Lecture 2C   Wed 6/28

Today:

- $A\vec{x} = \vec{b}$ as matrix-vector multiplication
- Define: span, linear dependence
- intro to PROOFS
- proof examples

Let's see a specific example of matrix-vector multiplication, which turns out is systems of equations (again)!

$$\begin{cases} 2x_1 + 3x_2 = 8 \\ 3x_1 - x_2 = 1 \end{cases}$$

We saw augmented matrix form: $\left[\begin{array}{cc|c} 2 & 3 & 8 \\ 3 & -1 & 1 \end{array}\right]$

Another form:

$$\underbrace{\begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 8 \\ 1 \end{bmatrix}}_{\vec{b}}$$

"matrix-vector form"

Each row of A represents one equation! "row view"

- each row represents how much each measurement is influenced by the different variables

What if we apply column view of matrix-vector multiplication?

Recall: $A\vec{x} = \begin{bmatrix} | & & | \\ \vec{a}_1 & \dots & \vec{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + \dots + x_n\vec{a}_n$

"linear combination" of cols of A

"weights" of the linear combo

Quick check: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 2\cdot -1 \\ 3\cdot 1 + 4\cdot -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

Col view: $1\cdot\begin{bmatrix} 1 \\ 3 \end{bmatrix} + -1\cdot\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ ✓

Two ways to matrix multiply are equivalent!

Apply column view of $A\vec{x}$ to our examples:

$$A\vec{x} = x_1\begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

each col = how much x_1 influences all the measurements

- imagine if one of the columns was all zero!
  - then nothing is measured about that variable
  - seems like a bad design that could lead to no unique solution!

It turns out that just looking at the columns of A will tell us if $A\vec{x} = \vec{b}$ has a unique solution!
- let's build up to that.

foreshadowing

If $A\vec{x} = x_1\vec{a_1} + \dots + x_n\vec{a_n} = \vec{b}$,

then the problem of "Find a solution $\vec{x}$ to $A\vec{x} = \vec{b}$."
is equivalent to "Find a linear combination of the columns of A that equals $\vec{b}$"

↳ find the weights $x_1, \dots x_n$

that's interesting.

Hold that thought. Let's define some terms.

$\boxed{\text{Def.}}$ The span of a set of vectors $\{\vec{v}_1, \dots \vec{v}_n\}$ is the set of ALL linear combinations of the vectors.

- math notation:

$$\text{span}\{\vec{v}_1, \dots \vec{v}_n\} = \{\alpha_1\vec{v}_1 + \dots + \alpha_n\vec{v}_n \mid \alpha_1, \dots \alpha_n \in \mathbb{R}\}$$

make a lin combo $\Leftarrow$ for any set of scalars

Ex 1. span $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$

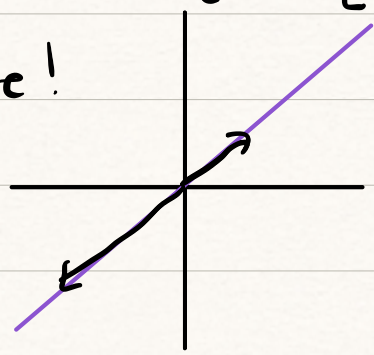
= set of all linear combos of $\begin{bmatrix}1\\1\end{bmatrix}$

nothing to add together, so just scalar multiples of $\begin{bmatrix}1\\1\end{bmatrix}$

e.g. $\left\{\begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}3\\3\end{bmatrix}, \begin{bmatrix}-0.5\\-0.5\end{bmatrix}, \begin{bmatrix}\sqrt{2}\\\sqrt{2}\end{bmatrix}, \dots\right\}$

math notation: $\left\{\alpha\begin{bmatrix}1\\1\end{bmatrix} \mid \alpha \in \mathbb{R}\right\}$

In a picture!



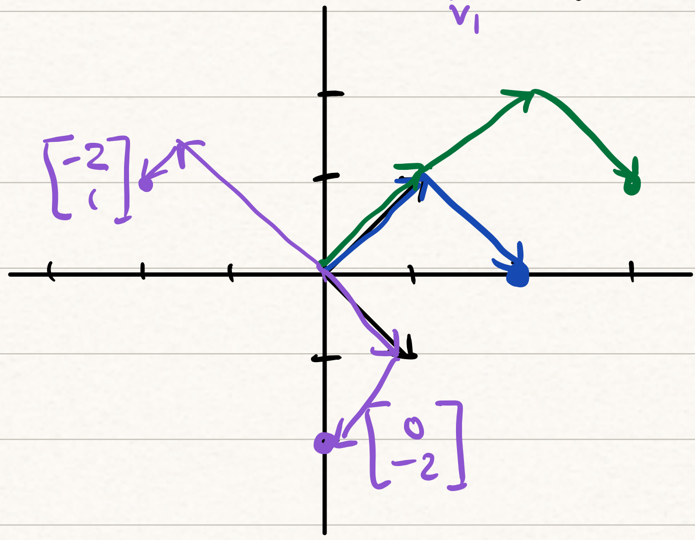
all vectors on this line!

Intuitively, span is about where you can "reach" using linear combos of the vectors.

- here, you can take steps forward or back in the direction of $\begin{bmatrix}1\\1\end{bmatrix}$, as many steps as you want

- and you can REACH anywhere on this line
- but you can't reach anywhere else!

