EECS 16A: Final Lecture

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Module 3 – GPS System: Acoustic Positioning System (APS)

Elements of GPS System we have used:

① Classification: Which satellite is transmitting?

$\vec{r} = \vec{s}_A^{(k_1)} + \vec{s}_B^{(k_2)} + \vec{n}$ ← noise

$\vec{r}$: received vector

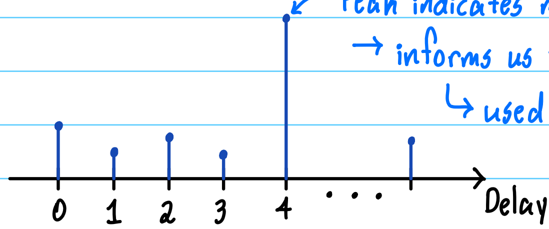
$\vec{s}_A^{(k_1)}$: Signature of satellite A at shift k_1

↳ Cross-correlation

cross-corr$_{\vec{s}_A}(\vec{r})$ > threshold if satellite A is present

< threshold if satellite A is not present

Cross-Correlate satellite signature w/ received signal:



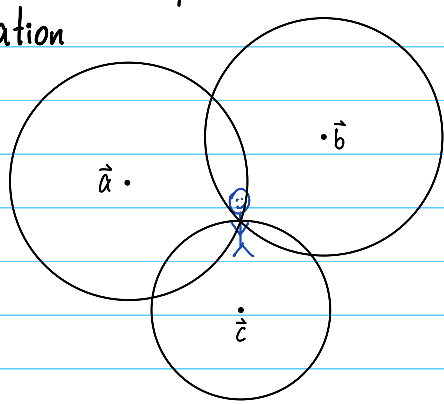
Peak indicates maximum matching/similarity

→ informs us that delay is 4

↳ used to capture distance to satellite (want to determine which satellite is transmitting + distance from it)

② Distance between you and satellite

③ Trilateration



n-dimensional world

⇒ n+1 beacons to locate

Challenge: noise can lead to no unique solution

④ Minimize impact of noise: Least-Squares Algorithm

Least Squares

$$A\vec{x}=\vec{b} \quad \begin{bmatrix} — \vec{a}_1^T — \\ — \vec{a}_2^T — \\ \vdots \\ — \vec{a}_m^T — \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$\{\vec{a}_i^T\vec{x} = b_i \leftarrow$ one experiment/measurement$\}$

m measurements

want to recover this

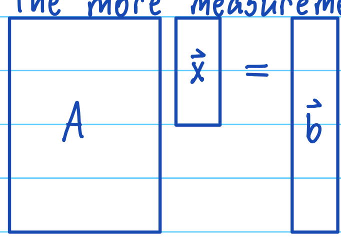
n columns

Recall: In Module 1, unknown quantity was image – project different masks until there were enough measurements (n unknowns ⇒ need n measurements, Gaussian Elimination to solve)

→ Guaranteed that A was invertible by choosing Hadamard set of masks in Imaging lab

→ Might not be able to guarantee this in the case of satellites, and noise is much more detrimental here

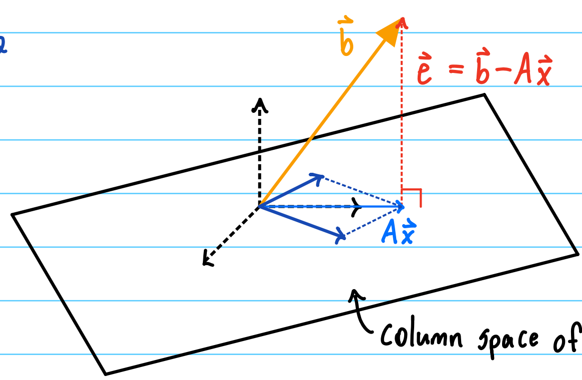
The more measurements you take, the more robust to noise you are:



Overdetermined System

Might not have unique solution; therefore, we aim to solve the following optimization problem:

$$\underset{\vec{x}}{\text{minimize}} \; \|A\vec{x}-\vec{b}\|^2$$



note: $\|A\vec{x}-\vec{b}\|^2 = \|\vec{b}-A\vec{x}\|^2$

Least-Squares solution: $\hat{\vec{x}} = \underbrace{(A^TA)}_{\text{square matrix}}{}^{-1}A^T\vec{b}$ {from yesterday's lecture}

$$A\hat{\vec{x}} = A(A^TA)^{-1}A^T\vec{b}$$

These are the fundamental ideas used in many modern ML algorithms

EECS 127 / CS 189 will start w/ LQ as the foundation of prediction

$\vec{x} = (A^TA)^{-1}A^T\vec{b}$

What if (A^TA) is not invertible? Need better experimental setup (e.g., not all measurements from the same satellite)

