EECS 16A   Designing Information Devices and Systems I
Summer 2023
Homework 2

This homework is due June 30th, 2023, at 23:59.
Self-grades are due July 7th, 2023, at 23:59.

Submission Format

Your homework submission should consist of a single PDF file that contains all of your answers (any hand-written answers should be scanned) as well as your IPython notebook saved as a PDF.

If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

1. Reading Assignment

For this homework, please read Note 1B, Note 2A, Note 2B, Note 3, Note 4, and Note 11A. Note 1B covers Gaussian Elimination. Notes 2A and 2B provide an overview of vectors, matrices, and operations among them. Note 3 covers the concepts of linear dependence and span, and Note 4 is a helpful guide for how to approach proofs. Note 11A is an introduction to circuit analysis.

Please answer the following questions:

  (a) Can you name a few applications where we might use vectors? What do the components represent in each application?

  (b) Are there special matrices where the order of their multiplication does *not* matter? If so come up with an example and explain why.

  (c) What is the mathematical definition of span?

  (d) List in your own words the steps used to construct a proof.

2. Image Masks

Learning Objective: *Learn to setup imaging problems with matrices.*

For these word problems, you only need to setup the problem in augmented matrix or matrix-vector notation. Of course, you may solve the system for practice (e.g., with Gaussian elimination), but no additional credit is awarded.

After your first EECS16A lecture, you decide to try to build a single-pixel camera. You want to take a 2x2 image, i.e. 4 tiles, and based on the first lecture, you choose to take 4 measurements. Recall that each measurement is the sum of the illuminated tiles. For each measurement, you will use a different mask.

  (a) Initially, you want to illuminate only one tile for each measurement. That is, you will first illuminate x_1, then you will illuminate x_2, etc. The outputs of your 4 measurements are $y_1, y_2, y_3,$ and y_4 respectively. The 4 measurements you take are shown in Figure 1. Explicitly setup the matrix problem for this in the $A\vec{x} = \vec{b}$ form.

x_1	x_2
x_3	x_4

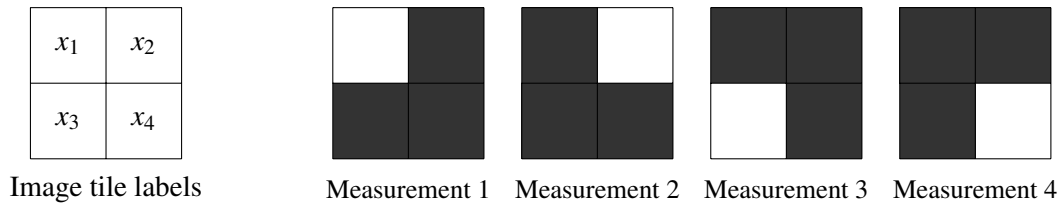Image tile labels   Measurement 1   Measurement 2   Measurement 3   Measurement 4

Figure 1: Four image masks.

(b) While setting up your code to create the masks, you forget to turn off the illuminated tiles from the previous measurement. As a result, measurement one contains x_1, measurement two contains $x_1 + x_2$, etc. The outputs of your 4 measurements are z_1, z_2, z_3, and z_4 respectively. The 4 measurements you take are shown in Figure 2. Explicitly setup the matrix problem for this in $A\vec{x} = \vec{b}$ form.

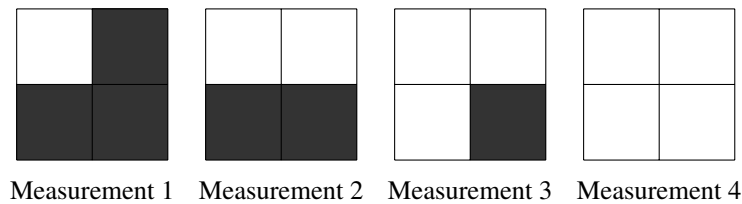Measurement 1   Measurement 2   Measurement 3   Measurement 4

Figure 2: Four image masks.

(c) Your friend is also building their own single pixel camera. However, they make a different mistake in their code and during each measurement, instead of lighting up one tile, you light up the other 3 tiles instead. That is, instead of measuring x_1, they measure $x_2 + x_3 + x_4$. The output of the 4 measurements are w_1, w_2, w_3, and w_4. The 4 measurements from their setup are shown in Figure 3. Explicitly setup the matrix problem for this in $A\vec{x} = \vec{b}$ form.
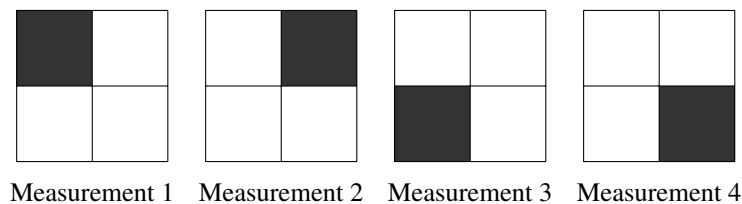

Measurement 1   Measurement 2   Measurement 3   Measurement 4

Figure 3: Four image masks.

