EECS 16A Designing Information Devices and Systems I
Summer 2023 Homework 4

This homework is due July 14, 2023, at 23:59.
Self-grades are due July 21, 2023, at 23:59.

Submission Format
Your homework submission should consist of **one** file.
`hw4.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).
Submit each file to its respective assignment on Gradescope.

1. Pre-lab Questions: Touch 1

These questions pertain to the pre-lab reading for the *Touch 1* lab. You can find the reading under the *Touch 1* Lab section on the 'Schedule' page of the website. We do not expect in-depth answers for the questions. Please limit your answers to a maximum of 2 sentences.

(a) What are the three terminals of a potentiometer?

(b) Why is the LED fader circuit that we're building in lab relevant to a touchscreen?

(c) What is the threshold voltage of the red LED we use in the lab?

Solution:

(a) A potentiometer (a type of variable resistor) has three terminals: two end terminals and one movable middle terminal that acts as the output.

(b) We can take advantage of its ability to adjust voltages using the voltage divider to identify touch locations on a touchscreen.

(c) $V_{\text{LED}} = +1.6\text{ V}$

2. Prelab Questions: Touch 2

These questions pertain to the pre-lab reading for the Touch 2 lab. You can find the reading under the Touch 2 Lab section on the 'Schedule' page of the website.

(a) How many layers are there in the resistive touchscreen and what are they made of?

(b) Provide 2 examples of resistive touchscreens (give one example not listed on the pre-lab reading).

(c) In the circuit given in the reading, what is the current i_3 flowing through resistor R_{h1}?

(d) How do we get touch coordinates in the horizontal direction if you have your circuit that works in the vertical direction?

Solution:

(a) The resistive touchscreen consists of two different layers - a flexible resistive layer on the top and a resistor circuit layer on the bottom.

(b) From the reading: old Nokias, Nintendo DS & Gameboy. Not from the reading: GPS displays, Printers, Airplane Entertainment Screens, Screens that require the use of a stylus.

(c) $0A$. There is no potential difference across the resistor R_{h1} and thus, no current flows through it.

(d) Rotate the orientation of the circuit by 90 degrees.

3. Reading Assignment

For this homework, please review and read Note 9, Note 11A, and Note 11B. Note 9 overviews eigenvalues and eigenvectors. Notes 11A/B introduce the basics of circuit analysis and node voltage analysis. You are always welcome and encouraged to read beyond this as well.

Please answer the following question:

(a) When identifying the eigenvalues and eigenvectors of a matrix, which do you find first? The eigenvalues? Or the eigenvectors?

(b) What are the steps for node voltage analysis (NVA)?

Solution:

(a) You have to find the eigenvalues first using the determinant. Then you can find the eigenvectors using the eigenvalues you derived.

(b) Note 11B outlines the steps for applying node voltage analysis to a circuit

Step 1: Pick a reference node

Step 2: Label all other nodes with node voltage variables

Step 3: Labal the current through every element

Step 4: Add +/- voltage labels to every element according to passive sign convention

Step 5: Identify all of the unknown node voltages and element currents. Simplify equations by recognizing currents in the same branch are the same. Substitute any known values for node voltages.

Step 6a: Write a KCL equation at each node with unknown voltage.

Step 6b: Use the I-V relationships of each element to express the voltage across each element as a function of current.

Step 7: Simplify equations and solve.

4. Mechanical Determinants

For each of the following matrices, compute their determinant and state whether they are invertible.

In lecture, we did not have time to go over the exact algorithm for computing the determinant of an $n \times n$ matrix. It turns out the algorithm is recursive and always boils down to computing determinants of 2×2 matrices.
Please watch this great YouTube video which calculates the determinant of a 3×3 matrix: https://www.youtube.com/watch?v=21LWuY8i6Hw.

(a) $\begin{bmatrix} 6 & 9 \\ 4 & 6 \end{bmatrix}$

Solution:

We can use the form of a 2×2 determinant from lecture:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Therefore,

$$\det\left(\begin{bmatrix} 6 & 9 \\ 4 & 6 \end{bmatrix}\right) = 6 \cdot 6 - 9 \cdot 4 = 0$$

Since the determinant is 0, the matrix is non-invertible. Note that the columns of the matrix are linearly dependent.

(b) $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

Solution:

$$\det\left(\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\right) = 2 \cdot 3 - 1 \cdot 0 = 6$$

Since the determinant is not 0, the matrix is invertible.

(c) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Solution:

$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right) = 2 \cdot 3 - 0 = 6$$

Since the determinant is not 0, the matrix is invertible.

(d) $\begin{bmatrix} -4 & 2 & 1 \\ 5 & 1 & -3 \\ 7 & 3 & 1 \end{bmatrix}$

Solution: To find the determinant of a 3 by 3 matrix, we can use the formula:

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a \cdot \det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b \cdot \det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c \cdot \det\left(\begin{bmatrix} d & d \\ g & h \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} -4 & 2 & 1 \\ 5 & 1 & -3 \\ 7 & 3 & 1 \end{bmatrix}\right) = -4 \cdot \det\left(\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}\right) - 2 \cdot \det\left(\begin{bmatrix} 5 & -3 \\ 7 & 1 \end{bmatrix}\right) + 1 \cdot \det\left(\begin{bmatrix} 5 & 1 \\ 7 & 3 \end{bmatrix}\right)$$

$$\begin{aligned} &= -4 \cdot [(1 \cdot 1) - (-3 \cdot 3)] - 2 \cdot [(5 \cdot 1) - (-3 \cdot 7)] + 1 \cdot [(5 \cdot 3) - (1 \cdot 7)] \\ &= -4 \cdot [10] - 2 \cdot [26] + 1 \cdot [8] \\ &= -84 \end{aligned}$$

Since the determinant is not 0, the matrix is invertible.