Recall: Matrix A is invertible if $Null(A)$ is trivial (i.e., $Null(M) = \{\vec{0}\}$)

Theorem: $Null(A^TA) = Null(A)$

Equality of two vector spaces:

① If $\vec{v} \in Null(A^TA)$ then $\vec{v} \in Null(A)$

② If $\vec{w} \in Null(A)$ then $\vec{w} \in Null(A^TA)$

$\Big\}$ need to prove this

Proof of ②:

Known: $A\vec{w} = \vec{0}$ Want to show: $(A^TA)\vec{w} = \vec{0}$

$$\begin{aligned}(A^TA)\vec{w} &= A^T(A\vec{w}) \quad \text{by Associativity of matrix multiplication} \\ &= A^T \cdot \vec{0} \\ &= \vec{0} \ \checkmark\end{aligned}$$

Recall: $(AB)^T = B^TA^T$

Dimensionality Check:

$A: m \times n$ $\quad B^T: k \times n$

$B: n \times k$ $\quad A^T: n \times m$

$AB: m \times k$ $\quad B^TA^T: k \times m \ \checkmark$

$(AB)^T: k \times m \ \checkmark$

Can start by proving for $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

and then generalize to higher dimensions to prove.

Recall: If $\vec{x} \in \mathbb{R}^n$ is such that $\|\vec{x}\| = 0$, then $\vec{x} = 0$

$\|\vec{x}\| = 0 \Rightarrow \|\vec{x}\|^2 = 0 \Rightarrow x_1^2 + x_2^2 + \cdots + x_n^2 = 0 \Rightarrow x_i = 0, \ i \in [0, n]$

Proof of ①:

Want to prove: $\vec{v} \in Null(A^TA) \Rightarrow \vec{v} \in Null(A)$

Known: $A^TA\vec{v} = \vec{0}$ Want to show: $A\vec{v} = \vec{0}$

If A^T was square and invertible:

$(A^T)^{-1}A^TA\vec{v} = (A^T)^{-1}\vec{0} \Rightarrow IA\vec{v} = \vec{0} \Rightarrow A\vec{v} = \vec{0}$

Does not work since A is tall and not square $\Rightarrow A^T$ wide + non-square

Consider: $\|A\vec{v}\|^2 = \langle A\vec{v}, A\vec{v}\rangle$

$= (A\vec{v})^T \cdot (A\vec{v})$

$= \vec{v}^TA^TA\vec{v}$

$= \vec{v}^T(A^TA\vec{v})$

$= \vec{v}^T \cdot \vec{0}$

$= 0$

$\|A\vec{v}\|^2 = 0 \Rightarrow A\vec{v} = 0$ ✓

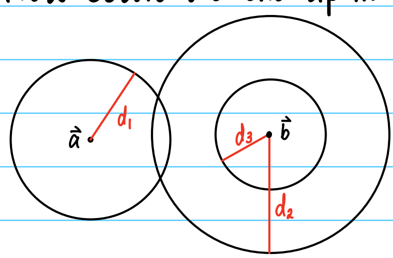
We have now proven that $Null(A^TA) = Null(A)$

Back to our original question: What if (A^TA) is not invertible?

$\Leftrightarrow$ A has a non-trivial nullspace

$\Leftrightarrow$ Columns of A are linearly dependent

How could we end up in this situation? Inconsistent/noisy/redundant measurements



① $\|\vec{x}-\vec{a}\|^2 = d_1^2$

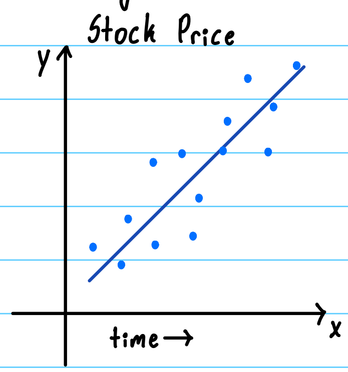
② $\|\vec{x}-\vec{b}\|^2 = d_2^2$

③ $\|\vec{x}-\vec{b}\|^2 = d_3^2$

e.g., two different distance readings from the same satellite in trilateration

Want to take independent measurements

Linear Regression: Collect data and make predictions



Want to generate a function to predict stock price

$y = \underset{\uparrow \text{slope}}{m}x + \underset{\uparrow \text{intercept}}{b}$

