & z & = & 8 \\ -4x & + & 5y & & & = & 7 \end{array}$$

- Back substitution:

Step V

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} \text{(1)} \\ \text{(2)} \\ \text{(3)} \end{array}$$

Step VI

$$\begin{array}{l} (1) - 2 \times (3) \\ \\ (2) + (3) \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} \text{(1)} \\ \text{(2)} \\ \text{(3)} \end{array}$$

Step VII

$$\begin{array}{l} (1) + (2) \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} \text{(1)} \\ \text{(2)} \\ \text{(3)} \end{array}$$

$$\qquad x \quad y \quad z$$

Diagonal/identity matrix
(reduced Row-Echelon form)

$\Longrightarrow$

$$\begin{array}{rcl} x & = & 2 \\ y & = & 3 \\ z & = & 1 \end{array}$$

Back to Tomography
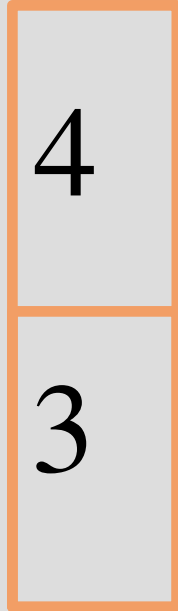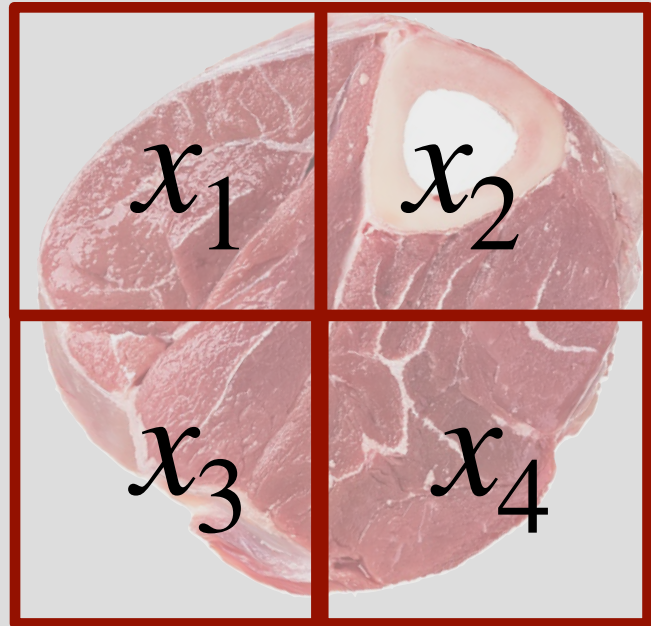
$$1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4$$

$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 = 3$$

$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2$$

$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & \bigm| & 4 \\ 0 & 0 & 1 & 1 & \bigm| & 3 \\ 1 & 0 & 1 & 0 & \bigm| & 2 \\ 0 & 1 & 0 & 1 & \bigm| & 5 \end{bmatrix}$$

How do we solve it?

Back to Tomography

Step I

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]\begin{array}{l}(1)\\(2)\\(3)\\(4)\end{array}$$

Step IV

$$\begin{array}{c}\\(3)+(2)\end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ & & & & \\ & & & & \end{array}\right]\begin{array}{l}(1)\\(2)\\(3)\\(4)\end{array}$$

Step II

$$\begin{array}{c}\\(3)-(1)\end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ & & & & \\ & & & & \end{array}\right]\begin{array}{l}(1)\\(2)\\(3)\\(4)\end{array}$$

Step V

$$\begin{array}{c}\\(4)-(3)\end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ & & & & \end{array}\right]\begin{array}{l}(1)\\(2)\\(3)\\(4)\end{array}$$

Step III

$$\begin{array}{c}\\(4)\\\\(2)\end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ & & & & \\ & & & & \\ & & & & \end{array}\right]\begin{array}{l}(1)\\(2)\\(3)\\(4)\end{array}$$

Back to Tomography

Step I

$$
\left[\begin{array}{cccc|c}
1 & 1 & 0 & 0 & 4 \\
0 & 0 & 1 & 1 & 3 \\
1 & 0 & 1 & 0 & 2 \\
0 & 1 & 0 & 1 & 5
\end{array}\right]
\begin{array}{l}(1)\\(2)\\(3)\\(4)\end{array}
$$

Step II

$$
(3) - (1) \left[\begin{array}{cccc|c}
1 & 1 & 0 & 0 & 4 \\
0 & 0 & 1 & 1 & 3 \\
0 & -1 & 1 & 0 & -2 \\
0 & 1 & 0 & 1 & 5
\end{array}\right]
\begin{array}{l}(1)\\(2)\\(3)\\(4)\end{array}
$$

Step III

$$
\begin{array}{l}\\(4)\\ \\(2)\end{array}
\left[\begin{array}{cccc|c}
1 & 1 & 0 & 0 & 4 \\
0 & 1 & 0 & 1 & 5 \\
0 & -1 & 1 & 0 & -2 \\
0 & 0 & 1 & 1 & 3
\end{array}\right]
\begin{array}{l}(1)\\(2)\\(3)\\(4)\end{array}
$$

Step IV

$$
(3) + (2) \left[\begin{array}{cccc|c}
1 & 1 & 0 & 0 & 4 \\
0 & 1 & 0 & 1 & 5 \\
0 & 0 & 1 & 1 & 3 \\
0 & 0 & 1 & 1 & 3
\end{array}\right]
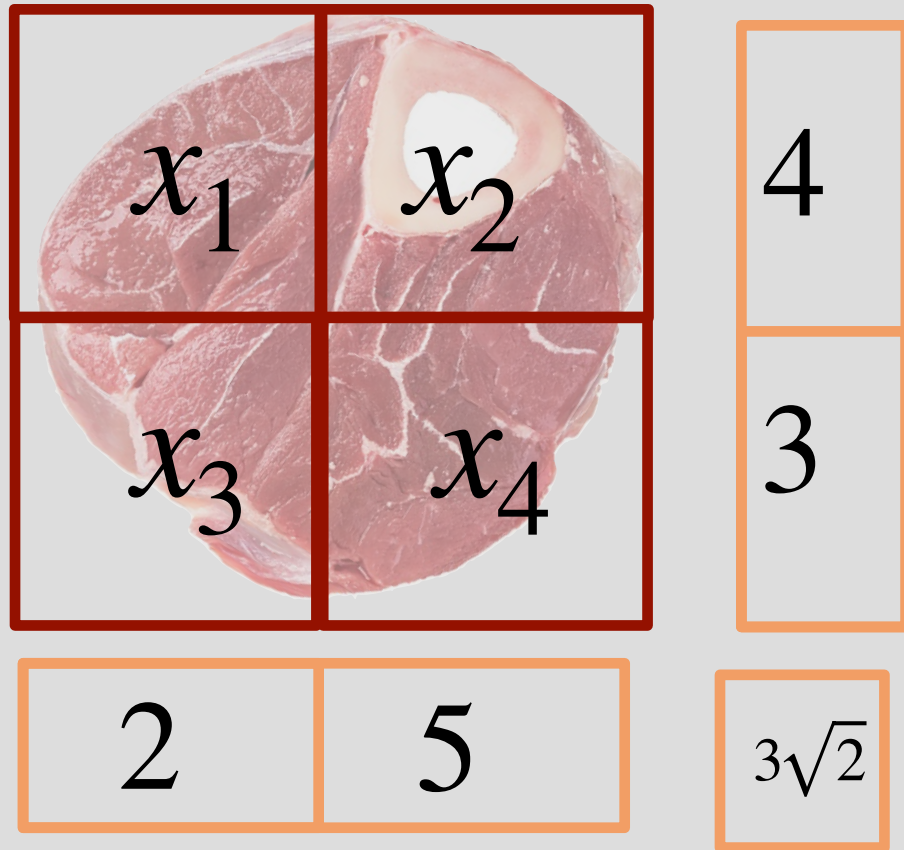\begin{array}{l}(1)\\(2)\\(3)\\(4)\end{array}
$$

