EECS 16A Designing Information Devices and Systems I
Summer 2023 Final Review

1. Some Proofs (Spring 2023 Midterm 1)

(a) Prove that if there are two unique solutions $\vec{x}_1$ and $\vec{x}_2$ to the system $A\vec{x} = \vec{b}$, then there are infinitely many solutions $\vec{x}$ to this system.

Solution:

$$A\vec{x}_1 = \vec{b}$$

$$A\vec{x}_2 = \vec{b}$$

$$A(\vec{x}_1 - \vec{x}_2) = \vec{b} - \vec{b} = \vec{0}$$

$$\implies \vec{x}_1 - \vec{x}_2 \in Null(A)$$

Vectors of the form $\vec{x} = \vec{x}_1 + \alpha(\vec{x}_1 - \vec{x}_2)$ are also solutions to the system $A\vec{x} = \vec{b}$ (for $\alpha \in \mathbb{R}$):

$$A(\vec{x}_1 + \alpha(\vec{x}_1 - \vec{x}_2)) = A\vec{x}_1 + \alpha A(\vec{x}_1 - \vec{x}_2) = \vec{b} + \vec{0} = \vec{b}$$

Note that $\vec{x}_1 - \vec{x}_2 \neq \vec{0}$ as $\vec{x}_1 \neq \vec{x}_2 \implies \vec{x} = \vec{x}_1 + \alpha(\vec{x}_1 - \vec{x}_2) \neq \vec{x}_1$ for $\alpha \neq 0$.
Therefore we have constructed an infinite number of solutions.

(b) A matrix $\mathbf{A} \in \mathbb{R}^{N\times N}$ has eigenvalues λ_1, λ_2, ... λ_n and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Let $\lambda_1 = 0$, $\lambda_2 = 0$ and λ_3, λ_4, ... $\lambda_n \neq 0$. Prove that $\vec{x} \in Null(A)$ for any $\vec{x} \in span\{\vec{v}_1, \vec{v}_2\}$.

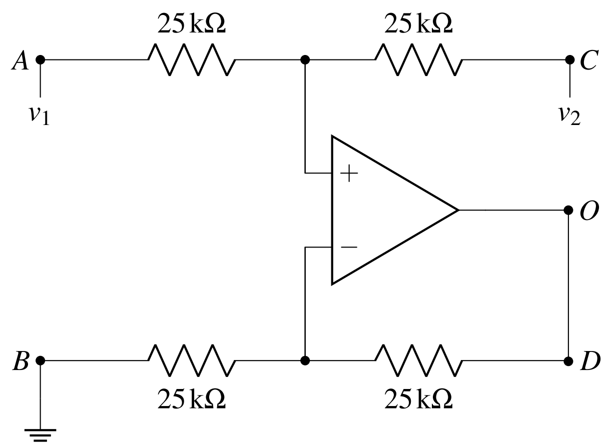
Solution:

$$\vec{x} = \alpha\vec{v}_1 + \beta\vec{v}_2 \text{ where } \alpha, \beta \in \mathbb{R}$$
$$A\vec{x} = A(\alpha\vec{v}_1 + \beta\vec{v}_2)$$
$$= \alpha A\vec{v}_1 + \beta A\vec{v}_2$$
$$= \alpha\lambda_1\vec{v}_1 + \beta\lambda_2\vec{v}_2$$
$$= \alpha(0)\vec{v}_1 + \beta(0)\vec{v}_2$$
$$= \vec{0} + \vec{0} = \vec{0}$$
$$\implies \vec{x} \in Null(A)$$

2. A Versatile Opamp Circuit

For each circuit configuration, determine the voltage at O, given that v_1 and v_2 are voltage sources. All circuit configurations are in negative feedback.