Ex 2. span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$
$\vec{v}_1$ $\vec{v}_2$



$\left\{ \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \middle| \alpha_1, \alpha_2 \in \mathbb{R} \right\}$

Some linear combos:

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Q. How would you reach $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$ or $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$?
- try figuring it out on the graph

$\begin{bmatrix} 0 \\ -2 \end{bmatrix} = \vec{v}_2 - \vec{v}_1$

$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = -1.5 \vec{v}_2 - \vec{v}_1$

Q. Where is everywhere you can reach? (the span!)
A. everywhere!

span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} = \mathbb{R}^2$

Ex 3. $\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$

Where can you go using these 2 vectors?

- still the line along $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So $\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Notice: $\vec{v}_2 = \alpha \vec{v}_1$

- $\vec{v}_2$ is "redundant"

Ex 4. $\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$

Where can we reach? $\mathbb{R}^2$ still.

"redundant"

$\vec{v}_3 \in \text{span}\{\vec{v}_1, \vec{v}_2\}$ already
"$\mathbb{R}^2$"

- doesn't let you reach anywhere new

Formal definition of this notion of redundancy:

$\boxed{\text{Def.}}$ A set of vectors $\{\vec{v}_1, \dots \vec{v}_n\}$ is called linearly dependent if one of the vectors is in the span of the others.

- math: for some i, $\vec{v}_i = \sum_{j \neq i} \alpha_j \vec{v}_j$

Σ "sigma notation"

- sum over all $j = 1$ to n, not including $j = i$

tinyurl.com/16aspan

Okay. it's time to talk about proofs! We will use some of the concepts we just learned in examples.

What is a proof ??

- It is a rigorous argument

  Given some assumptions $\Rightarrow$ show some conclusion is true.

- Proofs form the backbone of math!
  - how do we know anything is true in math? nothing is true unless we PROVE it.
- which is a pretty cool concept - we can build up knowledge that we are really sure is true!
  - but it's also a new way of thinking, almost a new language
  - it will take time to expose yourself and learn the language!
  - it can be intimidating at first, but today I'd like to give you a helpful guide + some tips.
  - and as you practice, soon you'll be a wizard!

Steps:

1. Identify the implication. has a direction

   IF P THEN Q. We also write $P \Rightarrow Q$

   - ex. IF $\underbrace{\text{I like cats}}_{P}$, THEN $\underbrace{\text{I like dogs}}_{Q}$.

- is not the same as

IF $\underbrace{\text{I like dogs}}_{P}$, THEN $\underbrace{\text{I like cats}}_{Q}$.

- sometimes it's less clear: Everyone who likes cats likes dogs.

  What is P? What is Q?

  - IF someone likes cats, THEN they like dogs.

- another example: I like cats only if I like dogs.

  Figure out yourself what P and Q are!

  - If I like cats, do I like dogs? maybe.

    If $\underbrace{\text{I like dogs}}_{P}$, do $\underbrace{\text{I like cats}}_{Q}$? YES always

- so as you can see it can be hard to figure out precisely what you're proving! but you must before you start!

2. Write everything you know (P) in math.
   - write down any relevant definitions.
3. Do the same for what you want to prove (Q).
4. Try to go from P to Q!
   - make sure every step is justified
   - can also manipulate Q to get closer to P

Some tips:

- try smaller example first, e.g. 2x2 matrix instead of general nxn matrix
- there are always many different approaches!
- don't skip steps 1-3!
- practice practice practice!

Ex 1. Recall: we claimed "$A\vec{x} = \vec{b}$ has a solution" is the same as "$\vec{b}$ can be written as a linear combination of the columns of A" (or in other words, $\vec{b} \in$ span {cols of A})

- can we prove it?
  - showing "equivalence" actually involves 2 directions!

    Statement 1 ⇒ Statement 2

    AND Statement 2 ⇒ Statement 1

    means Statement 1 and 2 are equivalent.
- let's do one direction first:

PROVE: IF $\vec{b} \in$ span {columns of A},

THEN $A\vec{x} = \vec{b}$ has a solution. (not necessarily unique)

- what is P? $\vec{b} \in$ span {cols of A}
  - how to write with math? name some variables!

Let $A = \begin{bmatrix} | & & | \\ \vec{a_1} & \cdots & \vec{a_n} \\ | & & | \end{bmatrix}$

$\vec{b} = \alpha_1 \vec{a_1} + \ldots + \alpha_n \vec{a_n}$ for some constants $\alpha_1, \ldots \alpha_n$.

- what is Q? $A\vec{x} = \vec{b}$ has a solution

  Math? There is some $\vec{x}$ such that $A\vec{x} = \vec{b}$.

- where should we go?
  - P is in terms of cols of A, Q is in terms of just A.
  - lets make P look more like Q.

Using the column view of matrix-vector multiplication,

$$\vec{b} = \alpha_1 \vec{a}_1 + \ldots + \alpha_n \vec{a}_n = A \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

look!

we found a solution $\vec{x}$ to $A\vec{x} = \vec{b}$! We did it!

$\blacksquare$

We draw this square to mean we proved it!

Can we prove the other direction?

PROVE: IF $A\vec{x} = \vec{b}$ has a solution,

THEN $\vec{b} \in \text{span}\{\text{cols of } A\}$

Try it yourself!

- P: There is an $\vec{x}$ such that $A\vec{x} = \vec{b}$.
- Q: $\vec{b} = \alpha_1 \vec{a}_1 + \ldots + \alpha_n \vec{a}_n$ for some $\alpha_1, \ldots \alpha_n$.
- Starting with P, $A\vec{x} = \vec{b}$ can also be written as

$$x_1 \vec{a}_1 + \ldots + x_n \vec{a}_n = \vec{b}$$

We found some coeffs to get a linear combo of cols of A that equals $\vec{b}$!

$\blacksquare$