(e) $\begin{bmatrix} -4 & 0 & 0 \\ 5 & 1 & -3 \\ 7 & 3 & 1 \end{bmatrix}$

Solution:

$$\det\left(\begin{bmatrix} -4 & 0 & 0 \\ 5 & 1 & -3 \\ 7 & 3 & 1 \end{bmatrix}\right) = -4\cdot\det\left(\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}\right) - 0\cdot\det\left(\begin{bmatrix} 5 & -3 \\ 7 & 1 \end{bmatrix}\right) + 0\cdot\det\left(\begin{bmatrix} 5 & 1 \\ 7 & 3 \end{bmatrix}\right)$$

$$= -4\cdot[(1\cdot 1) - (-3\cdot 3)] - 0 + 0$$

$$= -40$$

Since the determinant is not 0, the matrix is invertible.

5. Introduction to Eigenvalues and Eigenvectors

Learning Goal: *Practice calculating eigenvalues and eigenvectors. The importance of eigenvalues and eigenvectors will become clear in the following problems.*

For each of the following matrices, find their eigenvalues and the corresponding eigenvectors. For simple matrices, you may do this by inspection if you prefer.

(a) $\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

Solution:

Self-grading note: For this subproblem and the following subproblems which involve computing eigenvectors, give yourself full credit if the eigenvector(s) you calculated is/are a scaled (i.e, multiplied by a real valued α) version of the eigenvector(s) given in the solutions.

There are two ways to do this.

First, we can do it by inspection. We can see that this matrix multiplies everything in the first coordinate by 5 and everything in the second by 2. Consequently, when given $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, it will return 2 times the input. And when given $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, it will return 5 times the input vector.

Alternatively, we can use determinants.

$$\det(\begin{bmatrix} 5-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}) = 0$$

$$(5-\lambda)(2-\lambda) - 0 = 0$$

This is already factored for you! We see that, by definition, diagonal matrices have their eigenvalues on the diagonal.

$\lambda = 5$:

$$\mathbf{A}\vec{x} = 5\vec{x} \implies (\mathbf{A} - 5\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies y = 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

where x is a free variable.

Any vector in span$\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$ is an eigenvector of the matrix with corresponding eigenvalue $\lambda = 5$.

$\lambda = 2$:

$$\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x = 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$$

where y is a free variable.

Any vector in span$\{\begin{bmatrix}0\\1\end{bmatrix}\}$ is an eigenvector of the matrix with corresponding eigenvalue $\lambda = 2$.

(b) $\mathbf{A} = \begin{bmatrix}22 & 6\\6 & 13\end{bmatrix}$

Solution:

Here, it is hard to guess the answers.

$$\det(\begin{bmatrix}22-\lambda & 6\\6 & 13-\lambda\end{bmatrix}) = 0$$

$$\begin{aligned}(22-\lambda)(13-\lambda)-36 &= 0\\ 250-35\lambda+\lambda^2 &= 0\\ (\lambda-10)(\lambda-25) &= 0\\ \implies \lambda &= 10, 25\end{aligned}$$

$\lambda = 10$:

$$\mathbf{A}\vec{x} = 10\vec{x} \implies (\mathbf{A}-10\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix}22 & 6\\6 & 13\end{bmatrix} - \begin{bmatrix}10 & 0\\0 & 10\end{bmatrix}\right)\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies \begin{bmatrix}12 & 6\\6 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\implies \begin{bmatrix}2 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies 2x+y=0 \implies \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\-2x\end{bmatrix}$$

where x is a free variable.

Any vector that lies in span$\{\begin{bmatrix}1\\-2\end{bmatrix}\}$ is an eigenvector with corresponding eigenvalue $\lambda = 10$.

$\lambda = 25$:

$$\mathbf{A}\vec{x} = 25\vec{x} \implies (\mathbf{A}-25\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix}22 & 6\\6 & 13\end{bmatrix} - \begin{bmatrix}25 & 0\\0 & 25\end{bmatrix}\right)\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies \begin{bmatrix}-3 & 6\\6 & -12\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\implies \begin{bmatrix}-1 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies 2y=x \implies \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2y\\y\end{bmatrix}$$

where y is a free variable.

Any vector that lies in span$\{\begin{bmatrix}2\\1\end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda = 25$.

(c) $\mathbf{A} = \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$

Solution:

We can explicitly calculate:

$$\det(\begin{bmatrix}1-\lambda & 2\\2 & 4-\lambda\end{bmatrix}) = 0$$

$$(1-\lambda)(4-\lambda)-4=0$$
$$\lambda^2-5\lambda=0 \implies \lambda(\lambda-5)=0$$
$$\lambda=0,5$$

$\lambda=0$:

$$\mathbf{A}\vec{x}=0\vec{x} \implies \mathbf{A}\vec{x}=\vec{0}$$
$$\left(\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}-\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\right)\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \implies \begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\implies \begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \implies x=-2y \implies \begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-2y\\ y\end{bmatrix}$$

where y is a free variable.

Any vector that lies in $\text{span}\{\begin{bmatrix}-2\\ 1\end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda=0$.

$\lambda=5$:

$$\mathbf{A}\vec{x}=5\vec{x} \implies (\mathbf{A}-5\mathbf{I}_2)\vec{x}=\vec{0}$$
$$\left(\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}-\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}\right)\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \implies \begin{bmatrix}-4 & 2\\ 2 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\implies \begin{bmatrix}2 & -1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \implies y=2x \implies \begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}x\\ 2x\end{bmatrix}$$

where x is a free variable.

Any vector that lies in $\text{span}\{\begin{bmatrix}1\\ 2\end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda=5$.

