EECS 16A Designing Information Devices and Systems I
Summer 2023 Homework 3

This homework is due July 7th, 2023, at 23:59.
Self-grades are due July 14th, 2023, at 23:59.

Submission Format
Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned).

1. Prelab Questions

These questions pertain to the prelab reading for the Imaging 2 lab. You can find the reading under the Imaging 2 Lab section on the 'Schedule' page of the website. We do not expect in-depth answers for the questions. Please limit your answers to a maximum of 2 sentences.

(a) Briefly explain what the H matrix, $\vec{i}$ vector, and $\vec{s}$ vector each signify.

(b) How will we get the vector $\vec{i}$ from $\vec{s} = H\vec{i}$, the equation representing our imaging system?

2. Reading Assignment

For this homework, please read Notes 3, 4, 5, 6, 7, and 8. Note 3 provides an overview of linear dependence and span and Note 4 gives an introduction to thinking about and writing proofs. Note 5 provides an overview of multiplication of matrices with vectors, by considering the example of water reservoirs and water pumps. Note 6 introduces matrix inversion. Note 7 and Note 8 give an overview of matrix vector spaces and subspaces, as well as column spaces and nullspaces.

Please answer the following question:

(a) Why are there two definitions of linear dependence? What value does each definition provide?

(b) You have seen in Note 5 that the pump system can be represented by a state transition matrix. What constraint must this matrix satisfy in order for the pump system to obey water conservation?

(c) From Note 8, what are the three necessary properties for a vector space to be a *subspace*?

3. Linear Dependence

Learning Objectives: *Evaluate the linear dependency of a set of vectors.*

State if the following sets of vectors are linearly independent or dependent. If the set is linearly dependent, provide a linear combination of the vectors that sum to the zero vector.

(**Hint**: Consider using the given set of vectors to form the columns of a matrix A. What would it mean about the set of vectors if the equation $A\vec{x} = \vec{0}$ had a solution that was not just $\vec{x} = 0$? Recall the column view of matrix-vector multiplication and the definition of linear dependence.)

(a) $\left\{ \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

4. Linear Dependence in a Square Matrix

Learning Objective: *This is an opportunity to practice applying proof techniques. This question is specifically focused on linear dependence of rows and columns in a square matrix.*

Let A be a square $n \times n$ matrix, (i.e. both the columns and rows are vectors in $\mathbb{R}^n$). Suppose we are told that the columns of A are linearly dependent. Prove, then, that the rows of A must also be linearly dependent.

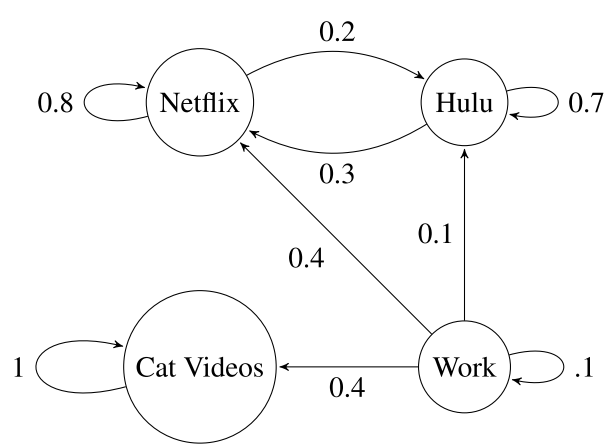
You can use the following conclusion in your proof:
If Gaussian elimination is applied to a matrix A, and the resulting matrix (in reduced row echelon form) has at least one row of all zeros, this means that the rows of A are linearly dependent.

(**Hint**: Can you use the linear dependence of the columns to say something about the number of solutions to $A\vec{x} = \vec{0}$? How does the number of solutions relate to the result of Gaussian elimination?)

5. Social Media

Learning Objective: *Practice setting up transition matrices from a diagram and understand how to compute subsequent states of the system.*

As a tech-savvy Berkeley student, the distractions of streaming services are always calling you away from productive stuff like homework for your classes. You're curious—are you the only one who spends hours switching between Netflix or Hulu? How do other students manage to get stuff done and balance staying up to date with the Bachelor? You conduct an experiment, collect some data, and notice Berkeley students tend to follow a pattern of behavior similar to the figure below. So, for example, if x = 100 students are on Netflix, in the next timestep, 20 (i.e., $0.2 \cdot x$) of them will click on a link and move to Hulu, and 80 (i.e, $0.8 \cdot x$) will remain on Netflix.