Step V

$$
(4) - (3) \left[\begin{array}{cccc|c}
1 & 1 & 0 & 0 & 4 \\
0 & 1 & 0 & 1 & 5 \\
0 & 0 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
\begin{array}{l}(1)\\(2)\\(3)\\(4)\end{array}
$$

Infinite solutions!

Back to Tomography

Back substitution:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(1)-(2)

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Row Echelon

Pivots

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced Row Echelon

Pivots

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Basic variables

Free variables

Back to Tomography

Perturbations in the measurements

x_1 x_2

x_3 x_4

4

3.1

2   5
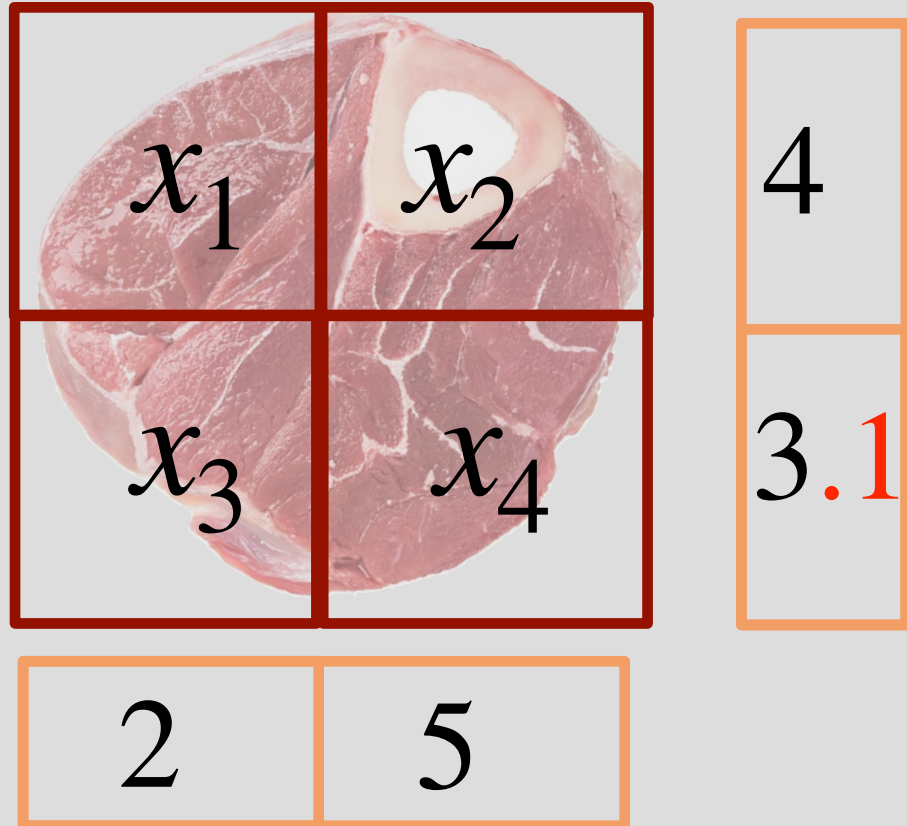
How do we solve it?

$$1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4$$

$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 = 3.1$$

$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2$$

$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3.1 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \end{array} \right]$$

Back to Tomography

Step I
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3.1 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]$$

Step II

$(3) - (1)$
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3.1 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]$$

Step III
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 3.1 \end{array}\right]$$

Step IV

$(3) + (2)$
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 3.1 \end{array}\right]$$

Step V

$(4) - (3)$
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]$$

No Solution!

Back to Tomography

$$1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4$$

$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 = 3$$

$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2$$

$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$

$$\sqrt{2}x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \sqrt{2}x_4 = 3\sqrt{2}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \\ \sqrt{2} & 0 & 0 & \sqrt{2} & 3\sqrt{2} \end{array} \right]$$

How do we solve it?

Gaussian Elimination

$$
\left[\begin{array}{cccc|c}
1 & 1 & 0 & 0 & 4 \\
0 & 0 & 1 & 1 & 3 \\
1 & 0 & 1 & 0 & 2 \\
0 & 1 & 0 & 1 & 5 \\
\sqrt{2} & 0 & 0 & \sqrt{2} & 3\sqrt{2}
\end{array}\right]
$$

$(3) + (2)$
$(4) - (2)$

$$\left[\begin{array}{cccc|c} & & & & \end{array}\right]$$

$(2) + (4)$

$(3) + (4)$

$$\left[\begin{array}{cccc|c} & & & & \end{array}\right]$$

$(3) - (1)$

$(5) - \sqrt{2} \times (1)$

$$\left[\begin{array}{cccc|c} & & & & \end{array}\right]$$

$(5) - (3)$

$$\left[\begin{array}{cccc|c} & & & & \end{array}\right]$$

$(1) - (2)$

$$\left[\begin{array}{cccc|c} & & & & \end{array}\right]$$

$-(5)/\sqrt{}(2)$

(2)

$$\left[\begin{array}{cccc|c} & & & & \end{array}\right]$$

$(4)/2$

$(5) - (4)$

$$\left[\begin{array}{cccc|c} & & & & \end{array}\right]$$

Gaussian Elimination

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \\ \sqrt{2} & 0 & 0 & \sqrt{2} & 3\sqrt{2} \end{array}\right]$$

$(3) + (2)$
$(4) - (2)$
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]$$

$(2) + (4)$
$(3) + (4)$
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$(3) - (1)$
$(5) - \sqrt{2} \times (1)$
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & -\sqrt{2} & 0 & \sqrt{2} & -\sqrt{2} \end{array}\right]$$

$(5) - (3)$
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right]$$

$(1) - (2)$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$-(5)/\sqrt(2)$
(2)
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]$$