Alternatively, this can also be seen by inspection. The matrix is not invertible since the first two rows are linearly dependent. Therefore, there must be a 0 eigenvalue. This has the eigenvector $\begin{bmatrix}-2\\ 1\end{bmatrix}$, which belongs in the **nullspace of the matrix**.

The other eigenvector can be seen by noticing that the second row is twice the first. So $\begin{bmatrix}1\\ 2\end{bmatrix}$ is a good guess to try and indeed it works with $\lambda=5$.

(d) Let $\mathbf{A}\in\mathbb{R}^{n\times n}$ be a general square matrix. Show that the set of eigenvectors corresponding to a particular eigenvalue of $\mathbf{A}$ is a subspace of $\mathbb{R}^n$. In other words, show that

$$\{\vec{x}\in\mathbb{R}^n : \mathbf{A}\vec{x}=\lambda\vec{x}, \lambda\in\mathbb{R}\}$$

is a subspace. You have to show that all three properties of a subspace (as mentioned in Note 8) hold.

Solution:

Recall the definition of a matrix subspace from Note 8. A subspace $\mathbb{U}$ consists of a subset of the vector space $\mathbb{V}$ if it contains the zero vector, is closed under scalar multiplication, and is closed under vector addition.

i. Zero vector: The zero vector is contained in this set since $\mathbf{A}\vec{0}=\vec{0}=\lambda\vec{0}$.

ii. Scalar multiplication: Let $\vec{v}_1$ be a member of the set. Let $\vec{u}=\alpha\vec{v}_1$. Note that $\vec{u}\in\mathbb{R}^n$, thus a possible value of $\vec{x}$. Now, $\mathbf{A}\vec{u}=\mathbf{A}\alpha\vec{v}_1=\alpha\mathbf{A}\vec{v}_1=\alpha\lambda\vec{v}_1=\lambda\vec{u}$. Hence, $\vec{u}$ is a member of the set as well and the set is closed under scalar multiplication.

iii. Vector addition: Let $\vec{v}_1$ and $\vec{v}_2$ be members of the set. Observe below that the set is closed under vector addition as well.

$$\mathbf{A}(\vec{v}_1+\vec{v}_2)=\mathbf{A}\vec{v}_1+\mathbf{A}\vec{v}_2=\lambda\vec{v}_1+\lambda\vec{v}_2=\lambda(\vec{v}_1+\vec{v}_2)$$

Note that $\vec{v}_1+\vec{v}_2$ is also a vector in $\mathbb{R}^n$, which corresponds to how $\vec{x}$ is defined in this setup.

Hence, the set defined in the question satisfies the properties of a subspace and is consequently a subspace of $\mathbb{R}^n$.

6. Properties of Pump Systems

Learning Objectives: *This problem builds on the pump examples we have been doing, but is meant to help you learn to do proofs in a step by step fashion. Can you generalize intuition from a simple example?*

We consider a system of reservoirs connected to each other through pumps. An example system is shown below in Figure 1, represented as a graph. Each node in the graph is marked with a letter and represents a reservoir. Each edge in the graph represents a pump which moves a fraction of the water from one reservoir to the next at every time step. The fraction of water moved is written on top of the edge.

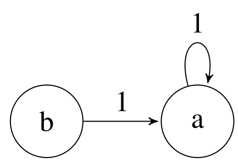


Figure 1: Pump system

We want to prove the following theorem. We will do this step by step.

Theorem: Consider a system consisting of k reservoirs such that the entries of each column in the system's state transition matrix sum to one. If s is the total amount of water in the system at timestep n, then total amount of water at timestep $n+1$ will also be s.

(a) Rewrite the theorem statement for a graph with only two reservoirs.

Solution: Consider a system consisting of 2 reservoirs such that the entries of each column in the system's state transition matrix sum to one. If s is the total amount of water in the system at timestep n, then total amount of water at timestep $n+1$ will also be s.

(b) Since the problem does not specify the transition matrix, let us consider the transition matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and the state vector $\vec{x}[n] = \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix}$. Write out what is "known" or what is given to you in the theorem statement in mathematical form.

Note: In general, it is helpful to write as much out mathematically as you can in proofs. It can also be helpful to draw the transition graph.

Solution: Each column of the transition matrix sums to one:

$$a_{11}+a_{21}=1, \qquad a_{12}+a_{22}=1$$

The total amount of water in the system is s at timestep n:

$$x_1[n]+x_2[n]=s$$

We know that the state vector at the next timestep is equal to the transition matrix applied to the state vector at the current timestep:

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n]$$

(c) Now write out the theorem we want to prove mathematically.

Solution: We want to prove that the total amount of water at timestep $n+1$ will also be s:

$$x_1[n+1] + x_2[n+1] = s$$

(d) Prove the statement for the case of two reservoirs. In other words, combine parts (b) and (c) to prove the theorem.

Solution: Consider the product $\mathbf{A}\vec{x}[n] = \vec{x}[n+1]$:

$$\mathbf{A}\vec{x}[n] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} = \begin{bmatrix} a_{11}x_1[n] + a_{12}x_2[n] \\ a_{21}x_1[n] + a_{22}x_2[n] \end{bmatrix}$$

Let's consider the sum of the elements in $\vec{x}[n+1]$:

$$\sum_{i=1}^{2} x_i[n+1] = (a_{11}x_1[n] + a_{12}x_2[n]) + (a_{21}x_1[n] + a_{22}x_2[n])$$

Regrouping terms:

$$(a_{11} + a_{21})x_1[n] + (a_{12} + a_{22})x_2[n] = 1 \cdot x_1[n] + 1 \cdot x_2[n] = x_1[n] + x_2[n] = s$$

(e) Now use what you learned to generalize to the case of k reservoirs. *Hint:* Think about $\mathbf{A}$ in terms of its columns, since you have information about the columns.

