EECS 16A Designing Information Devices and Systems I
Summer 2023 Homework 4

This homework is due July 14, 2023, at 23:59.
Self-grades are due July 21, 2023, at 23:59.

Submission Format
Your homework submission should consist of **one** file.
`hw4.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).
Submit each file to its respective assignment on Gradescope.

1. **Pre-lab Questions: Touch 1**

   These questions pertain to the pre-lab reading for the *Touch 1* lab. You can find the reading under the *Touch 1* Lab section on the 'Schedule' page of the website. We do not expect in-depth answers for the questions. Please limit your answers to a maximum of 2 sentences.

   (a) What are the three terminals of a potentiometer?

   (b) Why is the LED fader circuit that we're building in lab relevant to a touchscreen?

   (c) What is the threshold voltage of the red LED we use in the lab?

2. **Prelab Questions: Touch 2**

   These questions pertain to the pre-lab reading for the Touch 2 lab. You can find the reading under the Touch 2 Lab section on the 'Schedule' page of the website.

   (a) How many layers are there in the resistive touchscreen and what are they made of?

   (b) Provide 2 examples of resistive touchscreens (give one example not listed on the pre-lab reading).

   (c) In the circuit given in the reading, what is the current i_3 flowing through resistor R_{h1}?

   (d) How do we get touch coordinates in the horizontal direction if you have your circuit that works in the vertical direction?

3. **Reading Assignment**

   For this homework, please review and read Note 9, Note 11A, and Note 11B. Note 9 overviews eigenvalues and eigenvectors. Notes 11A/B introduce the basics of circuit analysis and node voltage analysis. You are always welcome and encouraged to read beyond this as well.

   Please answer the following question:

   (a) When identifying the eigenvalues and eigenvectors of a matrix, which do you find first? The eigenvalues? Or the eigenvectors?

   (b) What are the steps for node voltage analysis (NVA)?

4. Mechanical Determinants

For each of the following matrices, compute their determinant and state whether they are invertible.

In lecture, we did not have time to go over the exact algorithm for computing the determinant of an $n \times n$ matrix. It turns out the algorithm is recursive and always boils down to computing determinants of 2×2 matrices.
Please watch this great YouTube video which calculates the determinant of a 3×3 matrix: https://www.youtube.com/watch?v=21LWuY8i6Hw.

(a) $\begin{bmatrix} 6 & 9 \\ 4 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} -4 & 2 & 1 \\ 5 & 1 & -3 \\ 7 & 3 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} -4 & 0 & 0 \\ 5 & 1 & -3 \\ 7 & 3 & 1 \end{bmatrix}$

5. Introduction to Eigenvalues and Eigenvectors

Learning Goal: *Practice calculating eigenvalues and eigenvectors. The importance of eigenvalues and eigenvectors will become clear in the following problems.*

For each of the following matrices, find their eigenvalues and the corresponding eigenvectors. For simple matrices, you may do this by inspection if you prefer.

(a) $\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(d) Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a general square matrix. Show that the set of eigenvectors corresponding to a particular eigenvalue of $\mathbf{A}$ is a subspace of $\mathbb{R}^n$. In other words, show that

$$\{\vec{x} \in \mathbb{R}^n : \mathbf{A}\vec{x} = \lambda\vec{x}, \lambda \in \mathbb{R}\}$$

is a subspace. You have to show that all three properties of a subspace (as mentioned in Note 8) hold.

6. Properties of Pump Systems

Learning Objectives: *This problem builds on the pump examples we have been doing, but is meant to help you learn to do proofs in a step by step fashion. Can you generalize intuition from a simple example?*

We consider a system of reservoirs connected to each other through pumps. An example system is shown below in Figure 1, represented as a graph. Each node in the graph is marked with a letter and represents a reservoir. Each edge in the graph represents a pump which moves a fraction of the water from one reservoir to the next at every time step. The fraction of water moved is written on top of the edge.

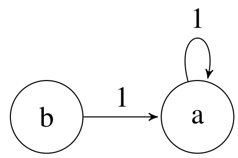


Figure 1: Pump system

We want to prove the following theorem. We will do this step by step.

