Lecture 3A Mon 7/3

Today:

- State transition systems
- Matrix Inverses

Announcements:

- Quest Mon 7/10, during lecture in this room
  - Scope: content up to and including this Wednesday.
    - includes lecture, dis, lab, hw
  - why is it called a "quest"?
    - "quiz/test"
    - it is a shorter exam than a normal midterm, meant to give you a sense of how the course is going for you
  - how to study?
    - review dis worksheets and hw problems
      - do not be surprised if some questions from these (or lecture) show up again on the exam
    - make a cheatsheet: 1 page front + back, handwritten
      - as much a way to study as a reference for the exam
    - do past exams
      - other semesters had different schedules and scopes for their exams - there may be material we haven't covered yet.
      - in particular, eigenvalues/eigenvectors are NOT in scope for our exam.

- ok. more information on Ed.

Last time: Viewing matrices as transformations on vectors

$A \vec{v}_1 = \vec{v}_2$ e.g. reflections, rotations, ...

input ↑ output ↑

The magic of linear algebra (and why it's so powerful and ubiquitous) is that matrices and vectors can be used for so many systems/problems!

- Today: another common use = modeling dynamic systems
  - changes over time

Use vectors to represent "states."

Ex. an autonomous vehicle. we want to know where it is and where it's gonna go.

- "state" at time t: contains some information representing what's going on at time t

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ v_x(t) \\ v_y(t) \end{bmatrix}$$

  - depending on your model, include different things in state vector!

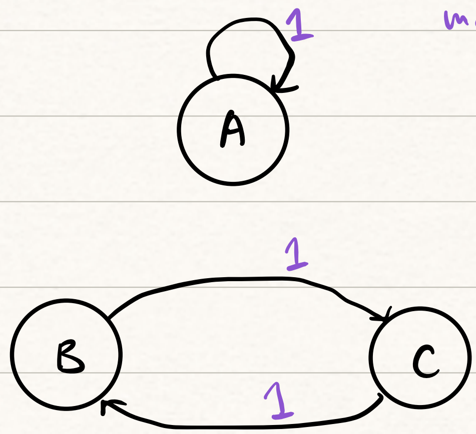
Autonomous vehicles are pretty complicated.

- the field of CONTROL builds on this idea of state with more linear algebra tools (more in EECS 16B)

Today we will consider a simpler model of a dynamic system.

- say you have a system of reservoirs and pumps.

manmade lakes to store water



- the pumps move water between different reservoirs
- it happens instantaneously at discrete time steps

This picture represents the system:

- the arrow means water moves from one reservoir to another
  - the number on the arrow is the fraction of water that moves
  - $B \to C$: 100% of the water in B at time t moves to C at time $t+1$
  - $A \to A$: A keeps 100% of its water.
  - we will have other fractions soon

This model can represent various things:

- city water systems
- flows of traffic
- how people move between cities
- energy storage and usage

Let's construct a state vector

$$\vec{x}(t) = \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

just keep track of how much water is in each reservoir at time t.

OK. What's $\vec{x}(t+1)$ given $\vec{x}(t)$?

$$\begin{cases} x_A(t+1) = x_A(t) \\ x_B(t+1) = x_C(t) \\ x_C(t+1) = x_B(t) \end{cases}$$

A keeps all its water.

B and C swap their water.

- it's a system of equations. Let's make a matrix out of it.

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

$$\vec{x}(t+1) = Q \, \vec{x}(t)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

"state transition matrix": tells you how to go from $t \to t+1$

Ex. Let's say

$$\vec{x}(0) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Then

$$\vec{x}(1) = Q\vec{x}(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

- initial state + Q tells you everything about the system (and its future!)

Q. What is $\vec{x}(t+2)$?

$\vec{x}(t+2) = Q\,\vec{x}(t+1)$ Q tells you how to go forward one step, at any time.

$= Q\left(Q\,\vec{x}(t)\right)$

$= \underbrace{Q\,Q}\,\vec{x}(t)$

just like transformations last time, Q^2 is the matrix that applies Q twice.

- Let's see what Q^2 is for our example.
  - intuitively, what would you expect?
    water just flip flops between B & C → after 2 time steps, just goes back to start!
  - Matrix multiplication time:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

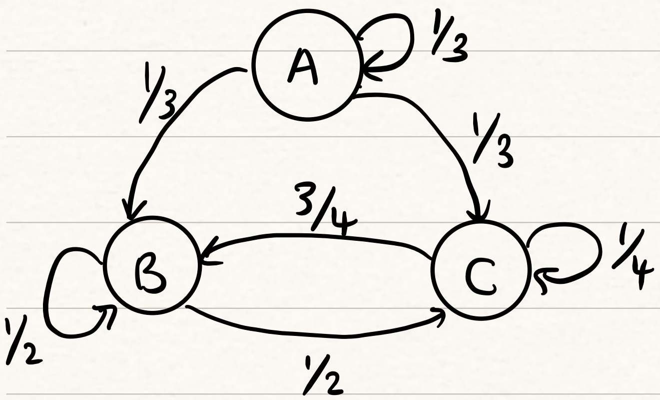
identity matrix! (the matrix that doesn't change its input)

- yay matches our intuition

Foreshadowing question: What is $\vec{x}(t+2000)$?
Does it depend on $\vec{x}(0)$?

- Does the system "converge" or have a "steady state"?
- this one doesn't really because B & C always flip flop
- We shall see!

Another example:



What is the state transition matrix?

