
EECS 16A Designing Information Devices and Systems I

Summer 2023 Homework 3

This homework is due July 7th, 2023, at 23:59.

Self-grades are due July 14th, 2023, at 23:59.

Submission Format

Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned).

1. Prelab Questions

These questions pertain to the prelab reading for the Imaging 2 lab. You can find the reading under the Imaging 2 Lab section on the ‘Schedule’ page of the website. We do not expect in-depth answers for the questions. Please limit your answers to a maximum of 2 sentences.

- (a) Briefly explain what the H matrix, \vec{i} vector, and \vec{s} vector each signify.
- (b) How will we get the vector \vec{i} from $\vec{s} = H\vec{i}$, the equation representing our imaging system?

Solution:

- (a) The matrix H is also known as the mask matrix. It allows us to selectively choose what pixels we want to read (scan) at a given time.
The vector \vec{i} represents the image that we are trying to reconstruct.
The vector \vec{s} represents the scan of the image.
- (b) $\vec{i} = H^{-1}\vec{s}$. Multiplying by H^{-1} on both sides of the equation gives us the vector \vec{i} .

2. Reading Assignment

For this homework, please read Notes 3, 4, 5, 6, 7, and 8. [Note 3](#) provides an overview of linear dependence and span and [Note 4](#) gives an introduction to thinking about and writing proofs. [Note 5](#) provides an overview of multiplication of matrices with vectors, by considering the example of water reservoirs and water pumps. [Note 6](#) introduces matrix inversion. [Note 7](#) and [Note 8](#) give an overview of matrix vector spaces and subspaces, as well as column spaces and nullspaces.

Please answer the following question:

- (a) Why are there two definitions of linear dependence? What value does each definition provide?
- (b) You have seen in Note 5 that the pump system can be represented by a state transition matrix. What constraint must this matrix satisfy in order for the pump system to obey water conservation?
- (c) From Note 8, what are the three necessary properties for a vector space to be a *subspace*?

Solution:

- (a) See Note 3.1 for the two definitions. Definition (I) is more useful for mathematically proving linear dependence while Definition (II) provides a more intuitive understanding of linear dependence and formalizes the notion of redundancy.

- (b) Each column in the state transition matrix must sum to one.
- (c) See Note 8 for definition of a subspace
 - i. Contains the zero vector: $\vec{0} \in \mathbb{U}$.
 - ii. Closed under vector addition: For any two vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{U}$, their sum $\vec{v}_1 + \vec{v}_2$ must also be in \mathbb{U} .
 - iii. Closed under scalar multiplication: For any vector $\vec{v} \in \mathbb{U}$ and scalar $\alpha \in \mathbb{R}$, the product $\alpha\vec{v}$ must also be in \mathbb{U} .

3. Linear Dependence

Learning Objectives: Evaluate the linear dependency of a set of vectors.

State if the following sets of vectors are linearly independent or dependent. If the set is linearly dependent, provide a linear combination of the vectors that sum to the zero vector.

(**Hint:** Consider using the given set of vectors to form the columns of a matrix A . What would it mean about the set of vectors if the equation $A\vec{x} = \vec{0}$ had a solution that was not just $\vec{x} = \vec{0}$? Recall the column view of matrix-vector multiplication and the definition of linear dependence.)

(a) $\left\{ \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$

Solution:

The vectors are linearly independent. A set of two vectors can only be linearly dependent if one of the vectors is a scaled version of the other. $\begin{bmatrix} -5 \\ 2 \end{bmatrix} \neq \alpha \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ for $\alpha \in \mathbb{R}$.

(b) $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Solution:

This set of vectors are linearly dependent. To both show this and find a linear combination that yields the vector $\vec{0}$, solve the equation $A\vec{x} = \vec{0}$ for a valid nonzero solution $\vec{x} \in \mathbb{R}^3$ and where the given vectors form the columns of the matrix A .

$$A\vec{x} = \vec{0} \rightarrow \begin{bmatrix} -1 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 3 & -1 \\ -1 & -2 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Gaussian elimination, we can solve this linear system of equations by finding the RREF of the

following augmented matrix:

$$\begin{aligned}
 \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ -1 & -2 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{array} \right] & \xrightarrow{R_1 \leftarrow -R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ -1 & -2 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{array} \right] & \xrightarrow{\substack{R_2 \leftarrow R_2 - R_1 \\ R_4 \leftarrow R_4 + R_1}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & -3 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{array} \right] \\
 & \xrightarrow{\substack{R_5 \leftarrow R_5 - R_1 \\ R_2 \leftarrow R_2/3}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 3 & -1 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] & \xrightarrow{\substack{R_3 \leftarrow R_3 - 3R_2 \\ R_4 \leftarrow R_4 + 3R_2}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]
 \end{aligned}$$

The free variable can be chosen to be the variable of third column $[1, -\frac{1}{3}, 0, 0, 0]^T$. We can assign the value of the free variable to be $x_3 = 3\alpha$ (chosen arbitrarily to make subsequent calculations easier) for some arbitrary scalar α . If we solve for the second column variable from the second row after substitution, we obtain the variable of the second column to be $x_2 = \alpha$.

$$x_2 - \frac{1}{3}x_3 = x_2 - \frac{1}{3}(3\alpha) = 0 \rightarrow x_2 = \alpha$$

We can do similarly with the first variable to obtain a value of $x_1 = -2\alpha$ from the first row.

$$x_1 - x_2 + x_3 = x_1 - (\alpha) + (3\alpha) = 0 \rightarrow x_1 = -2\alpha$$

Finally, we can express a linear combination of the original vectors as depending on a nonzero α and showing linear dependence.

$$\begin{aligned}
 x_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
 -2\alpha \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix} + 3\alpha \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
 \end{aligned}$$

If you have a set of coefficients that match a specific value of α , give yourself full credit.

$$(c) \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Solution:

Since this set contains $\vec{0}$, it is linearly dependent as we can take the linear combination

$$0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} + \alpha \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $\alpha \neq 0$ to get $\vec{0}$.