(a) Let us define $x_N[n]$ as the number of students on Netflix at time-step n; $x_H[n]$ as the number of students on Hulu at time-step n; $x_C[n]$ as the number of students watching any kind of cat video at time-step n; and $x_W[n]$ as the number of students working at time-step n.

Let the state vector be: $\vec{x}[n] = \begin{bmatrix} x_N[n] \\ x_H[n] \\ x_C[n] \\ x_W[n] \end{bmatrix}$. Derive the corresponding transition matrix $\mathbf{A}$.

Hint: A transition matrix, $\mathbf{A}$, is the matrix that transitions $\vec{x}[n]$, the vector at time-step n to $\vec{x}[n+1]$, the vector at time-step $n+1$. In other words: $\vec{x}[n+1] = \mathbf{A}\vec{x}[n]$.

(b) Assume that this class had 320 of you in total. Suppose on a given Friday evening (the day when HW is due), there are 110 EECS16A students on Netflix, 60 on Hulu, 10 watching Cat Videos, and 140 actually doing work. In the next timestep, how many people will be doing each activity? In other words, after you apply the matrix once to reach the next timestep, what is the state vector?

(c) Compute the sum of each column in the state transition matrix. What is the interpretation of this?

6. Inverse Transforms

Learning Objectives*: Matrices represent linear transformations, and their inverses (if they exist) represent the opposite transformation. Here we practice inversion, but are also looking to develop an intuition. Visualizing the transformations might help develop this intuition.*

For each of the following choices of matrix A:

i. Find the inverse, $\mathbf{A}^{-1}$, if it exists. If you find that the inverse does not exist, mention how you decided that. Solve this by hand.

ii. **For parts (a)-(b) only**, in addition to finding the inverse (if it exists), describe how the matrix $\mathbf{A}$ geometrically transforms an arbitrary vector $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in \mathbb{R}^2$.

For example, if $\mathbf{A} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 2x_0 \\ 2y_0 \end{bmatrix}$, then $\mathbf{A}$ could scale $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ by 2 to get $\begin{bmatrix} 2x_0 \\ 2y_0 \end{bmatrix}$. If $\mathbf{A} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ -y_0 \end{bmatrix}$, then $\mathbf{A}$ could reflect $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ across the x-axis, etc. *Hint: It may help to plot a few examples to recognize the pattern.*

iii. **Again, for parts (a)-(b) only**, if we use $\mathbf{A}$ to geometrically transform $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ to get $\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, **is it possible to reverse the transformation geometrically,** i.e. is it possible to retrieve $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ from $\begin{bmatrix} u \\ v \end{bmatrix}$ geometrically?

(a) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

(d) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

(e) **(OPTIONAL)** $\mathbf{A} = \begin{bmatrix} 3 & 0 & -2 & 1 \\ 0 & 2 & 1 & 3 \\ 3 & 1 & 0 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Hint 1: What do the linear (in)dependence of the rows and columns tell us about the invertibility of a matrix?

Hint 2: We're reasonable people!

7. Image Stitching

Learning Objective: *This problem is similar to one that students might experience in an upper division computer vision course. Our goal is to give students a flavor of the power of tools from fundamental linear algebra and their wide range of applications.*

Often, when people take pictures of a large object, they are constrained by the field of vision of the camera. This means that they have two options to capture the entire object:

- Stand as far away as they need to include the entire object in the camera's field of view (clearly, we do not want to do this as it reduces the amount of detail in the image)
- (This is more exciting) Take several pictures of different parts of the object and stitch them together like a jigsaw puzzle.

We are going to explore the second option in this problem. Daniel, who is a professional photographer, wants to construct an image by using "image stitching". Unfortunately, Daniel took some of the pictures from different angles as well as from different positions and distances from the object. While processing these pictures, Daniel lost information about the positions and orientations from which the pictures were taken. Luckily, you and your friend Marcela, with your wealth of newly acquired knowledge about vectors and matrices, can help him!

