$\boxed{\text{Lecture 2}}$ (Wed 6/21)

Today:

- what is a linear function really?
- little review of systems of linear equations
- intro to imaging lab!

Think of a function as a box that takes in inputs and gives you outputs. We will draw it like this:

$$\text{inputs} \rightarrow \boxed{f} \rightarrow \text{outputs}$$

"block diagram"

Ex. $f(x) = 4x$

$$2 \rightarrow \boxed{f} \rightarrow 8$$

Think concretely about a system that could be represented as a function: ex. a microphone / speaker system

you talk into mic at some volume $\rightarrow \boxed{f} \rightarrow$ (louder) copy of sound comes out of speaker

What would it mean if f was a "linear" function?

- it turns out that linearity is a very intuitive concept

2 properties:

$\boxed{1.}$ You talk 2x louder → you expect speaker output to be 2x louder too

$\boxed{2.}$ You and your friend talk at the same time → speaker output is sum of what you each would've sounded like individually

- this is generally how we expect the system to behave if it's working well!
  - but it is again a MODEL of the system
  - when might the model be wrong?
    - you scream too loud and the speaker output is saturated
    - ...

If a function satisfies these 2 properties, it is linear.
Formal math definitions (and fancy words):

$\boxed{1.}$ homogeneity: IF $f(x) = y$ THEN $f(ax) = ay$

output scaled by SAME constant

$\boxed{2.}$ superposition: IF $f(x_1) = y_1$ and $f(x_2) = y_2$
THEN $f(x_1 + x_2) = y_1 + y_2$

So to check if a function is linear, you just need to check that these 2 properties hold.

Let's do some examples of using the formal definition.

Ex 1. $f(x) = 2x$

plug in as x

1. homogeneity: $f(ax) = 2(ax) = a(2x) = af(x)$ ✓
2. superposition: $f(x_1+x_2) = 2(x_1+x_2) = 2x_1 + 2x_2$

$= f(x_1) + f(x_2)$ ✓

$\Rightarrow$ linear!

Ex 2. $f(x) = x^2$

1. $f(ax) = (ax)^2 = a^2x^2$

but $af(x) = ax^2$ ✗

2. $f(x_1+x_2) = (x_1+x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2$

$f(x_1) + f(x_2) = x_1^2 + x_2^2$ ✗

failed! $\Rightarrow$ not linear

Ex 3. $f(x) = 2x+3$

1. $f(ax) = 2(ax) + 3$

meanwhile, $af(x) = a(2x+3) = 2ax + 3a$ ✗

2. $f(x_1+x_2) = 2(x_1+x_2) + 3 = 2x_1 + 2x_2 + 3$

$f(x_1) + f(x_2) = 2x_1 + 3 + 2x_2 + 3 = 2x_1 + 2x_2 + 6$ ✗

not linear. actually called "affine"

Functions with multiple inputs:

1. homogeneity: $f(ax_1, ax_2, \dots ax_n) = af(x_1, \dots x_n)$
2. superposition: $f(x_1+z_1, \dots x_n+z_n) = f(x_1, \dots x_n) + f(z_1, \dots z_n)$

A pixel of a tomography projection is an example of a function with multiple inputs.

A big turns out: ALL functions that satisfy the 2 linearity properties have the same form:

$$f(x_1, \dots x_n) = \underline{a_1} x_1 + \dots + \underline{a_n} x_n$$

function of n variables

constants

- this is perhaps the definition of linear that you're more familiar with: all terms only have variables of degree one
- turns out: these 2 definitions are equivalent!

Let's check that this form satisfies homogeneity + superposition.

