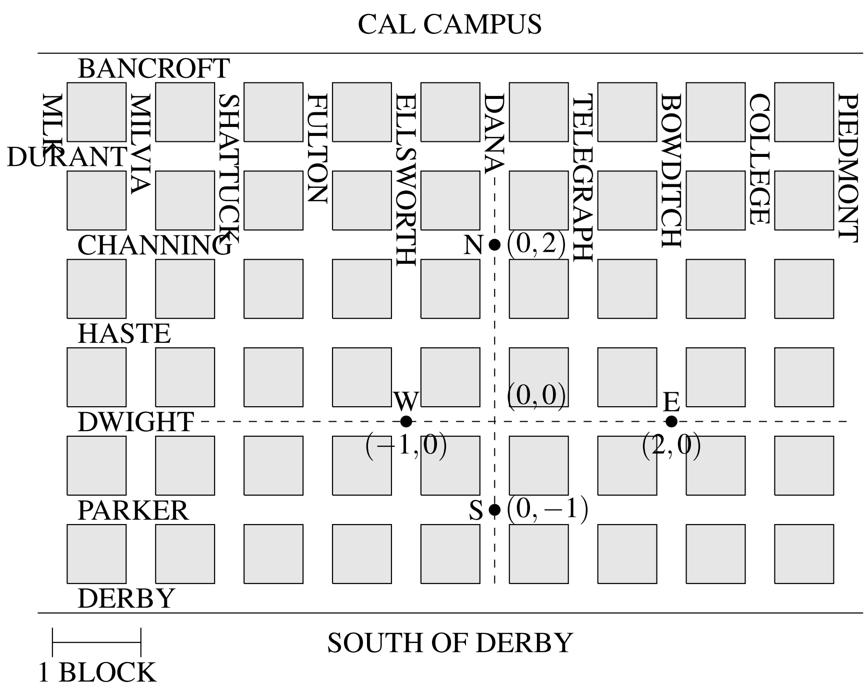
EECS 16A Designing Information Devices and Systems I
Summer 2023 Discussion 07B

1. Search and Rescue Cats

Berkeley's Kitten Shelter needs your help! While S'more the Kitten was being walked, the volunteer let go of her leash and she is now running wild in the streets of Berkeley (which are quite dangerous)! Thankfully, all of the kittens at the shelter have a collar that sends a Bluetooth signal to receiver towers, which are spread throughout the streets (pictured below). If the kitten/collar is within range of the receiver tower, the collar will send the tower a message: the distance of the collar to the tower. Each cell tower has a range of five city blocks. Can you help the shelter locate their lost kitten?

Note: A city block is defined as the middle of an intersection to the middle of an adjacent intersection (scale provided on map). S'more is constrained to running wild in the streets, meaning she won't be found in any buildings. If your TA asks 'Where is S'more?' it is sufficient to answer with her intersection or 'between these two intersections.'