4. Linear Dependence in a Square Matrix

Learning Objective: This is an opportunity to practice applying proof techniques. This question is specifically focused on linear dependence of rows and columns in a square matrix.

Let A be a square $n \times n$ matrix, (i.e. both the columns and rows are vectors in \mathbb{R}^n). Suppose we are told that the columns of A are linearly dependent. Prove, then, that the rows of A must also be linearly dependent.

You can use the following conclusion in your proof:

If Gaussian elimination is applied to a matrix A , and the resulting matrix (in reduced row echelon form) has at least one row of all zeros, this means that the rows of A are linearly dependent.

(Hint: Can you use the linear dependence of the columns to say something about the number of solutions to $A\vec{x} = \vec{0}$? How does the number of solutions relate to the result of Gaussian elimination?)

Solution:

Let $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ be the columns of A . By the definition of linear dependence, there exist scalars, c_1, c_2, \dots, c_n , not all zero, such that

$$c_1\vec{a}_1 + c_2\vec{a}_2 + \dots + c_n\vec{a}_n = \vec{0} \quad (1)$$

We define \vec{c} to be a vector containing the c_i 's as follows: $\vec{c} = [c_1 \ c_2 \ \dots \ c_n]^T$, where $\vec{c} \neq \vec{0}$ by the definition of linear dependence. We can write Eq. 1 in matrix vector form:

$$A\vec{c} = \vec{0} \quad (2)$$

Let's use the first hint: How many solutions are there to the equation $A\vec{x} = \vec{0}$? We know from Eq. 2 that \vec{c} is a solution, but we can also show that $\alpha\vec{c}$ is a solution for any α :

$$A(\alpha\vec{c}) = \alpha\vec{0} = \vec{0} \quad (3)$$

Since \vec{c} is not zero, every multiple of \vec{c} is a different solution. Therefore there are infinite solutions to the equation $A\vec{x} = \vec{0}$.

What can we say about the result of Gaussian elimination if there are infinite solutions? We know that if there are infinite solutions, there must be a free variable after Gaussian elimination. In other words, there must be a column in the row reduced matrix with no leading entry. Therefore, there must be fewer leading entries than the number of columns. Since the matrix A is square, it has the same number of rows as columns, so there must be fewer leading entries than the number of rows. That means there is at least one row with no leading entry, which is equivalent to saying there must be one row that's all zeros in the row reduced matrix.

For example consider performing elimination on the following square matrix with infinite solutions:

$$\left[\begin{array}{ccc|c} 2 & 2 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 2 & 0 & 1 & 1 \end{array} \right]$$

Subtracting row 1 from row 3 ($R_3 - R_1 \rightarrow R_3$):

$$\left[\begin{array}{ccc|c} 2 & 2 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 0 & -2 & -2 & -6 \end{array} \right]$$

Dividing row 1 by 2 ($R_1/2 \rightarrow R_1$):

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 1 & 1 & 3 \\ 0 & -2 & -2 & -6 \end{array} \right]$$

Adding row 2 multiplied by 2 to row 3 ($R_3 + 2 \cdot R_2 \rightarrow R_3$):

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

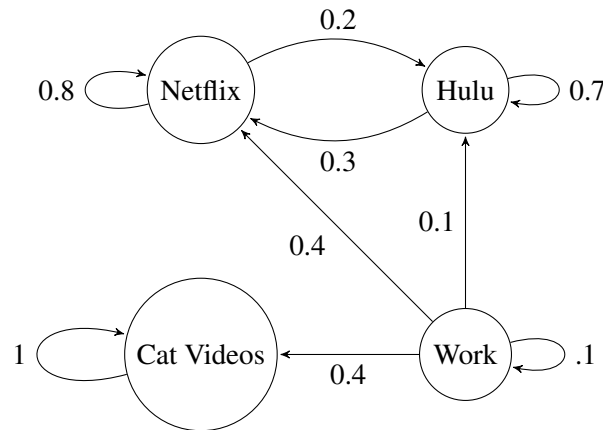
where we see that the last row is missing the leading entry and becomes all 0s.

Finally, we were given that if there is a row of all zeros in the row reduced matrix, then the rows of A must be linearly dependent. Thus, if the columns of an $n \times n$ matrix A are linearly dependent, then the rows are linearly dependent as well.

5. Social Media

Learning Objective: Practice setting up transition matrices from a diagram and understand how to compute subsequent states of the system.

As a tech-savvy Berkeley student, the distractions of streaming services are always calling you away from productive stuff like homework for your classes. You're curious—are you the only one who spends hours switching between Netflix or Hulu? How do other students manage to get stuff done and balance staying up to date with the Bachelor? You conduct an experiment, collect some data, and notice Berkeley students tend to follow a pattern of behavior similar to the figure below. So, for example, if $x = 100$ students are on Netflix, in the next timestep, 20 (i.e., $0.2 \cdot x$) of them will click on a link and move to Hulu, and 80 (i.e., $0.8 \cdot x$) will remain on Netflix.



- (a) Let us define $x_N[n]$ as the number of students on Netflix at time-step n ; $x_H[n]$ as the number of students on Hulu at time-step n ; $x_C[n]$ as the number of students watching any kind of cat video at time-step n ; and $x_W[n]$ as the number of students working at time-step n .

Let the state vector be: $\vec{x}[n] = \begin{bmatrix} x_N[n] \\ x_H[n] \\ x_C[n] \\ x_W[n] \end{bmatrix}$. Derive the corresponding transition matrix \mathbf{A} .

Hint: A transition matrix, \mathbf{A} , is the matrix that transitions $\vec{x}[n]$, the vector at time-step n to $\vec{x}[n+1]$, the vector at time-step $n+1$. In other words: $\vec{x}[n+1] = \mathbf{A}\vec{x}[n]$.

