EECS 16A Designing Information Devices and Systems I
Summer 2023 Discussion 06A

1. Dynamical Systems (Spring 2020 Midterm 1 Question 7)

Define matrices $Q, R \in \mathbb{R}^{2\times 2}$ according to

$$Q = \begin{bmatrix} 0 & 3/4 \\ 1 & 1/4 \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(a) Find the eigenvalues for the matrix Q.

Answer: Note that

$$\det(Q - \lambda I) = (-\lambda)(1/4 - \lambda) - 3/4 = (\lambda - 1)(\lambda + 3/4).$$

So, the eigenvalues are $\lambda_1 = 1, \lambda_2 = -3/4$.

Answer: The eigenvalues are $\lambda_1 = 1, \lambda_2 = -3/4$.

(b) Consider a system with state vector $\vec{x}[n] \in \mathbb{R}^2$ at time $n \geq 1$ given by

$$\vec{x}[n] = Q\vec{x}[n-1].$$

Is there a non-zero vector $\vec{x}$ satisfying $\vec{x} = Q\vec{x}$? If yes, give one such vector.

Answer: Yes, such a vector exists since the matrix has eigenvalue 1. To solve for it, we set up the system of equations $(Q - I)\vec{x} = 0$, which is explicitly written as

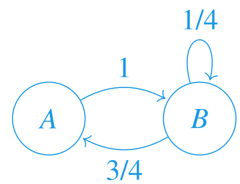
$$\begin{aligned} -x_1 + 3/4x_2 &= 0 \\ x_1 - 3/4x_2 &= 0 \end{aligned}$$

One solution is $x_1 = 3/4, x_2 = 1$, giving the desired vector $\vec{x} = [3/4,\ 1]^\top$.

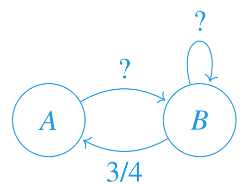
Answer: Yes; $\vec{x} = [3/4,\ 1]^\top$.

(c) Draw the state-transition diagram for the system in part (b). Label your nodes "A" and "B".

Answer:



Answer:



(d) Now, consider a system with state vector $\vec{w}[n] \in \mathbb{R}^2$ at time $n \geq 1$ given by:

$$\vec{w}[n] = \begin{cases} Q\,\vec{w}[n-1] & \text{if } n \text{ is odd} \\ R\,\vec{w}[n-1] & \text{if } n \text{ is even.} \end{cases}$$

Write expressions for $\vec{w}[1]$, $\vec{w}[2]$, $\vec{w}[3]$ and $\vec{w}[4]$ in terms of $\vec{w}[0]$ and Q and R. Write each answer in the form of a matrix-vector product.

Answer:

$$\vec{w}[1] = Q\vec{w}[0], \quad \vec{w}[2] = RQ\vec{w}[0], \quad \vec{w}[3] = Q(RQ)\vec{w}[0], \quad \vec{w}[4] = (RQ)^2\vec{w}[0].$$

Answer:

$$\vec{w}[1] = Q\vec{w}[0], \quad \vec{w}[2] = RQ\vec{w}[0], \quad \vec{w}[3] = Q(RQ)\vec{w}[0], \quad \vec{w}[4] = (RQ)^2\vec{w}[0].$$

(e) Suppose we start the system of part (d) with state $\vec{w}[0] = \begin{bmatrix} 11/14 & 3/14 \end{bmatrix}^\top$. Find expressions for $\vec{w}_{\text{even}}$ and $\vec{w}_{\text{odd}}$, which are defined according to

$$\vec{w}_{\text{even}} = \lim_{k\to\infty} \vec{w}[2k], \qquad \vec{w}_{\text{odd}} = \lim_{k\to\infty} \vec{w}[2k+1].$$

In words, $\vec{w}_{\text{even}}$ and $\vec{w}_{\text{odd}}$ describe the long-term behavior of the system at even and odd time-instants, respectively. (Hint: you can avoid computation by thinking about the system at even time-instants in terms of a state-transition diagram.)

