

Lecture 3C

Wed 7/5

To day:

- Conditions for invertibility + 1 proof
- Vector spaces, subspaces
- Column spaces, Null spaces

Today's material is the last material in scope for the Quest!

Last time we defined matrix inverses, and said that some matrices are invertible and some aren't.

Let's do a proof of when a matrix is invertible:

Thm For an $n \times n$ matrix A , if the columns of A are linearly dependent, then A is not invertible.

Pf. by contradiction: Assume A^{-1} exists.

What we know: cols of A linearly dependent

- in math: Let $A = \begin{bmatrix} \vec{a}_1 & \dots & \vec{a}_n \end{bmatrix}$

There exists constants $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, not all zero such that $\alpha_1 \vec{a}_1 + \dots + \alpha_n \vec{a}_n = \vec{0}$

(definition of linear dependence)

As usual, we will use the column view of matrix-vector mult:

$$\begin{bmatrix} \vec{a}_1 & \dots & \vec{a}_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A \cdot \vec{\alpha} = \vec{0}$$

We assumed A^{-1} exists, so let's multiply it on both sides:

$$A^{-1} (A \vec{x}) = A^{-1} \cdot \vec{0}$$

$$\underbrace{(A^{-1}A)}_{=I} \vec{x} = \vec{0}$$

matrix mult is associative.

$$I \vec{x} = \vec{0}$$

$$\vec{x} = \vec{0}$$

I · any vector = that vector

We said $\alpha_1, \dots, \alpha_n$ are constants that are NOT ALL ZERO,
i.e. $\vec{x} \neq \vec{0}$.

So we found a contradiction! A^{-1} must not exist and A must not be invertible. 

Nice! We've connected the invertibility of a matrix with the linear independence of its columns.

It turns out that a LOT of things are connected:

Thm For a square matrix A , the following conditions are EQUIVALENT:

↳ every condition implies every other condition

(a) A is invertible

(b) A has linearly independent columns

↳ we previously proved
a connection between
these 2

(c) $A\vec{x} = \vec{b}$ has a unique solution

(d) A has a trivial nullspace

(e) A is full rank

} today

(f) The determinant of A is nonzero.

} Next 2
lectures

(g) 0 is not an eigenvalue of A

Sometimes this is called the "invertible matrix theorem".

- intuitively, I think of an invertible matrix A as a "good" or "not broken" matrix
- and this matrix says all the ways you can tell whether A is "good" or whether it is "broken":
 - linearly indep columns are "good"!
 - trivial nullspace is "good"!
 - ... etc.
- this isn't even all the conditions... the "invertible matrix theorem" page on Wolfram Alpha lists 23 equivalent conditions!

How can we use A^{-1} to solve $A\vec{x} = \vec{b}$?

Consider $\vec{x}_0 = A^{-1}\vec{b}$

Check: $A\vec{x}_0 = A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I\vec{b} = \vec{b}$

definition of inverse
matrix mult is associative
 $I \cdot \text{any } \vec{v} = \vec{v}$

so $A\vec{x}_0 = \vec{b}$ meaning $A^{-1}\vec{b}$ is a solution to $A\vec{x} = \vec{b}$!

- so if you ever need to solve an $A\vec{x} = \vec{b}$...

you can now just do $A^{-1}\vec{b}$.

(though we just learned that to compute A^{-1} you also do Gaussian elimination... but $A^{-1}\vec{b}$ is easy to tell a computer to do)

Ok. Let's go on and define vector spaces.

- vector spaces are an abstract linear algebra object that satisfy a long list of properties ("axioms")
- abstract linear algebra underlies all the linear algebra we care about in this class, but it is also not the FOCUS of this class

Def A vector space is a set of vectors \mathbb{V} , a set of scalars \mathbb{F} , and 2 operations that satisfy these properties:

[think of $\mathbb{V} = \mathbb{R}^n$, $\mathbb{F} = \mathbb{R}$]

First operation: vector addition

- must be associative and commutative
- *- additive identity exists ($\vec{0}$ exists such that $\vec{v} + \vec{0} = \vec{v}$)
- additive inverses exist (for any \vec{v} , there's a $-\vec{v}$ such that $\vec{v} + -\vec{v} = \vec{0}$)
- *- closure under vector addition:

If $\vec{v}_1, \vec{v}_2 \in \mathbb{V}$, Then $\vec{v}_1 + \vec{v}_2 \in \mathbb{V}$.

Second operation: scalar multiplication:

- associative
- distributive: $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$
and $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$
- multiplicative identity (1 exists such that $1 \cdot \vec{v} = \vec{v}$)
- *- closure under scalar multiplication: if $\vec{v} \in \mathbb{V}$ and $\alpha \in \mathbb{F}$, then $\alpha\vec{v} \in \mathbb{V}$.