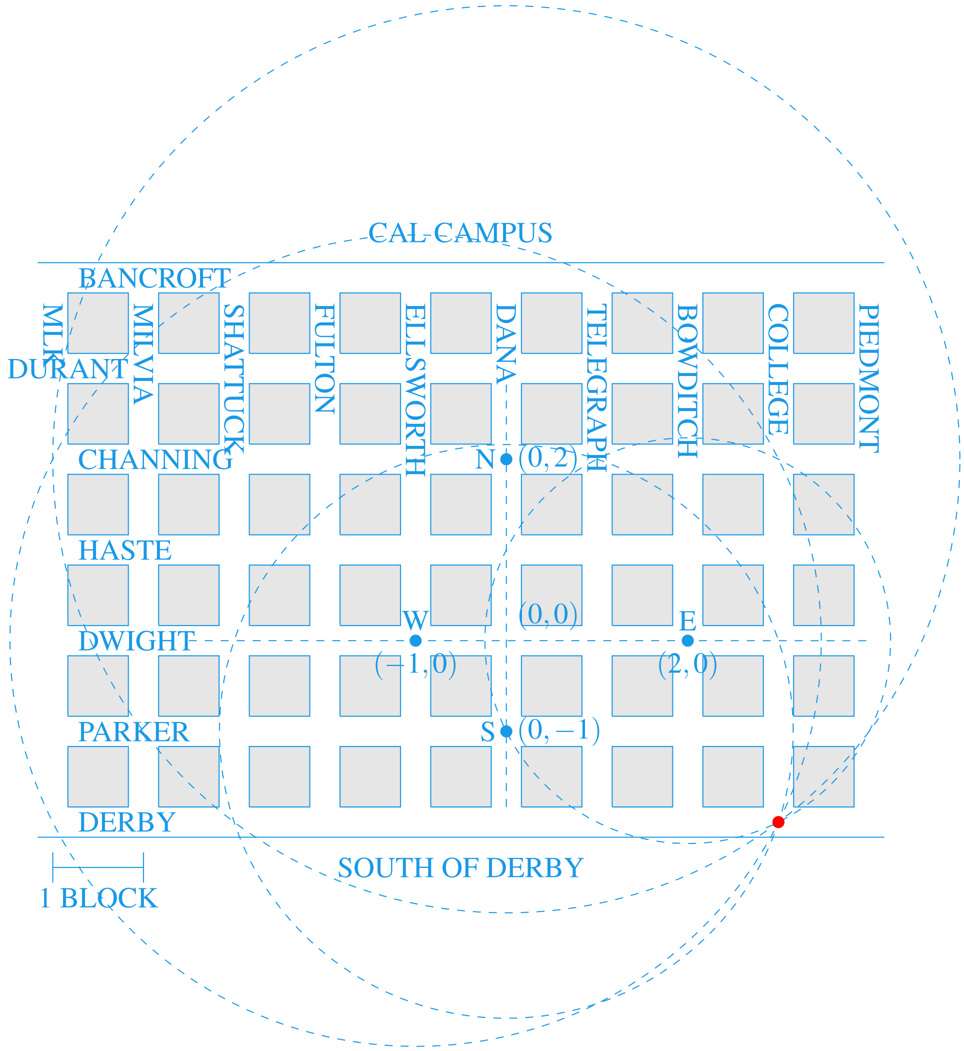


(a) You check the logs of the cell towers, and they have received the following messages:

Sensor	Distance
N	5
W	$\sqrt{20}$
E	$\sqrt{5}$
S	$\sqrt{10}$

On the map provided above, identify where S'more is!

Answer:



(b) Can you set this up as a system of equations? Are these equations linear? If not, can these equations be linearized? If you can linearize these equations, **write down a simplified form of your set of equations.**

Hint: Set $(0,0)$ to be Dwight and Dana.

Hint 2: Distance $= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

Hint 3: You don't need all 4 equations. You have two unknowns, x and y. You know from lecture that you need at least three circles to uniquely find a point on a 2D plane. How can you use the third circle/equation to get two equations and two unknowns?

Note: Remember to check for consistency for all nonlinear equations after finding the coordinates.

Answer: First, set up the system of equations:

$$(x-0)^2+(y-2)^2 = 5^2$$
$$(x+1)^2+(y-0)^2 = \sqrt{20}^2$$
$$(x-2)^2+(y-0)^2 = \sqrt{5}^2$$

Simplify by expanding the squared terms:

$$x^2 + y^2 - 4y + 4 = 25$$
$$x^2 + 2x + 1 + y^2 = 20$$
$$x^2 - 4x + 4 + y^2 = 5$$

Then subtract equation (1) from equations (2) and (3) to get linearized equations:

$$2x + 4y - 3 = 20 - 25 \quad \longrightarrow \quad 2x + 4y = -2$$
$$-4x + 4y = 5 - 25 \quad \longrightarrow \quad -4x + 4y = -20$$

This system of linear equations has a unique solution $x = 3, y = -2$, which is corresponds to the coordinate of the kitten at the intersection of College and Derby.