Solution: Let us explicitly write the transition equation for each state and then we can use these to identify the state transition matrix.

$$x_N[n+1] = 0.8 \cdot x_N[n] + 0.3 \cdot x_H[n] + 0.4 \cdot x_W[n]$$

$$x_H[n+1] = 0.2 \cdot x_N[n] + 0.7 \cdot x_H[n] + 0.1 \cdot x_W[n]$$

$$x_C[n+1] = 1 \cdot x_C[n] + 0.4 \cdot x_W[n]$$

$$x_W[n+1] = 0.1 \cdot x_W[n]$$

If $\vec{x}[n] = \begin{bmatrix} x_N[n] \\ x_H[n] \\ x_C[n] \\ x_W[n] \end{bmatrix}$, then for $\vec{x}[n+1] = \mathbf{A}\vec{x}[n]$ we can identify the state transition matrix \mathbf{A} as

$$\begin{bmatrix} 0.8 & 0.3 & 0 & 0.4 \\ 0.2 & 0.7 & 0 & 0.1 \\ 0 & 0 & 1 & 0.4 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}$$

- (b) Assume that this class had 320 of you in total. Suppose on a given Friday evening (the day when HW is due), there are 110 EECS16A students on Netflix, 60 on Hulu, 10 watching Cat Videos, and 140 actually doing work. In the next timestep, how many people will be doing each activity? In other words, after you apply the matrix once to reach the next timestep, what is the state vector?

Solution:

In order to calculate the state vector at the next timestep, we can use the equation $\vec{x}[n+1] = \mathbf{A}\vec{x}[n]$. Substituting the values for \mathbf{A} and $\vec{x}[n]$, we get the following:

$$\begin{bmatrix} 0.8 & 0.3 & 0 & 0.4 \\ 0.2 & 0.7 & 0 & 0.1 \\ 0 & 0 & 1 & 0.4 \\ 0 & 0 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} 110 \\ 60 \\ 10 \\ 140 \end{bmatrix} = \begin{bmatrix} 162 \\ 78 \\ 66 \\ 14 \end{bmatrix}$$

(c) Compute the sum of each column in the state transition matrix. What is the interpretation of this?

Solution:

Since each column's sum is equal to 1, the system is conservative. This means that we aren't losing (or gaining) students after each time step and the total number of students remains constant.

6. Inverse Transforms

Learning Objectives: Matrices represent linear transformations, and their inverses (if they exist) represent the opposite transformation. Here we practice inversion, but are also looking to develop an intuition. Visualizing the transformations might help develop this intuition.

For each of the following choices of matrix \mathbf{A} :

- i. Find the inverse, \mathbf{A}^{-1} , if it exists. If you find that the inverse does not exist, mention how you decided that. Solve this by hand.
- ii. **For parts (a)-(b) only**, in addition to finding the inverse (if it exists), describe how the matrix \mathbf{A} geometrically transforms an arbitrary vector $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in \mathbb{R}^2$.

For example, if $\mathbf{A} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 2x_0 \\ 2y_0 \end{bmatrix}$, then \mathbf{A} could scale $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ by 2 to get $\begin{bmatrix} 2x_0 \\ 2y_0 \end{bmatrix}$. If $\mathbf{A} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ -y_0 \end{bmatrix}$, then \mathbf{A} could reflect $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ across the x -axis, etc. *Hint: It may help to plot a few examples to recognize the pattern.*

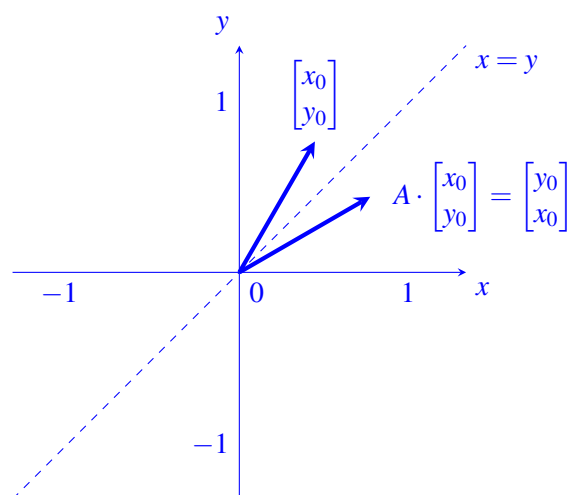
- iii. **Again, for parts (a)-(b) only**, if we use \mathbf{A} to geometrically transform $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ to get $\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, is it possible to reverse the transformation geometrically, i.e. is it possible to retrieve $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ from $\begin{bmatrix} u \\ v \end{bmatrix}$ geometrically?

(a) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Solution:

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\text{swap } R_1, R_2} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array} \right]$$

The inverse does exist: $\mathbf{A}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



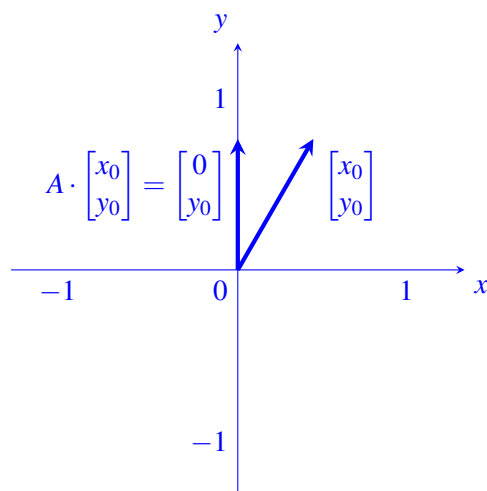
The original matrix \mathbf{A} flips the x and y components of the vector. Any correct equivalent sequence of operations (such as reflecting the vector across the $x = y$ line) warrants full credit. Notice how the inverse does the exact same thing—that is, it switches the x and y components of the vector it's applied to. This makes sense—switching x and y twice on a vector $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ gives us the same vector $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$. So the transformation done by \mathbf{A} is reversible.

(b) $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array} \right] \xrightarrow{\text{swap } R_1, R_2} \left[\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

We see here that *the inverse does not exist* because the second row represents an inconsistent equation. Another way to see that the inverse does not exist is by realizing that the first column (and first row) of the original matrix are the zero vector, so the columns are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.



The original matrix \mathbf{A} removes the x -component of the vector it's applied to and keeps the same y -component. Graphically speaking, this matrix can be thought of as taking the “shadow” of the vector on the y -axis if you were to shine a light perpendicular to the y -axis.

