
EECS 16A
Summer 2023

Designing Information Devices and Systems I

Discussion 3C

1. Identifying a Subspace: Proof

Is the set

$$V = \left\{ \vec{v} \mid \vec{v} = c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \text{ where } c, d \in \mathbb{R} \right\}$$

a subspace of \mathbb{R}^3 ? Why or why not?

Answer:

Yes, V is a subspace of \mathbb{R}^3 . We will *prove this* by using the definition of a subspace.

First of all, note that V is a subset of \mathbb{R}^3 – all elements in V are of the form $\begin{bmatrix} c+d \\ c \\ c+d \end{bmatrix}$, which is a 3-dimensional real vector.

Now, consider two elements $\vec{v}_1, \vec{v}_2 \in V$ and $\alpha \in \mathbb{R}$.

This means that there exists $c_1, d_1 \in \mathbb{R}$, such that $\vec{v}_1 = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Similarly, there exists $c_2, d_2 \in \mathbb{R}$,

such that $\vec{v}_2 = c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Now, we can see that

$$\vec{v}_1 + \vec{v}_2 = (c_1 + c_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (d_1 + d_2) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

so $\vec{v}_1 + \vec{v}_2 \in V$.

Also,

$$\alpha \vec{v}_1 = (\alpha c_1) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (\alpha d_1) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

so $\alpha \vec{v}_1 \in V$.

Furthermore, we observe that the zero vector is contained in V , when we set $c = 0$ and $d = 0$.

We have thus identified V as a subset of \mathbb{R}^3 , shown both of the no escape (closure) properties (closure under vector addition and closure under scalar multiplication), as well as the existence of a zero vector, so V is a subspace of \mathbb{R}^3 .

It's important to note that satisfying the subset property and the two forms of closure additionally implies that subspace V also satisfies the axioms of a vector space, and therefore is also a vector space.

2. Exploring Column Spaces and Null Spaces

- The **column space** is the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that when multiplied with the matrix result in the zero vector.

For the following matrices, answer the following questions:

- What is the column space of **A**? What is its dimension?
- What is the null space of **A**? What is its dimension?
- Do the columns of **A** span \mathbb{R}^2 ? Do they form a basis for \mathbb{R}^2 ? Why or why not?
- Are the column spaces of the row reduced matrix **A** and the original matrix **A** the same?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Answer: Column space: $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Null space: $\text{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

The column space does not span \mathbb{R}^2 and thus are not a basis for \mathbb{R}^2 .

The matrix is already row reduced. The column spaces of the row reduced matrix and the original matrix are the same.

(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

Answer:

Column space: $\text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Null space: $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Not a basis for \mathbb{R}^2 .

The two column spaces are not the same.

(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

Answer:

Column space: \mathbb{R}^2

Null space: $\text{span} \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

This is a basis for \mathbb{R}^2 .

The two column spaces are the same as the column span \mathbb{R}^2 , since they are two independent vectors.

(d) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$

Answer:

Column space: $\text{span} \left\{ \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} \right\}$

Null space: $\text{span} \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

Not a basis for \mathbb{R}^2 .

The two column spaces are not the same. We can also see that one of the columns is a scaled version of the other, and therefore, they are linearly dependent vectors.

(e) $\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

Answer:

- The column space of the columns is \mathbb{R}^2 . The columns of \mathbf{A} do not form a basis for \mathbb{R}^2 . This is because the columns of \mathbf{A} are linearly dependent.
- The following algorithm can be used to solve for the null space of a matrix. The procedure is essentially solving the matrix-vector equation $\mathbf{A}\vec{x} = \vec{0}$ by performing Gaussian elimination on \mathbf{A} . We start by performing Gaussian elimination on matrix \mathbf{A} to get the matrix into upper-triangular form.

$$\begin{aligned} \begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix} &\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{7}{2} \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \text{ reduced row echelon form} \end{aligned}$$

$$x_1 + \frac{1}{2}x_3 - \frac{7}{2}x_4 = 0$$

$$x_2 + \frac{5}{2}x_3 + \frac{1}{2}x_4 = 0$$

x_3 is free and x_4 is free

Now let $x_3 = s$ and $x_4 = t$. Then we have:

$$x_1 + \frac{1}{2}s - \frac{7}{2}t = 0$$

$$x_2 + \frac{5}{2}s + \frac{1}{2}t = 0$$

Now writing all the unknowns (x_1, x_2, x_3, x_4) in terms of the dummy variables:

$$\begin{aligned}
 x_1 &= -\frac{1}{2}s + \frac{7}{2}t \\
 x_2 &= -\frac{5}{2}s - \frac{1}{2}t \\
 x_3 &= s \\
 x_4 &= t
 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s + \frac{7}{2}t \\ -\frac{5}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s \\ -\frac{5}{2}s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{7}{2}t \\ -\frac{1}{2}t \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

So every vector in the null space of \mathbf{A} can be written as follows:

$$\text{Null space}(\mathbf{A}) = s \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Therefore the null space of \mathbf{A} is

$$\text{span} \left\{ \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$$

\mathbf{A} has a 2-dimensional null space.

- iii. No, the columns of \mathbf{A} do not form a basis for \mathbb{R}^2 .
 - iv. In this case, the column space of the row reduced matrix is also \mathbb{R}^2 , but this need not be true in general.
- (f) What do you notice about the relationship between the dimension of the column space, the dimension of the null space, and their sum in all of these matrices?

Answer:

The dimension of the column space indicates the number of linearly independent columns in the matrix, while the dimension of the null space indicates the number of linearly dependent columns in the matrix. In each of the above matrices, notice that summing the dimension of the column space and null space results in the total number of columns of the original matrix. This concept forms the rank-nullity theorem.