EECS 16A Designing Information Devices and Systems I
Summer 2023 Discussion 4B

1. Mechanical Eigenvalues and Eigenvectors

In each part, find the eigenvalues of the matrix $\mathbf{M}$ and their associated eigenvectors.

(a) $\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

Do you observe anything about the eigenvalues and eigenvectors?

Answer:

Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 9-\lambda \end{bmatrix}\right) = 0$$

The determinant of a diagonal matrix is the product of the entries.

$$(1-\lambda)(9-\lambda) = 0$$

From the above equation, we know that the eigenvalues are $\lambda = 1$ and $\lambda = 9$.

For the eigenvalue $\lambda = 1$:

$$(\mathbf{M} - 1\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix}\vec{x} = \vec{0}$$

From the second equation in the system, $x_2 = 0$, with any solution having the form $\begin{bmatrix} 1 \\ 0 \end{bmatrix} t$ for $t \in \mathbb{R}$. The eigenspace is thus $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.

For the eigenvalue $\lambda = 9$:

$$(\mathbf{M} - 9\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix} -8 & 0 \\ 0 & 0 \end{bmatrix}\vec{x} = \vec{0}$$

From the first equation in the system, $x_1 = 0$, so any solution must take the form $\begin{bmatrix} 0 \\ 1 \end{bmatrix} t$ for $t \in \mathbb{R}$. The eigenspace is $\operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.

We observe that the eigenvalues are just the diagonal entries. Since the matrix is diagonal, multiplying the diagonal matrix $\mathbf{D}$ with any standard basis vector $\vec{e}_i$ produces $d_i\vec{e}_i$, that is, $\mathbf{D}\vec{e}_i = d_i\vec{e}_i$. Therefore, the eigenvalues are the diagonal entries d_i of $\mathbf{D}$, and the corresponding eigenvector associated with $\lambda = d_i$ is the standard basis vector $\vec{e}_i$.

(b) $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}\right) = 0$$

$$\begin{aligned} -\lambda(-3-\lambda)+2 &= 0 \\ \lambda^2 + 3\lambda + 2 &= 0 \\ (\lambda+2)(\lambda+1) &= 0 \\ \lambda &= -1, -2 \end{aligned}$$

$\lambda = -1$:

$$\left[\begin{array}{cc|c} 0-(-1) & 1 & 0 \\ -2 & -3-(-1) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right] \overset{G.E.}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 &= t \end{aligned} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t$$

The eigenspace for $\lambda = -1$ is $\text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

$\lambda = -2$:

$$\left[\begin{array}{cc|c} 0-(-2) & 1 & 0 \\ -2 & -3-(-2) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -2 & -1 & 0 \end{array}\right] \overset{G.E.}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 + x_2/2 &= 0 \\ x_2 &= t \end{aligned} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t/2 \\ t \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} t$$

The eigenspace for $\lambda = -2$ is $\text{span}\left\{\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}\right\}$.

2. Steady and Unsteady States

You're given the matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

which generates the next state of a physical system from its previous state: $\vec{x}[k+1] = \mathbf{M}\vec{x}[k]$.

(a) The eigenvalues of $\mathbf{M}$ are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = \frac{1}{2}$. Define $\vec{x} = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$, a linear combination of the eigenvectors corresponding to the eigenvalues. For each of the cases in the table, determine if

$$\lim_{n\to\infty} \mathbf{M}^n \vec{x}$$

converges. If it does, what does it converge to?

α	β	γ	Converges?	$\lim_{n\to\infty} \mathbf{M}^n\vec{x}$
0	0	$\neq 0$		
0	$\neq 0$	0		
0	$\neq 0$	$\neq 0$		
$\neq 0$	0	0		
$\neq 0$	0	$\neq 0$		
$\neq 0$	$\neq 0$	0		
$\neq 0$	$\neq 0$	$\neq 0$		

Answer:

$$\begin{aligned}\mathbf{M}^n\vec{x} &= \mathbf{M}^n(\alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3)\\ &= \alpha\mathbf{M}^n\vec{v}_1 + \beta\mathbf{M}^n\vec{v}_2 + \gamma\mathbf{M}^n\vec{v}_3\\ &= 1^n\alpha\vec{v}_1 + 2^n\beta\vec{v}_2 + \left(\frac{1}{2}\right)^n\gamma\vec{v}_3\end{aligned}$$