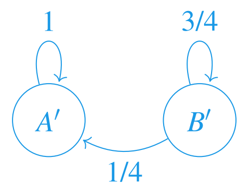
Answer: Following the hint, consider the system at even time-instants:

$$\vec{w}[2k] = (RQ)^k\vec{w}[0], \quad k \geq 0.$$

This looks like a dynamical system with transition matrix

$$RQ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3/4 \\ 1 & 1/4 \end{bmatrix} = \begin{bmatrix} 1 & 1/4 \\ 0 & 3/4 \end{bmatrix}.$$

The transition diagram for this system looks like:



This looks similar to the page rank example from lecture, where all traffic will end up on website A'. Hence, for the given choice of $\vec{w}[0]$ (whose entries add to one, and therefore can be thought of as representing fraction of traffic), we have

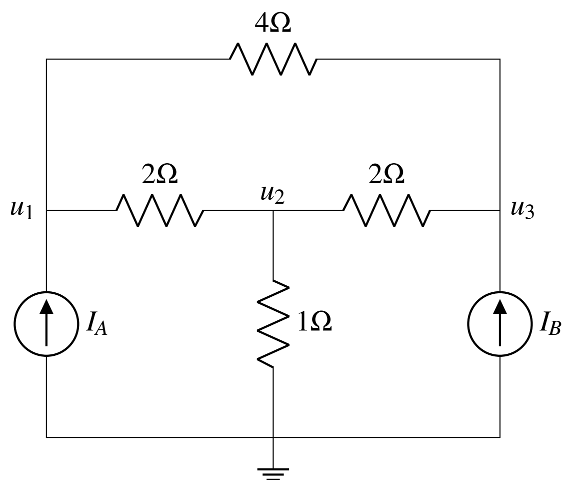
$$\vec{w}_{\text{even}} = \lim_{k\to\infty} \vec{w}[2k] = \begin{bmatrix}1\\0\end{bmatrix}, \qquad \text{and, } \vec{w}_{\text{odd}} = Q\vec{w}_{\text{even}} = \begin{bmatrix}0\\1\end{bmatrix}.$$

Answer:

$$\vec{w}_{\text{even}} = \lim_{k\to\infty} \vec{w}[2k] = \begin{bmatrix}1\\0\end{bmatrix}, \qquad \text{and, } \vec{w}_{\text{odd}} = Q\vec{w}_{\text{even}} = \begin{bmatrix}0\\1\end{bmatrix}.$$

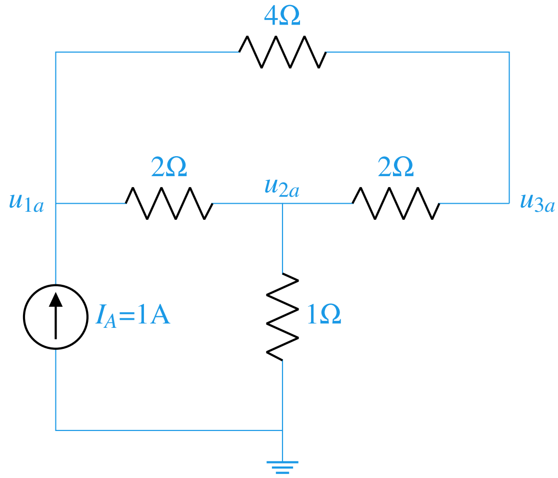
2. Superposition (Fall 2020 Midterm 2 Question 7)

For this question, we will analyze the circuit shown below with the two current sources of strength I_A and I_B as inputs. It may be observed that the network of resistors shown in the circuit is symmetric. We will first solve this circuit for symmetric inputs $I_A = I_B$, and then for anti-symmetric inputs $I_A = -I_B$. Using these two results, we we will solve the circuit for arbitrary inputs I_A, I_B.

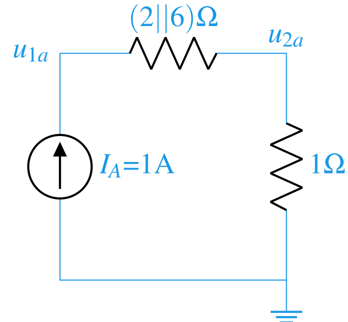


(a) Consider the circuit above with symmetric inputs, $I_A = I_B = 1\text{A}$. Using superposition, solve for the node voltages at the nodes marked u_1, u_2 and u_3.

Answer:
Current source I_B zeroed out.