Since the x -component of the vector is completely lost after the transformation, the process is not reversible.

$$(c) \mathbf{A} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution:

We can use Gaussian elimination to find the inverse of the matrix.

$$\begin{aligned} \left[\begin{array}{ccc|ccc} -1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] & \xrightarrow{-R_1 \rightarrow R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \\ & \xrightarrow{R_2 - R_1 \rightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \\ & \xrightarrow{\frac{1}{2}R_2 \rightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \\ & \xrightarrow{R_3 - R_2 \rightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 2 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array} \right] \\ & \xrightarrow{\frac{1}{2}R_3 \rightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{array} \right] \\ & \xrightarrow{R_1 + R_2 \rightarrow R_1, R_2 + R_3 \rightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{array} \right] \end{aligned}$$

The inverse does exist: $\mathbf{A}^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$

$$(d) \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_3 + R_2 \rightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right]$$

The inverse does not exist because the last equation is inconsistent. That is, we have a row of zeros on the left hand side, corresponding to which there is no row of zeros on the right hand side. An alternative reason is that the third column is the negative of the second column, i.e., they are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.

(e) (OPTIONAL)
$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -2 & 1 \\ 0 & 2 & 1 & 3 \\ 3 & 1 & 0 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Hint 1: What do the linear (in)dependence of the rows and columns tell us about the invertibility of a matrix?

Hint 2: We're reasonable people!

Solution:

Inverse does not exist because $\text{column}_1 + \text{column}_2 + \text{column}_3 = \text{column}_4$, which means that the columns are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.

7. Image Stitching

Learning Objective: This problem is similar to one that students might experience in an upper division computer vision course. Our goal is to give students a flavor of the power of tools from fundamental linear algebra and their wide range of applications.

Often, when people take pictures of a large object, they are constrained by the field of vision of the camera. This means that they have two options to capture the entire object:

- Stand as far away as they need to include the entire object in the camera's field of view (clearly, we do not want to do this as it reduces the amount of detail in the image)
- (This is more exciting) Take several pictures of different parts of the object and stitch them together like a jigsaw puzzle.

We are going to explore the second option in this problem. Daniel, who is a professional photographer, wants to construct an image by using "image stitching". Unfortunately, Daniel took some of the pictures from different angles as well as from different positions and distances from the object. While processing these pictures, Daniel lost information about the positions and orientations from which the pictures were taken. Luckily, you and your friend Marcela, with your wealth of newly acquired knowledge about vectors and matrices, can help him!

You and Marcela are designing an iPhone app that stitches photographs together into one larger image. Marcela has already written an algorithm that finds common points in overlapping images. **It's your job to figure out how to stitch the images together using Marcela's common points to reconstruct the larger image.**

We will use vectors to represent the common points which are related by an affine transformation. Your idea is to find this affine transformation. For this you will use a single matrix, \mathbf{R} , and a vector, \vec{t} , that transforms every common point in one image to their corresponding point in the other image. Once you find \mathbf{R} and \vec{t} you will be able to transform one image so that it lines up with the other image.

Suppose $\vec{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$ is a point in one image, which is transformed to $\vec{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$, the corresponding point in the other image (i.e., they represent the same object in the scene). For example, Fig. 1 shows how the points \vec{p}_1 ,

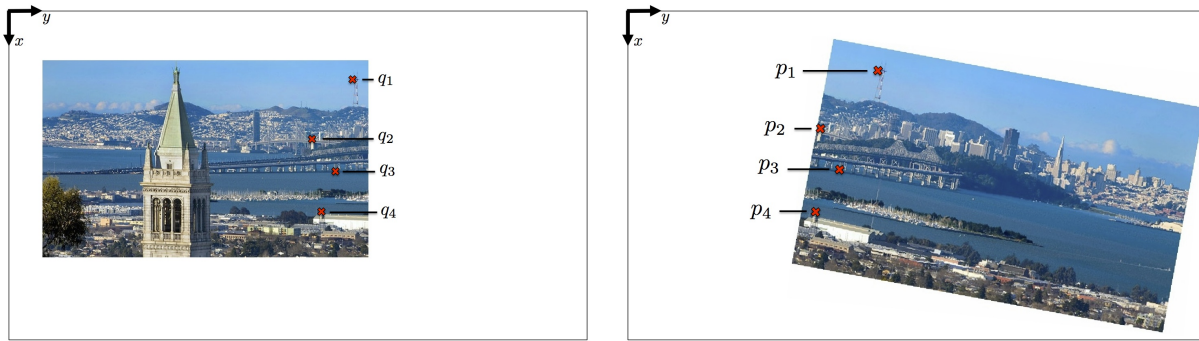


Figure 1: Two images to be stitched together with pairs of matching points labeled.

$\vec{p}_2 \dots$ in the right image are transformed to points $\vec{q}_1, \vec{q}_2 \dots$ on the left image. You write down the following relationship between \vec{p} and \vec{q} .

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \underbrace{\begin{bmatrix} r_{xx} & r_{xy} \\ r_{yx} & r_{yy} \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\vec{t}} \quad (4)$$

This problem focuses on finding the unknowns (i.e. the components of \mathbf{R} and \vec{t}), so that you will be able to stitch the image together. *Note that this is the opposite from our usual setting in which we would solve for \vec{p} given all other variables.*

- (a) To understand how the matrix \mathbf{R} and vector \vec{t} transforms any vector representing a point on a image, Consider this example equation similar to Equation (4),

$$\vec{v} = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \vec{u} + \vec{w} = \vec{v}_1 + \vec{w}. \quad (5)$$

Use $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for this part.

We want to find out what geometric transformation(s) can be applied on \vec{u} to give \vec{v} .

Step 1: Find out how $\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$ is transforming \vec{u} . Evaluate $\vec{v}_1 = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \vec{u}$.

What **geometric transformation(s)** might be applied to \vec{u} to get \vec{v}_1 ? Choose the options that answers the question and explain your choice.