$(4)/2$
$(5) - (4)$
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Tomography Solved!
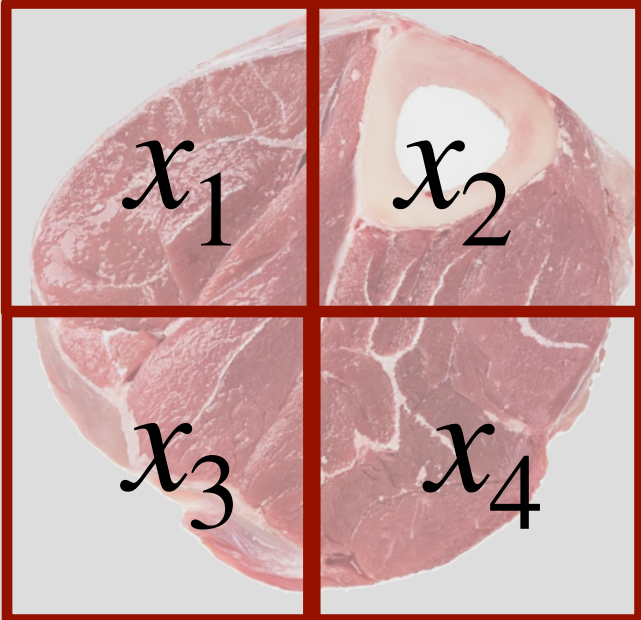
Possible reconstruction

1	3
1	2

$$
\begin{array}{cccc}
x_1 & x_2 & x_3 & x_4
\end{array}
$$

$$
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 3 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

Blurred version of :

Geometric Interpretation

(1) $x + 4y = 6$ $\qquad$ $x = 0 \Rightarrow y = 1.5$ $\quad$ $y = 0 \Rightarrow x = 6$

(2) $2x - y = 3$ $\qquad$ $x = 0 \Rightarrow y = -3$ $\quad$ $y = 0 \Rightarrow x = 1.5$

Single Solution!

$x = 2, \, y = 1$
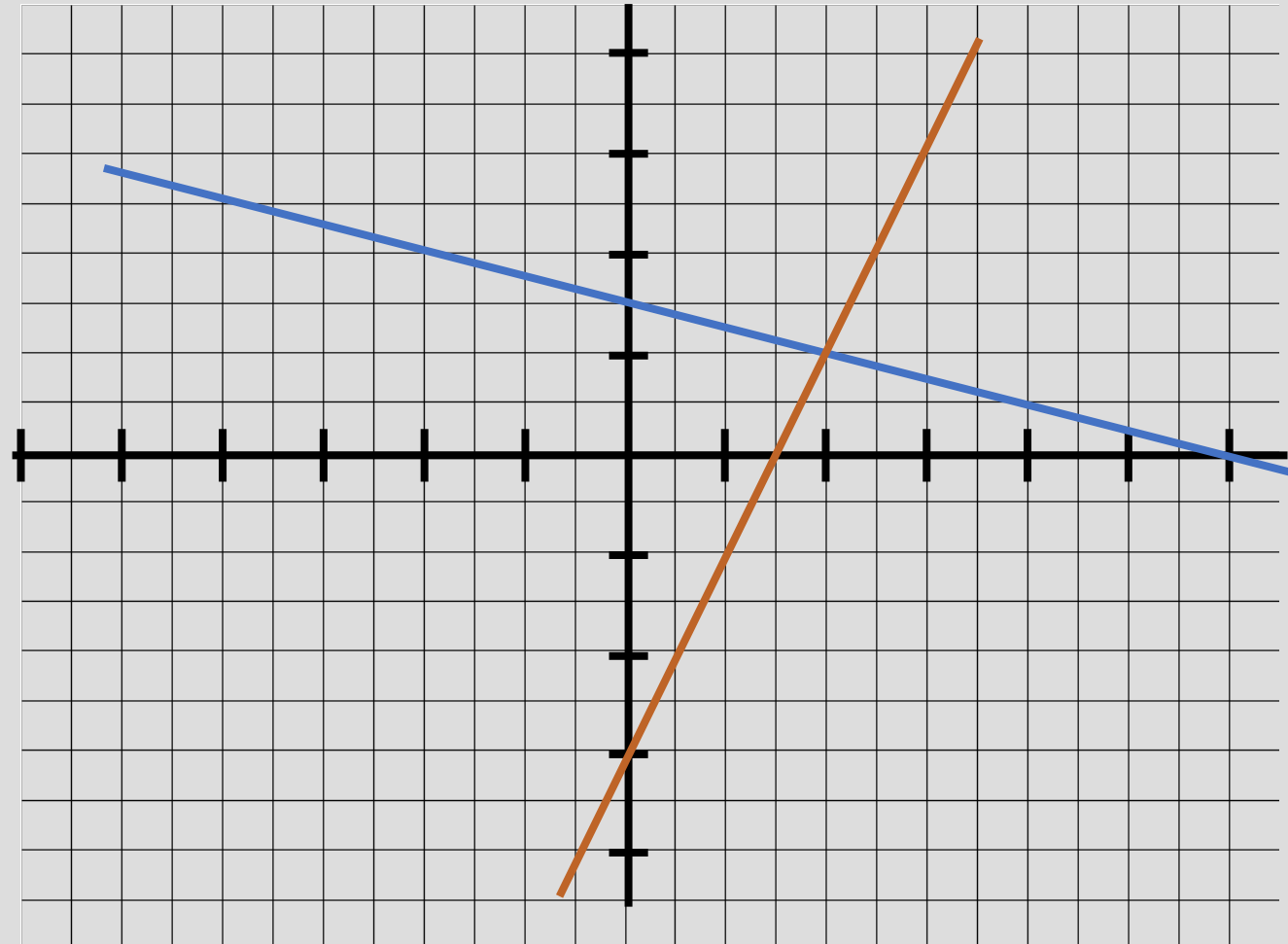
Geometric Interpretation

(1) $x + 4y = 6$ $\qquad$ $x = 0 \Rightarrow y = 1.5$ $\quad$ $y = 0 \Rightarrow x = 6$

(2) $2x + 8y = 12$ $\qquad$ $x = 0 \Rightarrow y = 1.5$ $\quad$ $y = 0 \Rightarrow x = 6$

Infinite Solutions!
anything that satisfies:

$$x = 6 - 4y_0$$
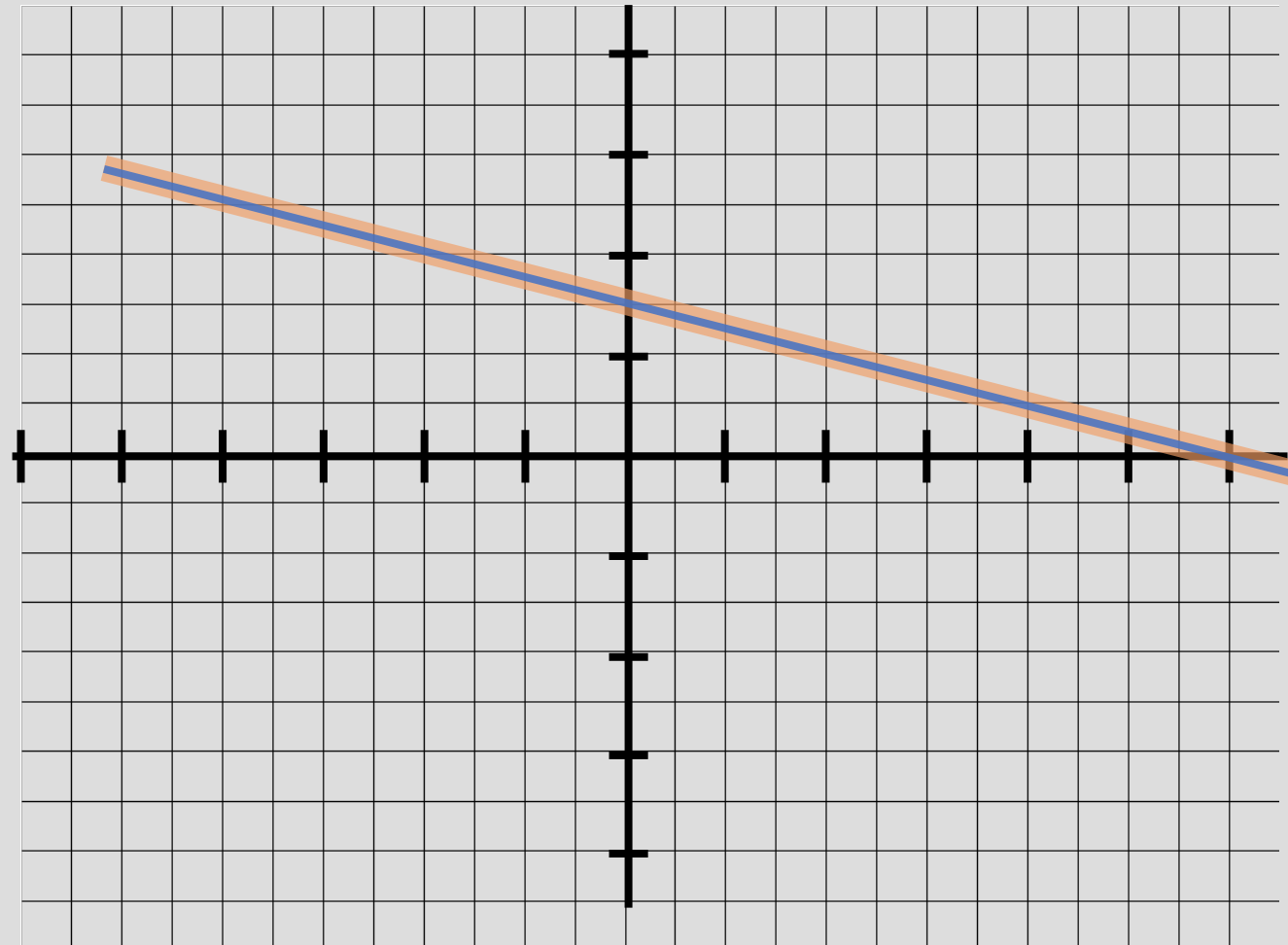
Geometric Interpretation

(1) $x + 4y = 6$ $x = 0 \Rightarrow y = 1.5$ $y = 0 \Rightarrow x = 6$

(2) $2x + 8y = \underline{8}$ $x = 0 \Rightarrow y = 1$ $y = 0 \Rightarrow x = 4$

No Solutions!
Parallel lines do not intersect!

$\varnothing$
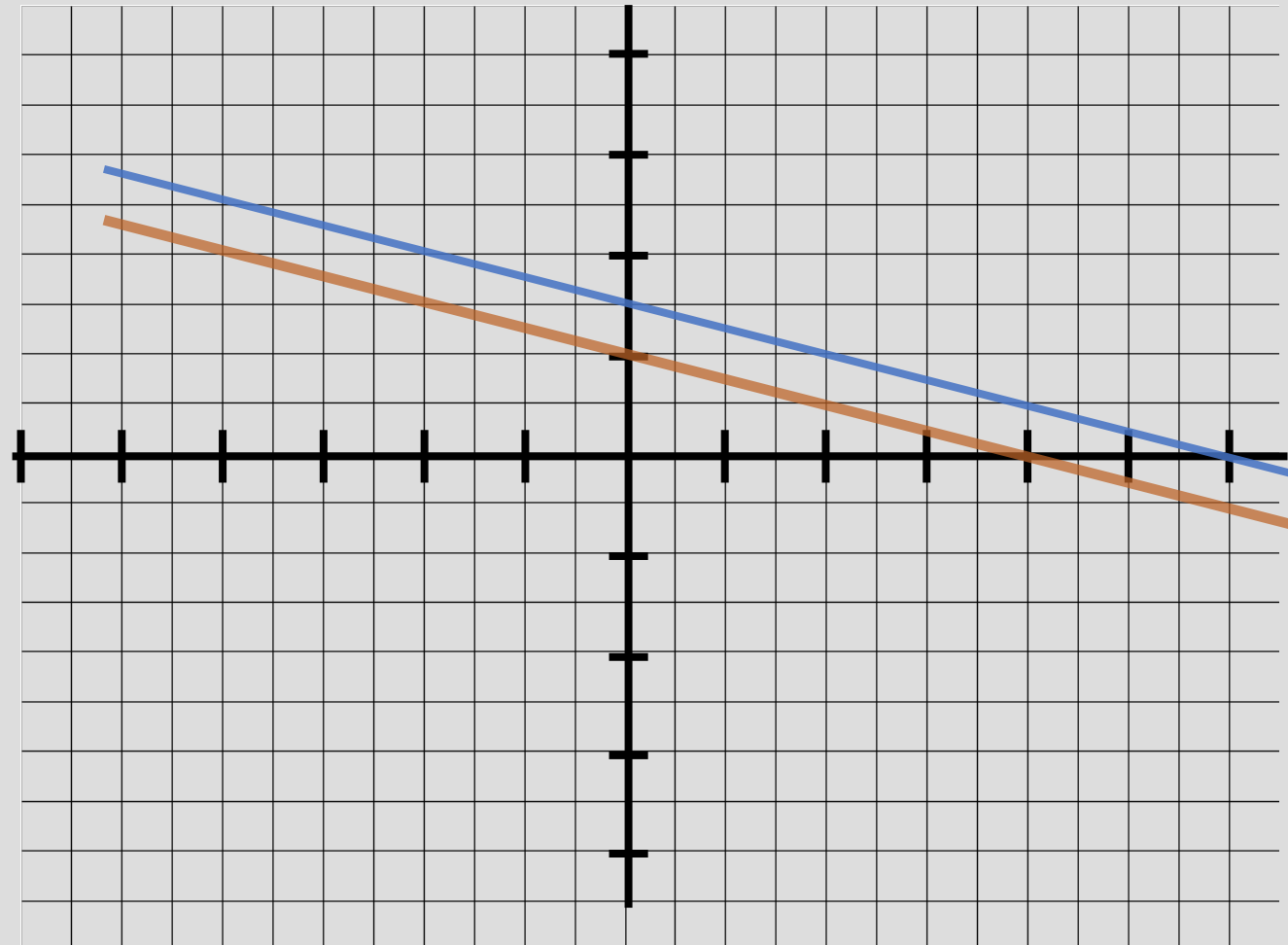
Gaussian Elimination Summary

