Lecture 2D Thurs 6/29

Today:

1. A second definition of linear dependence and a proof that it's equivalent to the first definition
2. A proof of a theorem that says you can tell if the system you design will have a unique solution without taking any measurements!
3. Viewing matrices as transformations

We had one definition of linear dependence last time, but I'm going to give a second equivalent one.

- because turns out this one is sometimes easier to work with!
- and then we can prove their equivalence

Def 2 A set of vectors $\{\vec{v}_1, \ldots \vec{v}_n\}$ is linearly dependent if $\alpha_1 \vec{v}_1 + \ldots + \alpha_n \vec{v}_n = \vec{0}$ for some $\alpha_1, \ldots \alpha_n \in \mathbb{R}$ where not all the α_i's are zero.

- there is a "nontrivial" linear combo of $\vec{v}_n$'s that add to $\vec{0}$.
  ↳ not all coeffs zero
- it's not immediately clear this is equivalent! let's prove it.

To show the 2 definitions are equivalent, we need to prove 2 directions:

1. If Def 1. holds for some $\{\vec{v}_1, \dots \vec{v}_n\}$ then Def 2. holds.
2. Vice versa.

$1 \Rightarrow 2$:

P: For the set of vectors $\{\vec{v}_1, \dots \vec{v}_n\}$,

$$\vec{v}_i = \sum_{j \neq i} \alpha_j \vec{v}_j \quad \text{for some } i.$$

Q: There exists scalars $\beta_1, \dots \beta_n$, not all zero, such that

$$\beta_1 \vec{v}_1 + \dots + \beta_n \vec{v}_n = \vec{0}$$

Strategy: Manipulate the equation in P to look like the equation in Q.

Let's start with P and try to to create a $\vec{0}$.

Move $\vec{v}_i$ to RHS: $\vec{0} = \underbrace{-1}_{\neq 0} \cdot \vec{v}_i + \sum_{j \neq i} \alpha_j \vec{v}_j$

We found a nontrivial linear combo of the vectors that adds up to $\vec{0}$! $\Rightarrow$ Def. 2 holds. ■

- another way to think about this: we FOUND some $\beta_1, \dots \beta$ that satisfies the equation in Q: $\beta_i = -1$, $\beta_{j \neq i} = \alpha_j$

$2 \Rightarrow 1$:

P: There exists $\alpha_1, \ldots \alpha_n \in \mathbb{R}$, not all zero, such that
$$\alpha_1 \vec{v}_1 + \ldots + \alpha_n \vec{v}_n = \vec{0}$$
Q: One of the vectors is a linear combo of the others:
there exists an index i such that $\vec{v}_i = \sum_{j \neq i} \beta_j \vec{v}_j$

Strategy: again, manipulate P to look like Q.

"Without loss of generality", let's say $\alpha_1 \neq 0$.

↳ If $\alpha_1 = 0$, we could just reorder and rename the vectors so α_1 is one that is not zero.

From here, try to complete the proof on your own!

- start with the equation in P and manipulate it to fit into the form of Q!

Manipulate:
$$\alpha_1 \vec{v}_1 = -\alpha_2 \vec{v}_2 - \ldots - \alpha_n \vec{v}_n$$
$$\vec{v}_1 = -\frac{\alpha_2}{\alpha_1} \vec{v}_2 - \ldots - \frac{\alpha_n}{\alpha_1} \vec{v}_n$$

careful: allowed because $\alpha_1 \neq 0$!

We wrote one vector as a linear combo of the others!

$\Rightarrow$ Def. 1 also holds for this set of vectors. ∎

- another way to think about it: we FOUND the index i and the coefficients β to satisfy the eq: $i=1$, $\beta_j = -\frac{\alpha_j}{\alpha_1}$

Now that we've proved that the 2 definitions of linear dependence are equivalent, we can use either of them!

- Def. 2 is often easier to work with in proofs.

One more proof example. We're going to use a proof technique called "proof by contradiction".

Say you want to prove: Unicorns don't exist.