3. **Gaussian Elimination**

Learning Goal: *Understand the relationship between Gaussian elimination and the graphical representation of linear equations, and explore different types of solutions to a system of equations. You will also practice determining the parametric solutions when there are infinitely many solutions.*

(a) In this problem we will investigate the relationship between Gaussian elimination and the geometric interpretation of linear equations. You are welcome to draw plots by hand or using software. Please be sure to label your equations with a legend on the plot.

   i. Plot the following set of linear equations in the *x-y* plane. If the lines intersect, write down the point or points of intersection.

$$x+2y=4 \tag{1}$$
$$2x-4y=4 \tag{2}$$
$$3x-2y=8 \tag{3}$$

   ii. Write the above set of linear equations in augmented matrix form and do the first step of Gaussian elimination to eliminate the x variable from equation 2. Now, the second row of the augmented matrix has changed. Plot the corresponding new equation created in this step on the same graph as above. What do you notice about the new line you draw?

   iii. Complete all of the steps of Gaussian elimination including back substitution. Now plot the new equations represented by the rows of the augmented matrix in the last step (after completing back substitution) on the same graph as above. What do you notice about the new line you draw?

(b) Write the following set of linear equations in augmented matrix form and use Gaussian elimination to determine if there are no solutions, infinite solutions, or a unique solution. If any solutions exist, determine what they are. You may do this problem by hand or use a computer. We encourage you to try it by hand to ensure you understand Gaussian elimination. Remember that it is possible to end up with fractions during Gaussian elimination.

$$x+2y+5z=3$$
$$x+12y+6z=1$$
$$2y+z=4$$
$$3x+16y+16z=7$$

(c) Consider the following system:

$$4x+4y+4z+w+v=1$$
$$x+y+2z+4w+v=2$$
$$5x+5y+5z+w+v=0$$

If you were to write the above equations in augmented matrix form and use Gaussian elimination to solve the system, you would get the following (for extra practice, you can try and do this yourself):

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 3 & 16 \\ 0 & 0 & 1 & 0 & -3 & -17 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

How many variables are free variables? Which ones? Find the general form of the solutions in terms of real constants (e.g., $s \in \mathbb{R}$).

4. **Vector-Vector, Matrix-Vector, and Matrix-Matrix Multiplication**

   Learning Objective: *Practice evaluating vector-vector, matrix-vector, and matrix-matrix multiplication.*

(a) For the following multiplications, state the dimensions of the result. If the product is not defined and thus has no solution, state this and justify your reasoning. For this problem $\vec{x} \in \mathbb{R}^N, \vec{y} \in \mathbb{R}^N, \vec{z} \in \mathbb{R}^M$, with $N \neq M$.

    i. $\vec{x}^T \cdot \vec{z}$

    ii. $\vec{x} \cdot \vec{x}^T$

    iii. $\vec{x} \cdot \vec{y}^T$

    iv. $\vec{x} \cdot \vec{z}^T$

For questions (b) through (d), complete the matrix-vector multiplication. If the product is not defined and thus has no solution, state this and justify your reasoning:

(b)

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(e) Compute $\mathbf{AB}$ by hand, where $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -3 & 0 & 2 & -1 \end{bmatrix}$$

What are the dimensions of $\mathbf{AB}$? Compute $\mathbf{BA}$ too if the operation is valid. If it is invalid, explain why. Make sure you show the work for your calculations.

(f) Compute $\mathbf{AB}$ by hand, where $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{bmatrix} 3 & 21 & 9 \\ -1 & 14 & 4 \\ 7 & -8 & 2 \end{bmatrix}, \text{ and } \quad \mathbf{B} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \\ 3 & -6 \end{bmatrix}$$

Compute $\mathbf{BA}$ too if the operation is valid. If it is invalid, explain why. Make sure you show the work for your calculations.

5. **Basic Circuit Components**

   Learning Objectives: *Review basics of cucuit components and current-voltage relationships*

   In the laboratory, you are tasked with identifying a single unknown component within a circuit. You can use a piece of electrical equipment called a multimeter to measure either voltage (voltmeter) across or the current (ammeter) through the component (you can measure both quantities simultaneously using two multimeters).

   For each part of the problem, deduce the most likely type of circuit component based on the provided voltage and current measurements. Also draw the component circuit symbol and sketch the IV curve.