Solution:

Let $\vec{x}[n] = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}$ be the amount of water in each reservoir at timestep n. We know:

$$x_1[n] + x_2[n] + \cdots + x_k[n] = s$$

Let $\vec{a}_j$ be the j-th column of the state transition matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_k \end{bmatrix}$$

We know that every column of $\mathbf{A}$ sums to one, so we know for all j,

$$a_{1j} + a_{2j} + \cdots + a_{kj} = 1$$

Now, consider the product $\mathbf{A}\vec{x}[n]$:

$$\mathbf{A}\vec{x}[n] = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_k \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ \vdots \\ x_k[n] \end{bmatrix} = x_1[n]\vec{a}_1 + x_2[n]\vec{a}_2 + \cdots + x_k[n]\vec{a}_k = \vec{x}[n+1]$$

Let's consider the sum of the elements in $\vec{x}[n+1]$:

$$\begin{aligned} x_1[n+1] + x_2[n+1] + \cdots + x_k[n+1] =& (a_{11}x_1[n] + a_{12}x_2[n] + \cdots + a_{1k}x_k[n]) \\ &+ (a_{21}x_1[n] + a_{22}x_2[n] + \cdots + a_{2k}x_k[n]) \\ &+ \dots \\ &+ (a_{k1}x_1[n] + a_{k2}x_2[n] + \cdots + a_{kk}x_k[n]) \end{aligned}$$

Factoring out each element of $x[n]$ gives:

$$\begin{aligned} &x_1[n](a_{11} + a_{21} + \cdots + a_{k1}) + x_2[n](a_{12} + a_{22} + \cdots + a_{k2}) + \cdots + x_k[n](a_{1k} + a_{2k} + \cdots + a_{kk}) \\ &= 1 \cdot x_1[n] + 1 \cdot x_2[n] + \cdots + 1 \cdot x_k[n] = x_1[n] + x_2[n] + \cdots + x_k[n] = s \end{aligned}$$

7. Page Rank

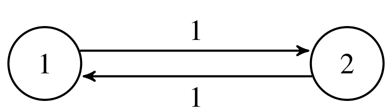
Learning Goal: *This problem highlights the use of transition matrices in modeling dynamical linear systems. Predictions about the steady state of a system can be made using the eigenvalues and eigenvectors of this matrix.*

In homework and discussion, we have discussed the behavior of water flowing in reservoirs and the people flowing in social networks. We now consider the setting of web traffic where the dynamical system can be described with a directed graph, also known as state transition diagram.

As we have seen in lecture and discussion, the "transition matrix", $\mathbf{T}$, can be constructed using the state transition diagram as follows: entries t_{ji} represent the *proportion* of the people who are at website i that click the link for website j.

The **steady-state frequency** (i.e. fraction of visitors in steady-state) for a graph of websites is related to the eigenspace associated with eigenvalue 1 for the "transition matrix" of the graph. Once computed, an eigenvector with eigenvalue 1 will have values which correspond to the steady-state frequency for the fraction of people for each webpage. When the elements of this eigenvector are made to **sum to one** (to conserve population), the i^{th} element of the eigenvector will correspond to the fraction of people on the i^{th} website.

(a) For graph A shown below, what are the steady-state frequencies (i.e. fraction of visitors in steady-state) for the two webpages? Graph A has weights in place to help you construct the transition matrix. Remember to ensure that your steady state-frequencies sum to 1 to maintain conservation.



Graph A

Solution:
To determine the steady-state frequencies for the two pages, we need to find the appropriate eigenvector of the transition matrix. In this case, we are trying to determine the proportion of people who would be on a given page at steady state.
The transition matrix of graph A:

$$\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1}$$

To determine the eigenvalues of this matrix:

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 - 1 = 0 \tag{2}$$

$\lambda = 1, -1$. The steady state vector is the eigenvector that corresponds to $\lambda = 1$. To find the eigenvector,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{3}$$

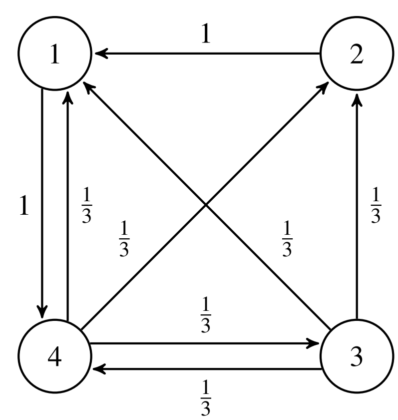
The sum of the values of the vector should equal 1 since the number of people is conserved, so our conditions are:

$$v_1 + v_2 = 1$$
$$v_1 = v_2$$

The steady-state frequency eigenvector is $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ and each webpage has a steady-state frequency of 0.5.

(b) For graph B, what are the steady-state frequencies for the webpages? You may use IPython and the Numpy command `numpy.linalg.eig` for this. We have set up a template IPython notebook `prob7.ipynb` for you (you **do not** need to turn in this notebook to receive full credit). Graph B is shown below, with weights in place to help you construct the transition matrix.
Hint: The steady-state frequencies are fractions or percentages of the total (i.e., 1). Make sure your frequencies sum to one.
Hint: `numpy.linalg.eig` returns eigenvectors and eigenvalues. The eigenvectors are arranged in a matrix in *column-major* order. In other words, given eigenvectors
$\vec{v}_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}$
NumPy will return:

$$\begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} \tag{4}$$



Graph B

Solution:

To determine the steady-state frequencies, we need to create the transition matrix $\mathbf{T}$ first.