- Reduce to row-echelon form, from left-to-right by using:

    1. Multiply an equation with *nonzero* scalar

    2. Adding a scalar constant multiple of one equation to another

    3. Swapping equations

Single solution
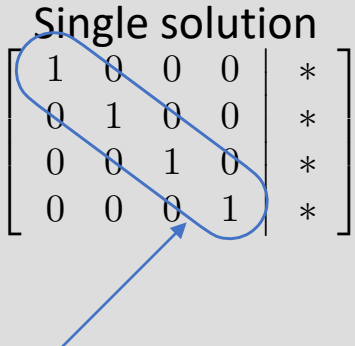
$$\begin{bmatrix} 1 & * & * & * & | & * \\ 0 & 1 & * & * & | & * \\ 0 & 0 & 1 & * & | & * \\ 0 & 0 & 0 & 1 & | & * \end{bmatrix}$$

Infinite solutions

$$\begin{bmatrix} 1 & * & * & * & | & * \\ 0 & 1 & * & * & | & * \\ 0 & 0 & 0 & 1 & | & * \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

No solution

$$\begin{bmatrix} 1 & * & * & * & | & * \\ 0 & 1 & * & * & | & * \\ 0 & 0 & 0 & 1 & | & * \\ 0 & 0 & 0 & 0 & | & * \end{bmatrix}$$
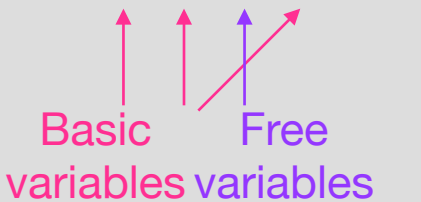
- Back substitute to reduced row-echelon form, from right-to-left

Single solution

$$\begin{bmatrix} 1 & 0 & 0 & 0 & | & * \\ 0 & 1 & 0 & 0 & | & * \\ 0 & 0 & 1 & 0 & | & * \\ 0 & 0 & 0 & 1 & | & * \end{bmatrix}$$

Infinite solutions

$$\begin{bmatrix} 1 & 0 & * & 0 & | & * \\ 0 & 1 & * & 0 & | & * \\ 0 & 0 & 0 & 1 & | & * \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Pivots

Basic variables   Free variables

Data: Augmented matrix $A \in \mathbb{R}^{m \times (n+1)}$, for a system of m equations with n variables
Result: Reduced form of augmented matrix
 # Forward elimination procedure:
for *each variable index i from 1 to n* **do**
   if *entry in row i, column i of A is 0* **then**
      if *all entries in column i and row > i of A are 0* **then**
         proceed to next variable index;
      else
         find j, the smallest row index $> i$ of A for which entry in column $i \neq 0$;
         # The following rows implement the "swap" operation:
         `old_row_j` ⟵ row j of A;
         row j of A ⟵ row i of A;
         row i of A ⟵ `old_row_j`;
      end
   end
   divide row i of A by entry in row i, column i of A;
   for *each row index k from i + 1 to m* **do**
      `scaled_row_i` ⟵ row i of A times entry in row k, column i of A;
      row k of A ⟵ row k of A − `scaled_row_i`;
   end
end
 # Back substitution procedure:
for *each variable index u from n − 1 to 1* **do**
   if *entry in row u, column u of A $\neq$ 0* **then**
      for *each row v from u − 1 to 1* **do**
         `scaled_row_u` ⟵ row u of A times entry in row v, column u of A;
         row v of A ⟵ row v of A − `scaled_row_u`;
      end
   end
end

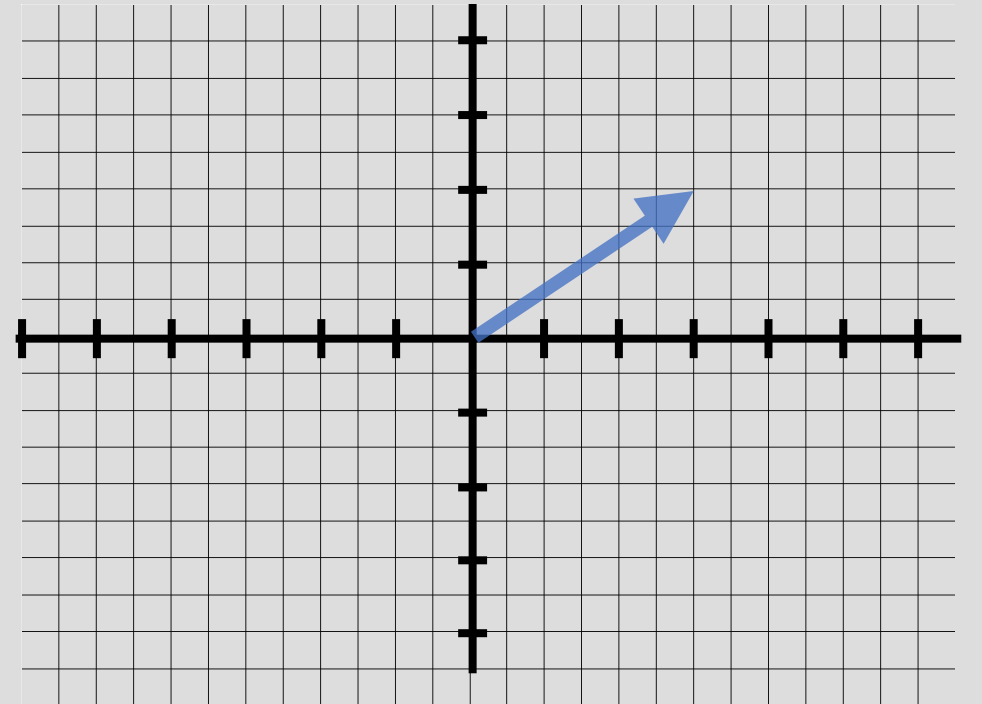**Algorithm 1:** The Gaussian elimination algorithm.

Vectors

- An array of N numbers
  - Represents coordinates in an N-dimensional space

$$\vec{x} = \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \qquad \vec{x} \in \mathbb{R}^N$$
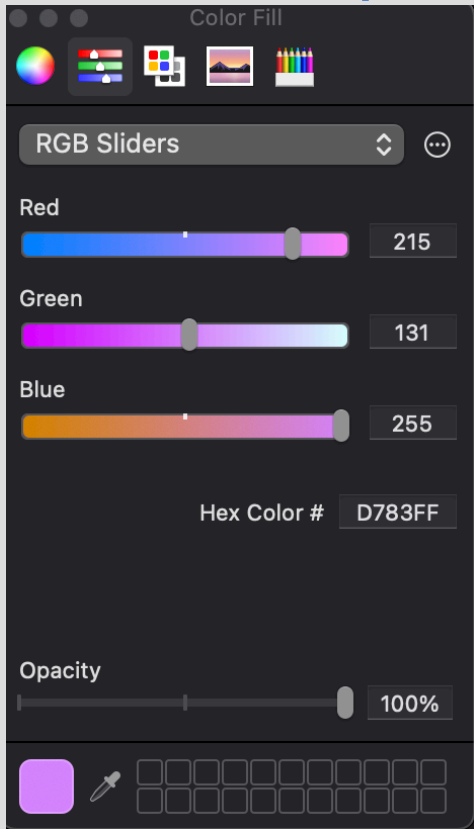
- For example:

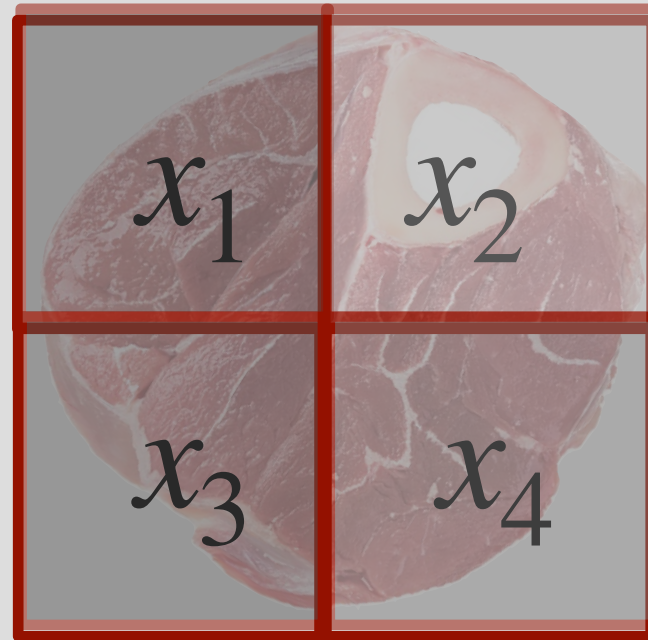$$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad \vec{x} \in \mathbb{R}^2$$

Vectors

- Since it's an array of numbers, it can represent other things….

pixel color

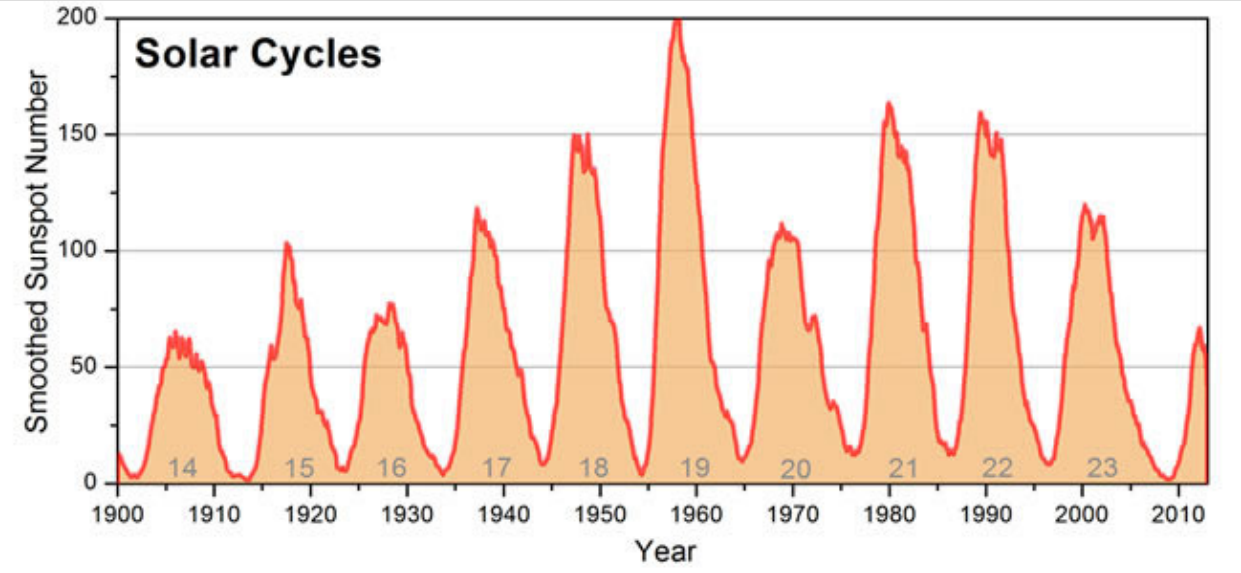$$\vec{x} = \begin{bmatrix} 215 \\ 131 \\ 25 \end{bmatrix}$$

pixel values in an image

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Vectors

Data

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{120} \end{bmatrix}$$

Special Vectors

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_N = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Matrices

- A collection of numbers in a rectangular form

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1M} \\ x_{21} & x_{22} & \cdots & x_{2M} \\ \vdots & \vdots & \cdots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{NM} \end{bmatrix}, \quad X \in \mathbb{R}^{\mathbb{N} \times \mathbb{M}}$$

- Or a collection of M, N-length vectors

$$X = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_M \end{bmatrix}, \quad X \in \mathbb{R}^{\mathbb{N} \times \mathbb{M}}$$

Vectors as Matrices

- A vector is a degenerate matrix

$$\vec{x} = \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \vec{x} \in \mathbb{R}^{N \times 1}$$

- A scalar is a degenerate vector or matrix

$$a \in \mathbb{R}^{1 \times 1}$$

Vector addition

- Two vectors of the same length can be added
  - Addition is element-wise

$$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \ \vec{y} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\vec{x} + \vec{y} =$
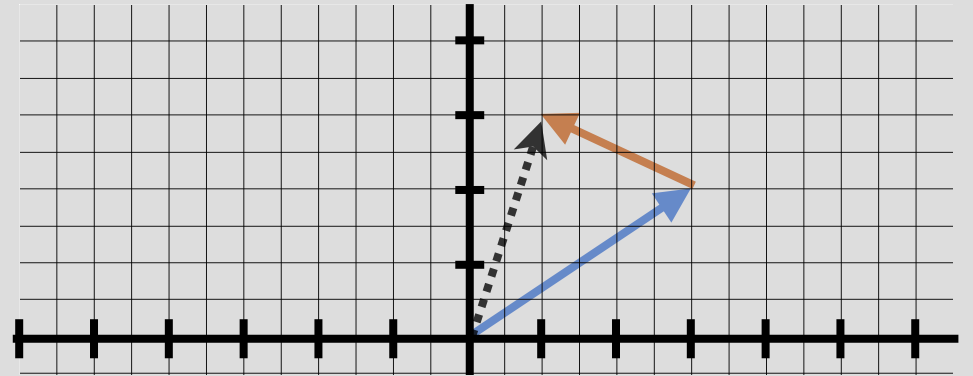
$\vec{y} + \vec{x} =$
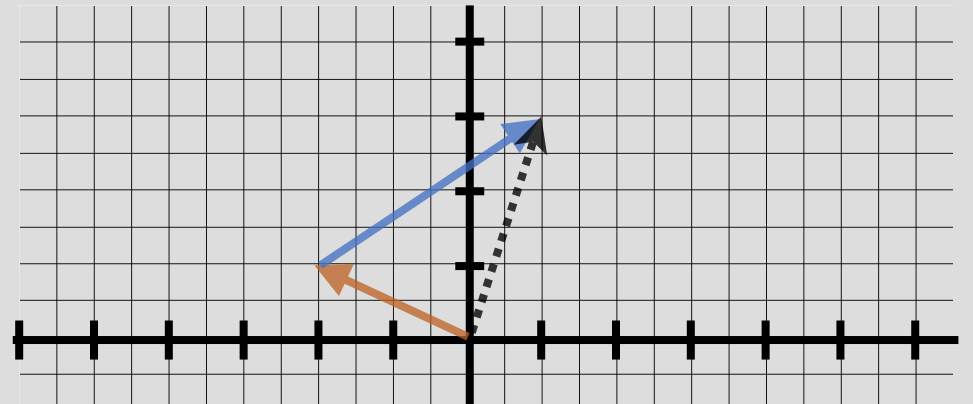
Properties of vector addition