In this class, the main vector space we care about is $(\mathbb{R}^n, \mathbb{R})$.

* we won't always write this.

Sometimes we may say " \mathbb{R}^n is a vector space", which implies \mathbb{R} is the associated set of scalars.

- you can check that $(\mathbb{R}^n, \mathbb{R})$ under the normal operations of vector addition and scalar multiplication satisfies all these properties and therefore is a vector space.
- I think of a vector space as a set of vectors that "behaves nicely" under the 2 operations
- it would be annoying to deal with a set of vectors where adding 2 vectors takes you out of the set!

The properties marked with \star are perhaps the most important / interesting ones

- because we might want to change up the set V but use the same \mathbb{F} and operations.

Ex. Is $V = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} ? \\ ? \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ a vector space?

- contains zero? yes ✓

- closure under scalar multiplication?

$$2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \notin V \quad \text{No } X$$

- can also check closure under vector addition:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} ? \\ ? \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix} \notin V \quad \text{No } X$$

Ex. Is $V = \text{span} \{ [1] \}$ a vector space?

- contains zero? yes. ✓

- closure under scalar multiplication?

If $\vec{v} \in V$, it has the form $\begin{bmatrix} c \\ c \end{bmatrix}$ for some $c \in \mathbb{R}$.

Then is $\alpha\vec{v} \in V$?

$$\alpha\vec{v} = \begin{bmatrix} \alpha c \\ \alpha c \end{bmatrix} \in V \quad \text{yes. ✓}$$

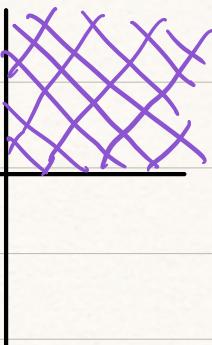
- closure under vector addition?

If $\vec{v}_1, \vec{v}_2 \in V$, is $\vec{v}_1 + \vec{v}_2 \in V$?

$$\begin{bmatrix} c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} c_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 + c_2 \end{bmatrix} \quad c_1, c_2 \in \mathbb{R}$$

$$\text{Then } \vec{v}_1 + \vec{v}_2 = \begin{bmatrix} c_1 + c_2 \\ c_1 + c_2 \end{bmatrix} \in V \quad \text{yes ✓}$$

Is this a vector space?



i.e. all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$
where $x \geq 0, y \geq 0$.

- contains zero? yes. ✓

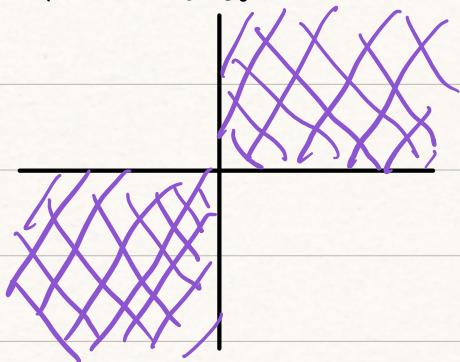
- closure under scalar multiplication?

- yes for positive scalars

- ... but not for negative ones. NO X

- also, additive inverses don't exist.

What about



- closed under scalar multiplication
now

- but closure under vector addition doesn't:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \notin V$$

NO

Def A subspace U of a vector space V is a subset of V that is still a vector space.

- you only need to check:

1. $\vec{0} \in U$

2. Closure under scalar multiplication

3. Closure under vector addition.

Every vector space can be represented by a basis.

Def $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for a vector space V if

1. $\text{span } \{\vec{v}_1, \dots, \vec{v}_n\} = V$

2. $\{\vec{v}_1, \dots, \vec{v}_n\}$ are linearly independent.

A basis is the MINIMUM set of vectors that still span V

Ex. $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ is a basis for \mathbb{R}^2 .

- spans \mathbb{R}^2 and is linearly independent.

Ex. Is $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ a basis for \mathbb{R}^2 ?

No, because linearly dependent.

Intuitively, linear dependence implies redundancy, and a basis cannot contain any "extra" vectors.

Q. Is $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ the only basis for \mathbb{R}^2 ?

No. We said any 2 linearly independent vectors can span \mathbb{R}^2 . So any of those are also bases!

- as long as every vector in \mathbb{R}^2 can be written as a linear combination of the vectors in the basis
 - like the "building blocks" of \mathbb{R}^2

Q. How many vectors are in a basis of \mathbb{R}^3 ?

A. 3. Need 3 linearly indep vectors to span \mathbb{R}^3 , just as you need 2 for \mathbb{R}^2 .

Even though the basis for a certain vector space is not unique, the NUMBER of vectors in each set stays the same. This is called the dimension of the space:

Def The dimension of a vector space V is the number of vectors in any basis of V .

