
EECS 16A
Summer 2023

Designing Information Devices and Systems I

Discussion 2C

1. Span Basics

- (a) What is $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$?

Answer:

$\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ contains any vector \vec{u} that can be written as

$$\vec{u} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Any vector whose last component is zero can be written in this form and any vector whose last component is nonzero cannot. Hence, the required span is the set of all vectors that can be written in the

form $\begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$.

- (b) Is $\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ in $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$?

Answer:

From the definition of span, we know that if we can express $\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and

$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, it is in the span of those two vectors. Assume such a linear combination exists, then we can set up a vector equation

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}.$$

We can solve this system of equations for the coefficients α_1 and α_2 by expressing it as an augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 1 & 5 \\ 0 & 0 & 0 \end{array} \right]$$

Solving with Gaussian Elimination, we get

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{5}{3} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 \end{array} \right]$$

From this result, $\alpha_1 = \frac{5}{3}$ and $\alpha_2 = \frac{5}{3}$, so we can conclude the statement is true.

- (c) What is a possible choice for \vec{v} that would make $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \vec{v}\right\} = \mathbb{R}^3$?

Answer: From part (a), any vector whose last component is zero can be written as a linear combination of the two vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. Hence if we include, for example, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ into the set, then we should be able to reach any vector in \mathbb{R}^3 . Any vector \vec{v} whose last component is non-zero is a valid choice to achieve the desired span of \mathbb{R}^3 .

- (d) For what values of b_1, b_2, b_3 is the following system of linear equations consistent? *Note: “Consistent” means there is at least one solution.*

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Answer: For the system of linear equations to be consistent, there must exist some \vec{x} such that the equality above holds. Performing matrix vector multiplication, we can rewrite the above expression as

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \vec{b}$$

The question now becomes: which vectors \vec{b} can be written in the above form, i.e as a linear combination of the columns of A ? This is exactly the definition of span, and the answer must be the same as that from part (a).

2. Span Proofs

Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, show the following:

(a)

$$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}, \text{ where } \alpha \text{ is a non-zero scalar}$$

In other words, we can scale our spanning vectors and not change their span.

Answer:

(a) Suppose we have some arbitrary $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. For some scalars a_i :

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n = \left(\frac{a_1}{\alpha}\right)\alpha\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n.$$

Scalar multiplication cancels out. Thus, we have shown that $\vec{q} \in \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. Therefore, we have $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subseteq \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. Now, we must show the other direction. Suppose we have some arbitrary $\vec{r} \in \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. For some scalars b_i :

$$\vec{r} = b_1(\alpha\vec{v}_1) + b_2\vec{v}_2 + \dots + b_n\vec{v}_n = (b_1\alpha)\vec{v}_1 + b_2\vec{v}_2 + \dots + b_n\vec{v}_n.$$

Thus, we have shown that $\vec{r} \in \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. Therefore, we now have $\text{span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. Combining this with the earlier result, the spans are thus the same.