Theorem: Consider a system consisting of k reservoirs such that the entries of each column in the system's state transition matrix sum to one. If s is the total amount of water in the system at timestep n, then total amount of water at timestep $n+1$ will also be s.

(a) Rewrite the theorem statement for a graph with only two reservoirs.

(b) Since the problem does not specify the transition matrix, let us consider the transition matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and the state vector $\vec{x}[n] = \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix}$. Write out what is "known" or what is given to you in the theorem statement in mathematical form.
Note: In general, it is helpful to write as much out mathematically as you can in proofs. It can also be helpful to draw the transition graph.

(c) Now write out the theorem we want to prove mathematically.

(d) Prove the statement for the case of two reservoirs. In other words, combine parts (b) and (c) to prove the theorem.

(e) Now use what you learned to generalize to the case of k reservoirs. *Hint:* Think about $\mathbf{A}$ in terms of its columns, since you have information about the columns.

7. Page Rank

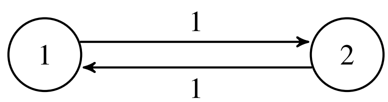
Learning Goal: *This problem highlights the use of transition matrices in modeling dynamical linear systems. Predictions about the steady state of a system can be made using the eigenvalues and eigenvectors of this matrix.*

In homework and discussion, we have discussed the behavior of water flowing in reservoirs and the people flowing in social networks. We now consider the setting of web traffic where the dynamical system can be described with a directed graph, also known as state transition diagram.

As we have seen in lecture and discussion, the "transition matrix", $\mathbf{T}$, can be constructed using the state transition diagram as follows: entries t_{ji} represent the *proportion* of the people who are at website i that click the link for website j.

The **steady-state frequency** (i.e. fraction of visitors in steady-state) for a graph of websites is related to the eigenspace associated with eigenvalue 1 for the "transition matrix" of the graph. Once computed, an eigenvector with eigenvalue 1 will have values which correspond to the steady-state frequency for the fraction of people for each webpage. When the elements of this eigenvector are made to **sum to one** (to conserve population), the i^{th} element of the eigenvector will correspond to the fraction of people on the i^{th} website.

(a) For graph A shown below, what are the steady-state frequencies (i.e. fraction of visitors in steady-state) for the two webpages? Graph A has weights in place to help you construct the transition matrix. Remember to ensure that your steady state-frequencies sum to 1 to maintain conservation.



Graph A

(b) For graph B, what are the steady-state frequencies for the webpages? You may use IPython and the Numpy command `numpy.linalg.eig` for this. We have set up a template IPython notebook `prob7.ipynb` for you (you **do not** need to turn in this notebook to receive full credit). Graph B is shown below, with weights in place to help you construct the transition matrix.

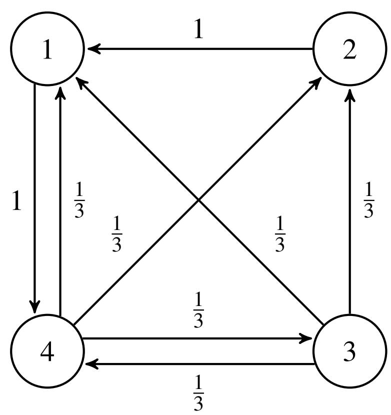
Hint: The steady-state frequencies are fractions or percentages of the total (i.e., 1). Make sure your frequencies sum to one.