- (i) Rotation
- (ii) Scaling
- (iii) Shifting/Translation

Drawing the vectors \vec{u} , and \vec{v}_1 in two dimensions on a single plot might help you to visualize the transformations.

Step 2: Find out $\vec{v} = \vec{v}_1 + \vec{w}$. Find out how **addition of \vec{w} is geometrically transforming \vec{v}_1** . Choose the option that answers the question and explain your choice.

- (i) Rotation
- (ii) Scaling
- (iii) Shifting/Translation

Drawing the vectors \vec{v} , \vec{w} , and \vec{v}_1 in two dimensions on a single plot might help you to visualize the transformations.

Solution: Plugging in the given vectors and performing the matrix vector multiplication,

$$\vec{v}_1 = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}. \quad (6)$$

It is observable that \vec{v}_1 is a scaled, rotated version of \vec{u} .

We get $\vec{v} = \vec{v}_1 + \vec{w} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. We can also see that \vec{v} is a shifted version of \vec{v}_1 .

Hence \vec{u} is scaled, rotated and shifted to get \vec{v} .

Note: We can only use matrix transformations to scale and/rotate a vector. We cannot translate a vector through matrix transformations; instead we must use vector addition for this.

- (b) Now back to the main problem. First, multiply Equation (4) out into **two equations**.
- (i) What are the known values and what are the unknown values in each equation (recall what we are trying to solve for in Equation (4))?
 - (ii) How many unknown values are there?
 - (iii) How many independent equations do you need to solve for all the unknowns?
 - (iv) How many pairs of common points \vec{p} and \vec{q} will you need in order to write down a system of equations that you can use to solve for the unknowns? **Hint:** Remember that each pair of \vec{p} and \vec{q} is related by two equations, one for each coordinate.

Solution:

We can rewrite the above matrix equation as the following two scalar equations:

$$q_x = p_x r_{xx} + p_y r_{xy} + t_x$$

$$q_y = p_x r_{yx} + p_y r_{yy} + t_y$$

Here, the known values are each pair of points' elements: q_x , q_y , p_x , p_y , and the scaling factor of the \vec{r} vector (1). The unknowns are elements of \mathbf{R} and \vec{t} : r_{xx} , r_{xy} , r_{yx} , r_{yy} , t_x , and t_y . There are 6 unknowns, so we need a total of 6 equations to solve for them. For every pair of points we add, we get two more equations. Thus, we need 3 pairs of common points to get 6 equations.

- (c) Use what you learned in the above two subparts to explicitly write out **just enough** equations of these transformations as you need to solve the system. Assume that the four pairs of points from Fig. 1 are labeled as:

$$\vec{q}_1 = \begin{bmatrix} q_{1x} \\ q_{1y} \end{bmatrix}, \vec{p}_1 = \begin{bmatrix} p_{1x} \\ p_{1y} \end{bmatrix} \quad \vec{q}_2 = \begin{bmatrix} q_{2x} \\ q_{2y} \end{bmatrix}, \vec{p}_2 = \begin{bmatrix} p_{2x} \\ p_{2y} \end{bmatrix} \quad \vec{q}_3 = \begin{bmatrix} q_{3x} \\ q_{3y} \end{bmatrix}, \vec{p}_3 = \begin{bmatrix} p_{3x} \\ p_{3y} \end{bmatrix} \quad \vec{q}_4 = \begin{bmatrix} q_{4x} \\ q_{4y} \end{bmatrix}, \vec{p}_4 = \begin{bmatrix} p_{4x} \\ p_{4y} \end{bmatrix}.$$

Solution: From the previous part, we know that we will need six equations, as we have six unknowns. Recalling that each point provides us with two equations, we arbitrarily select the first three points:

$$\vec{q}_1 = \begin{bmatrix} q_{1x} \\ q_{1y} \end{bmatrix}, \quad \vec{p}_1 = \begin{bmatrix} p_{1x} \\ p_{1y} \end{bmatrix} \quad \vec{q}_2 = \begin{bmatrix} q_{2x} \\ q_{2y} \end{bmatrix}, \quad \vec{p}_2 = \begin{bmatrix} p_{2x} \\ p_{2y} \end{bmatrix} \quad \vec{q}_3 = \begin{bmatrix} q_{3x} \\ q_{3y} \end{bmatrix}, \quad \vec{p}_3 = \begin{bmatrix} p_{3x} \\ p_{3y} \end{bmatrix}$$

which yield the following system:

$$r_{xx}p_{1x} + r_{xy}p_{1y} + t_x = q_{1x} \quad (7)$$

$$r_{yx}p_{1x} + r_{yy}p_{1y} + t_y = q_{1y} \quad (8)$$

$$r_{xx}p_{2x} + r_{xy}p_{2y} + t_x = q_{2x} \quad (9)$$

$$r_{yx}p_{2x} + r_{yy}p_{2y} + t_y = q_{2y} \quad (10)$$

$$r_{xx}p_{3x} + r_{xy}p_{3y} + t_x = q_{3x} \quad (11)$$

$$r_{yx}p_{3x} + r_{yy}p_{3y} + t_y = q_{3y} \quad (12)$$

- (d) Remember that we are ultimately solving for the components of the \mathbf{R} matrix and the vector \vec{t} . This is different from our usual setting and so we need to reformulate the problem into something we are more used to (i.e., $A\vec{x} = \vec{b}$ where x is the unknown). In order to do this, let's view the components of \mathbf{R} and \vec{t} as the unknowns in the equations from part c). We then have a system of linear equations which we should be able to write in the familiar matrix-vector form. Specifically, we can store the unknowns

in a vector $\vec{\alpha} = \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{yx} \\ r_{yy} \\ t_x \\ t_y \end{bmatrix}$ and specify 6×6 matrix \mathbf{A} and vector \vec{b} such that $A\vec{\alpha} = \vec{b}$. Please write out the

entries of \mathbf{A} and \vec{b} to match your equations from part c). To get you started, we provide the first row of \mathbf{A} and first entry of \vec{b} which corresponds to one possible equation from part c):

$$\begin{bmatrix} p_{1x} & p_{1y} & 0 & 0 & 1 & 0 \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{yx} \\ r_{yy} \\ t_x \\ t_y \end{bmatrix} = \begin{bmatrix} q_{1x} \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Your job is to fill in the remaining entries according to the other equations.