- Start with the system of equations:

$$x_A(t+1) = ?$$
$$x_B(t+1)$$
$$x_C(t+1)$$

$$x_A(t+1) = \frac{1}{3} x_A(t)$$
$$x_B(t+1) = \frac{1}{3} x_A(t) + \frac{1}{2} x_B(t) + \frac{3}{4} x_C(t)$$
$$x_C(t+1) = \frac{1}{3} x_A(t) + \frac{1}{2} x_B(t) + \frac{1}{4} x_C(t)$$

With equations, it's easy to fill in matrix:

$$\rightarrow \begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

Match each arrow to an entry:

each row = water INTO each reservoir

$$\begin{bmatrix} A \to A & B \to A & C \to A \\ A \to B & B \to B & C \to B \\ A \to C & B \to C & C \to C \end{bmatrix}$$

each column = water OUT from each reservoir

- easy to get confused between $A \to B$ and $B \to A$, but the way to get it right is start with the system of equations.

Then Q is the matrix that satisfies

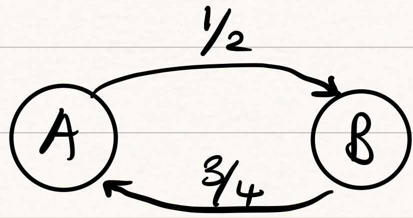
$$\vec{x}(t+1) = Q\,\vec{x}(t)$$

Notice in our example: A loses $\frac{2}{3}$ of its water at every time and never gets more water!

- what do we expect to happen in long term?

(just more foreshadowing...)

Another example:



$$x_A(t+1) = \frac{3}{4} x_B(t)$$
$$x_B(t+1) = \frac{1}{2} x_A(t)$$
$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \end{bmatrix} = \begin{bmatrix} 0 & \frac{3}{4} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \end{bmatrix}$$

Not all the water is accounted for!

- maybe it evaporated or something

This is called a "non-conservative" system.

- how can we tell in general if a system is conservative?

A. Fractions of water OUT of each reservoir must add up to 1.

↳ every COLUMN sums to one.

A non-conservative system can also "create" water (if column sums up to >1).

Remember: this is a model! What real-world systems might not be conservative?

What if we want to go back in time?

- Q goes forward in time one step: $\vec{x}(t+1) = Q\vec{x}(t)$
- What is the matrix that goes $\vec{x}(t-1) = P\vec{x}(t)$?

Q. Could we just flip the arrows?

A. In general, no. Check the 2nd example above.

We need the idea of INVERSES.

- a system / function is called "invertible" if we can recover the input from the output.

Ex. Is $f(x) = 0$ invertible? No.

- all information about x is lost.

Ex. Is writing on the chalkboard invertible?

Yes because can erase (but not for the chalk...)

$\boxed{\text{Def}}$ For $n \times n$ matrices P and Q,

P is called the <u>inverse</u> of Q if $PQ = I$

We write $P = Q^{-1}$.

$(PQ)\vec{v} = I\vec{v} = \vec{v}$

- if compose 2 transformations, goes back to original vector!

- note: inverses only defined for square matrices (in 16B you will learn about "pseudo inverses")
- not all (square) matrices are invertible! Some represent non-reversible transformations!

How do we calculate an inverse?

Ex.

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix}$$

Want: $QP = I$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{23} & P_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

it's a system of equations! with 9 unknowns.

(and 9 equations)

Side note: An alternative view of matrix-matrix mult.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} A\begin{bmatrix} -1 \\ 2 \end{bmatrix} & A\begin{bmatrix} 0 \\ 3 \end{bmatrix} \end{bmatrix}$$

"stacked matrix-vector mult"

General case:

$$A \begin{bmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \\ | & | & & | \end{bmatrix}$$

ok so $QP = I$

$$Q \begin{bmatrix} \vec{P}_1 & \vec{P}_2 & \vec{P}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We've seen $A\vec{x} = \vec{b}$. This is just 3 of them!

$$Q\underline{\vec{P}_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad Q\underline{\vec{P}_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad Q\underline{\vec{P}_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

solve for this vector! 3 times! (use Gaussian elimination)

Hmm. Could there be a shortcut?

- Notice: same Q matrix for 3 equations
- contemplate: the row operations you choose to do in GE depend only on elements of Q!
  $\Rightarrow$ solving the 3 systems would be exactly the same!

$\rightarrow$ can do them all at the same time:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

3 "$\vec{b}$ vectors" stacked

- Now you do GE (with these very long rows)

$$\xrightarrow{\text{swap } R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1/2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{2R_1 \rightarrow R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - \frac{1}{2}R_1 \rightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{-R_3 \rightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_3 \rightarrow R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

RREF! must be $P = Q^{-1}$!

Started with: $Q \underline{P} = I$

unknown to solve for

- key insights: 1. Matrix multiplication is stacked matrix-vector mult.

2. GE steps depend only on Q matrix.

You can check that $QQ^{-1} = I$ indeed.

That was called the "Gauss-Jordan" method of calculating an inverse. To summarize:

$$\left[\begin{array}{c|c} A & I \end{array}\right] \xrightarrow{GE} \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

- but the "stacked $A\vec{x} = \vec{b}$" interpretation is why it works!
- if the system has no solution, then inverse doesn't exist!

Properties of inverses:

1. Left and right inverses are the same.

   If $P = Q^{-1}$, $QP = PQ = I$

   - special case where matrix multiplication order doesn't matter!

2. Inverse is unique.