We show one solution approach using series/parallel equivalence and Ohm's law. We can redraw this circuit as follows:



Using Ohm's Law and noting that $2||6 = 1.5$ we find

$$\begin{aligned} u_{2a} &= I_A \cdot 1\,\Omega & &= 1\,\text{V} \\ u_{1a} &= u_{2a} + I_A \cdot 1.5\,\Omega & &= 2.5\,\text{V} \end{aligned}$$

To find u_{3a}, we note that the current flowing through the $4\,\Omega$ and $2\,\Omega$ resistors in the top branch clockwise is (using Ohm's Law):

$$I_{top} = \frac{(u_{1a} - u_{2a})\text{V}}{6\,\Omega} = \frac{1.5\,\text{V}}{6\,\Omega} = 0.25\,\text{A}.$$

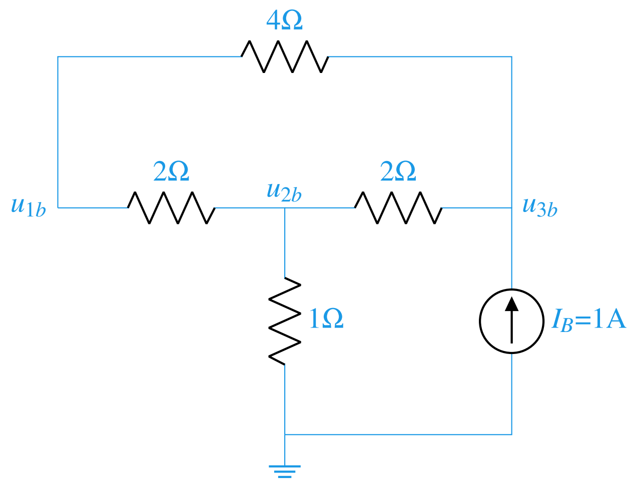
And then applying Ohm's Law again across the $4\,\Omega$ resistor, we find:

$$u_{1a} - u_{3a} = I_{top} \cdot 4\,\Omega = 0.25\,\text{A} \cdot 4\,\Omega = 1\,\text{V}$$
$$\Rightarrow u_{3a} = u_{1a} - 1\,\text{V} = 2.5\,\text{V} - 1\,\text{V} = 1.5\,\text{V}.$$

So we have found:

$$\begin{aligned} u_{1a} &= 2.5\,\text{V} \\ u_{2a} &= 1\,\text{V} \\ u_{3a} &= 1.5\,\text{V}. \end{aligned}$$

Current source I_A zeroed out.



This circuit is very similar to when I_B is zeroed out, just with the relative positions of the nodes with respect to the active current source modified. So we can observe that:

$$u_{1b} = u_{3a} = 1.5\,\text{V}$$
$$u_{2b} = u_{2a} = 1\,\text{V}$$
$$u_{3b} = u_{1a} = 2.5\,\text{V}.$$

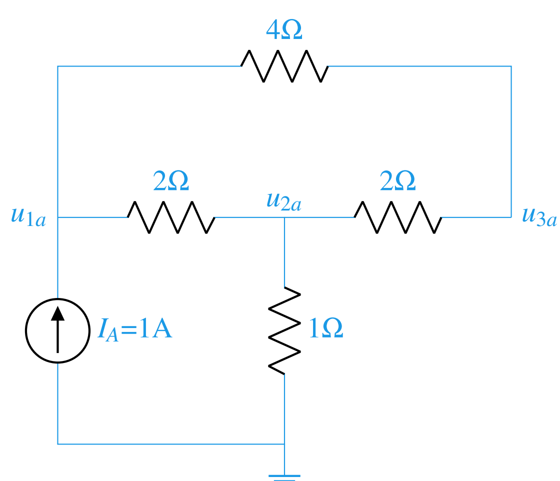
Both current sources I_A and I_B active. We simply find the sum to see what happens when both I_A and I_B are active.

$$u_1 = u_{1a} + u_{1b} = 2.5\,\text{V} + 1.5\,\text{V} = 4\,\text{V}$$
$$u_2 = u_{2a} + u_{2b} = 1\,\text{V} + 1\,\text{V} = 2\,\text{V}$$
$$u_3 = u_{3a} + u_{3b} = 1.5\,\text{V} + 2.5\,\text{V} = 4\,\text{V}.$$

(b) Consider the same circuit as before with anti-symmetric inputs, $I_A = 1\,\text{A}$ and $I_B = -1\,\text{A}$. Using superposition solve for the node voltages at the nodes marked u_1, u_2 and u_3.

Answer: This solution mostly follows the same logic as in part (a). We simply need to account for the different direction of the current source I_B.