α	β	γ	Converges?	$\lim_{n\to\infty} \mathbf{M}^n\vec{x}$
0	0	$\neq 0$	Yes	$\vec{0}$
0	$\neq 0$	0	No	-
0	$\neq 0$	$\neq 0$	No	-
$\neq 0$	0	0	Yes	$\alpha\vec{v}_1$
$\neq 0$	0	$\neq 0$	Yes	$\alpha\vec{v}_1$
$\neq 0$	$\neq 0$	0	No	-
$\neq 0$	$\neq 0$	$\neq 0$	No	-

(b) (**Practice**) Find the eigenspaces associated with the eigenvalues:

i. span($\vec{v}_1$), associated with $\lambda_1 = 1$

ii. span($\vec{v}_2$), associated with $\lambda_2 = 2$

iii. span($\vec{v}_3$), associated with $\lambda_3 = \frac{1}{2}$

Answer:

i. $\lambda = 1$:

$$\left[\begin{array}{c|c} \mathbf{M}-\mathbf{I} & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{G.E.}{\to} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_1 = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}$$

This means that

$$span\{\vec{v}_1\} = span\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

ii. $\lambda = 2$:

$$\left[\begin{array}{c|c} \mathbf{M}-2\mathbf{I} & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} \frac{-3}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{G.E.}{\to} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_2 = \beta \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \beta \in \mathbb{R}$$

This means that

$$span\{\vec{v}_2\} = span\left\{\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right\}$$

iii. $\lambda = \frac{1}{2}$:

$$\left[\begin{array}{c|c} \mathbf{M}-\frac{1}{2}\mathbf{I} & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -2 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \end{array}\right] \overset{G.E.}{\to} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_3 = \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \gamma \in \mathbb{R}$$

This means that

$$span\{\vec{v}_3\} = span\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

3. Are eigenvectors linearly independent?

Suppose we have a square matrix $\mathbf{A}^{n\times n}$ with n distinct eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ (meaning that $\lambda_i \neq \lambda_j$ when $i \neq j$) and n corresponding eigenvectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Prove that any two eigenvectors $\vec{v}_i, \vec{v}_j$ (for $i \neq j$) are linearly independent.

HINT: Begin proof by contradiction: Suppose that $\vec{v}_i$ and $\vec{v}_j$ correspond to distinct eigenvalues, so that $(\lambda_i - \lambda_j) \neq 0$, and are linearly dependent. Show this leads to a nonsensical equality after applying $\mathbf{A}$.

If you still feel stuck, apply the definition of linear dependence to $\vec{v}_i$ and $\vec{v}_j$. What happens when we apply $\mathbf{A}$ to eigenvectors, and more importantly to the definition you found in the last sentence? If you need help understanding proof by contradiction, Example 4.4 in Note 4 gives a good explanation and example.

Answer:

PROOF BY CONTRADICTION:

Suppose $\vec{v}_i$ and $\vec{v}_j$ correspond to distinct eigenvalues such that $(\lambda_i - \lambda_j) \neq 0$ and are linearly dependent, meaning $\alpha\vec{v}_i + \beta\vec{v}_j = \vec{0}$.

NOTE: We know that both $\alpha \neq 0$ and $\beta \neq 0$ since any zero constant would imply that one of the eigenvectors is $\vec{0}$, which by definition of an eigenvector cannot be true.

Let $\vec{u} = \alpha\vec{v}_i + \beta\vec{v}_j = \vec{0}$. By definition $\mathbf{A}\,\vec{u} = \mathbf{A}\,\vec{0} = \vec{0}$.
However ...

$$
\begin{aligned}
\mathbf{A}\,\vec{u} &= \mathbf{A}(\alpha\vec{v}_i + \beta\vec{v}_j) = \alpha\,\mathbf{A}\,\vec{v}_i + \beta\,\mathbf{A}\,\vec{v}_j \\
&= \alpha\,\lambda_i\,\vec{v}_i + \beta\,\lambda_j\,\vec{v}_j \\
&= \lambda_i(\alpha\vec{v}_i + \beta\vec{v}_j) + (\lambda_j - \lambda_i)\,\beta\,\vec{v}_j \\
&= \lambda_i\,\vec{u} + (\lambda_j - \lambda_i)\,\beta\,\vec{v}_j \\
&= (\lambda_j - \lambda_i)\,\beta\,\vec{v}_j \qquad\qquad = \vec{0}
\end{aligned}
$$

Since all three components $(\lambda_j - \lambda_i)$, β, and $\vec{v}_j$ cannot be zero by construction (and/or definition), we've arrived at a contradiction suggesting that the eigenvectors $\vec{v}_i$ and $\vec{v}_j$ MUST be linearly independent! $\square$

