EECS 16A Designing Information Devices and Systems I
Summer 2023 Homework 7

This homework is due Saturday, August 5, 2023, at 23:59.
Self-grades are due Friday, August 11, 2023, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw7.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

Submit the file to the appropriate assignment on Gradescope.

1. **Reading Assignment**

For this homework, please read Note 20 (Circuit Design), Note 21 (Inner Products and GPS), Note 22 (Trilateration and Correlation), and read Note 23 (Least Squares). You are always encouraged to read beyond this as well.

(a) In trilateration, the distances between the beacons and the unknown location $\vec{x}$ involve quadratic terms of $\vec{x}$. What trick can we use to get a system of linear equations in $\vec{x}$?

(b) Suppose the signal $x[n]$ is only defined for timesteps $0, 1, \cdots, 5$. For the purpose of computing linear cross-correlation, what value of $x[n]$ do we assume when n is a timestep out of the range: $0 \leq n \leq 5$ (e.g. $n = 6$ or $n = -1$)?

2. Transistor Equivalent

Consider the amplifier circuit in Fig. 1 which amplifies input V_{in} to output V_{out}. The circuit accomplishes this by using a bipolar junction transistor or BJT. The BJT is a three-terminal circuit element with nodes B, C, and E.

In some situations, the BJT can be modeled with an equivalent linear circuit containing a voltage-dependent current source as shown in Fig. 2.

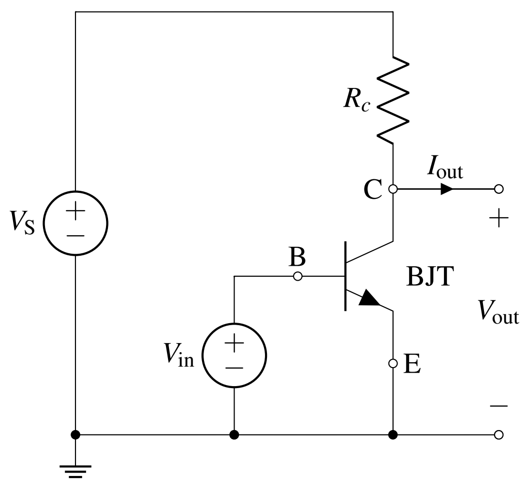


Figure 1: Amplifier circuit with BJT

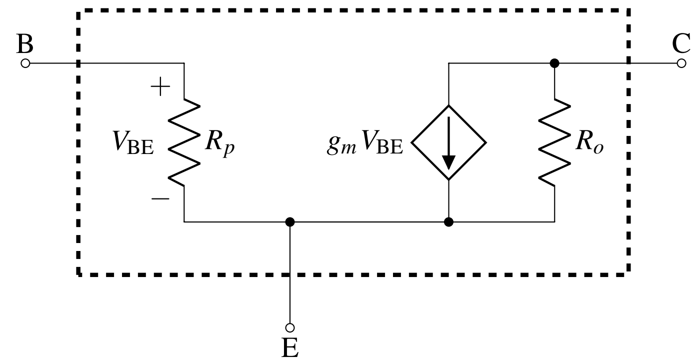


Figure 2: Equivalent circuit model for BJT

We want to find the Thevenin and Norton equivalent of the amplifier circuit across the terminals V_{out} (i.e., between nodes C and E).

Note: You can use the parallel operator (||) in your final answers.

(a) Redraw the original amplifier circuit in Fig. 1 but with the BJT equivalent circuit model in Fig. 2 substituted.

(b) Use the open circuit test to find the Thevenin voltage, V_{th}, between nodes C and E.

Recall the open circuit test finds $V_{out} = V_{oc}$ when an open circuit is connected across the terminals, then $V_{th} = V_{oc}$.

(c) Use the short circuit test to find the Norton current, I_{no}, between nodes C and E.

Recall the short circuit test finds $I_{out} = I_{sc}$ when a short circuit is connected between the terminals, then $I_{no} = I_{sc}$.