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} & 0 \end{bmatrix}$$

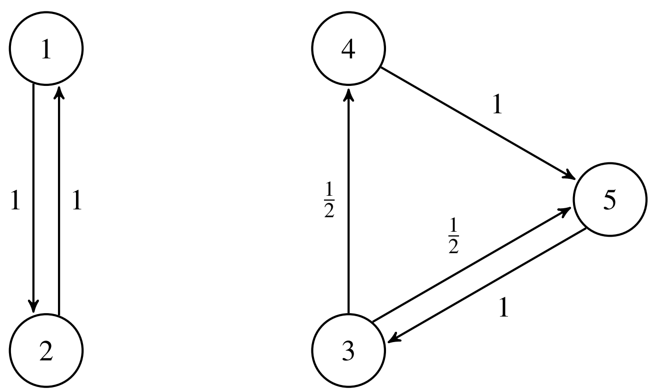
One possible eigenvector associated with eigenvalue 1 is $\begin{bmatrix} -0.61 & -0.31 & -0.23 & -0.69 \end{bmatrix}^T$ (found using IPython). Scaling it by

$$\frac{1}{(-0.61 + (-0.31) + (-0.23) + (-0.69))}$$

so the elements sum to 1, we get $\begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{8} & \frac{3}{8} \end{bmatrix}^T$

These are the steady-state frequencies for the pages.

(c) Graph C with weights in place is shown below. Find the eigenspace that corresponds to the steady-state for graph C. How many independent systems (disjoint sets of webpages) are there in graph C versus in graph B? What is the dimension of the eigenspace corresponding to the steady-state for graph C? You may use IPython to compute the eigenvalues and eigenvectors again.



Graph C

Solution:

The transition matrix for graph C is

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 \end{bmatrix}$$

Using IPython, we find that the eigenspace associated with $\lambda = 1$ is spanned by the vectors $\begin{bmatrix} 0 & 0 & 0.4 & 0.2 & 0.4 \end{bmatrix}^T$ and $\begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \end{bmatrix}^T$. While any linear combination of these vectors is an eigenvector, these two particular vectors have a nice interpretation.

The first eigenvector describes the steady-state frequencies for the last three webpages, and the second vector describes the steady-state frequencies for the first two webpages. This makes sense since there are essentially “two internets”, or two disjoint sets of webpages. Surfers cannot transition between the two, so you cannot assign steady-state frequencies to webpage 1 and webpage 2 relative to the rest. This is why the eigenspace corresponding to the steady-state has dimension 2.

Assuming that each set of steady-state frequencies needs to add to 1, the first assigns steady-state frequencies of 0.4, 0.2, 0.4 to webpage 3, webpage 4, and webpage 5, respectively. The second assigns steady-state frequencies of 0.5 to both webpage 1 and webpage 2.

8. Reverse Eigenvalues

Learning Goal: *Understand how to construct a matrix with a particular set of eigenvalues and eigenvectors.*

In lecture, homework, and section, we have seen a number of ways to compute eigenvalues and eigenvectors from a particular matrix, and explored what they mean in terms of how the matrix transforms vectors. In this problem, we will explore this in the reverse direction by designing it to have a desired set of eigenvalues. Recall the fundamental eigenvector/eigenvalue equation:

$$A\vec{v} = \lambda\vec{v} \tag{5}$$

(a) Suppose you are given the following eigenvalue/eigenvector pairs:

$$\lambda_1 = 1, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda_2 = -1, \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Explicitly write out the matrix-vector equations for the two eigenvector/eigenvalue pairs. Make sure to identify each component of the $\mathbf{A}$ matrix and fill in the relevant values for the eigenvector and eigenvalue. Assume the unkown components of $\mathbf{A}$ are $a_{11}, a_{12}, a_{21}, a_{22}$.

Solution:

As suggested by the question, we begin with the eigenvector/eigenvalue equation:

$$Av = \lambda v$$

Let us select the first pair, λ_1 and $\vec{v}_1$, and explicitly write out the components of the $\mathbf{A}$ matrix in the equation.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The second pair can be written similarly:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Simplifying:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(b) Reformat the equations you wrote above into a system of linear equations, with $a_{11}, a_{12}, a_{21}, a_{22}$ as unknowns.

Solution:

Working out the matrix multiplication yields 4 equations (2 from each set), as follows:

$$\begin{aligned} 1a_{11} + 1a_{12} &= 1 \\ 1a_{21} + 1a_{22} &= 1 \\ -1a_{11} + 1a_{12} &= 1 \\ -1a_{21} + 1a_{22} &= -1 \end{aligned}$$

(c) Now, setup a matrix-vector system of equations and solve for the $\mathbf{A}$ matrix. Think carefully about what the unknowns in your system are when setting it up.

Solution:

The unknowns are the components of the $\mathbf{A}$ matrix (the a_{ij}'s) We can create the matrix-vector formulation as follows:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

You can then solve this system however you would like, with the result:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(d) Describe the geometric transformation represented by the matrix $\mathbf{A}$. How does the matrix graphically transform its eigenvectors? How does this relate to the associated eigenvalues?

Solution:

The matrix

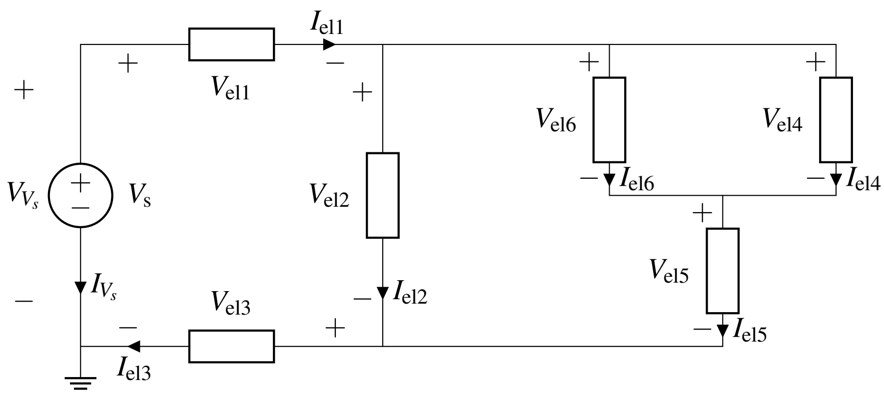
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

reflects a vector across the line $y = x$ or along the vector $\vec{v} = [1, 1]$. In other words, the matrix swaps the x-component with the y-component of an applied vector.