(a) Configuration 1:



Answer:

By recognizing a voltage summer (or by using superposition), we note that the voltage at v^+ is given by:

$$v^+ = \frac{25\,\text{k}\Omega}{25\,\text{k}\Omega + 25\,\text{k}\Omega} v_1 + \frac{25\,\text{k}\Omega}{25\,\text{k}\Omega + 25\,\text{k}\Omega} v_2 = \frac{v_1 + v_2}{2}$$

Method 1: Voltage Divider

The circuit connected to the negative op-amp terminal v^- also forms a voltage divider since the input current $I^- = 0$, thus

$$v^- = \frac{25\,\text{k}\Omega}{25\,\text{k}\Omega + 25\,\text{k}\Omega} v_O = \frac{v_O}{2}$$

Since the circuit is in negative feedback, we can apply the second Golden Rule: $v^+ = v^-$. Thus,

$$\begin{aligned} v^- &= v^+ \\ \frac{v_O}{2} &= \frac{v_1 + v_2}{2} \\ v_O &= v_1 + v_2 \end{aligned}$$

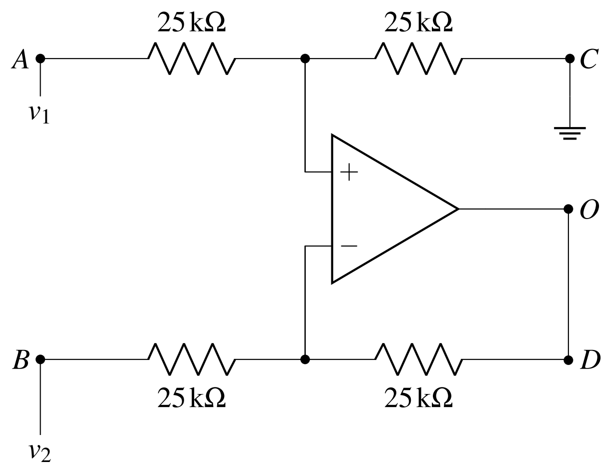
Method 2: Non-Inverting Amplifier

Alternatively, the rest of the circuit looks like a non-inverting amplifier with a gain of $1 + \frac{25\,\text{k}\Omega}{25\,\text{k}\Omega} = 2$. Therefore, the output voltage is:

$$v_O = 2v^+ = v_1 + v_2$$

This circuit configuration takes the sum of v_1 and v_2.

(b) Configuration 2:



Answer:

By recognizing a voltage divider and applying the formula, we note that the voltage at the positive input terminal v^+ of the op-amp is given by

$$v^+ = \frac{25\,\mathrm{k}\Omega}{25\,\mathrm{k}\Omega + 25\,\mathrm{k}\Omega} v_1 = \frac{v_1}{2}$$

By applying the Golden Rule: $I^- = 0$, we can write a KCL equation at the negative input terminal v^-

$$\frac{v^- - v_2}{25\,\mathrm{k}\Omega} + \frac{v^- - v_O}{25\,\mathrm{k}\Omega} + 0 = 0$$

Solving for v_O yields

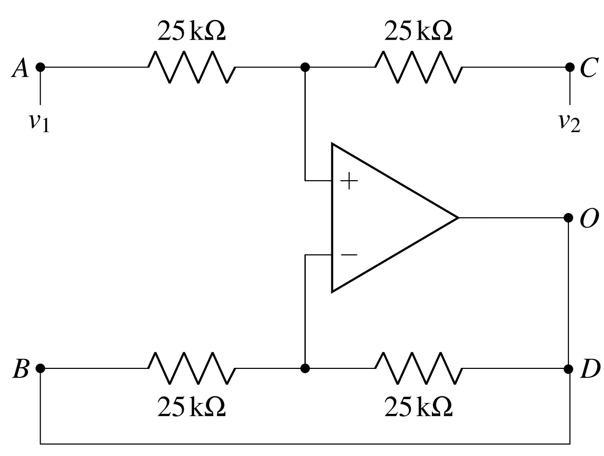
$$v_O = 2v^- - v_2$$

Since the op-amp is in negative feedback, we can apply the second Golden Rule $v^+ = v^-$. Thus

$$v_O = 2v^- - v_2 = 2v^+ - v_2 = 2\left(\frac{v_1}{2}\right) - v_2$$
$$v_O = v_1 - v_2$$

This circuit configuration takes the difference of v_1 and v_2.