(d) Find the Thevenin/Norton resistance $R_{th} = R_{no}$ using $R_{th} = \frac{V_{th}}{I_{no}}$.

(e) We can also find R_{th} by turning off all of the independent sources (but *not* the dependent sources) and deriving the equivalent resistance seen from the terminals. Derive R_{th} with this method. Does it match your answer from part (c)?

Hint: To simplify the dependent source, focus on first finding V_{BE}.

3. Inner Product Properties

Learning Goal: *The objective of this problem is to exercise useful identities for inner products.*

Our definition of the inner product in $\mathbb{R}^n$ is:

$$\langle \vec{x}, \vec{y} \rangle \; = \; x_1 y_1 + x_2 y_2 + \ldots + x_n y_n \; = \vec{x}^\intercal \vec{y} \,, \quad \text{for any } \; \vec{x}, \vec{y} \in \mathbb{R}^n$$

Prove the following identities in $\mathbb{R}^n$:

(a) $\langle \vec{x}, \vec{x} \rangle = \|\vec{x}\|^2$

(b) $\langle -\vec{x}, \vec{y} \rangle = -\langle \vec{x}, \vec{y} \rangle$.

(c) $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

(d) $\langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle = \langle \vec{x}, \vec{x} \rangle + 2\langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{y} \rangle$

4. Inner Products

For each of the following functions, show whether it defines an inner product on the given vector space. If not, give a counterexample, i.e., find a pair of vectors p and q such that the given function fails to satisfy one of the inner product properties.

(a) For $\mathbb{R}^2$:

$$\langle \vec{p}, \vec{q} \rangle = \vec{p}^T \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \vec{q}$$

(b) For $\mathbb{R}^2$:

$$\langle \vec{p}, \vec{q} \rangle = \vec{p}^T \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} \vec{q}$$

(c) For $\mathbb{R}^2$:

$$\langle \vec{p}, \vec{q} \rangle = \vec{p}^T \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \vec{q}$$

(d) For $\mathbb{R}^{2x2}$, the space of all 2x2 real matrices, the *Frobenius* inner product is defined as:

$$\langle A, B \rangle_F = Tr(A^T B)$$

Where A and B are 2x2 real matrices, and Tr represents the *trace* of a matrix, or the sum of its diagonal entries. Prove that the Frobenius inner product is valid over $\mathbb{R}^{2x2}$.

5. Orthonormal Matrices

Definition: A matrix $U \in \mathbb{R}^{n \times n}$ is called an orthornomal matrix if $U^{-1} = U^T$ and each column of U is a unit vector.

Orthornomal matrices represent linear transformations that preserve angles between vectors and the lengths of vectors. Rotations and reflections, useful in computer graphics, are examples of transformations that can be represented by orthonormal matrices.

Hint: The transpose of a product of matrices is equivalent to the product of the transposes of the matrices in reverse order. For example $(U\vec{x})^T = \vec{x}^T U^T$.

(a) Let U be an orthonormal matrix. For two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, show that $\langle \vec{x}, \vec{y} \rangle = \langle U\vec{x}, U\vec{y} \rangle$, assuming we are working with the Euclidean inner product.

(b) Show that $\|U\vec{x}\| = \|\vec{x}\|$, where $\|\cdot\|$ is the Euclidean norm.

(c) How does multiplying $\vec{x}$ by U affect the length of the vector? That is, how do the lengths of $U\vec{x}$ and $\vec{x}$ compare?

6. Audio File Matching

Learning Goal: *This problem motivates the application of correlation for pattern matching applications such as Shazam. Note: Shazam is an application that identifies songs playing around you.*

Many audio processing applications rely on representing audio files as vectors, referred to as audio *signals*. Every component of the vector determines the sound we hear at a given time. We can use inner products to determine if a particular audio clip is part of a longer song, similar to an application like *Shazam*.

Let us consider a very simplified model for an audio signal, $\vec{x}$. At each timestep k, the audio signal can be either $x[k] = -1$ or $x[k] = 1$.