You and Marcela are designing an iPhone app that stitches photographs together into one larger image. Marcela has already written an algorithm that finds common points in overlapping images. **It's your job to figure out how to stitch the images together using Marcela's common points to reconstruct the larger image.**

We will use vectors to represent the common points which are related by a affine transformation. Your idea is to find this affine transformation. For this you will use a single matrix, $\mathbf{R}$, and a vector, $\vec{t}$, that transforms every common point in one image to their corresponding point in the other image. Once you find $\mathbf{R}$ and $\vec{t}$ you will be able to transform one image so that it lines up with the other image.

Suppose $\vec{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$ is a point in one image , which is transformed to $\vec{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$, the corresponding point in the other image (i.e., they represent the same object in the scene). For example, Fig. 1 shows how the points $\vec{p}_1$, $\vec{p}_2$ ... in the right image are transformed to points $\vec{q}_1$, $\vec{q}_2$ ... on the left image. You write down the following relationship between $\vec{p}$ and $\vec{q}$.

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \underbrace{\begin{bmatrix} r_{xx} & r_{xy} \\ r_{yx} & r_{yy} \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\vec{t}} \tag{1}$$

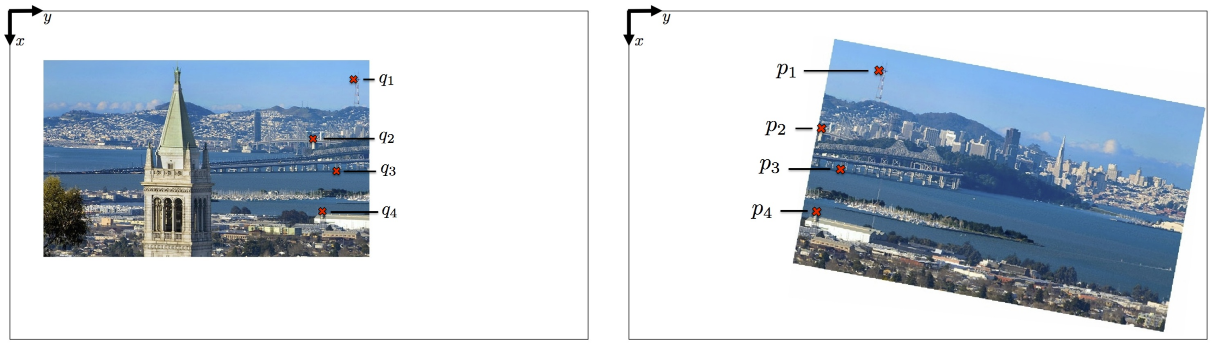


Figure 1: Two images to be stitched together with pairs of matching points labeled.

This problem focuses on finding the unknowns (i.e. the components of $\mathbf{R}$ and $\vec{t}$), so that you will be able to stitch the image together. *Note that this is the opposite from our usual setting in which we would solve for $\vec{p}$ given all other variables.*

(a) To understand how the matrix $\mathbf{R}$ and vector $\vec{t}$ transforms any vector representing a point on a image, Consider this example equation similar to Equation (1),

$$\vec{v} = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \vec{u} + \vec{w} = \vec{v}_1 + \vec{w}. \tag{2}$$

Use $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for this part.
We want to find out what geometric transformation(s) can be applied on $\vec{u}$ to give $\vec{v}$.

Step 1: Find out how $\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$ is transforming $\vec{u}$. Evaluate $\vec{v}_1 = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \vec{u}$.
What **geometric transformation(s)** might be applied to $\vec{u}$ to get $\vec{v}_1$? Choose the options that answers the question and explain your choice.

(i) Rotation
(ii) Scaling
(iii) Shifting/Translation

Drawing the vectors $\vec{u}$, and $\vec{v}_1$ in two dimensions on a single plot might help you to visualize the transformations.

Step 2: Find out $\vec{v} = \vec{v}_1 + \vec{w}$. Find out how **addition of $\vec{w}$ is geometrically transforming** $\vec{v}_1$. Choose the option that answers the question and explain your choice.