Solution: We write the system of linear equations from the previous part in matrix form.

$$\begin{bmatrix} p_{1x} & p_{1y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{1x} & p_{1y} & 0 & 1 \\ p_{2x} & p_{2y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{2x} & p_{2y} & 0 & 1 \\ p_{3x} & p_{3y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{3x} & p_{3y} & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{yx} \\ r_{yy} \\ t_x \\ t_y \end{bmatrix} = \begin{bmatrix} q_{1x} \\ q_{1y} \\ q_{2x} \\ q_{2y} \\ q_{3x} \\ q_{3y} \end{bmatrix}$$

- (e) In the IPython notebook `prob4.ipynb`, you will have a chance to test out your solution. Plug in the values that you are given for p_x , p_y , q_x , and q_y for each pair of points into your system of equations to solve for the matrix, \mathbf{R} , and vector, \vec{t} . The notebook will solve the system of equations, apply your transformation to the second image, and show you if your stitching algorithm works. **You are NOT**

responsible for understanding the image stitching code or Marcela's algorithm. What are the values for \mathbf{R} and \vec{t} which correctly stitch the images together?

Solution:

The parameters for the transformation from the coordinates of the first image to those of the second image are $\mathbf{R} = \begin{bmatrix} 1.1954 & .1046 \\ -.1046 & 1.1954 \end{bmatrix}$ and $\vec{t} = \begin{bmatrix} -150 \\ -250 \end{bmatrix}$.

8. Subspaces, Bases and Dimension

Learning Objective: Explore how to recognize and show if a subset of a vector space is or is not a subspace. Further practice identifying a basis for (i.e., a minimal set of vectors which span) an arbitrary subspace.

For each of the sets \mathbb{U} (which are subsets of \mathbb{R}^3) defined below, state whether \mathbb{U} is a subspace of \mathbb{R}^3 or not. If \mathbb{U} is a subspace, find a basis for it and state the dimension.

Note:

- To show \mathbb{U} is a subspace, you have to show that all three properties of a subspace hold.
- To show \mathbb{U} is not a subspace, you only have to show at least one property of a subspace does not hold.

$$(a) \mathbb{U} = \left\{ \begin{bmatrix} 2(x+y) \\ x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$$

Solution: We test the three properties of a subspace:

i. Let $\vec{v}_1 = \begin{bmatrix} 2(x_1 + y_1) \\ x_1 \\ y_1 \end{bmatrix}$ be a member of the set \mathbb{U} . Assume $\vec{v}_2 = \alpha \vec{v}_1$, where α is a scalar. Here

$$\vec{v}_2 = \alpha \vec{v}_1 = \alpha \begin{bmatrix} 2(x_1 + y_1) \\ x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2(\alpha x_1 + \alpha y_1) \\ \alpha x_1 \\ \alpha y_1 \end{bmatrix} = \begin{bmatrix} 2(x_i + y_i) \\ x_i \\ y_i \end{bmatrix},$$

where $x_i = \alpha x_1$ and $y_i = \alpha y_1$. Hence, $\vec{v}_2 = \alpha \vec{v}_1$ is a member of the set as well and the set is closed under scalar multiplication.

ii. Let $\vec{v}_1 = \begin{bmatrix} 2(x_1 + y_1) \\ x_1 \\ y_1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2(x_2 + y_2) \\ x_2 \\ y_2 \end{bmatrix}$ be members of the set \mathbb{U} . Now, let us assume $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$:

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 2(x_1 + y_1) + 2(x_2 + y_2) \\ x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} 2(x_1 + x_2 + y_1 + y_2) \\ x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} 2(x_3 + y_3) \\ x_3 \\ y_3 \end{bmatrix},$$

where $x_3 = x_1 + x_2$ and $y_3 = y_1 + y_2$. Hence, \vec{v}_3 is a member of the set as well and the set is closed under vector addition.

iii. Let $\vec{v}_0 = \begin{bmatrix} 2(x_0 + y_0) \\ x_0 \\ y_0 \end{bmatrix}$ be a member of the set, where we choose $x_0 = 0$ and $y_0 = 0$. So the vector

$$\vec{v}_0 = \begin{bmatrix} 2(0+0) \\ 0 \\ 0 \end{bmatrix} = \vec{0}. \text{ So the zero vector is contained in this set.}$$

Hence we can decide that \mathbb{U} is a subspace of \mathbb{R}^3 . Any vector in the subspace can be written as:

$$\begin{bmatrix} 2(x+y) \\ x \\ y \end{bmatrix} = x \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},$$

where x and y are free variables. So \mathbb{U} can be expressed as $\text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$. Hence the basis is

given by the set: $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$. Dimension = 2.

$$(b) \mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ z+1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

Solution:

Again we check the three properties of a subspace:

i. Now let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1+1 \end{bmatrix}$ be a member of the set \mathbb{U} . Assume $\vec{v}_2 = \alpha \vec{v}_1$, where α is a scalar. Here

$$\vec{v}_2 = \alpha \vec{v}_1 = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha z_1 + \alpha \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ (\alpha z_1 + \alpha - 1) + 1 \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \\ z_i + 1 \end{bmatrix},$$

where $x_i = \alpha x_1$, $y_i = \alpha y_1$ and $z_i = \alpha z_1 + \alpha - 1$. Hence, $\vec{v}_2 = \alpha \vec{v}_1$ is a member of the set as well and the set is closed under scalar multiplication.

ii. Let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1+1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2+1 \end{bmatrix}$ be members of the set \mathbb{U} . Now, let us assume $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$:

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 + 2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ (z_1 + z_2 + 1) + 1 \end{bmatrix} = \begin{bmatrix} x_3 \\ y_3 \\ z_3 + 1 \end{bmatrix},$$

where $x_3 = x_1 + x_2$, $y_3 = y_1 + y_2$ and $z_3 = z_1 + z_2 + 1$. Hence, \vec{v}_3 is a member of the set as well and the set is closed under vector addition.

iii. Let $\vec{v}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0+1 \end{bmatrix}$ be a member of the set, where we choose $x_0 = 0$, $y_0 = 0$ and $z_0 = -1$. So the

vector $\vec{v}_0 = \begin{bmatrix} 0 \\ 0 \\ -1+1 \end{bmatrix} = \vec{0}$. So the zero vector is contained in this set.