Note that we only needed three out of the four equations to find the location of S'more and could omit any one of the four. We can verify that this solution is valid with our fourth equation:

$$(x - 0)^2 + (y + 1)^2 = \sqrt{10}^2$$
$$(3 - 0)^2 + (-2 + 1)^2 = 10$$

Alternative Vector Solution: The above was done in scalar form. You can also do it in vector form, which is easier to generalize to higher dimensions. It's unlikely a student will do it in vector form, but you might want to show it to your students. We have below Sensor = North, East, South, West:

The distance d from the kitten to each sensor is represented as

$$\begin{aligned} d^2 &= \| \text{Kitten} - \text{Sensor} \|^2 \\ &= \langle \text{Kitten} - \text{Sensor}, \text{Kitten} - \text{Sensor} \rangle \\ &= \langle \text{Kitten, Kitten} \rangle + \langle \text{Sensor, Sensor} \rangle - 2\langle \text{Kitten, Sensor} \rangle \\ &= \| \text{Kitten} \|^2 + \| \text{Sensor} \|^2 - 2\langle \text{Kitten, Sensor} \rangle \end{aligned}$$

Next we construct the vector equation for each sensor

$$\| \text{Kitten} \|^2 + \| \text{North} \|^2 - 2\langle \text{Kitten, North} \rangle = 5^2$$
$$\| \text{Kitten} \|^2 + \| \text{West} \|^2 - 2\langle \text{Kitten, West} \rangle = \sqrt{20}^2$$
$$\| \text{Kitten} \|^2 + \| \text{East} \|^2 - 2\langle \text{Kitten, East} \rangle = \sqrt{5}^2$$

Then to construct a linear equation, subtract any two of the nonlinear equations above. Repeat for a different choice of two equations to construct a second linear equation. Then solve!

(c) Suppose S'more is moving fast, and by the time you get to destination in part (a) she's already run off! You check the logs of the cell towers again, and see the following updated messages:

Sensor	Distance
N	5
W	$\sqrt{20}$
E	Out of Range
S	Out of Range

Can you find S'more? With a system of linear equations? Other methods? If so, on the map provided above, identify where S'more is!

Answer: Once again, we can set up a system of equations, but we only have two known values for the distances, so we have information for two equations:

$$(x-0)^2+(y-2)^2=5^2$$
$$(x+1)^2+(y-0)^2=\sqrt{20}^2$$

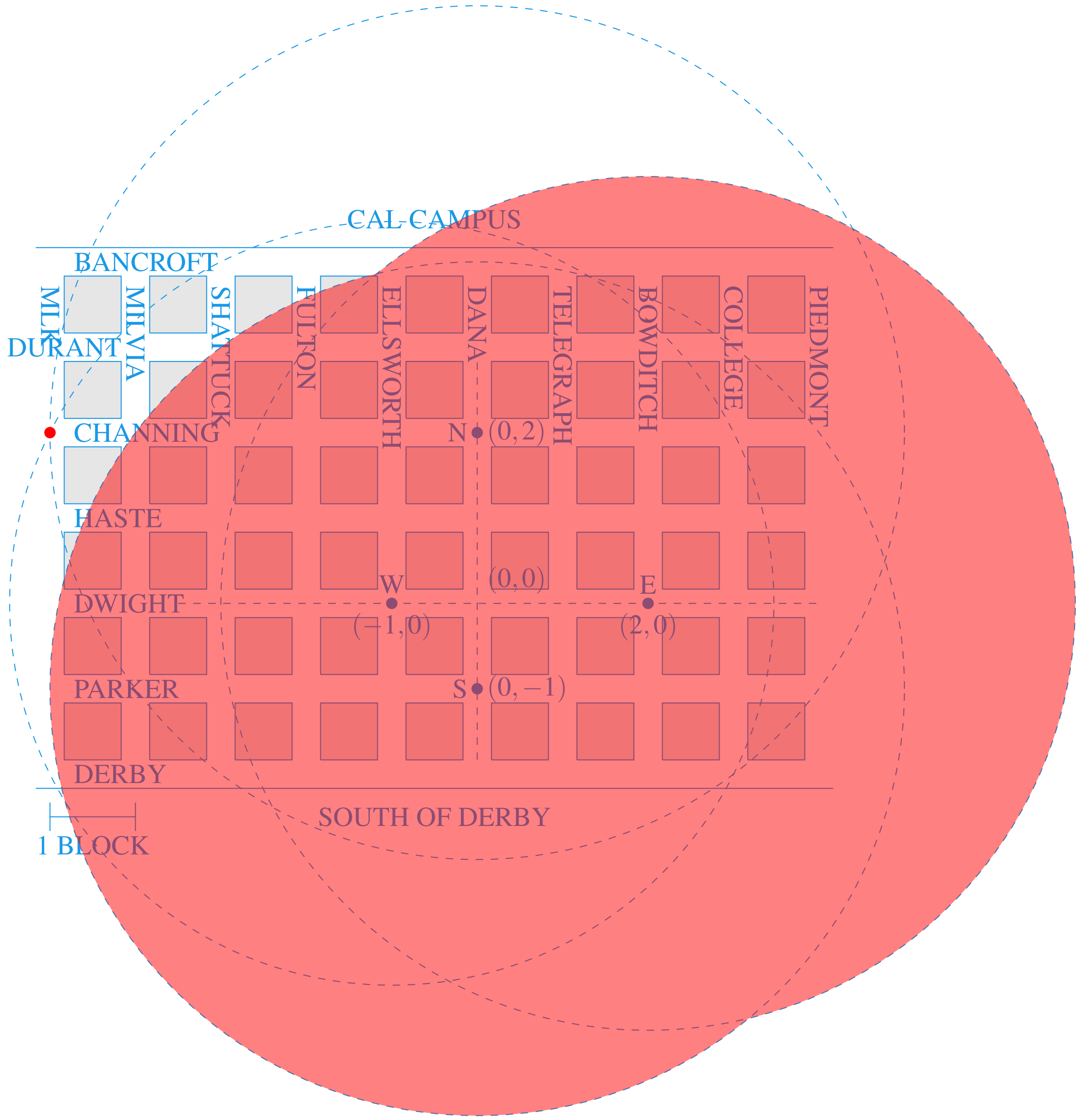
We can expand the squared terms:

$$x^2+y^2-4y+4=25$$
$$x^2+2x+1+y^2=20$$

To get linear equations, we would have to subtract one equation from the other, leaving us with just one linear equation and two unknowns, which would not give us one unique solution for where S'more is:

$$2x+4y-3=20-25 \quad \longrightarrow \quad 2x+4y=-2$$

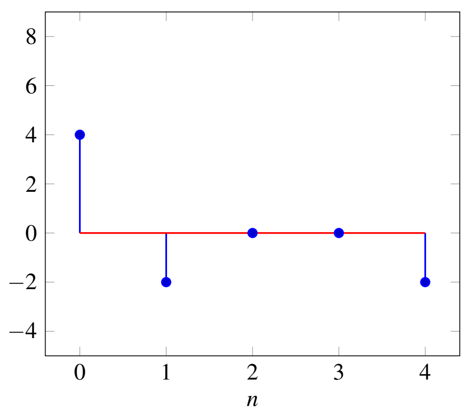
Thus, in this question, by linearizing our system, we are losing information that could have helped us solve the system. Let's go back to a non-linear approach, which is drawing circles. With the information we have from the two sensors in range, we can draw two circles, which can find at most two possible solutions. With two out-of-range sensors, it might seem like we will not be able to find a unique solution (since we need three circles to intersect at a point). The trick is that out-of-range sensors still provide information on where S'more is NOT located. See the diagram below - S'more cannot be in the shaded region. Therefore, S'more should be in Channing and MLK.


CAL CAMPUS
BANCROFT
MLK
MILVIA
SHATTUCK
FULTON
ELLSWORTH
DANA
TELEGRAPH
BOWDITCH
COLLEGE
PIEDMONT
DURANT
CHANNING
N
(0,2)
HASTE
W
(0,0)
E
DWIGHT
(−1,0)
(2,0)
PARKER
S
(0,−1)
DERBY
SOUTH OF DERBY
1 BLOCK

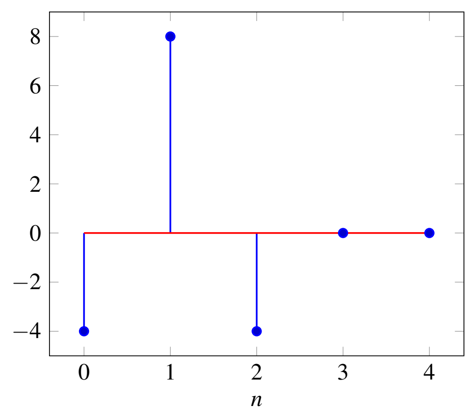
2. Correlation

(a) Sketch the linear cross-correlation of Signal 1 with Signal 2. That is, find: $\text{corr}(\vec{s}_1, \vec{s}_2)[k]$ for $k = 0, 1, .., 4$. Do not assume the signals are periodic.