(c) Configuration 3:



Answer:
Like Configuration 1 in part (a), we note that the circuit connected to node v^+ is a voltage summer, thus

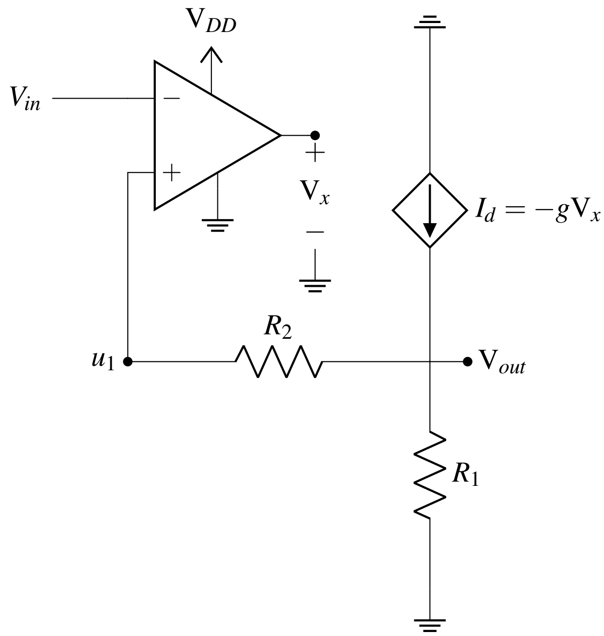
$$v^+ = \frac{v_1 + v_2}{2}$$

We note that the resistors connected to B and D do not affect the circuit as no current is flowing through them. Therefore,

$$v_O = v^- = v^+ = \frac{v_1 + v_2}{2}$$

3. Can I Give You Some Feedback? (Spring 2023 Final Question 5)

The following circuit is a linear voltage regulator.



V_{DD} and V_{in} are both connected to ideal voltage sources. g is the gain factor of the dependent current source. The opamp has finite gain A.

Using the method for negative feedback analysis, if V_{out} **increases**, determine what happens to the following values. Circle one of the two options for each line below. *Note that if a quantity is getting more negative, that means it is decreasing.*

Voltage at u_1 will: Increase Decrease

V_x will: Increase Decrease

Dependent current I_d will: Increase Decrease

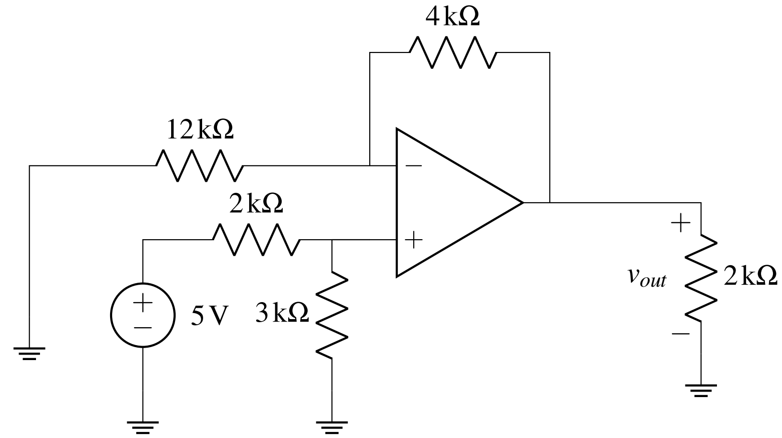
The circuit is in: Negative Feedback Positive Feedback

Solution: Increase, increase, decrease, and negative feedback.
When V_{out} increases, because no current can go into the positive terminal of the op-amp, there cannot be voltage drop across R_2. As a result,voltage at u_1 must follow V_{out} and increase.
By definition of op-amp output voltage, $V_x = A(u_1 - V_{in})$, as u_1 increases, V_x increases.