- Commutativity: $\quad \vec{x} + \vec{y} = \vec{y} + \vec{x}$

- Associativity: $\quad (\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$

- Additive negative: $\quad \vec{x} + (-\vec{x}) = \vec{0}$

- Additive identity: $\quad \vec{x} + \vec{0} = \vec{x}$

Scalar Vector Multiplication

- Multiplying with a scalar result in multiplying each element.

$$a\vec{x} = \begin{bmatrix} ax_1 \\ ax_1 \\ \vdots \\ ax_{N \times 1} \end{bmatrix}$$

$$2 \cdot \vec{x} = 2 \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$
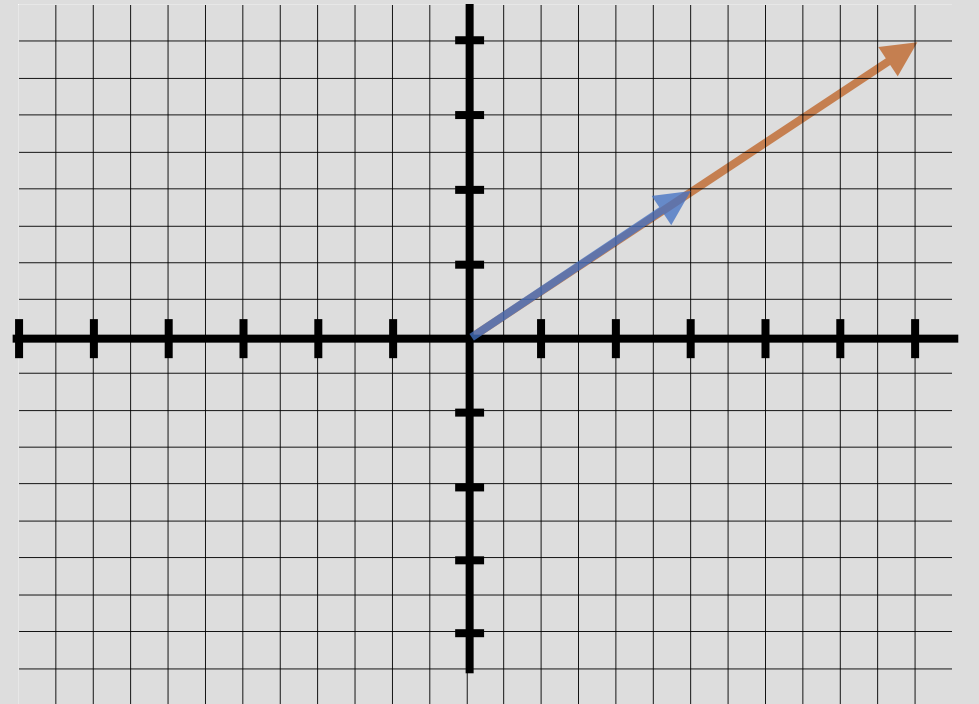
Vector Transpose

- $\vec{x}^T$ is the transpose of $\vec{x}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \vec{x} \in \mathbb{R}^{N \times 1} \qquad\qquad \vec{x}^T = \begin{bmatrix} x_1 & x_1 & \cdots & x_N \end{bmatrix}, \quad \vec{x}^T \in \mathbb{R}^{1 \times N}$$

- $\vec{x}$ is always a column vector
- To represent a row vector, write: $\vec{x}^T$

Vector Matrix Multiplication

- Let $\vec{x} \in \mathbb{R}^{N \times 1}$, $\vec{y} \in \mathbb{R}^{N \times 1}$, $\vec{z} \in \mathbb{R}^{M \times 1}$

- Multiplication is valid only for specific matching dimensions!

Like this....
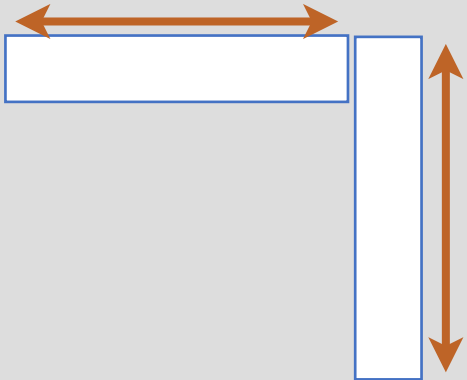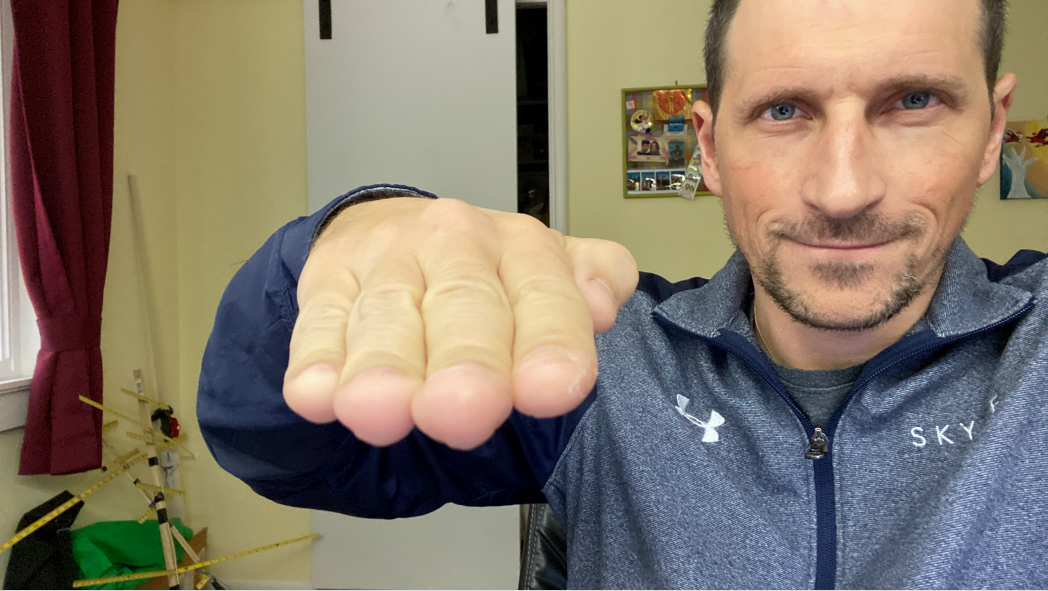
and like that!
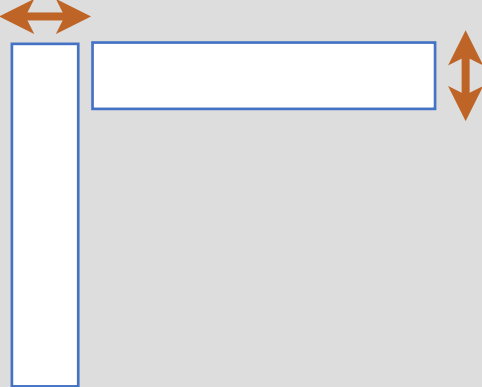
Or

