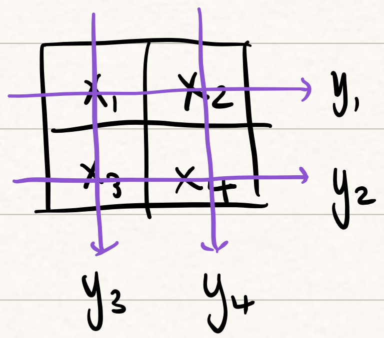
Lecture 3 (Thu 6/22)

Today:

- close the tomography cliffhanger
- Gaussian elimination: an ALGORITHM to solve systems of equations or find that they have ∞ or no solution

First, recall the tomography system of equations we came up with.



$$\begin{cases} y_1 = x_1 + x_2 = 3 \\ y_2 = x_3 + x_4 = 4 \\ y_3 = x_1 + x_3 = 1 \\ y_4 = x_2 + x_4 = 5 \end{cases}$$

let's say we take these measurements.

unknowns to solve for

measurements (known)

Consider: Add 1st and 2nd equation, and 3rd + 4th.

$$\begin{array}{r} x_1 + x_2 = 3 \\ + \quad x_3 + x_4 = 4 \\ \hline x_1 + x_2 + x_3 + x_4 = 7 \end{array} \qquad \begin{array}{r} x_1 + x_3 = 1 \\ + \quad x_2 + x_4 = 5 \\ \hline x_1 + x_2 + x_3 + x_4 = 6 \end{array} \qquad ?!$$

What does this inconsistency mean?

There is no way to choose x_1, x_2, x_3, x_4 to make both true $\Rightarrow$ NO SOLUTION

What if the measurements were

$$\begin{cases} x_1 + x_2 = 3 & (1) \\ x_3 + x_4 = 4 & (2) \\ x_1 + x_3 = 2 & (3) \\ x_2 + x_4 = 5 & (4) \end{cases}$$

eq. labels

Then there's no inconsistency.

But notice that (1) + (2) − (3) = (4)

$$(x_1 + x_2) + (x_3 + x_4) - (x_1 + x_3) = x_2 + x_4$$

Which means that (4) gives us <u>NO new information</u>

- we could've known what (4) was just by adding and subtracting the measurements we already had!
- so we effectively only have 3 "useful" measurements

There isn't enough information to get a unique solution (4 unknowns, 3 equations), so there are ∞ solutions.

- We can try choosing some value for x_1, say $x_1 = 2$.
- then we can get what all the other variables are:

(1) $x_1 + x_2 = 3$

$2 + x_2 = 3 \rightarrow x_2 = 1$

(3) $x_1 + x_3 = 2$

$2 + x_3 = 2 \rightarrow x_3 = 0$

(2) $x_3 + x_4 = 4$

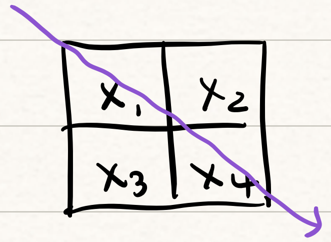
$0 + x_4 = 4 \rightarrow x_4 = 4$

- we could do this for ANY choice of x_1 and get values that satisfy all the equations

= INFINITE SOLUTIONS

We need more information! Is there another measurement we could take?

what about a diagonal projection?



$y_5 = x_1 + x_4$

Is the system of equations solveable now? you try!

$$\begin{cases} x_1 + x_2 = 3 \\ x_3 + x_4 = 4 \\ x_1 + x_3 = 2 \\ x_2 + x_4 = 5 \\ x_1 + x_4 = 4 \end{cases}$$

the answer is yes :)

- note that with 4 unknowns we only need 4 equations, so you can throw at any of the first 4.

It was kind of annoying to solve that system because it had 4 variables.

And it required some cleverness to notice when the system had infinite or no solutions.

There were many different paths to the answer.

What if we want our computer to do it for us?

- We need a SYSTEMATIC set of steps that a computer could follow = an *algorithm*

Gaussian elimination is exactly that.

Let's demonstrate on a small system.

$$\begin{cases} 2x + 3y = 8 & (E1) \\ 3x - y = 1 & (E2) \end{cases}$$

Systematic steps of Gaussian Elimination:

1. Start with the 1st equation.

   Set the coefficient of x to 1.

