EECS 16A Designing Information Devices and Systems I
Summer 2023 Discussion 3D

1. Matrix Multiplication Proof

(a) Given that matrix A is square and has linearly independent columns, which of the following are true? (You do not need to prove everything)

i. A is full rank

ii. A has a trivial nullspace

iii. A$\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$

iv. A is invertible

v. The determinant of A is non-zero

Solution/Answer: They are all true. Below are some informal explanations of why.

i.: If a square matrix has linearly independent columns then the dimension of its columnspace is equal to the number of columns in the matrix, which by definition means it is full rank.

ii.: Since A is square and has linearly independent columns, the dimension of it's columnspace is equal to the number of columns in the matrix. Thus, by the rank-nullity theorem, the dimension of its nullspace is 0 meaning that it is a trivial nullspace.

iii. : Let's say $A \in \mathbb{R}^{n\times n}$. Since A has linearly independent columns, then its columns form a basis for $\mathbb{R}^n$. This means any vector for any vector $\vec{b} \in \mathbb{R}^n$ there exists a unique $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{b}$.

iv. : From iii. we saw that $A\vec{x} = \vec{b}$ has a unique solution for any $\vec{b}$. Consequently $AB = I$ must have a unique solution B where I is the identity matrix. By definition, $B = A^{-1}$ and so A is invertible.

v. : Recall the geometric interpretation of the determinant of A as the volume of the parallelepiped formed by its columns. Since A has linearly independent columns, none of the sidelengths of the parallelepiped will be 0 and so the determinant will always be nonzero.

(b) Let two square matrices $\mathrm{M}_1, \mathrm{M}_2 \in \mathbb{R}^{2x2}$ each have linearly independent columns. Prove that $\mathrm{G} = \mathrm{M}_1\mathrm{M}_2$ also has linearly independent columns.

Solution/Answer: If $\mathrm{M_i}$ is square and has linearly independent columns, then it is also invertible.

Now, let's consider the toy case of $\mathrm{G} = \mathrm{M}_1$. Since we've established that $\mathrm{M_i}$ is invertible, we get ${\mathrm{M}_1}^{-1}\mathrm{G} = \mathrm{I}$. By definition, we've found that G has an inverse, namely ${\mathrm{M}_1}^{-1}$. Since G has an inverse, it must have linearly independent columns.

We're now in a position to extend this argument:

$$\begin{aligned} \mathrm{G} &= \mathrm{M}_1\mathrm{M}_2 \\ \implies {\mathrm{M}_1}^{-1}\mathrm{G} &= I\mathrm{M}_2 \\ \implies {\mathrm{M}_2}^{-1}{\mathrm{M}_1}^{-1}\mathrm{G} &= I \\ \implies \mathrm{G}^{-1} &= {\mathrm{M}_2}^{-1}{\mathrm{M}_1}^{-1} \end{aligned}$$

Thus, we've shown that G has an inverse composed of two invertible matrices, which implies that it has linearly independent columns.

Note that there are many other ways to prove this. For example, consider a proof by contradiction in which you assume G has a nontrivial nullspace. What does this imply about M_1 and/or M_2?

2. Exploring Dimension and Linear Independence

In this problem, we are going to talk about the connections between several concepts we have learned about in linear algebra – linear independence and dimension of a vector space/subspace.

Let's consider the vector space $\mathbb{R}^k$ (the k-dimensional real-world) and a set of n vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in $\mathbb{R}^k$.

(a) For the first part of the problem, let $k > n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ span the full $\mathbb{R}^k$ space? If so, prove it. If not, what conditions does it violate/what is missing?

Answer:

No, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ cannot span the space $\mathbb{R}^k$. The dimension of $\mathbb{R}^k$ is k, so you would need k linearly independent vectors to describe the vector space. Since $n < k$, this is not possible.

(b) Let $k = n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ span the full $\mathbb{R}^k$ space? Why/why not? What conditions would we need?

Answer:

Fact: matrix V is invertible $\iff$ V is square and has linearly independent columns

Note you have not proven this, but you will see the proof in EECS16B. Yes, this is possible. The only condition we need is that $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is linearly independent. If the vectors are linearly independent, since there are k of them, we can put them into a square matrix V:

$$V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

This matrix is square because the number of entries in the column vectors (k) is equal to the number of column vectors (n).

Using the fact from above, we know that if the square matrix V has n linearly independent columns, it will be invertible. If the matrix V is invertible, the matrix vector equation $V\vec{x} = \vec{b}$ will always have a unique solution for all vectors $\vec{b}$. Thus all possible $\vec{b} \in \mathbb{R}^k$ are in the span of the columns of matrix V: $(\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\})$.

To summarize, we can conclude then that if $n = k$ and $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is linearly independent, $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ can span the full $\mathbb{R}^k$ space.

(c) Finally, let $k < n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ span the full $\mathbb{R}^k$ space?

Hint: Think about whether the vectors can be linearly independent.

Answer:

If k out of the n vectors are linearly independent, then the set of vectors will be able to span the full $\mathbb{R}^k$ space. However, if we have less than k linearly independent vectors in the set, then there is too much linear dependence for this set of vectors to be able to span the full $\mathbb{R}^k$ space.

The two regimes—one where $n > k$ and one where $n < k$—give rise to two different classes of interesting problems. You might learn more about them in upper division courses!

3. Row Space

Consider:

$$\mathbf{V} = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 0 & 4 \\ 6 & 4 & 10 \\ -2 & 4 & 2 \end{bmatrix}$$

Row reducing this matrix yields:

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Show that the row spaces of $\mathbf{U}$ and $\mathbf{V}$ are the same. Argue that in general, Gaussian elimination preserves the row space.

Answer:

To show that the spans of two sets of vectors are the same, show that each vector in one set can be written as a linear combination of the vectors in the other. Let's call the rows of $\mathbf{U}$ $\vec{u}_i^T$ and the rows of $\mathbf{V}$ $\vec{v}_i^T$. Because of the structure of $\mathbf{U}$, it is easier to write the $\vec{v}_i$ in terms of the $\vec{u}_i$:

$$\vec{v}_1 = 2\vec{u}_1 + 4\vec{u}_2$$
$$\vec{v}_2 = 4\vec{u}_1$$
$$\vec{v}_3 = 6\vec{u}_1 + 4\vec{u}_2$$
$$\vec{v}_4 = -2\vec{u}_1 + 4\vec{u}_2$$

We have now shown that all the $\vec{v}_i$ lie in the span of the $\vec{u}_i$. By rearranging the equations to be in terms of the $\vec{u}_i$, we can then show that all the $\vec{u}_i$ lie in the span of the $\vec{v}_i$

In general: the valid Gaussian elimination operations are scaling rows, swapping rows, and adding rows to each other. Since the span of a set of vectors consists of all linear combinations of the vectors, it is not affected by these operations.

(b) Show that the null spaces of $\mathbf{U}$ and $\mathbf{V}$ are the same. Argue that in general, Gaussian elimination preserves the null space.

Answer:

To show the null spaces are the same, compute them and check. The null space for both is span $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$.

In general: if there is a vector $\vec{x}$ in the null space of $\mathbf{V}$, then $\mathbf{V}\vec{x} = \vec{0}$. One interpretation is that for every row $\vec{v}_i^T$, $\vec{v}_i^T\vec{x} = 0$. During Gaussian elimination, we scale/swap/add rows to each other, and all of them satisfy $\vec{v}_i^T\vec{x} = 0$. Therefore, all the rows will always continue to satisfy this relation during GE.