(a) Say we want to compare two audio files of the same length N to decide how similar they are. First, consider two vectors that are exactly identical, namely $\vec{x}_1 = \begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$ and $\vec{x}_2 = \begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$. What is the inner product of these two vectors? What if $\vec{x}_1 = \begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$ but $\vec{x}_2$ oscillates between 1 and -1 (i.e., $\vec{x}_1 = \begin{bmatrix}1 & -1 & 1\cdots & -1\end{bmatrix}^T$)? Assume that N, the length of the two vectors, is an even number.

Use this to suggest a method for comparing the similarity between a generic pair of length-N vectors.

(b) Next, suppose we want to find a short audio clip in a longer one. We might want to do this for an application like *Shazam*, which is able to identify a song from a short clip. Consider the vector of length 8, $\vec{x} = \begin{bmatrix}-1 & 1 & 1 & -1 & 1 & 1 & -1 & 1\end{bmatrix}^T$.

We want to find the short segment $\vec{y} := \begin{bmatrix}y[0] & y[1] & y[2]\end{bmatrix}^T = \begin{bmatrix}1 & 1 & -1\end{bmatrix}^T$ in the longer vector. To do this, perform the linear cross correlation between these two finite length sequences and identify at what shift(s) the linear cross correlation is maximized. Apply the same technique to identify what shift(s) gives the best match for $\vec{y} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$.

(If you wish, you may use iPython to do this part of the question, but you do not have to.)

(c) Now suppose our audio vector is represented using integers beyond simply just 1 and -1. Find the short audio clip $\vec{y} = \begin{bmatrix}1 & 2 & 3\end{bmatrix}^T$ in the song given by $\vec{x} = \begin{bmatrix}1 & 2 & 3 & 1 & 2 & 2 & 3 & 10\end{bmatrix}^T$. Where do you expect to see the peak in the correlation of the two signals? Is the peak where you want it to be, i.e. does it pull out the clip of the song that you intended? Why?

(If you wish, you may use iPython to do this part of the question, but you do not have to.)

(d) Let us think about how to get around the issue in the previous part. We applied cross-correlation to compare segments of $\vec{x}$ of length 3 (which is the length of $\vec{y}$) with $\vec{y}$. Instead of directly taking the cross correlation, we want to normalize each inner product computed at each shift by the magnitudes of both segments, i.e. we want to consider the inner product $< \frac{\vec{x}_k}{\|\vec{x}_k\|}, \frac{\vec{y}}{\|\vec{y}\|} >$, where $\vec{x}_k$ is the length 3 segment starting from the k-th index of $\vec{x}$. This is referred to as normalized cross correlation. Using this procedure, now which segment matches the short audio clip best?

(e) We can use this on a more 'realistic' audio signal – refer to the IPython notebook, where we use normalized cross-correlation on a real song. Run the cells to listen to the song we are searching through, and add a simple comparison function `vector_compare` to find where in the song the clip comes from (i.e. write down the matching timestamp of the long audio clip). Running this may take a couple minutes on your machine, but note that this computation can be highly optimized and run super fast in the real world! Also note that this is not exactly how Shazam works, but it draws heavily on some of these basic ideas.

Note: if the script is running super slowly on Datahub, we recommend running it locally by installing Jupyter Notebook. An explanation for how to do this can be found here.

7. Mechanical Projections

Learning Goal: *The objective of this problem is to practice calculating projection of a vector and the corresponding squared error.*

(a) Find the projection of $\vec{b} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ onto $\vec{a} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. What is the squared error between the projection and $\vec{b}$, i.e. $\|e\|^2 = \|\text{proj}_{\vec{a}}(\vec{b}) - \vec{b}\|^2$?

(b) Find the projection of $\vec{b} = \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix}$ onto the column space of $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. What is the squared error between the projection and $\vec{b}$, i.e. $\|e\|^2 = \|\text{proj}_{\text{Col}(\mathbf{A})}(\vec{b}) - \vec{b}\|^2$?

