EECS 16A Designing Information Devices and Systems I
Summer 2023 Discussion 1B

1. Solving Systems of Equations

While we'd love every system of linear equations to have a unique solution, in reality we can either have (a) one unique solution, (b) an infinite number of solutions, or (c) no solution at all. We are going to walk through some examples to see what sorts of equations create these types of solutions.

(a) Let's consider the system (where each $a, b \in \mathbb{R}$ can be any real number):

$$\begin{aligned} ax + y &= 3 \\ -x + 2y &= b \end{aligned}$$

For each of the selected values for a and b, sketch or plot out the lines $y(x)$ each of these equations form.
Can you conclude which values result in a unique solution? Infinite solutions? No solutions?

- i. $a = 1$, $b = 0$
- ii. $a = 0$, $b = 2$
- iii. $a = -1/2$, $b = 6$
- iv. $a = -1/2$, $b = 4$

Answer: First, we re-write our system into line equations by isolating y on one side:

$$y = 3 - ax \qquad\qquad y = \frac{b}{2} + \frac{1}{2}x$$

By directly inserting the values for a, b we can get a collection of lines to plot

i. $y = 3 - x$	ii. $y = 3 + 0x$	iii. $y = 3 + x/2$	iv. $y = 3 + x/2$
. $y = 0 + x/2$	$y = 1 + x/2$	$y = 3 + x/2$	$y = 2 + x/2$

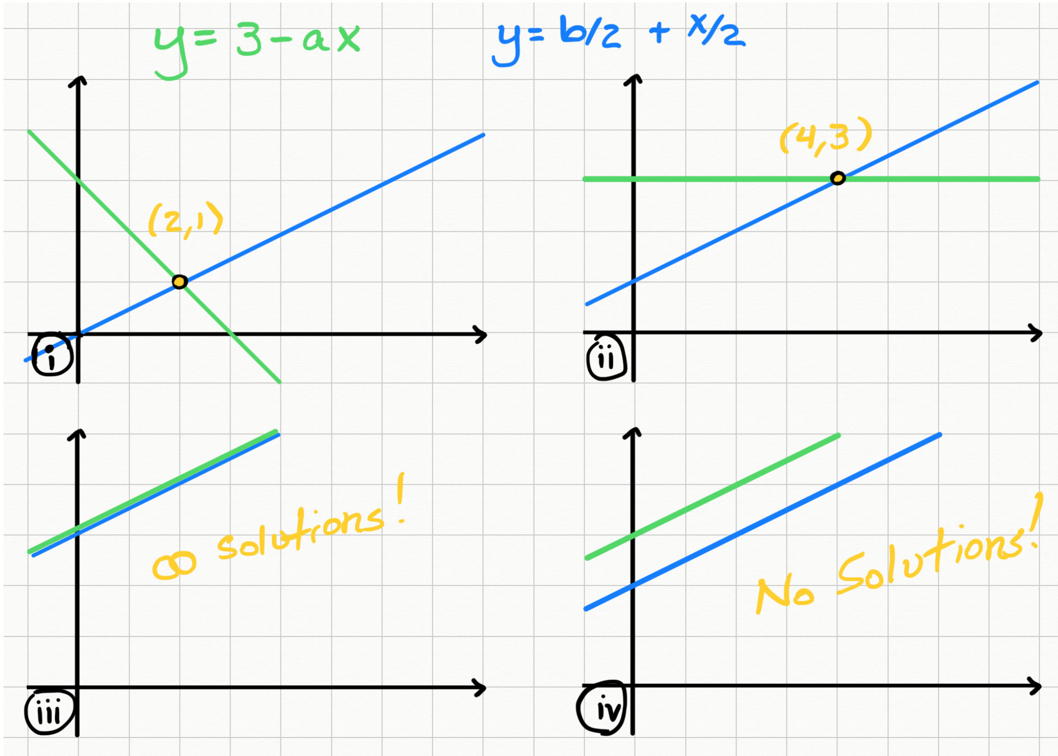
Essentially, each line represents the values of (x,y) that satisfy that equation, so we need values that satisfy both equations (i.e. the intersection).

The key take-away here is that for i. and ii. we have different slopes, but for iii. and iv. the slopes are both $1/2$, which means that they either never intersect (they're parallel) OR they're the same line.

Thus, for i. and ii. there must be one point of intersection somewhere and so there is one solution! We can get this by equating the two equations, $3 - x = x/2$, solving for $x = 2$, and substituting in for $y = 1$. A similar method works on ii. $3 = 1 + x/2 \ \rightarrow x = 4 \ \rightarrow y = 1 + 4/2 = 3$.

However, iii. produces just one line, meaning there are infinite solutions along the line $y = 3 + x/2$. Problem iv. has no solution, since the lines are parallel (the lines have the same slope with different intercepts) and never intersect. Thus, we have *inconsistent* equations.

See the sketched plots below:



(b) Now, assume we are using the tomography imaging technique described in lecture to image a 2x2 grid, as shown below.

2x2 Tomography Example

x_1	x_2
x_3	x_4

i. We record the following measurements:

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 + x_3 &= 2 \\ x_2 + x_4 &= 2 \\ x_3 + x_4 &= 3 \end{aligned}$$

Solve this system using any method. Is this a valid set of measurements? Why or why not?

Answer: No, this is not valid. For instance, $(x_1 + x_2) + (x_3 + x_4) = 2 + 3 = 5$, however, $(x_1 + x_3) + (x_2 + x_4) = 2 + 2 = 4$, which is a contradiction.

ii. We are led to believe our measurements might be faulty, so we record the following new measurements:

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 + x_3 &= 3 \\ x_2 + x_4 &= 2 \\ x_3 + x_4 &= 3 \end{aligned}$$

Now, solve this system using any method. Is this a valid set of measurements? Why or why not?

Answer: These are valid measurements. Doing the check from the previous part, we get $(x_1 + x_2) + (x_3 + x_4) = 2 + 3 = (x_1 + x_3) + (x_2 + x_4)$. By plugging in a random initial value for x_1, we can in fact find infinite solutions — for instance, if we set $x_1 = 1$, we get the solution $(x_1, x_2, x_3, x_4) = (1, 1, 2, 1)$, and if we set $x_1 = 0.5$, we get the solution $(x_1, x_2, x_3, x_4) = (0.5, 1.5, 2.5, 0.5)$. This system wasn't that different from the one in part i. – instead of 2, the value of $x_1 + x_3$ was 3. This goes to show that even a little noise or interference in our measurements can cause our real-life data to be mathematically inconsistent. Some level of noise is inevitable in most real-life scenarios, so we will learn techniques to combat that later in this course.

Now, while this given system is technically mathematically consistent, there are infinite solutions, so there is no way to tell exactly which solution is true and in fact, this will always be the case with our current measurement technique (measuring the sum of the rows and columns).