The first eigenvector $\vec{v}_1 = [1, 1]$ lies along the line of reflection, so it stays unchanged after the transformation since the associated eigenvalue $\lambda_1 = 1$. The second eigenvector $\vec{v}_2 = [-1, 1]$ is perpendicular to the line of reflection, meaning that vectors in that direction will be exactly flipped in sign due to the associated eigenvalue $\lambda_2 = -1$. These two eigenvectors (and their scalar multiples) are the only vectors which purely scale (i.e., keep their original direction) after the transformation. All other vectors in $\mathbb{R}^2$ will rotate around the origin (i.e., change their original direction) after the transformation.

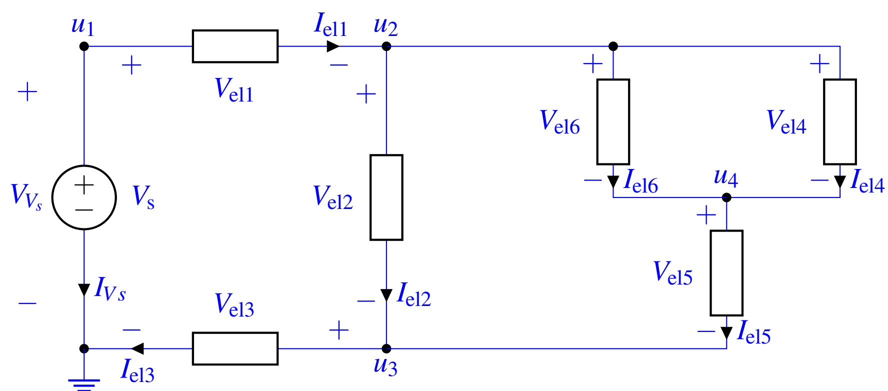
9. Intro to Circuits

Learning Goal: *This problem will help you practice labeling circuit elements and setting up KVL equations.*



(a) How many nodes does the above circuit have? Label them.
Note: The reference/0V node has already been selected for you, so you do not need to label it, but you do need to include it in your node count.

Solution: There are a total of 5 nodes in the circuit, including the reference node. They are labeled u_1 - u_4 below:



(b) Express all element voltages (including the element voltage across the voltage source, V_s) as a function of node voltages. This will depend on the node labeling you chose in part (a).

Solution: For our specific node labeling we can write:

$$V_{V_s} = u_1 - 0 = u_1 = V_s$$

$$V_{\text{el1}} = u_1 - u_2$$

$$V_{\text{el2}} = u_2 - u_3$$

$$V_{\text{el3}} = u_3 - 0 = u_3$$

$$V_{\text{el4}} = u_2 - u_4$$

$$V_{\text{el5}} = u_4 - u_3$$

$$V_{\text{el6}} = u_2 - u_4$$

Notice that the element voltage is always of the form: $V_{\text{el}} = u_+ - u_-$.

(c) Write a KVL equation for all the loops that contain the voltage source V_s. These equations should be a function of element voltages and the voltage source V_s.

Solution: Notice that there are in fact 3 loops that contain the voltage source V_s, for which we can write the following equations, starting each time from the reference node and ending at the reference node:

$$V_s - V_{\text{el}1} - V_{\text{el}2} - V_{\text{el}3} = 0$$
$$V_s - V_{\text{el}1} - V_{\text{el}6} - V_{\text{el}5} - V_{\text{el}3} = 0$$
$$V_s - V_{\text{el}1} - V_{\text{el}4} - V_{\text{el}5} - V_{\text{el}3} = 0$$

10. It's a Triforce!

Learning Goal: *This problem explores passive sign convention and nodal analysis in a slightly more complicated circuit.*

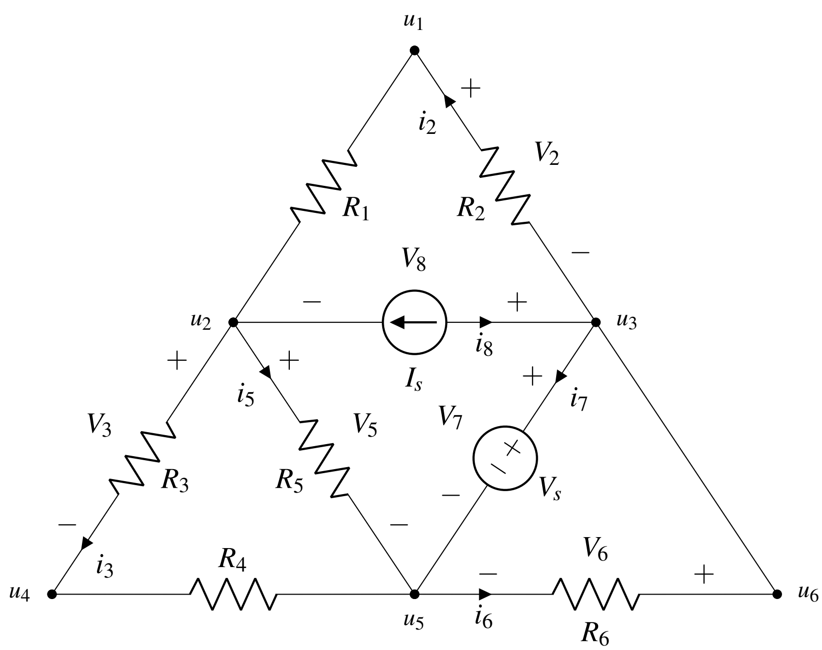


Figure 2: A triangular circuit consisting of a voltage source V_s, current source I_s, and resistors R_1 to R_6.