Thus, because V_x increases, $I_d = -gV_x$ must decrease. Finally, because I_d decreases, the current through R_1 decreases, therefore V_{out} ends decreases. This is the expected result of negative feedback.

4. Op-amps and Comparators (Spring 2022 Midterm 2 Question 10)

(a) You are given the following op-amp in negative feedback. Find v_{out}.



Solution: First, we can find V+ using a voltage divider.

$$V^+ = 5V * \frac{3k\Omega}{2k\Omega + 3k\Omega}$$
$$V^+ = 3V$$

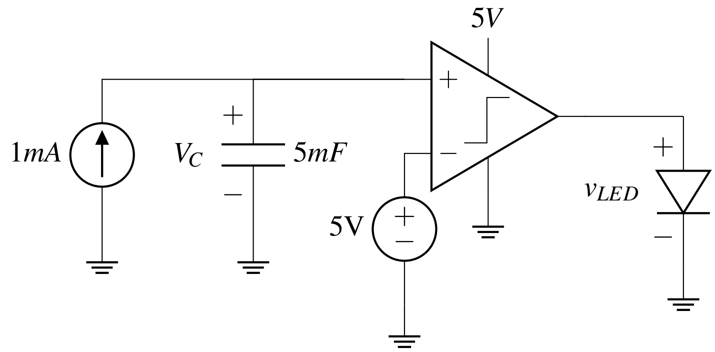
As op-amp is in negative feedback and it is an ideal opamp, by golden rules, V- = 3V too.

Next, using node voltage analysis at v^-, we can write:

$$\frac{V^- - 0}{12k\Omega} + \frac{V^- - v_{out}}{4k\Omega} = 0$$

Using V^- = 3V, we solve that $v_{out} = 4V$.

(b) You are given the circuit below. The capacitor is initially uncharged. At time t = 0, the current source is turned on. Find V_c(t).



Solution: We know that the IV relationship of a capacitor is $I = C\frac{dV}{dt}$, By integrating both sides, we can derive that

$$V_C(t) = \frac{I * t}{C} + V(0)$$

In this case, we know V(0) = 0V because it is initially uncharged. Thus, $V(t) = \frac{1mA}{5mF}t = 0.2t$.

(c) The LED turns on when the voltage across it is greater than 3.3V. Using the same setup as part (b), at what time t does the LED turn on?

Solution: When V+ is greater than V- for the comparator, the op-amp will output Vdd (5V), which will turn on the LED. In order to have V+ > V-, V(t) > 5V. This happens when $V_C = 5V$, 0.2t = 5, so t = 25s.

5. An Easier Way To Do Math Homework (Spring 2023 Final Question 13)

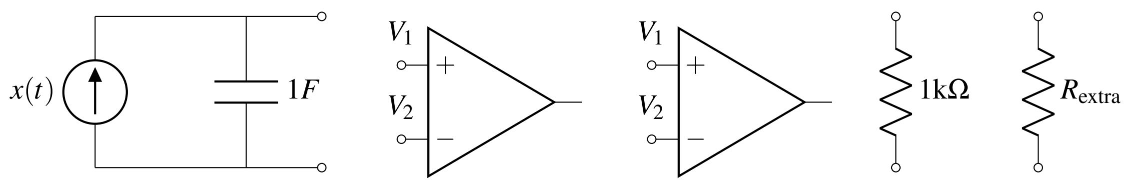
You're working on your Math 1B homework and you don't know how to calculate an integral. Instead, you decide to put your circuit skills to use to solve this problem!
The integral that you're trying to solve is of the form

$$-\frac{1}{5}\int_0^\tau x(t)\,dt$$

Your helpful lab TA, Raghav, gives you several circuit elements that you can use.