   $(E1)/2 \rightarrow x + \frac{3}{2}y = 4 \quad (E1^*)$

2. Use $(E1^*)$ to eliminate x from $(E2)$

   $(E2) - 3(E1^*)$

$$(\cancel{3x} - y) - 3(\cancel{x} + \tfrac{3}{2}y) = 1 - 3 \cdot 4$$

$$-y - \tfrac{9}{2}y = -11$$

$$-\tfrac{11}{2}y = -11 \quad (E2^*)$$

3. Solve for y

(by making its coefficient 1)

$(E2^*) \cdot -\frac{2}{11} \rightarrow \boxed{y = 2}$ $(E2^{**})$

4. Solve for x by "backsubstituting" y in.

$(E1^*)$: $x + \frac{3}{2}y = 4$

$x + \frac{3}{2} \cdot 2 = 4 \rightarrow \boxed{x = 1}$

- You can also view this as using $(E2^{**})$ to eliminate y from $(E1^*)$

$(E1^*) - \frac{3}{2}(E2^{**})$

$$\left(x + \cancel{\tfrac{3}{2}y}\right) - \cancel{\tfrac{3}{2}y} = 4 - \tfrac{3}{2} \cdot 2$$

$$x = 1$$

That's Gaussian elimination!

- notice that the steps are all things you know how to do.

Let's make it a little more convenient with some notation.

$$\begin{cases} 2x + 3y = 8 \\ 3x - y = 1 \end{cases} \rightarrow \left[\begin{array}{cc|c} 2 & 3 & 8 \\ 3 & -1 & 1 \end{array}\right]$$

We're lazy and don't want to write variables

Just remember that 1st column is x

and 2nd column is y.

We call this "augmented matrix form".

Each row is an equation.

- Gaussian elimination is a series of "row operations" on the augmented matrix.

Allowed operations (PRESERVES the set of solutions)

1. Multiply both sides of an equation by a scalar
2. Add equations together (possibly scalar multiples).
3. Also, swapping order of equations is allowed.

Let's try.

make 1

$$\left[\begin{array}{cc|c} 2 & 3 & 8 \\ 3 & -1 & 1 \end{array}\right] \xrightarrow{\frac{R_1}{2} \to R_1} \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 4 \\ 3 & -1 & 1 \end{array}\right]$$

make 0

$$\xrightarrow{R_2 - 3R_1 \to R_2} \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 4 \\ 0 & -\frac{11}{2} & -11 \end{array}\right] \xrightarrow{-\frac{2}{11}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 4 \\ 0 & 1 & 2 \end{array}\right]$$

make 1

make 0

"upper triangular form"

"pivots"

$$\xrightarrow{R_1 - \frac{3}{2}R_2 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

$x = 1$

$y = 2$

"reduced row echelon form"

- these are the same steps we did above, just with different notation.
- notice that each step targets changing one number.
  - that can help you keep track of what to do.

You can also think of Gaussian elimination as 2 "phases"

- PHASE 1: "row reduction"

  goal: upper triangular form $\left[\begin{array}{ccc|c} 1 & & * & \\ & 1 & & * \\ 0 & & \ddots \; 1 & \end{array}\right]$ can be anything

- PHASE 2: "backsubstitution"

  goal: reduced row echelon form $\left[\begin{array}{ccc|c} 1 & & 0 & \\ & 1 & & * \\ 0 & & \ddots \; 1 & \end{array}\right]$

That was a system with a unique solution. How does the algorithm behave when that's not the case?

Ex. 2 $\begin{cases} 2x + 3y = 8 \\ 2x + 3y = 6 \end{cases}$

$$\left[\begin{array}{cc|c} 2 & 3 & 8 \\ 2 & 3 & 6 \end{array}\right] \xrightarrow{\frac{R_1}{2} \to R_1} \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 4 \\ 2 & 3 & 6 \end{array}\right] \xrightarrow{R_2 - 2R_1 \to R_2}$$

$$\left[\begin{array}{cc|c} 1 & \frac{3}{2} & 4 \\ 0 & 0 & -2 \end{array}\right]$$ wat does this mean?

- remember what the augmented matrix notation means.

  $0x + 0y = -2$

  - there is no x, y that could satisfy this!

    $0 = -2$ is a contradiction $\Rightarrow$ NO SOLUTION

Ex. 3 : $\begin{cases} x + 4y = 6 \\ 2x + 8y = 12 \end{cases}$ (notice: $E2 = 2 \cdot E1$.

what do you predict is the solution?)

$$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 2 & 8 & 12 \end{array}\right] \xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & 0 & 0 \end{array}\right]$$

- again, what does augmented matrix notation mean?

  $0x + 0y = 0$

  $0 = 0$ Not a useful equation... means that we didn't have enough useful equations!

  $\Rightarrow$ INFINITE SOLUTIONS!

So getting a row of zeros suggests one of these problem cases.

- Check rightmost value to distinguish.

Okay. Do you feel like a robot yet?

- Doing Gaussian elimination is mind-numbing which is why computers can do it! But sorry about the mind-numbing...

Let's try a 3-variable system.