- One strategy could be:
  1. Assume unicorns DO exist.
  2. Show that making that assumption leads to a contradiction: something you know is false.
  3. Then you've shown the assumption CANNOT be true!
- that's pretty clever, right?

This is often a useful strategy when you need to prove something DOESN'T exist, like in the next proof.

Recall our motivating tomography example: how can we design a system to take a set of measurements that we know will be solveable?

- it would be great if we could figure that out without having to take any measurements!
- this next theorem will give us a very nice mathematical condition on the A matrix (without knowing $\vec{b}$)!

"theorem": something important we want to prove

Thm If the columns of the matrix A are linearly dependent, then $A\vec{x} = \vec{b}$ does not have a unique solution.

(either no solution or ∞ solutions)

P: what we know: cols of A linearly dependent

Let $A = \begin{bmatrix} | & & | \\ \vec{a_1} & \cdots & \vec{a_n} \\ | & & | \end{bmatrix}$ give some names to variables

Using Def. 2 of linear dependence,

there exist scalars $\alpha_1, \ldots \alpha_n$, not all zero,

such that $\alpha_1 \vec{a_1} + \ldots + \alpha_n \vec{a_n} = \vec{0}$

Q: what we want to prove: $A\vec{x} = \vec{b}$ has no unique solution.

- but we are going to use proof by contradiction:

ASSUME $A\vec{x} = \vec{b}$ does have a unique solution.

- let's call that solution $\vec{x}^*$
- it's a solution so we know $A\vec{x}^* = \vec{b}$
- P and this assumption are what we know.

Start with equation in P:

$\alpha_1 \vec{a_1} + \ldots + \alpha_n \vec{x_n} = \vec{0}$ for some $\alpha_1, \ldots \alpha_n$ not all zero

Rewrite this as a matrix-vector multiplication.

$$\rightarrow \begin{bmatrix} | & & | \\ \vec{a_1} & \cdots & \vec{a_n} \\ | & & | \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A \cdot \vec{\alpha} = \vec{0}$$

- the A matrix takes $\vec{\alpha}$ to $\vec{0}$!
  - $\vec{\alpha}$ is "invisible" to this matrix

Now we assumed x^* was the unique solution to $A\vec{x}=\vec{b}$.

- Consider $x^* + \vec{\alpha}$:

$$A(x^* + \vec{\alpha}) = A\vec{x}^* + \underbrace{A\vec{\alpha}}_{=\vec{0}!} = A\vec{x}^* = \vec{b}$$

That means $\vec{x}^* + \vec{\alpha}$ is also a solution to $A\vec{x}=\vec{b}$!

- and we know $\vec{\alpha} \neq \vec{0}$ so $\vec{x}^* + \vec{\alpha}$ is a distinct solution

Therefore, $\vec{x}^*$ is NOT a unique solution, violating our assumption.

- we started by assuming something was true, and that led us to conclude that the something was false.
- it can't be BOTH true and false! so it's a contradiction!

Our assumption led to a contradiction, so it must be false!

$\Rightarrow$ $A\vec{x}=\vec{b}$ must have no unique solution after all. ∎

how exciting. take some time to digest the logic of the proof, make sure you follow all the steps.

To reiterate, this theorem says that if we design our system to take some measurements (represented in A), we can just check if the columns of A are linearly dependent. If they are, we know there will be no unique solution, so we better try a different design!

Q. Is it true that if A has linearly INDEPENDENT columns, then $A\vec{x}=\vec{b}$ HAS a unique solution? That's NOT what we proved. But turns out IF A is square, then it is true. (imagine all tall A matrix)

tinyurl.com/16a lindep

ok let's move on from proofs for now.

We've been talking a lot about matrices representing a system of equations, i.e. $A\vec{x} = \vec{b}$.

- but let's return to another way to think about matrices (that we touched on briefly in the YouTube video)

Let's think of a matrix as a function (or "operator") that takes as input a vector and outputs another vector.