1. $f(\alpha x_1, \dots \alpha x_n) = a_1(\alpha x_1) + \dots + a_n(\alpha x_n)$
$$= \alpha(a_1 x_1 + \dots + a_n x_n)$$
$$= \alpha f(x_1, \dots x_n) \checkmark$$

2. $f(x_1 + z_1, \dots x_n + z_n) = a_1(x_1 + z_1) + \dots + a_n(x_n + z_n)$
$$= a_1 x_1 + \dots + a_n x_n + a_1 z_1 + \dots + a_n z_n$$
$$= f(x_1, \dots x_n) + f(z_1, \dots z_n) \checkmark$$

Note: I did not show that functions of this form are the ONLY functions that are linear. Turns out that is true! Proof in Note 1A.

- but that means you can check linearity EITHER using the 2 properties OR just looking at the form.

Let's take a step back.

We like linear functions because turns out they are easier to solve and mathematicians have developed a whole set of tools to solve them aka <u>LINEAR ALGEBRA</u>

- but in reality, most of the world is nonlinear.
  "saying you study nonlinear systems is like a zoologist saying they only study non-elephant animals."
- but turns out models that are linear approximations are often good enough!

Was our tomography model linear?

- yes. conveniently.

Def. A system of equations is linear if all the equations can be written as $f(x_1, \dots x_n) = b$ where f is a linear function and b is a constant.

Ex. $\begin{cases} x_1 + 2x_2 = 4 \\ 3x_1 - x_2 = 2 \end{cases}$ $\quad$ $f(x_1, x_2) = b_1$, $g(x_1, x_2) = b_2$ ✓

What if an equation has an affine left-hand side?

$f(x_1, \dots x_n) = b$ but f is AFFINE.

- $x_1 + 2x_2 - 4 = 0$

can just move to other side!

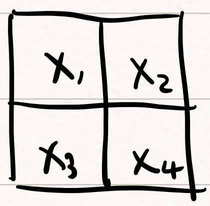
- as long as you can make it look like $f(x) = b$ with linear f, you're good.

You will explore another linear system of equations setup in the imaging lab: a single-pixel camera.

- most cameras have many pixels so you can form an image of the world.
- in some cases, you're actually only able to have one pixel e.g. non-visible light wavelengths, super super fast frame rate (faster readout)
- the pixel just adds up all the light that hits it.

Q. how can we get back an image of a scene when we only have one pixel as our detector?

- what if we can make a "mask" that only collects light from some parts of the scene?

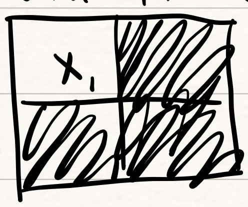


scene

In lab, a projector illuminates the scene and we can project any pattern.

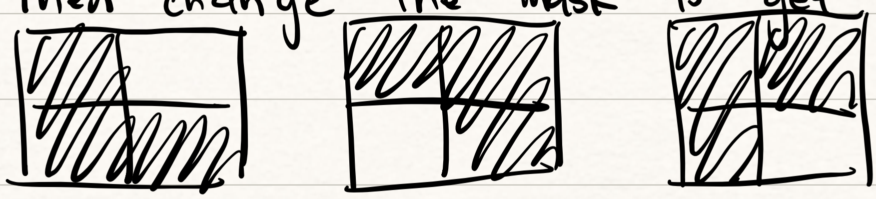
measure: $y = x_1 + x_2 + x_3 + x_4$

What if block out all pixels except x_1?



then $y_1 = x_1$

Then change the mask to get each of the other pixels.



$y_2 = x_2$ $\qquad$ $y_3 = x_3$ $\qquad$ $y_4 = x_4$

This is not the only set of masks that would work!
- There are 4 unknowns, so you just need 4 equations!

Could do tomography style:

But in this case we have so much freedom!

This leads to the question of: what would be the BEST set of measurements to take?

↳ hard to define!

One consideration: noise

- your single pixel isn't perfect

  Imagine how "grainy" pictures look in the dark.

  - when less light hits the pixel, the error on the value read out will be more significant.
  - so maybe it's better to light up more parts of the scene at once!

Another consideration: the thing you're imaging gets damaged by too much light.

POPCORN (see gradescope)