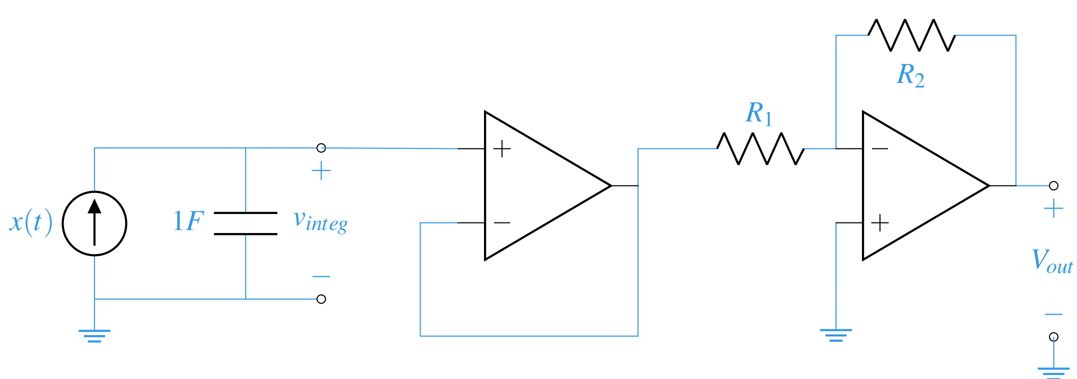
These elements are:

- A current source $I_s = x(t)$ amps in parallel with a capacitance of 1F.
- **Two** op-amps (assume that the supply voltages to the op-amps are provided).
- A resistor $R_{\text{fixed}} = 1\text{k}\Omega$.
- One additional resistor R_{extra} that can have **any value**. Be sure to specify the resistance you use.



Design a circuit to have the above output using the provided elements. Clearly specify the resistance of R_{extra} if used.

Solution:



The circuit consists of three main blocks: the current source and capacitor which integrate our signal, a buffer, and an inverting amplifier. The current source charges our capacitor resulting in:

$$v_{integ} = \frac{1}{1F}\int_0^\tau x(t)dt$$

Next, in order to scale and invert our signal we can chose $R_1 = 5R_2$. Since we have a fixed resistor of 1Ω, we pick $R_2 = R_{fixed} = 1\text{k}\Omega$ and $R_1 = R_{extra} = 5\text{k}\Omega$.

Our overall result becomes:

$$v_{out} = \frac{-R_2}{R_1}v_{integ} = \frac{-1}{5}\int_0^\tau x(t)dt$$

6. Least Squares (Fall 2022 Final Question 3)

(a) Consider the system of equations $\vec{a}x = \vec{b}$ where $\vec{a}, \vec{b} \in \mathbb{R}^2$ and $x \in \mathbb{R}$. When applying least squares, we want to find the $\vec{v} \in \text{span}(\vec{a})$ that is closest to $\vec{b}$ in Euclidean distance.
Hint: It might be helpful to draw the vectors.

i. When solving for vector $\vec{v}$, which of the following operations are required?

◯ Projecting $\vec{b}$ onto $\vec{a}$
◯ Projecting $\vec{a}$ onto $\vec{b}$
◯ Subtracting $\vec{b}$ from $\vec{a}$
◯ Subtracting $\vec{a}$ from $\vec{b}$
◯ None of the above

Solution:
Projecting $\vec{b}$ onto $\vec{a}$.
When we are finding $\vec{v}$, or the best approximation of $\vec{b}$ in the span of $\vec{a}$, we project $\vec{b}$ onto $\vec{a}$.

ii. The vector $\vec{v}$ can also be determined by minimizing the length of the error vector, represented as

◯ $\vec{v} = \underset{\vec{b}}{\text{argmin}} \|\vec{a} - \vec{b}\|$
◯ $\vec{v} = \underset{\vec{v}}{\text{argmin}} \|\vec{a} - \vec{v}\|$
◯ $\vec{v} = \underset{\vec{b}}{\text{argmin}} \|\vec{b} - \vec{v}\|$
◯ $\vec{v} = \underset{\vec{v}}{\text{argmin}} \|\vec{b} - \vec{v}\|$