Ex. 4. $\begin{cases} 2y + z = 1 \\ 2x + 6y + 4z = 10 \\ x - 3y + 3z = 14 \end{cases}$

no x in first equation

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 1 \\ 2 & 6 & 4 & 10 \\ 1 & -3 & 3 & 14 \end{array}\right]$$

We want to start by making 1st row's x coefficient 1 by multiplying/dividing, but we can't! This is when we can SWAP.

$$\xrightarrow{\text{swap } R_1 + R_2} \left[\begin{array}{ccc|c} 2 & 6 & 4 & 10 \\ 0 & 2 & 1 & 1 \\ 1 & -3 & 3 & 14 \end{array}\right]$$

$$\xrightarrow{\frac{R_1}{2} \to R_1} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 2 & 1 & 1 \\ 1 & -3 & 3 & 14 \end{array}\right]$$

R_1 done (in upper triangular)

move to R_2.

$$\xrightarrow{\frac{R_2}{2} \to R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & -3 & 3 & 14 \end{array}\right]$$

R_2 done.

move to R_3.

$$\xrightarrow{R_3 - R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -6 & 1 & 9 \end{array}\right]$$

eliminate everything before pivot.

$$\xrightarrow{R_3 + 6R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 4 & 12 \end{array}\right]$$

almost there!

$$\xrightarrow{\frac{R_3}{4} \to R_3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

upper triangular!!

time to backsubstitute.

start with second-to-last row.

$$\xrightarrow{R_2 - \frac{1}{2}R_3 \to R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{R_1 - 3R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_3 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

RREF!!! we did it.

I like to think of the 2 phases of Gaussian elimination with directionality:

- row reduction is left → right, top → bottom.
- backsubstitution is bottom → top, right → left.

Gaussian elimination works when # unknowns ≠ # equations.

Ex 5. $\begin{cases} 2y + 3z = 2 \\ x + y = 1 \end{cases}$

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 2 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{\text{swap } R_1 \,\&\, R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 3 & 2 \end{array}\right]$$

$$\xrightarrow{\frac{R_2}{2} \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} & 1 \end{array}\right]$$

this also counts as upper triangular!

missing 3rd pivot. can't eliminate in 3rd col.

$$\xrightarrow{R_1 - R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 1 \end{array}\right]$$

counts as RREF!

doesn't matter what's happening in 3rd column because it has no pivot.

But what is the answer in this case?

- We didn't find a contradiction, so there isn't no solution.
- But we have 3 equations, 2 unknowns so there must be INFINITE solutions!

Let's see if we can write what the set of solutions is

- because infinite solutions doesn't mean ANYTHING is a solution!

The way we're going to do this is by starting with which column doesn't have a pivot : the z column.

- we're gonna call z a "free variable".
- the first row gives a relationship between $x + z$, and the 2^{nd} between $y + z$.
- So if we just choose z to be something, (anything! "free"!) x and y will be determined!

Let's move back to equation form.

$$x - \tfrac{3}{2}z = 0 \rightarrow \qquad x = \tfrac{3}{2}z$$

$$y + \tfrac{3}{2}z = 1 \rightarrow \begin{cases} y = 1 - \tfrac{3}{2}z \\ z = \text{anything} \end{cases}$$

(infinite) set of solutions to this system

That was a wide matrix. Let's try a tall one (more equations than unknowns)

Ex. 6
$$\begin{cases} x + y = 2 \\ x - y = 1 \\ 2x - 2y = 2 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & -2 & 2 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -1 \\ 2 & -2 & 2 \end{array}\right]$$

$$\xrightarrow{-\frac{R_2}{2} \to R_2} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & \frac{1}{2} \\ 2 & -2 & 2 \end{array}\right] \xrightarrow{R_3 - 2R_1 \to R_3} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & -4 & -2 \end{array}\right]$$

$$-\frac{R_3}{4} \to R_3 \quad \longrightarrow \quad \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{array}\right] \quad \xrightarrow{R_3 - R_2 \to R_3} \quad \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right]$$

upper triangular!

$0 = 0$

Let's finish the top 2 rows.

$$R_1 - R_2 \to R_1 \quad \longrightarrow \quad \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right]$$

RREF

Are there infinite solutions?

- $[0 \quad 0 \mid 0]$ tells us there's a redundant equation.
- but there's 3 equations, 2 unknowns, so of course!
- throw away one equation and we actually have a unique solution!

Q. Is it possible for a system with more equations than unknowns to have no solution?

A. Yes.

$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & -2 \end{array}\right]$$

is still inconsistent!

What about infinite solutions?

A. Yes.

$$\left[\begin{array}{cc|c} 1 & 0 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

enough. more examples in discussion.