   *Hint: You are told the possible choices are **short circuit (wire)**, **open circuit**, **resistor**, **voltage source**, and **current source**. Moreover, each part in this problem has a unique component, there are no repeats.*
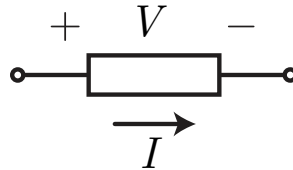
   (a) First, to familiarize yourself with common quantities of *voltage*, *current*, and *resistance*, fill in the unit name and unit symbol for each:

Quantity	Symbol	Unit Name	Unit Symbol
Voltage	V		
Current	I		
Resistance	R		

   (b) You take one measurement and find the voltage is $V = 10$ V and current is $I = 1$ A. After changing a part of the circuit (not the part you are measuring), you take another measurement and find $V = 10$ V and $I = 2$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

   (c) You take one measurement and find the voltage is $V = 0$ V and current is $I = 1$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

   (d) You take one measurement and find the voltage is $V = 5$ V and current is $I = 0$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

   (e) You take one measurement and find the voltage is $V = 10$ V and current is $I = 2$ A. After changing a part of the circuit (not the part you are measuring), you take another measurement and find $V = -5$ V and $I = -1$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

   (f) You take one measurement and find the voltage is $V = 10$ V and current is $I = 2$ A. After changing a part of the circuit (not the part you are measuring), you take another measurement and find $V = -3$ V and $I = 2$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

6. **Easing into Proofs**

   Learning Objectives: *This is an opportunity to practice your proof development skills.*

   Proof: Show that if the system of linear equations, $\mathbf{A}\vec{x} = \vec{b}$, has infinitely many solutions, then columns of $\mathbf{A}$ are linearly dependent.

To approach this proof, let us use a simplified version of the methodology delineated in Note 4. Although the final proof would read sequentially as in the *Proof Steps* indicated in Table 1, each part of this question will tackle each proof step but in a non-sequential order.

Table 1: Fundamental steps to a proof

Proof Steps	Description	Question Part
1	Identify what is known	(a)
2	Manipulate what is known	(c)
3	Connecting it up	(d)
4	Identify what is to be proved	(b)

(a) **Proof Step 1: Write what is known**

Think about the information we already know from the problem statement. Every detail could be important and some details could be implicit.

We know that system of equations, $A\vec{x} = \vec{b}$, has infinitely many solutions which can be difficult to work with, but perhaps we can simplify to a case that we can work with. It turns out that if a linear system has at least **two** distinct solutions, then it must also have an infinite number of solutions.

We can also construct arbitrary vectors $\vec{u}$ and $\vec{v}$ which, in this case, are each a solution to the system of equations $A\vec{x} = \vec{b}$ but not the same vector. Express the previous sentence in a mathematical form (just writing the equations will suffice; no need to do further mathematical manipulation).

(b) **Proof Step 4: Identify what is to be proved**

We have to show that the columns of A are linearly dependent. The matrix A can always be deconstructed and the columns explicitly denoted as vectors $\vec{c}_1$, $\vec{c}_2$, ..., and $\vec{c}_n$, i.e. $A = \begin{bmatrix} | & | & \cdots & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & \cdots & | \end{bmatrix}$.

Using the Definition 3.1 of linear dependence from Note 3.1, write a mathematical equation that conveys linear dependence of $\vec{c}_1$, $\vec{c}_2$, ..., and $\vec{c}_n$.

(c) **Proof Step 2: Manipulating what is known**

Now let us try to start from the givens in part (a) and make mathematically logical steps to reach the final result in part (b). Since your answer to (b) is expressed in terms of the column vectors of A, try to express the mathematical equations from part (a) in terms of the column vectors too. Using the column interpretation of matrix-vector multiplication, we can write

$$A\vec{x} = \vec{b} \implies \begin{bmatrix} | & | & \cdots & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \vec{b} \implies x_1\vec{c}_1 + x_2\vec{c}_2 + \ldots + x_n\vec{c}_n = \vec{b}$$

Now rewrite your answer to part (a) using the above formulation to get equations for distinct solutions $\vec{u}$ and $\vec{v}$ in terms of the column vectors of A.

(d) **Proof Step 3: Connecting it up**

Now think about how you can mathematically manipulate your answer from part (c) to match the pattern of your desired final proof statement in part (b).

7. Span Proofs

Learning Objectives: *This is an opportunity to practice your proof development skills. Keep in mind the method and tips given in Note 4!*

(a) Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$$

In other words, we can replace one vector with the sum of itself and another vector and not change their span.

In order to show this, you have to prove the two following statements:

- If a vector $\vec{q}$ belongs in $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, then it must also belong in $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$.
- If a vector $\vec{r}$ belongs in $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$, then it must also belong in $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$.