Hint: `numpy.linalg.eig` returns eigenvectors and eigenvalues. The eigenvectors are arranged in a matrix in *column-major* order. In other words, given eigenvectors

$\vec{v}_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}$

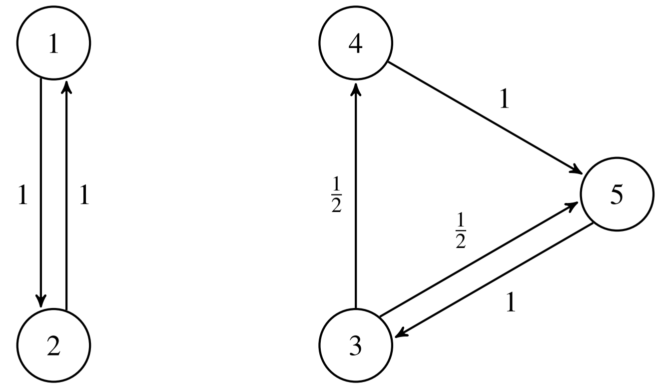
NumPy will return:

$$\begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} \tag{1}$$



Graph B

(c) Graph C with weights in place is shown below. Find the eigenspace that corresponds to the steady-state for graph C. How many independent systems (disjoint sets of webpages) are there in graph C versus in graph B? What is the dimension of the eigenspace corresponding to the steady-state for graph C? You may use IPython to compute the eigenvalues and eigenvectors again.



Graph C

8. Reverse Eigenvalues

Learning Goal: *Understand how to construct a matrix with a particular set of eigenvalues and eigenvectors.*

In lecture, homework, and section, we have seen a number of ways to compute eigenvalues and eigenvectors from a particular matrix, and explored what they mean in terms of how the matrix transforms vectors. In this problem, we will explore this in the reverse direction by designing it to have a desired set of eigenvalues. Recall the fundamental eigenvector/eigenvalue equation:

$$A\vec{v} = \lambda\vec{v} \tag{2}$$

(a) Suppose you are given the following eigenvalue/eigenvector pairs:

$$\lambda_1 = 1, \vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda_2 = -1, \vec{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Explicitly write out the matrix-vector equations for the two eigenvector/eigenvalue pairs. Make sure to identify each component of the **A** matrix and fill in the relevant values for the eigenvector and eigenvalue. Assume the unkown components of **A** are $a_{11}, a_{12}, a_{21}, a_{22}$.

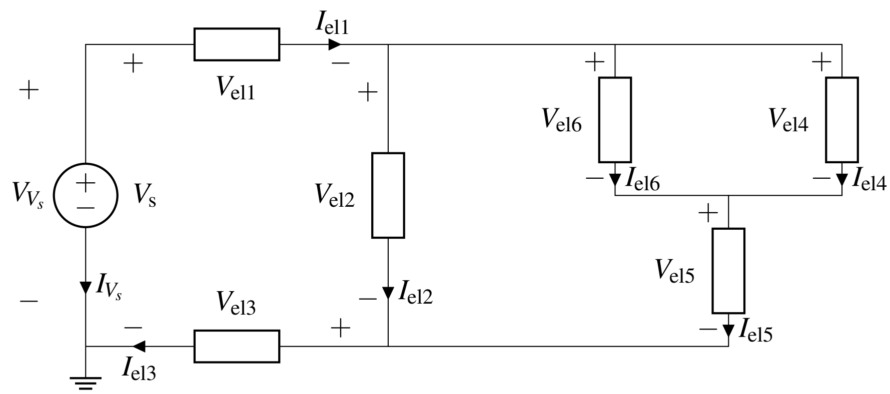
(b) Reformat the equations you wrote above into a system of linear equations, with $a_{11}, a_{12}, a_{21}, a_{22}$ as unknowns.

(c) Now, setup a matrix-vector system of equations and solve for the **A** matrix. Think carefully about what the unknowns in your system are when setting it up.

(d) Describe the geometric transformation represented by the matrix **A**. How does the matrix graphically transform its eigenvectors? How does this relate to the associated eigenvalues?

9. Intro to Circuits

Learning Goal: *This problem will help you practice labeling circuit elements and setting up KVL equations.*



(a) How many nodes does the above circuit have? Label them.
Note: The reference/0V node has already been selected for you, so you do not need to label it, but you do need to include it in your node count.

(b) Express all element voltages (including the element voltage across the voltage source, V_s) as a function of node voltages. This will depend on the node labeling you chose in part (a).