Solution:
$\vec{v} = \underset{\vec{v}}{\text{argmin}} \|\vec{b} - \vec{v}\|$
In the least squares problem, we minimize the length of the error vector, $\vec{e}$, defined as the difference between the known vector $\vec{b}$ and the span of possible vectors $\vec{a}x = \vec{v}$. Thus the error vector is $\vec{e} = \vec{b} - \vec{v}$. And the vector $\vec{v}$ is the minimization argument.

(b) For the following systems of $A\vec{x} = \vec{b}$, determine if they have a unique least squares solution.

i. $A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

◯ Yes
◯ No

Solution:
Yes. There is a unique least squares solution since A has linearly independent columns.

ii. $A = \begin{bmatrix} 1 & 4 \\ 3 & 12 \\ 2 & 8 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$

◯ Yes

◯ No

Solution:
No. There is not a unique least squares solution as A does not have linearly independent columns.

(c) For the following three questions, consider the system of $A\vec{x} = \vec{b}$ with $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

i. Can we apply the least squares formula?

◯ Yes

◯ No

Solution:
No. The fat matrix A does not have linearly independent columns. Additionally, $A^T A$ is not invertible since its determinant is zero.

ii. What is the determinant of $A^T A$?

$\det(A^T A) = \boxed{}$

Solution:

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\det\left(A^T A\right) = \det\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\right) = 0$$

The zero determinant can be inspected since $A^T A$ is not invertible (i.e., not full column rank, not linearly independent columns).

iii. (1 point) Does $A\vec{x} = \vec{b}$ have zero, one, or more than one solution for $\vec{x}$?

◯ No solutions

◯ One unique solution

◯ More than one solution

Solution:
More than one solution. There are less equations (rows) than unknowns (columns).

(d) Find the best approximation $x = \hat{x}$ to this system of equations:

$$a_1 x = b_1$$
$$a_2 x = b_2$$

i. Write the problem into $A\vec{x} \approx \vec{b}$ format and solve for $\hat{x}$ using least squares. Choose the correct $\hat{x}$.

○ $\hat{x} = \dfrac{a_1 b_1 + a_2 b_2}{a_1^2 + a_2^2}$

○ $\hat{x} = \dfrac{a_1 b_1 - a_2 b_2}{a_1^2 + a_2^2}$

○ $\hat{x} = \dfrac{a_1 b_2 + a_2 b_1}{a_1^2 + a_2^2}$

○ $\hat{x} = \dfrac{a_1 b_2 - a_2 b_1}{a_1^2 + a_2^2}$

○ None of the above

Solution:

$$A x = \vec{b} \quad \rightarrow \quad \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\hat{x} = \left(A^T A\right)^{-1} A^T \vec{b} = \left(\begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$
$$= \frac{a_1 b_1 + a_2 b_2}{a_1^2 + a_2^2}$$

ii. Suppose the inner product is defined instead as a non-Euclidean $\langle x, y \rangle = x^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} y$. Which of the following expressions must be true with respect to the minimized least squares error vector, $\vec{e}$?

○ $\vec{e}^T A = \vec{0}$

○ $A^T \vec{e} = \vec{0}$

○ $A^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{e} = \vec{0}$

○ $\left(A^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} A \right)^{-1} \vec{e} = \vec{0}$

○ None of the above

Solution:
The least squares error, $\vec{e}$, is minimized when it is orthogonal to every column of A (i.e., colspace(A)). Orthogonality occurs when the inner product (in this case the non-Euclidean inner product) of two vectors is zero.
Mathematically, $\langle \vec{a}_i, \vec{e} \rangle = 0$ for every column $\vec{a}_i$ of A. Thus, $A^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{e} = \vec{0}$.