In summary, you have to prove the problem statement from both directions.

(b) Consider the span of the set $(\vec{v}_1, \ldots, \vec{v}_n, \vec{u})$. Suppose $\vec{u}$ is in the span of $\{\vec{v}_1, \ldots, \vec{v}_n\}$. Then, show that any vector $\vec{r}$ in $span\{\vec{v}_1, \ldots, \vec{v}_n, \vec{u}\}$ is in $span\{\vec{v}_1, \ldots, \vec{v}_n\}$.

8. Filtering Out The Troll

Learning Objectives: *The goal of this problem is to explore the problem of sound reconstruction by solving a system of linear equations.*

You were attending the 16A lecture the day before the first exam, and decided to record it using two directional microphones (one microphone receives sound from the 'x' direction and the other from the 'y' direction). However, someone in the audience was trolling around loudly, adding interference to the recording! The troll's interference dominates both of your microphones' recordings, so you cannot hear the recorded speech. Fortunately, since your recording device contained two microphones, you can combine the two individual microphone recordings to remove the troll's interference.

The diagram shown in Figure 4 shows the locations of the speaker, the troll, and you and your two microphones (at the origin).
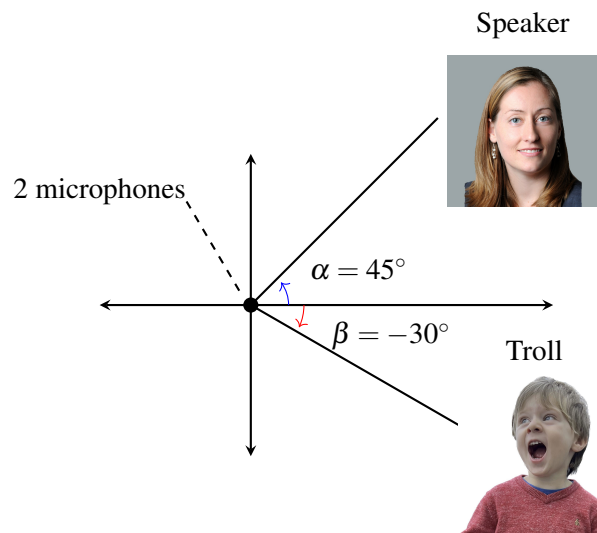

Figure 4: Locations of the speaker and the troll.

Since the microphones are directional, the strength of the recorded signal depends on the angle from which the sound arrives. Suppose that the sound arrives from an angle θ relative to the *x*-axis (in our case, these angles are 45° and $-30°$, labeled as α and β, respectively). The first microphone scales the signal by $\cos(\theta)$, while the second microphone scales the signal by $\sin(\theta)$. Each microphone records the weighted sum (or linear combination) of all received signals.

The speech signal can be represented as a vector, $\vec{s}$, and the troll's interference as vector $\vec{r}$, with each entry representing an audio sample at a given time. The recordings of the two microphones are given by $\vec{m}_1$ and $\vec{m}_2$:

$$\vec{m}_1 = \cos(\alpha) \cdot \vec{s} + \cos(\beta) \cdot \vec{r} \tag{4}$$

$$\vec{m}_2 = \sin(\alpha) \cdot \vec{s} + \sin(\beta) \cdot \vec{r} \tag{5}$$

where α and β are the angles at which the professor and the troll respectively are located with respect to the x-axis, and variables $\vec{s}$ and $\vec{r}$ are the audio signals produced by the professor and the troll respectively.

(a) Plug in $\alpha = 45° = \frac{\pi}{4}$ and $\beta = -30° = -\frac{\pi}{6}$ to Equations 4 and 5 to write the recordings of the two microphones $\vec{m}_1$ and $\vec{m}_2$ as a linear combination (i.e. a weighted sum) of $\vec{s}$ and $\vec{r}$.

(b) Solve the system from part (a) using any convenient method you prefer to recover the important speech $\vec{s}$ as a weighted combination of $\vec{m}_1$ and $\vec{m}_2$. In other words, write $\vec{s} = c \cdot \vec{m}_1 + k \cdot \vec{m}_2$ (where c and k are scalars). What are the values of c and k?

(c) Partial IPython code can be found in `prob2.ipynb`, which you can access through the Datahub link associated with this assignment on the course website. Complete the code to get the signal of the important speech. Write out what the speaker says. (Optional: Where is the speech taken from?)

Note: You may have noticed that the recordings of the two microphones sound remarkably similar. This means that you could recover the real speech from two "trolled" recordings that sound almost identical! Leave out the fact that the recordings are actually different, and have some fun with your friends who aren't lucky enough to be taking EECS16A.

9. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.