(a) Which elements I_s, V_s, R_2, R_3, R_5, or R_6 in Figure 2 have current-voltage labeling that violates *passive sign convention*? Explain your reasoning.

Solution:

Recall *passive sign convention* dictates that positive current should *enter* the positive voltage terminal and *exit* the negative voltage terminal.

The elements associated with I_s, R_2, and R_6 have (external) voltage/current labelings which violate passive sign convention as the current is depicted leaving the positive terminal (or entering the negative terminal). This could be corrected either by swapping the voltage polarity or the current direction.

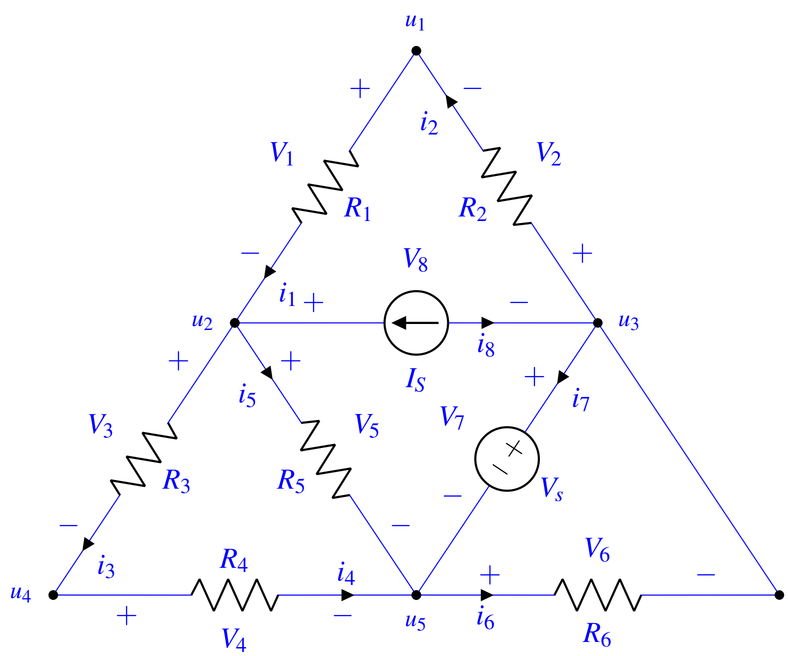
(b) In Figure 2, the nodes are labeled with u_1, u_2, ... etc. There is a subset of u_i's in the given circuit that are redundant, i.e. there might be more than one label for the same node. Which node(s) do not have a unique label? Justify your answer.

Solution:

The nodes u_3 and u_6 are redundant and can be represented with the same node since there is a short connecting them.

(c) Redraw the circuit diagram by correctly labeling *all* the element voltages and element currents according to passive sign convention. The component labels that were violating passive sign convention in part (a) should be corrected by *swapping the element voltage polarity*. Additionally, label the elements that have not been labeled yet.

Solution:



This is one of the possible correct labelings. Since the problem statement disallows swapping the current direction, the only other valid answers would include variations of the completely unlabeled elements R_1 and R_4. The flipping of *both* current *and* voltage labelings for R_1 and/or R_4 is also a correct answer.

(d) Write an equation to describe the current-voltage relationship for element R_4 in terms of the relevant i's, R's, and node voltages in this circuit. Your final expression should include u_4, u_5, and i_4.

Solution:

The resulting equation should look like:

$$R_4 = \frac{u_4 - u_5}{i_4}$$

Alternatively, if you labelled the current i_4 going in the opposite direction (going from u_5 to u_4), the resulting equation should look like:

$$R_4 = \frac{u_5 - u_4}{i_4}$$

(e) Write the KCL equation for node u_2 in terms of the node voltages and other circuit elements.

Solution:

KCL at u_2 gives us:

$$i_1 - i_8 - i_3 - i_5 = 0$$

The equations for the current through each of the branches are:

$$\begin{aligned} i_1 &= \frac{u_1 - u_2}{R_1} \\ i_3 &= \frac{u_2 - u_4}{R_3} \\ i_8 &= -I_s \\ i_5 &= \frac{u_2 - u_5}{R_5} \end{aligned}$$

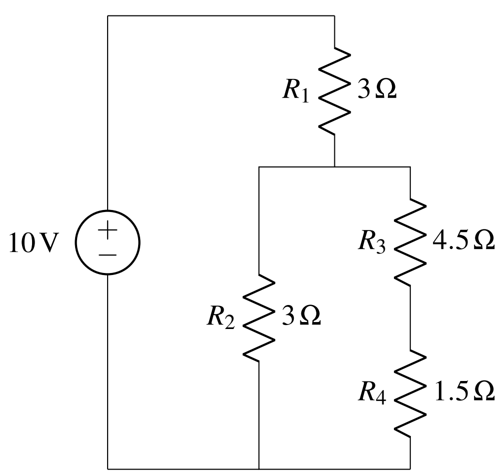
The final expression is then:

$$\frac{u_1 - u_2}{R_1} + I_s - \frac{u_2 - u_5}{R_5} - \frac{u_2 - u_4}{R_3} = 0$$

Or any equivalent equation.