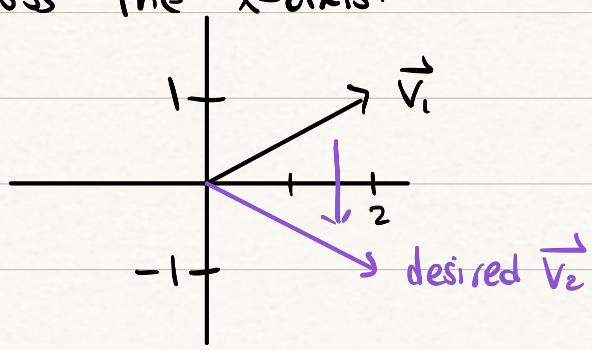
- It "transforms" the input vector into the output vector

Ex. $\underset{\text{input}}{A \vec{v_1}} = \underset{\text{output}}{\vec{v_2}}$

transformed into by A

Some matrices represent transformations that are very nice to visualize!

Ex. let's derive the 2×2 matrix that reflects vectors across the x-axis.



$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \underset{\vec{v_1}}{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}} = \underset{\vec{v_2}}{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}$$

Notice: x coordinate stays the same, y gets x-1.

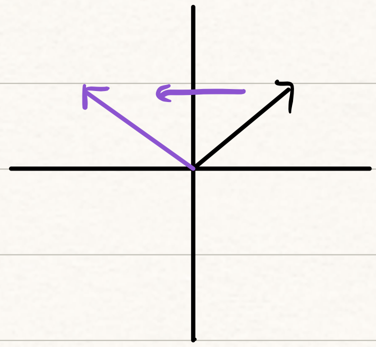
So should be $x_2 = x_1$, $y_2 = -y_1$.

- can you fill in the matrix?

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

(multiply it out to check!)

Q. What is the matrix that reflects vectors across y-axis?



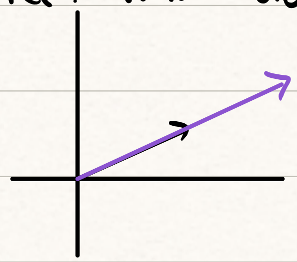
$x_2 = -x_1$

$y_2 = y_1$

In matrix vector form:

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Nice. What about a matrix that stretches all vectors 2x?



$x_2 = 2x_1$

$y_2 = 2y_1$

$\rightarrow \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

Let's consider a slightly more complicated transformation: a rotation!

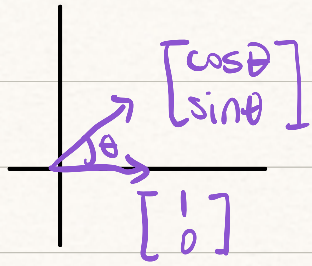
$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

turns out: this matrix will rotate any vector counter clockwise by an angle θ.

- you'll talk more about the derivation in dis, but let's apply it to a simple vector: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

- Picture:



Recall: points on the unit circle have coordinates $(\cos\theta, \sin\theta)$ where θ is the angle counter-clockwise from the x-axis.

So the matrix indeed rotated $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by θ!
(and didn't change its length)

What do we expect to happen if you apply the matrix multiple times?

$$R_\theta \cdot (R_\theta \vec{x})$$
② ①

Apply the matrix to $\vec{x}$, then apply it again to the first output.

- we expect it to rotate by 2θ, right? Let's check on $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ again.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

take output & apply R_θ again.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta \\ 2\sin\theta\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta) \end{bmatrix}$$

(it's okay if you don't remember trig identities)

In general, if you want to apply transform A_1 and then transform A_2, you should first do $A_1\vec{x}$, and then apply A_2 to the output of $A_2\vec{x}$.
- i.e. $A_2(A_1\vec{x})$

But what if we want a single matrix that does A_1 then A_2?
- $A_2(A_1\vec{x}) = (\underbrace{A_2 A_1})\vec{x}$

if we do this matrix-matrix multiply, we should get that matrix!

Careful! Notice that $A_2 A_1$ is the matrix that does A_1 first then A_2. (kind of backwards!)
- matrix-matrix multiplication is not commutative in general!

You will do a bunch more in discussion.
But the point of this is to build up your intuition of what a matrix is! (not just a box of numbers).

POPCORN time!