(c) Write a KVL equation for all the loops that contain the voltage source V_s. These equations should be a function of element voltages and the voltage source V_s.

10. It's a Triforce!

Learning Goal: *This problem explores passive sign convention and nodal analysis in a slightly more complicated circuit.*

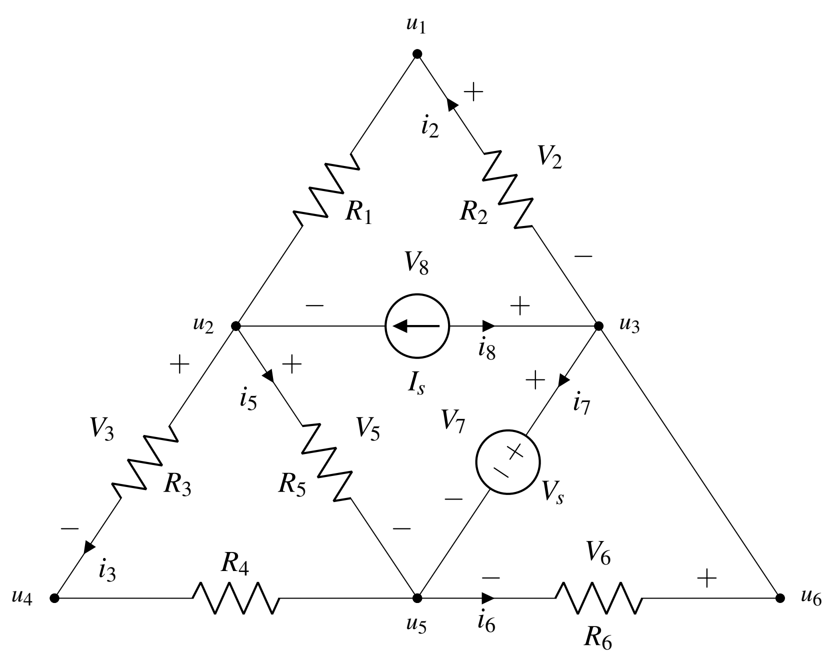


Figure 2: A triangular circuit consisting of a voltage source V_s, current source I_s, and resistors R_1 to R_6.

(a) Which elements I_s, V_s, R_2, R_3, R_5, or R_6 in Figure 2 have current-voltage labeling that violates *passive sign convention*? Explain your reasoning.

(b) In Figure 2, the nodes are labeled with $u_1, u_2, \ldots$ etc. There is a subset of u_i's in the given circuit that are redundant, i.e. there might be more than one label for the same node. Which node(s) do not have a unique label? Justify your answer.

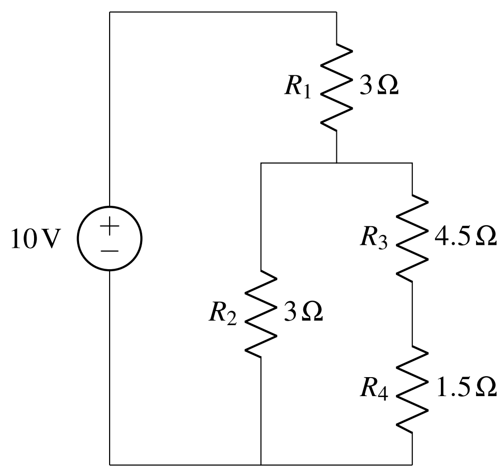
(c) Redraw the circuit diagram by correctly labeling *all* the element voltages and element currents according to passive sign convention. The component labels that were violating passive sign convention in part (a) should be corrected by *swapping the element voltage polarity*. Additionally, label the elements that have not been labeled yet.

(d) Write an equation to describe the current-voltage relationship for element R_4 in terms of the relevant i's, R's, and node voltages in this circuit. Your final expression should include u_4, u_5, and i_4.

(e) Write the KCL equation for node u_2 in terms of the node voltages and other circuit elements.

11. Mechanical Circuits

Find the voltages across and currents flowing through all of the resistors. *Hint: Use the seven steps of node voltage analysis.*



12. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.