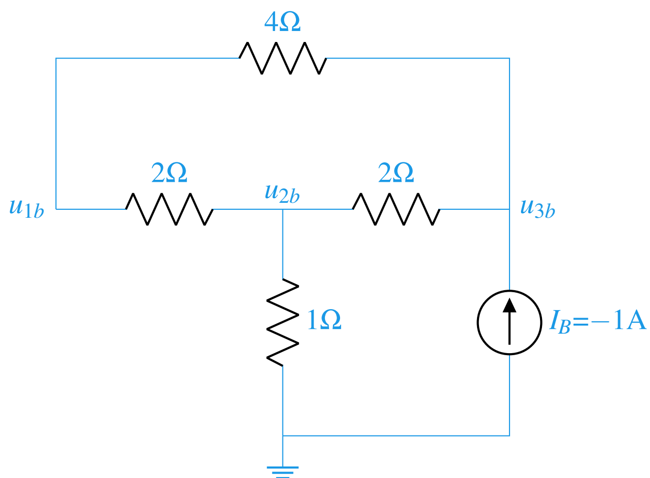
Current source I_B zeroed out.



We analyzed this same circuit in part (a) and arrive at the same results:

$$u_{1a} = 2.5\,\text{V}$$
$$u_{2a} = 1\,\text{V}$$
$$u_{3a} = 1.5\,\text{V}.$$

Current source I_A zeroed out.



As with part (a), due to symmetry, we can use the results from the case where only the current source I_A is active. However, since I_B is "flipped" relative to part (a) – i.e. the value of I_B is negative – we simply need to scale our answers from part (a) by -1. Taking the symmetry and sign information into account, we arrive at:

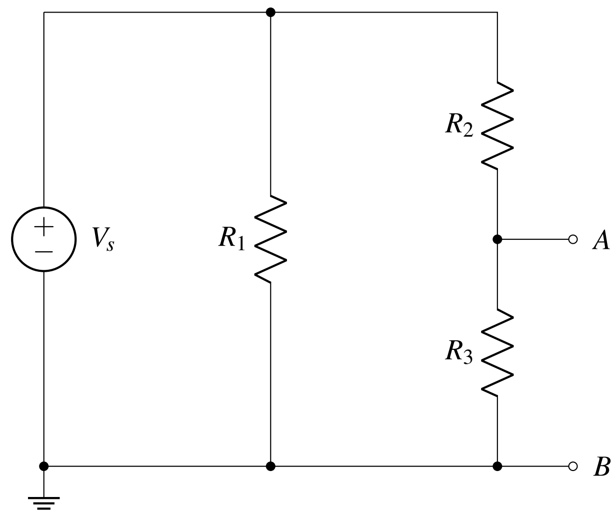
$$\begin{aligned} u_{1b} &= -u_{3a} &&= -1.5\,\text{V} \\ u_{2b} &= -u_{2a} &&= -1\,\text{V} \\ u_{3b} &= -u_{1a} &&= -2.5\,\text{V}. \end{aligned}$$

Both current sources I_A and I_B active. We simply find the sum to see what happens when both I_A and I_B are active.

$$\begin{aligned} u_1 &= u_{1a} + u_{1b} &&= 2.5\,\text{V} - 1.5\,\text{V} = 1\,\text{V} \\ u_2 &= u_{2a} + u_{2b} &&= 1\,\text{V} - 1\,\text{V} = 0\,\text{V} \\ u_3 &= u_{3a} + u_{3b} &&= 1.5\,\text{V} - 2.5\,\text{V} = -1\,\text{V}. \end{aligned}$$

3. Thévenin/Norton Equivalence

(a) Find the Thévenin resistance R_{th} of the circuit shown below, with respect to its terminals A and B.

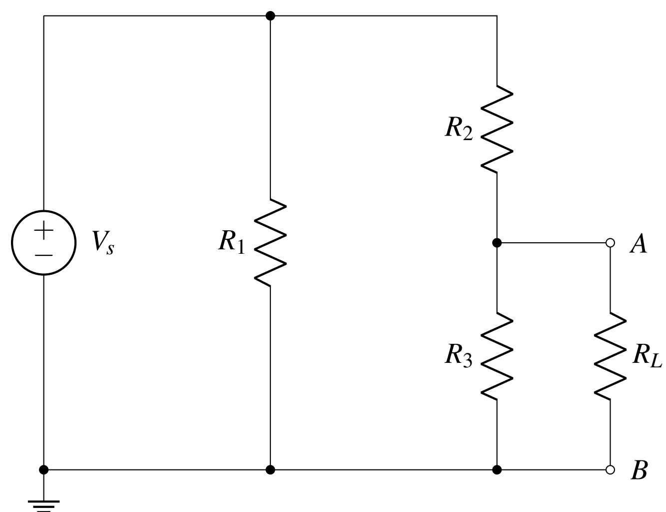


Answer: To find the Thévenin resistance, we null out the voltage source (which shorts out R_1) and find the equivalent resistance, which is:

$$R_{th} = R_2 \parallel R_3$$

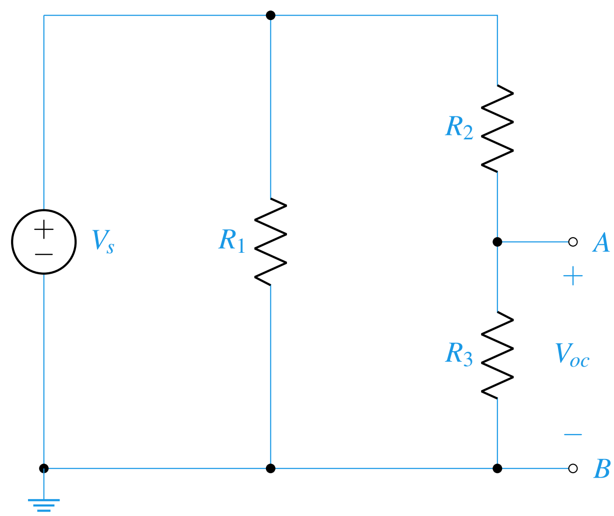
since resistors R_2 and R_3 are in parallel.

(b) Now a load resistor, R_L, is connected across terminals A and B, as shown in the circuit below. Using Thévenin equivalence, find the power dissipated in the load resistor in terms of the given variables.



Answer:

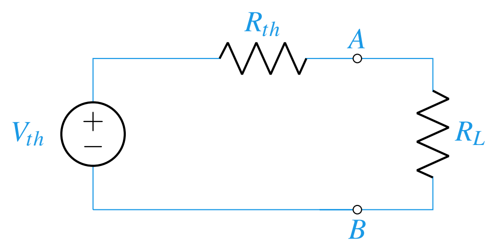
To help simplify the analysis, we replace the circuit with its Thévenin equivalent circuit. In order to do so, we first need to find the Thévenin voltage, V_{th}. One way to determine V_{th} is to find the open circuit voltage, $V_{AB} = V_{oc}$, in the original circuit when an open circuit is connected externally to terminals A and B.



The open circuit can be derived from a voltage divider:

$$V_{th} = V_{oc} = \frac{R_3}{R_2 + R_3} V_s$$

We also already know the Thévenin resistance $R_{th} = R_2 \parallel R_3$ from part (a).
Thus, the circuit can be simplified to:

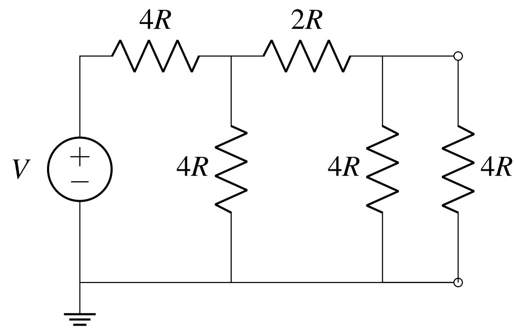


The power through the load resistor is then given by:

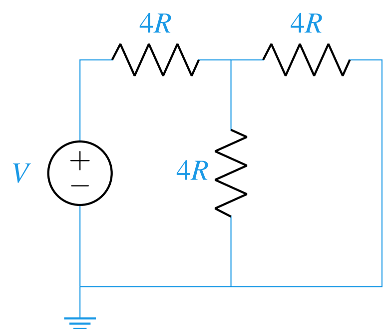
$$P_{R_L} = V_{R_L} \cdot I_{R_L} = I_{R_L}^2 \cdot R_L = \left(\frac{V_{th}}{R_L + R_{th}} \right)^2 R_L = \left(\frac{R_3}{R_2 + R_3} V_s \cdot \frac{1}{R_L + R_2 \parallel R_3} \right)^2 R_L$$

4. OPTIONAL: Power to Resist (from Spring 2018 midterm 2)

Find the power dissipated by the voltage source in the circuit below. Be sure to use passive sign convention.



Answer: We want to find the equivalent resistance across the voltage source in Figure 6.2. Start by reducing the two resistors on the right to $4R \parallel 4R = 2R$. Then combine the other $2R$ resistor with this to get a new resistor of value $4R$ as in the circuit below.



Once again we have $4R \parallel 4R = 2R$. This is finally in series with $4R$ giving us a total resistance of $4R + 2R = 6R$

$$P = VI = V\frac{-V}{6R} = -\frac{V^2}{6R}$$

The negative sign is present because the voltage source actually provides power, which can also be seen by using passive sign convention.