11. Mechanical Circuits

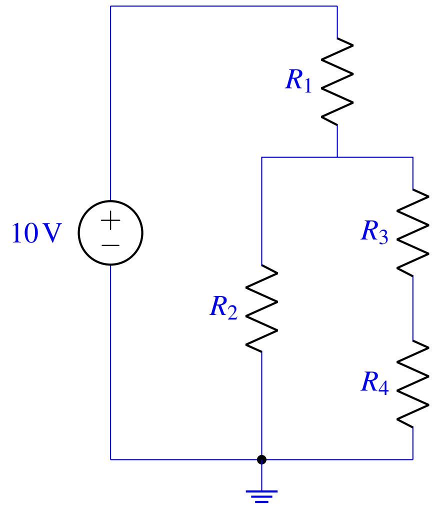
Find the voltages across and currents flowing through all of the resistors. *Hint: Use the seven steps of node voltage analysis.*



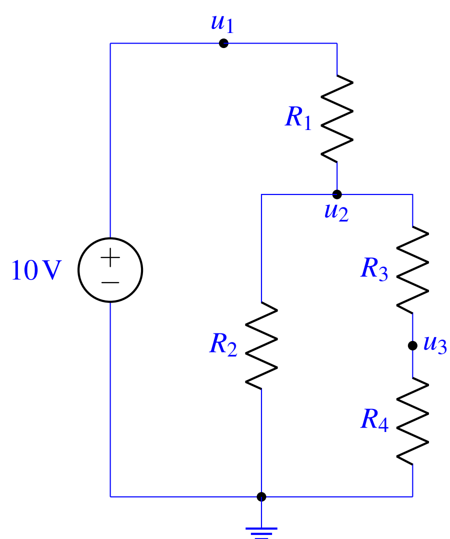
Solution:

Node Voltage Analysis (Seven Step Method):

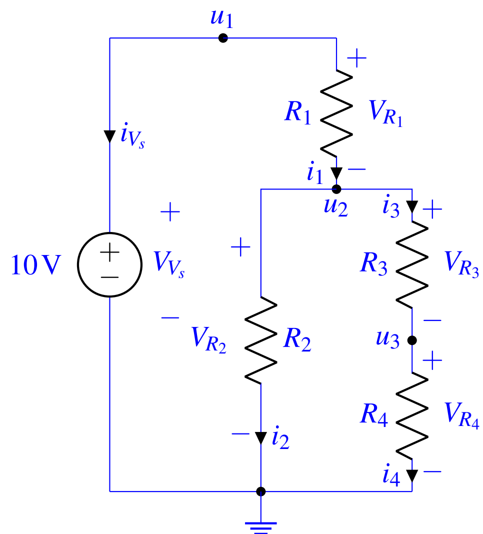
Step 1) Select a ground node. Any choice of ground is valid, but we choose the bottom node:



Step 2) Label all remaining nodes:



Step 3) & 4) Label element voltages and currents. Currents can be labeled in arbitrary directions, but labeling must follow passive sign convention.



Step 5) Identify and reduce unknowns. The voltage source is connected to the top node and ground node. We know that the value of the voltage source $V_s = 10$ V, and that a voltage source forces the difference between its nodes to be V_s. We also know the ground node is 0 V by definition. Therefore, we know that the top node must hold a potential of 10V.

$$u_1 - 0 = V_s = 10\,\text{V}$$

Step 6) Write KCL equations for all nodes with unknowns (u_2 and u_3). The definition of KCL is that the sum of all currents entering and leaving the node must equal 0 A. We will call current entering positive, and current leaving negative. Going in order from node u_2 to node u_3, we set up our KCL expressions:

$$i_1 - i_2 - i_3 = 0 \tag{6}$$
$$i_3 - i_4 = 0 \tag{7}$$

Use element I-V relationships to find equations relating the branch currents to the node voltages. Looking at the differences of node potentials, the top node and u_2 are separated by a resistor, and Ohm's law relates the potential *difference* between each side of the resistor to the current through it, so we have:

$$10V - u_2 = V_{R_1} = i_1 R_1 \implies i_1 = \frac{10V - u_2}{R_1} \tag{8}$$

Similarly for the other resistors

$$u_2 - 0 = V_{R_2} = i_2 R_2 \quad \implies \quad i_2 = \frac{u_2}{R_2} \tag{9}$$
$$u_2 - u_3 = V_{R_3} = i_3 R_3 \quad \implies \quad i_3 = \frac{u_2 - u_3}{R_3} \tag{10}$$
$$u_3 - 0 = V_{R_4} = i_4 R_4 \quad \implies \quad i_4 = \frac{u_3}{R_4} \tag{11}$$

Step 7) Solve the system of equations set up in step 6. Note that we have 6 unknowns $(i_1, i_2, i_3, i_4, u_2, u_3)$ and 6 equations. Substituting equations (3-6) into our KCL equations (1-2), we get

$$\begin{aligned} \left(\frac{10V - u_2}{R_1}\right) - \frac{u_2}{R_2} - \left(\frac{u_2 - u_3}{R_3}\right) &= 0 \\ \left(\frac{u_2 - u_3}{R_3}\right) - \frac{u_3}{R_4} &= 0 \end{aligned}$$

which gives us

$$\begin{aligned} u_2 &= 4\,\mathrm{V} \\ u_3 &= 1\,\mathrm{V} \end{aligned}$$

Using the node voltages, we can derive the voltage across and current through every resistor

$$\begin{aligned} V_{R_1} &= 10V - u_2 = 6\,\mathrm{V}, & i_1 &= \frac{V_{R_1}}{R_1} = 2\,\mathrm{A} \\ V_{R_2} &= u_2 = 4\,\mathrm{V}, & i_2 &= \frac{V_{R_2}}{R_2} = 1.33\,\mathrm{A} \\ V_{R_3} &= u_2 - u_3 = 3\,\mathrm{V}, & i_3 &= \frac{V_{R_3}}{R_3} = 0.67\,\mathrm{A} \\ V_{R_4} &= u_3 = 1\,\mathrm{V}, & i_4 &= \frac{V_{R_4}}{R_1} = 0.67\,\mathrm{A} \end{aligned}$$

12. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.