8. Mechanical Trilateration

Learning Goal: *The objective of this problem is to practice using trilateration to find the position based on distance measurements and known beacon locations.*

Trilateration is the problem of finding one's coordinates given distances from known beacon locations. For each of the following trilateration problems, you are given 3 beacon locations ($\vec{s}_1$, $\vec{s}_2$, $\vec{s}_3$) and the corresponding distance (d_1, d_2, d_3) from each beacon to your location.

For each problem, **graph** (by hand, with a graphing calculator, or iPython) the set of coordinates indicating your possible location for each beacon and find any coordinate solutions where they all intersect. Then **solve the trilateration problem algebraically** using the method introduced in lecture, to find your location or possible locations. If a solution does not exist, state that it does not.

For any solutions found using trilateration, be sure to check that they are consistent with the beacon measurements.

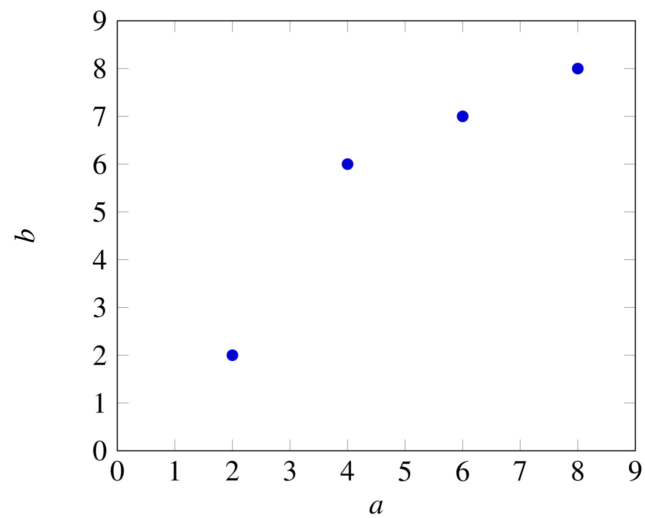
(a) $\vec{s}_1 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, d_1 = 5; \vec{s}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, d_2 = 2; \vec{s}_3 = \begin{bmatrix} -11 \\ 6 \end{bmatrix}, d_3 = 13$

(b) $\vec{s}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, d_1 = 5\sqrt{2}; \vec{s}_2 = \begin{bmatrix} 10 \\ 0 \end{bmatrix}, d_2 = 5\sqrt{2}; \vec{s}_3 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}, d_3 = 5$

Why can't we precisely determine our location, even though we have the same number of measurements as part (a)? Can we use our original constraints to narrow down our set of possible solutions we got from trilateration?

(c) $\vec{s}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, d_1 = 5; \vec{s}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, d_2 = 2; \vec{s}_3 = \begin{bmatrix} -12 \\ 5 \end{bmatrix}, d_3 = 12$

9. Mechanical: Linear Least Squares



a	2	4	6	8
b	2	6	7	8

(a) Consider the above data points. Find the linear model of the form

$$\vec{b} = \vec{a}x$$

that best fits the data, i.e. find the scalar value of $x = \hat{x}$ that minimizes the squared error

$$\|\vec{e}\|^2 = \|\vec{b} - \vec{a}x\|^2 = \left\| \begin{bmatrix} b_1 \\ \vdots \\ b_4 \end{bmatrix} - \begin{bmatrix} a_1 \\ \vdots \\ a_4 \end{bmatrix} x \right\|^2. \tag{1}$$

Note: By using this linear model, we are implicitly forcing the fit equation to go through the origin.

Do not use IPython for this calculation and show your work. Once you've computed $\hat{x}$, compute the squared error between your model's prediction and the actual $\vec{b}$ values as shown in Equation 1. Plot the best fit line along with the data points to examine the quality of the fit. (It is okay if your plot of $\vec{b} = \vec{a}x$ is approximate.)