- we write $\dim(V)$.

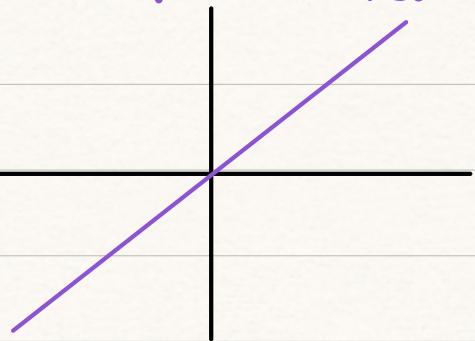
Ex. $\dim(\mathbb{R}^2) = ?$ 2. $\dim(\mathbb{R}^n) = ?$ n.

Ex. $\dim (\text{span} \{ [1] \})$

First find a basis for $\text{span} \{ [1] \}$:

$\{ [1] \}$ is a basis.

Number of vectors in basis? $1 = \dim$



this space is represented by a line: a "one-dimensional" space

Ex. $\dim (\text{span} \{ [1], [0] \})$

Try it yourself.

These vectors span the xy plane in 3D space.

$\{ [1], [0] \}$ is a basis for the spanned space.

$$\rightarrow \dim = 2.$$

Graphically, it's a "2-dimensional plane"

What about $\dim (\text{span} \{ [1], [0], [1] \})$?

- these vectors span \mathbb{R}^2 , so $\dim = 2$.

- $\{ [1], [0] \}$ is a basis for \mathbb{R}^2 . Last vector redundant.

Careful: $[1]$ and $[0]$ are "3 dimensional" vectors,
i.e. they are in \mathbb{R}^3 . That's different than the "dimension"
of the space they span.

Okay. Let's go back to matrices and define some vector spaces related to them.

My plan is to give you definitions and mechanically how to compute stuff today.

- tomorrow, I will fill in more of the intuitive understanding, so be patient if these definitions feel arbitrary.

Def The column space of a matrix A is the span of its columns. Written $\text{Col}(A)$.

Recall: $\text{span}\{\text{cols of } A\}$ was relevant to if $\vec{Ax} = \vec{b}$ had a solution.

Ex. What is the column space of $\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$?

$$\text{Col}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right\} = \mathbb{R}^2$$

$$\dim(\text{Col}(A)) = 2$$

Ex. $\text{Col}\left(\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}\right)$?

$$\text{Col}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}\right\} = \mathbb{R}^2$$

$$\dim(\text{Col}(A)) = 2$$

Notice: dimension of column space can't exceed number of columns.

Ex. $\text{Col}\left(\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}\right)$?

$$\text{Col}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

$$\dim(\text{Col}(A)) = 1$$

tinyurl.com/1bacolspace

Def The null space of a matrix A is the set of vectors \vec{x} that satisfy $A\vec{x} = \vec{0}$. It is written $N(A)$.

Let's look at the equation $A\vec{x} = \vec{0}$.

like $A\vec{x} = \vec{b}$ but $\vec{b} = \vec{0}$.

Notice that $\vec{x} = \vec{0}$ is ALWAYS a solution.

But for some matrices, there are nonzero \vec{x} that satisfy this too!

$$\text{Ex. } \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 - 2x_2 = 0 \rightarrow x_1 = 2x_2$$

So $A\vec{x} = \vec{0}$ is satisfied by $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \dots$

$$\rightarrow N(A) = \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

This is different from how scalars behave!

If $ab = 0$, either $a=0$ or $b=0$.

If $A\vec{x} = \vec{0}$, neither A nor \vec{x} have to be zero!

$$\text{Ex. } \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{What is the null space?}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \quad \begin{array}{l} \text{already in RREF} \\ \text{pivots} \\ x_3 \text{ is free} \end{array}$$

Nullspace is the set of solutions to $A\vec{x} = \vec{0}$.

Let's set our free variable: $x_3 = t$

$$x_1 + x_3 = 0 \rightarrow \begin{cases} x_1 = -t \\ x_2 = 0 \\ x_3 = t \end{cases}$$

$$\rightarrow N(A) = \text{span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Notice: $\dim \{N(A)\} = \text{number of free variables}$
when solving $A\vec{x} = \vec{0}$.

Def A matrix whose nullspace only contains $\vec{0}$
is said to have a trivial nullspace.

If $A\vec{x} = \vec{0}$ is satisfied by nonzero \vec{x} , then we say A
has a nontivial nullspace.

Turns out: if A has a nontivial nullspace, it is
not invertible!

- sending lots of vectors to $\vec{0}$ is not "good"

We learned a LOT of words today:

- vector space
- subspace
- basis
- dimension
- column space
- null space

I would recommend going back
and considering whether you
understand all the definitions
of these words.