(i) Rotation
(ii) Scaling
(iii) Shifting/Translation

Drawing the vectors $\vec{v}$, $\vec{w}$, and $\vec{v}_1$ in two dimensions on a single plot might help you to visualize the transformations.

(b) Now back to the main problem. First, multiply Equation (1) out into **two equations**.

(i) What are the known values and what are the unknown values in each equation (recall what we are trying to solve for in Equation (1))?

(ii) How many unknown values are there?

(iii) How many independent equations do you need to solve for all the unknowns?

(iv) How many pairs of common points $\vec{p}$ and $\vec{q}$ will you need in order to write down a system of equations that you can use to solve for the unknowns? ***Hint**: Remember that each pair of $\vec{p}$ and $\vec{q}$ is related by two equations, one for each coordinate.*

(c) Use what you learned in the above two subparts to explicitly write out **just enough** equations of these transformations as you need to solve the system. Assume that the four pairs of points from Fig. 1 are labeled as:

$$\vec{q}_1 = \begin{bmatrix} q_{1x} \\ q_{1y} \end{bmatrix}, \vec{p}_1 = \begin{bmatrix} p_{1x} \\ p_{1y} \end{bmatrix} \qquad \vec{q}_2 = \begin{bmatrix} q_{2x} \\ q_{2y} \end{bmatrix}, \vec{p}_2 = \begin{bmatrix} p_{2x} \\ p_{2y} \end{bmatrix} \qquad \vec{q}_3 = \begin{bmatrix} q_{3x} \\ q_{3y} \end{bmatrix}, \vec{p}_3 = \begin{bmatrix} p_{3x} \\ p_{3y} \end{bmatrix} \qquad \vec{q}_4 = \begin{bmatrix} q_{4x} \\ q_{4y} \end{bmatrix}, \vec{p}_4 = \begin{bmatrix} p_{4x} \\ p_{4y} \end{bmatrix}.$$

(d) Remember that we are ultimately solving for the components of the **R** matrix and the vector $\vec{t}$. This is different from our usual setting and so we need to reformulate the problem into something we are more used to (i.e., $A\vec{x} = \vec{b}$ where x is the unknown). In order to do this, let's view the components of **R** and $\vec{t}$ as the unknowns in the equations from from part c). We then have a system of linear equations which we should be able to write in the familiar matrix-vector form. Specifically, we can store the unknowns in a vector $\vec{\alpha} = \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{yx} \\ r_{yy} \\ t_x \\ t_y \end{bmatrix}$ and specify 6×6 matrix **A** and vector $\vec{b}$ such that $A\vec{\alpha} = \vec{b}$. Please write out the entries of **A** and $\vec{b}$ to match your equations from part c). To get you started, we provide the first row of **A** and first entry of $\vec{b}$ which corresponds to one possible equation from part c):

$$\begin{bmatrix} p_{1x} & p_{1y} & 0 & 0 & 1 & 0 \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{yx} \\ r_{yy} \\ t_x \\ t_y \end{bmatrix} = \begin{bmatrix} q_{1x} \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Your job is the fill in the remaining entries according to the other equations.

(e) In the IPython notebook `prob4.ipynb`, you will have a chance to test out your solution. Plug in the values that you are given for p_x, p_y, q_x, and q_y for each pair of points into your system of equations to solve for the matrix, **R**, and vector, $\vec{t}$. The notebook will solve the system of equations, apply your transformation to the second image, and show you if your stitching algorithm works. **You are NOT responsible for understanding the image stitching code or Marcela's algorithm.** What are the values for **R** and $\vec{t}$ which correctly stitch the images together?

8. Subspaces, Bases and Dimension

***Learning Objective**: Explore how to recognize and show if a subset of a vector space is or is not a subspace. Further practice identifying a basis for (i.e., a minimal set of vectors which span) an arbitrary subspace.*

For each of the sets $\mathbb{U}$ (which are *subsets* of $\mathbb{R}^3$) defined below, state whether $\mathbb{U}$ is a *subspace* of $\mathbb{R}^3$ or not. If $\mathbb{U}$ is a subspace, find a basis for it and state the dimension.

Note:

- To show $\mathbb{U}$ is a subspace, you have to show that all three properties of a subspace hold.
- To show $\mathbb{U}$ is not a subspace, you only have to show at least one property of a subspace does not hold.