Signal 1, $\vec{s}_1[n]$



Signal 2, $\vec{s}_2[n]$



$k = 0$:

n	0	1	2	3	4	5	6	7	8
$\vec{s}_1[n]$									
$\vec{s}_2[n]$									
$\langle \vec{s}_1[n], \vec{s}_2[n] \rangle$	+	+	+	+	+	+	+	+	=

$k = 1$:

$\vec{s}_1[n]$									
$\vec{s}_2[n-1]$									
$\langle \vec{s}_1[n], \vec{s}_2[n-1] \rangle$	+	+	+	+	+	+	+	+	=

$k = 2$:

$\vec{s}_1[n]$									
$\vec{s}_2[n-2]$									
$\langle \vec{s}_1[n], \vec{s}_2[n-2] \rangle$	+	+	+	+	+	+	+	+	=

$k = 3$:

$\vec{s}_1[n]$									
$\vec{s}_2[n-3]$									
$\langle \vec{s}_1[n], \vec{s}_2[n-3] \rangle$	+	+	+	+	+	+	+	+	=

$k = 4$:

$\vec{s}_1[n]$									
$\vec{s}_2[n-4]$									
$\langle \vec{s}_1[n], \vec{s}_2[n-4] \rangle$	+	+	+	+	+	+	+	+	=

Answer:

Represent Signal 1 as the vector $\vec{s}_1 = \begin{bmatrix} 4 & -2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}^T$, zero-padded so that we compute only the linear correlation. Similarly, represent Signal 2 as the vector
$\vec{s}_2 = \begin{bmatrix} -4 & 8 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, where we once again zero pad the vector. Notice that we zero pad the vectors $\vec{s}_1$ and $\vec{s}_2$ to represent the signals from $n = 0, 1, ..., 8$. This is because we are only interested in calculating the cross-correlation for for $k = 0, 1, .., 4$, therefore we will only need to shift the vector $\vec{s}_2$ four times.

The cross-correlation between two vectors is defined as follows:

$$\text{corr}(\vec{x}, \vec{y})[k] = \sum_{n=-\infty}^{\infty} \vec{x}[n]\vec{y}[n-k]$$

To compute the cross-correlation $\text{corr}(\vec{s}_1, \vec{s}_2)$, we shift the vector $\vec{s}_2$ and compute the inner product of the shifted $\vec{s}_2$ and the vector $\vec{s}_1$.

$k = 0$:

n	0		1		2		3		4		5		6		7		8		
$\vec{s}_1[n]$	4		−2		0		0		−2		0		0		0		0		
$\vec{s}_2[n]$	−4		8		−4		0		0		0		0		0		0		
$\langle \vec{s}_1[n], \vec{s}_2[n] \rangle$	−16	+	−16	+	0	+	0	+	0	+	0	+	0	+	0	+	0	=	−32

$k = 1$:

$\vec{s}_1[n]$	4		−2		0		0		−2		0		0		0		0		
$\vec{s}_2[n-1]$	0		−4		−8		−4		0		0		0		0		0		
$\langle \vec{s}_1[n], \vec{s}_2[n-1] \rangle$	0	+	8	+	0	+	0	+	0	+	0	+	0	+	0	+	0	=	8

$k = 2$:

$\vec{s}_1[n]$	4		−2		0		0		−2		0		0		0		0		
$\vec{s}_2[n-2]$	0		0		−4		8		−4		0		0		0		0		
$\langle \vec{s}_1[n], \vec{s}_2[n-2] \rangle$	0	+	0	+	0	+	0	+	8	+	0	+	0	+	0	+	0	=	8

$k = 3$:

$\vec{s}_1[n]$	4		−2		0		0		−2		0		0		0		0		
$\vec{s}_2[n-3]$	0		0		0		−4		8		−4		0		0		0		
$\langle \vec{s}_1[n], \vec{s}_2[n-3] \rangle$	0	+	0	+	0	+	0	+	−16	+	0	+	0	+	0	+	0	=	−16