Vector Vector Multiplication
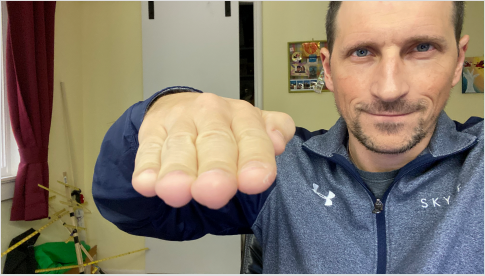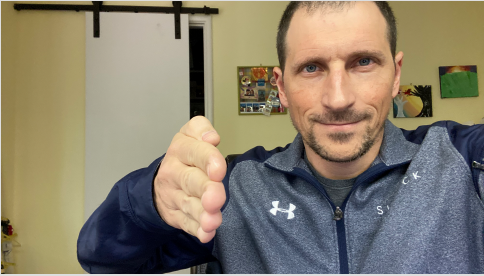
$$\vec{x}, \vec{y} \in \mathbb{R}^{N \times 1}$$

$$1 \times N$$

$$\vec{y}^T \vec{x} =$$

$$N \times 1$$

Vector Vector Multiplication

$$\vec{x}, \vec{y} \in \mathbb{R}^{N \times 1}$$

$$1 \times N$$

$$\vec{y}^T \vec{x} = \qquad = y_1 x_1 + y_2 x_2 + y_3 x_3 + \cdots + y_N x_N$$

$$N \times 1$$
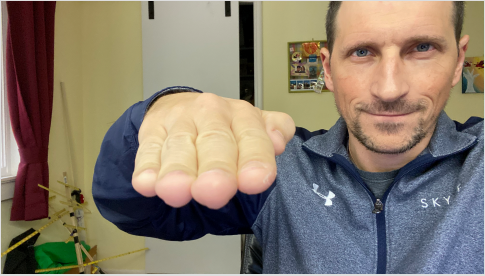
scalar 1×1

Also known as "inner product" or "dot product"

Vector Vector Multiplication

$$\vec{x}, \vec{y} \in \mathbb{R}^{N \times 1}$$
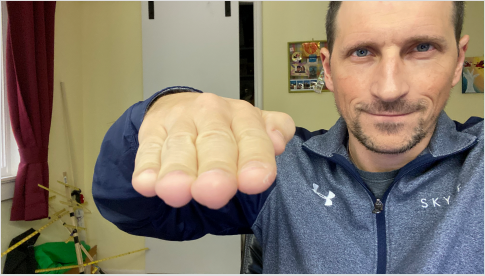
Like this....   and like that!

$$\vec{y}^T \vec{x} = \qquad = y_1 x_1 + y_2 x_2 + y_3 x_3 + \cdots + y_N x_N$$

scalar 1×1

Also known as "inner product" or "dot product"

$1 \times N$

$$\vec{x}\, \vec{y}^T = \qquad\qquad =$$

$N \times 1$

$N \times N$

Vector Vector Multiplication

$$\vec{x}, \vec{y} \in \mathbb{R}^{N \times 1}$$

$$\vec{y}^T \vec{x} = \qquad = y_1 x_1 + y_2 x_2 + y_3 x_3 + \cdots + y_N x_N$$

scalar 1×1

Also known as "inner product" or "dot product"

$$\vec{x}\, \vec{y}^T = \quad 1 \times N$$

$$= \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_N \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_N \\ \vdots & \vdots & \cdots & \vdots \\ x_N y_1 & x_N y_2 & \cdots & x_N y_N \end{bmatrix}$$
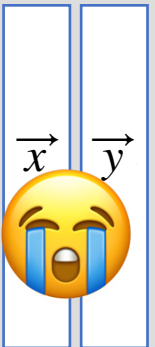
$N \times 1$

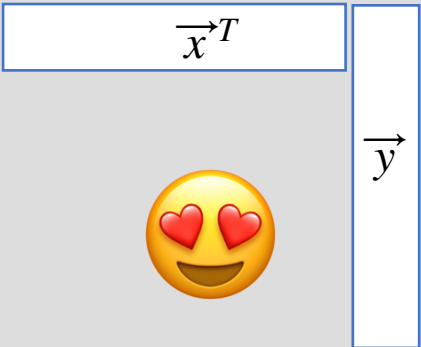$N \times N$

Do not commute!

Also known as "outer product"

Vector Matrix Multiplication

- Let $\vec{x} \in \mathbb{R}^{N \times 1}, \vec{y} \in \mathbb{R}^{N \times 1}, \vec{z} \in \mathbb{R}^{M \times 1}$

- $N \times 1$ with $1 \times M$ with output: $N \times M$

- $M \times 1$ with $1 \times N$ with output: $M \times N$

- $1 \times N$ with $N \times 1$ with output: 1×1
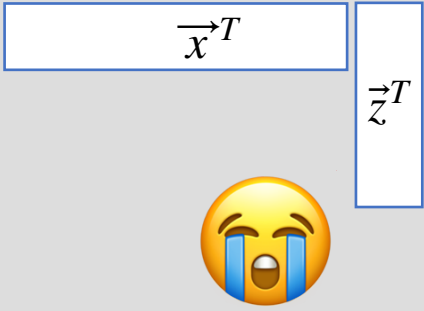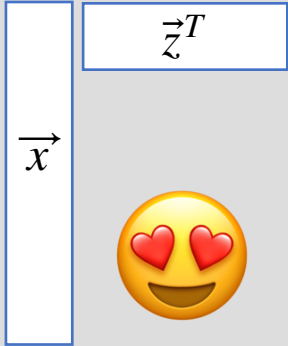
$$\vec{x} \cdot \vec{y}$$
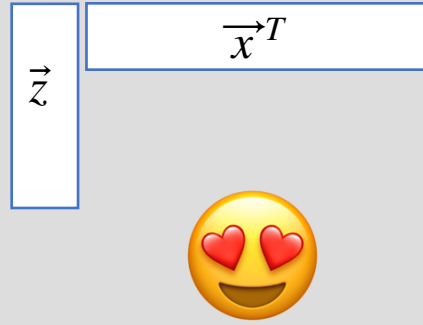
$$\vec{x}^T \cdot \vec{y}$$

$$\vec{x}^T \cdot \vec{z}$$

$$\vec{x} \cdot \vec{z}^T$$

$$\vec{z} \cdot \vec{x}^T$$