Hence we can decide that \mathbb{U} is a subspace of \mathbb{R}^3 . Any vector in the subspace can be written as:

$$\begin{bmatrix} x \\ y \\ z+1 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (z+1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z_{new} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where x , y and $z_{new} = z + 1$ are free variables. So \mathbb{U} can be expressed as $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Hence the basis is given by the set: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. The dimension is 3, which makes \mathbb{U} the same as \mathbb{R}^3 .

$$(c) \mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ x+1 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$$

Solution: Again we check the three properties of a subspace:

i. Now let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ x_1 + 1 \end{bmatrix}$ be a member of the set \mathbb{U} . Assume $\vec{v}_2 = \alpha \vec{v}_1$, where α is a scalar. Here

$$\vec{v}_2 = \alpha \vec{v}_1 = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha x_1 + \alpha \end{bmatrix} \neq \begin{bmatrix} x_i \\ y_i \\ x_i + 1 \end{bmatrix},$$

where $x_i = \alpha x_1$ and $y_i = \alpha y_1$. Hence, $\vec{v}_2 = \alpha \vec{v}_1$ is not a member of the set and the set is not closed under scalar multiplication.

ii. Let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ x_1 + 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ x_2 + 1 \end{bmatrix}$ be members of the set \mathbb{U} . Now, let us assume $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$:

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ x_1 + x_2 + 2 \end{bmatrix} \neq \begin{bmatrix} x_3 \\ y_3 \\ x_3 + 1 \end{bmatrix},$$

where $x_3 = x_1 + x_2$, and $y_3 = y_1 + y_2$. Hence, \vec{v}_3 is not a member of the set and the set is not closed under vector addition.

iii. Let $\vec{v}_0 = \begin{bmatrix} x_0 \\ y_0 \\ x_0 + 1 \end{bmatrix}$ be a member of the set. The first and third elements cannot both be zero

regardless of the value chosen for x_0 . So the zero vector is not contained in this set.

Hence we can decide that \mathbb{U} is not a subspace of \mathbb{R}^3 . **Note that for full credit you only have to show that one of the properties is violated, you don't have to show all three.**

9. Finding Null Spaces and Column Spaces

Learning Objectives: Null spaces and column spaces are two fundamental vector spaces associated with matrices and they describe important attributes of the transformations that these matrices represent. This problem explores how to find and express these spaces.

Definition (Null space): The null space of a matrix, $\mathbf{A} \in \mathbb{R}^{m \times n}$, is the set of all vectors $\vec{x} \in \mathbb{R}^n$ such that $\mathbf{A}\vec{x} = \vec{0}$. The null space is notated as $\text{Null}(\mathbf{A})$ and the definition can be written in set notation as:

$$\text{Null}(\mathbf{A}) = \{\vec{x} \mid \mathbf{A}\vec{x} = \vec{0}, \vec{x} \in \mathbb{R}^n\}$$

Definition (Column space): The column space of a matrix, $\mathbf{A} \in \mathbb{R}^{m \times n}$, is the set of all vectors $\mathbf{A}\vec{x} \in \mathbb{R}^m$ for all choices of $\vec{x} \in \mathbb{R}^n$. Equivalently, it is also the span of the column vector of \mathbf{A} . The column space can be notated as $Col(\mathbf{A})$ or $Range(\mathbf{A})$ and the definition can be written in set notation as:

$$Col(\mathbf{A}) = \{\mathbf{A}\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

Definition (Dimension): The dimension of a vector space is the number of basis vectors — i.e. the minimum number of vectors required to span the vector space.

- (a) Consider a matrix $\mathbf{A} \in \mathbb{R}^{3 \times 5}$. What is the maximum possible number of linearly independent column vectors (i.e. the maximum possible dimension) of $Col(\mathbf{A})$?

Solution:

There are a maximum of 3 linearly independent columns of \mathbf{A} .

If you are stuck solving a problem like this, consider concrete examples. We want to find the maximum possible number of linearly independent column vectors, so we look for examples and check if we can exceed certain values.

Consider the following example matrix, where the entries marked with * are arbitrary values:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \end{bmatrix}$$

Here all 5 columns are $\in \mathbb{R}^3$. The first three columns are linearly independent, so at least three linearly independent columns are achievable. The first three columns span \mathbb{R}^3 , therefore any choice of fourth and fifth columns, also in \mathbb{R}^3 , can be written as a linear combination of the first three columns. This means that we cannot exceed three linearly independent columns. Thus the maximum number of linearly independent column vectors is 3. In general, if $m < n$, then the columns of $\mathbf{A} \in \mathbb{R}^{m \times n}$ will always be linearly dependent, since you cannot have more than m linearly independent columns in \mathbb{R}^m .

- (b) You are given the following matrix \mathbf{A} .

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a *minimum* set of vectors that span $Col(\mathbf{A})$ (i.e. a basis for $Col(\mathbf{A})$). (This problem does not have a unique answer, since you can choose many different sets of vectors that fit the description here.) What is the dimension of $Col(\mathbf{A})$?

Hint: You can do this problem by observation. Alternatively, use Gaussian Elimination on the matrix to identify how many columns of the matrix are linearly independent. The columns with pivots (leading ones) in them correspond to the columns in the original matrix that are linearly independent.

Solution: $Col(\mathbf{A})$ is the space spanned by its columns, so the set of all columns is a valid span for $Col(\mathbf{A})$. However, we are asking you to choose a subset of the columns and still span $Col(\mathbf{A})$, as we showed in part (a). To find the minimum number of columns needed and determine the dimension of $Col(\mathbf{A})$, we can remove vectors from the set of columns until we are left with a linearly independent set.