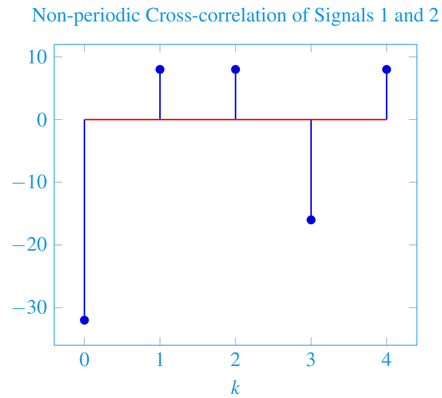
$k = 4$:

$\vec{s}_1[n]$	4		−2		0		0		−2		0		0		0		0		
$\vec{s}_2[n-4]$	0		0		0		0		−4		8		−4		0		0		
$\langle \vec{s}_1[n], \vec{s}_2[n-4] \rangle$	0	+	0	+	0	+	0	+	8	+	0	+	0	+	0	+	0	=	8

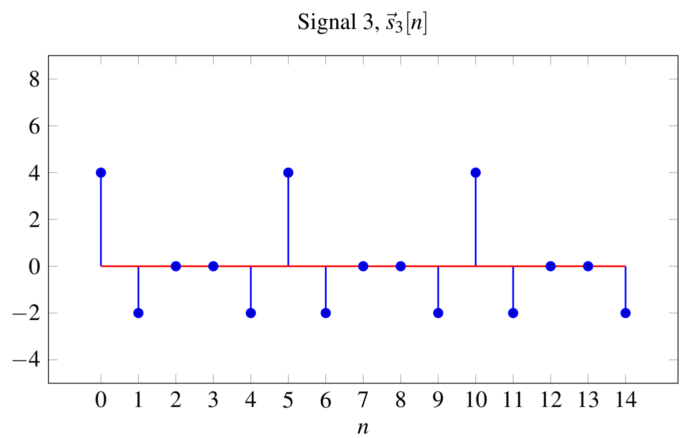
The correlation is then

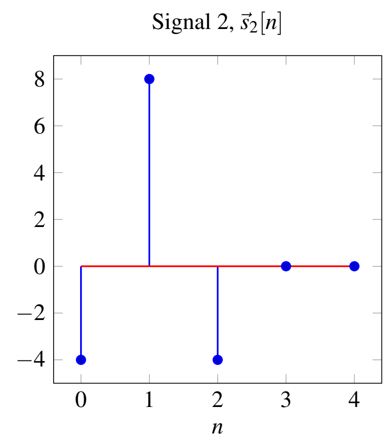
$$\begin{aligned}\text{corr}(\vec{s}_3, \vec{s}_2)[k] &= \sum_{n=-\infty}^{\infty} \vec{s}_3[n]\vec{s}_2[n-k] \\ &= \begin{bmatrix} -32 & 8 & 8 & -16 & 8 \end{bmatrix}^T\end{aligned}$$

from $k = 0, 1, \ldots, 4$.



(b) Now, the pattern in $\vec{s}_1$ is repeated three times and we call this Signal 3





Sketch the linear cross-correlation of Signal 3 with Signal 2, $\text{corr}(\vec{s}_3, \vec{s}_2)[k]$, for $k = 0, 1, .., 4$.

$k = 0$:

n	0	1	2	3	4	5	6	7	8	9
$\vec{s}_3[n]$										
$\vec{s}_2[n]$										
$\langle \vec{s}_3[n], \vec{s}_2[n] \rangle$	+	+	+	+	+	+	+	+	+	=

$k = 1$:

$\vec{s}_3[n]$										
$\vec{s}_2[n-1]$										
$\langle \vec{s}_3[n], \vec{s}_2[n-1] \rangle$	+	+	+	+	+	+	+	+	+	=

$k = 2$:

$\vec{s}_3[n]$										
$\vec{s}_2[n-2]$										
$\langle \vec{s}_3[n], \vec{s}_2[n-2] \rangle$	+	+	+	+	+	+	+	+	+	=

$k = 3$:

$\vec{s}_3[n]$										
$\vec{s}_2[n-3]$										
$\langle \vec{s}_3[n], \vec{s}_2[n-3] \rangle$	+	+	+	+	+	+	+	+	+	=

$k = 4$:

$\vec{s}_3[n]$										
$\vec{s}_2[n-4]$										
$\langle \vec{s}_3[n], \vec{s}_2[n-4] \rangle$	+	+	+	+	+	+	+	+	+	=

Answer:

Recall that $\text{corr}(\vec{x}, \vec{y})[k] = \sum_{n=-\infty}^{\infty} \vec{x}[n]\vec{y}[n-k]$

As we did in part (a) to compute the cross-correlation $\text{corr}(\vec{s}_1, \vec{s}_2)$, we shift the vector $\vec{s}_2$ and compute the inner product of the shifted $\vec{s}_2$ and the vector $\vec{s}_3$. Since we are interested in $\text{corr}(\vec{s}_3, \vec{s}_2)[n]$, for $k = 0, 1, .., 4$, here we have shown the two signals for $n = 0, 1, .., 8$.

$k=0$:

n	0	1	2	3	4	5	6	7	8	9	
$\vec{s}_3[n]$	4	−2	0	0	−2	4	−2	0	0	−2	
$\vec{s}_2[n]$	−4	8	−4	0	0	0	0	0	0	0	
$\langle \vec{s}_3[n], \vec{s}_2[n] \rangle$	−16	+ −16	+ 0	+ 0	+ 0	+ 0	+ 0	+ 0	+ 0	+ 0	= −32

$k=1$:

$\vec{s}_3[n]$	4	−2	0	0	−2	4	−2	0	0	−2	
$\vec{s}_2[n-1]$	0	−4	8	−4	0	0	0	0	0	0	
$\langle \vec{s}_3[n], \vec{s}_2[n-1] \rangle$	0	+ 8	+ 0	+ 0	+ 0	+ 0	+ 0	+ 0	+ 0	+ 0	= 8

$k=2$:

$\vec{s}_3[n]$	4	−2	0	0	−2	4	−2	0	0	−2	
$\vec{s}_2[n-2]$	0	0	−4	8	−4	0	0	0	0	0	
$\langle \vec{s}_3[n], \vec{s}_2[n-2] \rangle$	0	+ 0	+ 0	+ 0	+ 8	+ 0	+ 0	+ 0	+ 0	+ 0	= 8

$k=3$:

$\vec{s}_3[n]$	4	−2	0	0	−2	4	−2	0	0	−2	
$\vec{s}_2[n-3]$	0	0	0	−4	8	−4	0	0	0	0	
$\langle \vec{s}_3[n], \vec{s}_2[n-3] \rangle$	0	+ 0	+ 0	+ 0	+ −16	+ −16	+ 0	+ 0	+ 0	+ 0	= −32

$k=4$:

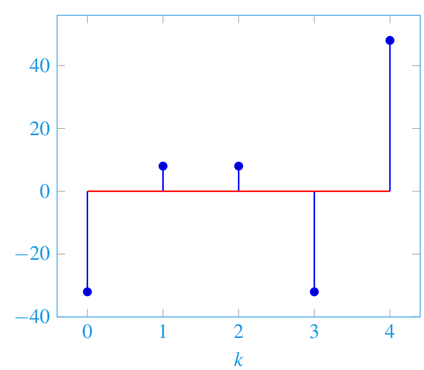
$\vec{s}_3[n]$	4	−2	0	0	−2	4	−2	0	0	−2	
$\vec{s}_2[n-4]$	0	0	0	0	−4	8	−4	0	0	0	
$\langle \vec{s}_3[n], \vec{s}_2[n-4] \rangle$	0	+ 0	+ 0	+ 0	+ 8	+ 32	+ 8	+ 0	+ 0	+ 0	= 48

The correlation is then

$$\begin{aligned} \operatorname{corr}(\vec{s}_3, \vec{s}_2)[k] &= \sum_{n=-\infty}^{\infty} \vec{s}_3[n]\vec{s}_2[n-k] \\ &= \begin{bmatrix} -32 & 8 & 8 & -32 & 48 \end{bmatrix}^T \end{aligned}$$

from $k = 0, 1, \ldots, 4$.

Cross-correlation of Signals 3 and 2



Since $\vec{s}_3$ is a repeated version of $\vec{s}_1$, notice that when $\vec{s}_1$ is periodic we don't simply get the result from part (a) repeated.