(a) $\mathbb{U} = \left\{ \begin{bmatrix} 2(x+y) \\ x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

(b) $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ z+1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$

(c) $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ x+1 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

9. Finding Null Spaces and Column Spaces

Learning Objectives*: Null spaces and column spaces are two fundamental vector spaces associated with matrices and they describe important attributes of the transformations that these matrices represent. This problem explores how to find and express these spaces.*

Definition (Null space): The null space of a matrix, $\mathbf{A} \in \mathbb{R}^{m\times n}$, is the set of all vectors $\vec{x} \in \mathbb{R}^n$ such that $\mathbf{A}\vec{x} = \vec{0}$. The null space is notated as $Null(\mathbf{A})$ and the definition can be written in set notation as:

$$Null(\mathbf{A}) = \{\vec{x} \mid \mathbf{A}\vec{x} = \vec{0}, \vec{x} \in \mathbb{R}^n\}$$

Definition (Column space): The column space of a matrix, $\mathbf{A} \in \mathbb{R}^{m\times n}$, is the set of all vectors $\mathbf{A}\vec{x} \in \mathbb{R}^m$ for all choices of $\vec{x} \in \mathbb{R}^n$. Equivalently, it is also the span of the column vector of $\mathbf{A}$. The column space can be notated as $Col(\mathbf{A})$or $Range(\mathbf{A})$ and the definition can be written in set notation as:

$$Col(\mathbf{A}) = \{\mathbf{A}\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

Definition (Dimension): The dimension of a vector space is the number of basis vectors — i.e. the minimum number of vectors required to span the vector space.

(a) Consider a matrix $\mathbf{A} \in \mathbb{R}^{3\times 5}$. What is the maximum possible number of linearly independent column vectors (i.e. the maximum possible dimension) of $Col(\mathbf{A})$?

(b) You are given the following matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a *minimum* set of vectors that span $Col(\mathbf{A})$ (i.e. a basis for $Col(\mathbf{A})$). (This problem does not have a unique answer, since you can choose many different sets of vectors that fit the description here.) What is the dimension of $Col(\mathbf{A})$?

Hint: You can do this problem by observation. Alternatively, use Gaussian Elimination on the matrix to identify how many columns of the matrix are linearly independent. The columns with pivots (leading ones) in them correspond to the columns in the original matrix that are linearly independent.

(c) Find a *minimum* set of vectors that span $Null(\mathbf{A})$ (i.e. a basis for $Null(\mathbf{A})$), where $\mathbf{A}$ is the same matrix as in part (b). What is the dimension of $Null(\mathbf{A})$?

(d) For the following matrix $\mathbf{D}$, find $Col(\mathbf{D})$ and its dimension, and $Null(\mathbf{D})$ and its dimension. Using inspection or Gaussian elimination are both valid methods to solve the problem.

$$\mathbf{D} = \begin{bmatrix} 1 & -1 & -3 & 4 \\ 3 & -3 & -5 & 8 \\ 1 & -1 & -1 & 2 \end{bmatrix}$$

(e) Find the sum of the dimensions of $Null(\mathbf{A})$ and $Col(\mathbf{A})$. Also find the sum of the dimensions of $Null(\mathbf{D})$ and $Col(\mathbf{D})$. What do you notice about these sums in relation to the dimensions of $\mathbf{A}$ and $\mathbf{D}$, respectively?

10. Prelab Questions

These questions pertain to the prelab reading for the Imaging 3 lab. You can find the reading under the Imaging 3 Lab section on the 'Schedule' page of the website. We do not expect in-depth answers for the questions. Please limit your answers to a maximum of 2 sentences.

(a) What properties does the mask matrix H need to have for us to reconstruct the image?

(b) Briefly describe why averaging multiple signals/measurements is a good idea.

(c) What is the new equation that models our system?

(d) How do we get the image back from the new equation that models our system? Note that your answer cannot contain the image vector, $\vec{i}$ (since it is an unknown). You may, however, give an answer that includes $\vec{i}_{est}$.

(e) What term allows us to control the effect of noise in our system? *Hint*: Look at the terms in the equation that contains $\vec{i}_{est}$.

11. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.