By inspection, the second, fourth, and fifth columns can be omitted from a set of columns as they can be expressed as linear combinations of the first and third columns. Thus the dimension of \mathbf{A} is 2.

One set spanning $Col(\mathbf{A})$ is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Another valid set of vectors which span $Col(\mathbf{A})$ is:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

Note with this second set, none of the columns of \mathbf{A} appear. Despite this, the span of this set will still be equal to $Col(\mathbf{A})$, which for this matrix is the set of all vectors in \mathbb{R}^3 with zero third entry. Geometrically, both of these solutions span the same plane, i.e. the xy -plane in the 3D space.

Give yourself full credit if you recognized that the dimension was 2, and if you had a *minimum* set of vectors that spans $Col(\mathbf{A})$.

- (c) Find a *minimum* set of vectors that span $Null(\mathbf{A})$ (i.e. a basis for $Null(\mathbf{A})$), where \mathbf{A} is the same matrix as in part (b). What is the dimension of $Null(\mathbf{A})$?

Solution:

Finding $Null(\mathbf{A})$ is the same as solving the following system of linear equations:

$$\begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{aligned} x_1 + x_2 - 2x_4 + 3x_5 &= 0 \\ x_3 - x_4 + x_5 &= 0 \end{aligned}$$

We observe that x_2 , x_4 , and x_5 are free variables, since they correspond to the columns with no pivots. Thus, we let $x_2 = a$, $x_4 = b$, and $x_5 = c$. Now we rewrite the equations as:

$$x_1 = -a + 2b - 3c$$

$$x_2 = a$$

$$x_3 = b - c$$

$$x_4 = b$$

$$x_5 = c$$

We can then write this in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, $Null(\mathbf{A})$ is spanned by the vectors:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The dimension of $Null(\mathbf{A})$ is 3, as it is the minimum number of vectors we need to span it.

- (d) For the following matrix \mathbf{D} , find $\text{Col}(\mathbf{D})$ and its dimension, and $\text{Null}(\mathbf{D})$ and its dimension. Using inspection or Gaussian elimination are both valid methods to solve the problem.

$$\mathbf{D} = \begin{bmatrix} 1 & -1 & -3 & 4 \\ 3 & -3 & -5 & 8 \\ 1 & -1 & -1 & 2 \end{bmatrix}$$

Solution:

To find $\text{Col}(\mathbf{D})$, we identify the linearly independent columns of \mathbf{D} by inspection. The second column is a scaled version of the first column. The third column is linearly independent from the first and second columns, since it is not a scaled version of the first column. Finally, the fourth column is the first column minus the third column and thus is linearly dependent with respect to prior columns. If the linear dependence of the fourth column is not clear by inspection, we can instead perform Gaussian elimination. By doing so, we would find that the second and fourth column lack pivots, which also indicates their linear dependence.

So we conclude that the linearly independent columns of \mathbf{D} are the first and third columns so that a basis for $\text{Col}(\mathbf{D})$ is:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -1 \end{bmatrix} \right\}$$

and thus the dimension of $\text{Col}(\mathbf{D})$ is 2.

To find $\text{Null}(\mathbf{D})$, we can row reduce the matrix to find solutions to $\mathbf{D}\vec{x} = \vec{0}$.

$$\left[\begin{array}{cccc|c} 1 & -1 & -3 & 4 & 0 \\ 3 & -3 & -5 & 8 & 0 \\ 1 & -1 & -1 & 2 & 0 \end{array} \right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Since we only have pivots in the first and third columns, we can assign the free variables $x_2 = s$ and $x_4 = t$. We can write all solutions to $\mathbf{D}\vec{x} = \vec{0}$ as:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s-t \\ s \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} t$$

A basis for $\text{Null}(\mathbf{D})$ is:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and thus the dimension of $\text{Null}(\mathbf{D})$ is 2.

- (e) Find the sum of the dimensions of $\text{Null}(\mathbf{A})$ and $\text{Col}(\mathbf{A})$. Also find the sum of the dimensions of $\text{Null}(\mathbf{D})$ and $\text{Col}(\mathbf{D})$. What do you notice about these sums in relation to the dimensions of \mathbf{A} and \mathbf{D} , respectively?

Solution:

The dimensions of $\text{Col}(\mathbf{A})$ and $\text{Null}(\mathbf{A})$ add up to the number of columns in \mathbf{A} . The same is true of \mathbf{D} . This is true of all matrices and relates to what is known as the ‘rank-nullity theorem’; however we will not be covering this in 16A. You’ll get to explore this in 16B.

10. Prelab Questions

These questions pertain to the prelab reading for the Imaging 3 lab. You can find the reading under the Imaging 3 Lab section on the ‘Schedule’ page of the website. We do not expect in-depth answers for the questions. Please limit your answers to a maximum of 2 sentences.

- What properties does the mask matrix H need to have for us to reconstruct the image?
- Briefly describe why averaging multiple signals/measurements is a good idea.
- What is the new equation that models our system?
- How do we get the image back from the new equation that models our system? Note that your answer cannot contain the image vector, \vec{i} (since it is an unknown). You may, however, give an answer that includes \vec{i}_{est} .
- What term allows us to control the effect of noise in our system? *Hint:* Look at the terms in the equation that contains \vec{i}_{est} .

Solution:

- Invertible, linearly independent columns, a trivial nullspace, non-zero determinant, unique solution for $\mathbf{A}\vec{x} = \vec{b}$. All these properties are equivalent!
- Averaging is good because it reduces noise over multiple measurements especially if we have one or more bad measurements. (can talk about the given single-pixel vs multi-pixel example in the reading)
- $\vec{s} = H\vec{i} + \vec{w}$
- $\vec{i}_{est} = H^{-1}\vec{s}_{ideal} + H^{-1}\vec{w}$
- The term $H^{-1}\vec{w}$ allows us to control the noise term. [OPTIONAL: A smaller $H^{-1}\vec{w}$ reduces the effect of noise on our system.]

11. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.