(b) You will notice from your graph that you can get a better fit by adding a b-intercept. That is we can get a better fit for the data by assuming a linear model of the form

$$\vec{b} = x_1\vec{a} + x_2.$$

In order to do this, we need to augment our $\mathbf{A}$ matrix for the least squares calculation with a column of 1's (do you see why?), so that it has the form

$$\mathbf{A} = \begin{bmatrix} a_1 & 1 \\ \vdots & \vdots \\ a_4 & 1 \end{bmatrix}.$$

Find x_1 and x_2 that minimize the squared error

$$\|\vec{e}\|^2 = \left\|\vec{b} - \mathbf{A}\vec{x}\right\|^2 = \left\| \begin{bmatrix} b_1 \\ \vdots \\ b_4 \end{bmatrix} - \begin{bmatrix} a_1 & 1 \\ \vdots & \vdots \\ a_4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|^2. \tag{2}$$

Do not use IPython for this calculation and show your work.

Compute the squared error between your model's prediction and the actual $\vec{b}$ values as shown in Equation 2. Plot your new linear model. Is it a better fit for the data?

10. Proof: Least Squares

Let $\vec{x}$ be the solution to a least squares problem. Show that the minimizing least squares error vector $\vec{e} = \vec{b} - \mathbf{A}\vec{x}$ is orthogonal to the columns of $\mathbf{A}$ by direct manipulation (i.e. plug the formula for the least squares solution $\vec{x}$ into the error vector and then check if $\mathbf{A}^T\vec{e} = \vec{0}$.)

11. Trilateration With Noise!

Learning Goal: *This problem will help to understand how noise affects the accuracy of trilateration and consistency of the corresponding system of equations.*

In this question, we will explore how various types of noise affect the quality of triangulating a point on the 2D plane to see when trilateration works well and when it does not.

First, we will remind ourselves of the fundamental equations underlying trilateration.

(a) There are four beacons at the known coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$. You are located at some unknown coordinate (x, y) that you want to determine. The distance between your location and each of the four beacons are d_1 through d_4, respectively. Write down one equation for each beacon that relates the coordinates to the distances using the Pythagorean Theorem.

(b) Unfortunately, the above system of equations is nonlinear, so we can't use least squares or Gaussian Elimination to solve it. We will use the technique discussed in lecture to obtain a system of linear equations. In particular, we can subtract the first of the above equations (involving x_1, y_1 and d_1) from the other three to obtain three linear equations (cancel out the nonlinear terms). Write down these three linear equations.

Combine the three equations in the above system into a single matrix equation of the form

$$\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \vec{b}.$$

(c) Now, go to the IPython notebook. In the notebook we are given three possible sets of measurements for the distances of each beacon from the receiver:

i. `ideal_distances`: the ideal set of measurements, the true distances of our receiver to the beacons. $d_1 = d_2 = d_3 = d_4 = 5$.

ii. `imperfect_distances`: imperfect measurements. $d_1 = 5.5, d_2 = 4.5, d_3 = 5, d_4 = 5$.

iii. `one_bad_distances`: mostly perfect measurements, but d_1 is a very bad measurement. $d_1 = 7.5$ and $d_2 = d_3 = d_4 = 5$.

Plot the graph illustrating the case when the receiver has received `ideal_distances` and visually solve for the position of the observer (x, y). What is the coordinate?

(d) You will now set up the above linear system using IPython. Fill in each element of the matrix $\mathbf{A}$ that you found in part (c).

(e) Similarly, fill in the entries of $\vec{b}$ from part (c) in the `make_b` function.

(f) Now, you should be able to plot the estimated position of (x, y) using the supplied code for the `ideal_distances` observations. Modify the code to estimate (x, y) for `imperfect_distances` and `one_bad_distances`, and comment on the results.

In particular, for `one_bad_distances` would you intuitively have chosen the same point that our trilateration solution did knowing that only one measurement was bad?

(g) We define the "cost" of a position (x, y) to be the sum of the squares of the differences in distance of that position from the observation, as defined symbolically in the notebook. Study the heatmap of the cost of various positions on the plane, and make sure you see why $(0,0)$ appears to be the point with the lowest cost.

Now, compare the cost of $(0,0)$ with the cost of your estimated position obtained from the least-squares solution in all three cases. For which cases does least squares do worse?

12. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