Known Data

(x_1, y_1)
(x_2, y_2)
$\vdots$
(x_n, y_n)

Assume that $y = mx + b$ is the model

$$\underbrace{\begin{aligned} y_1 &= m x_1 + b \\ y_2 &= m x_2 + b \\ &\vdots \\ y_n &= m x_n + b \end{aligned}}$$

Knowns: (x_i, y_i)
Unknowns: m, b

↓ Matrix-vector form

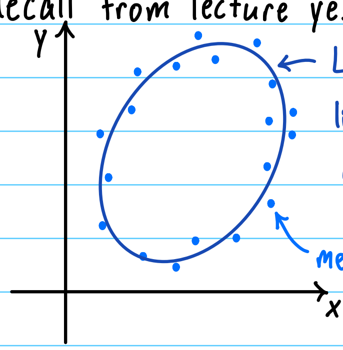
$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$n > 2 \Rightarrow$ tall matrix

What can we use to solve? Least-Squares!

Once m, b are found, can make predictions

Recall from lecture yesterday: planetary motion example (orbit of planet Ceres)



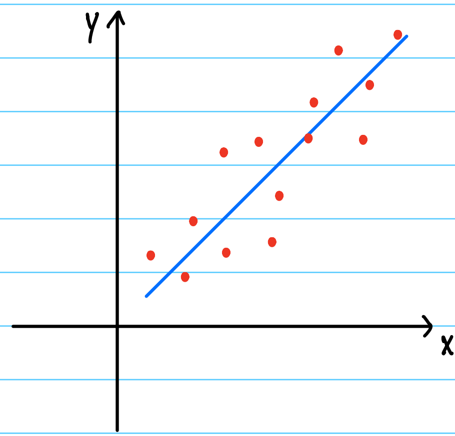
Least squares is not limited to lines; it can help fit higher order polynomials too!

Assume model: $\alpha x^2 + \beta xy + \gamma y^2 + \delta x + \varepsilon y = 1$

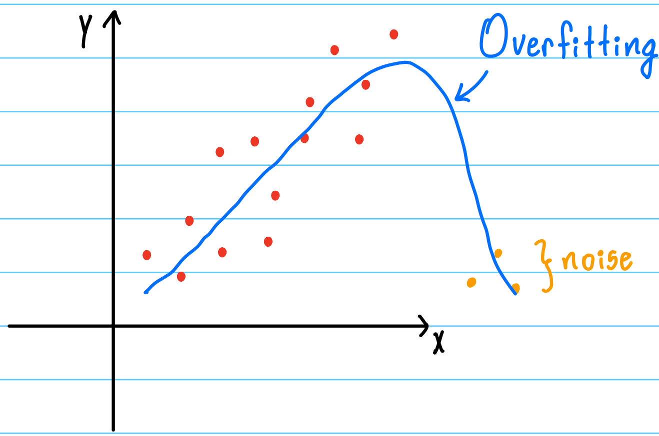
Linear in unknowns → can use LQ

Non-linearity lies in known values

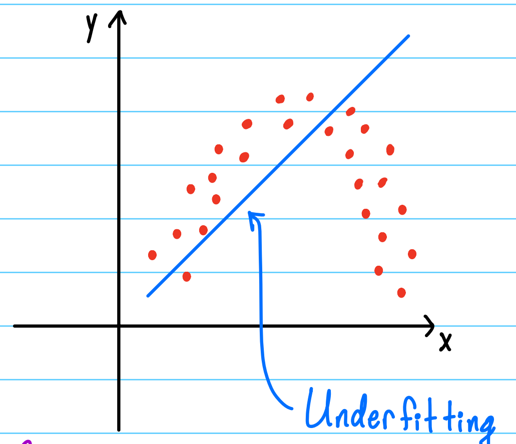
★ Refer to Note 23 section 23.4 for full application walkthrough



$y = mx + b$ ✓ Correct fit



Can fix this by using a lower order/degree model

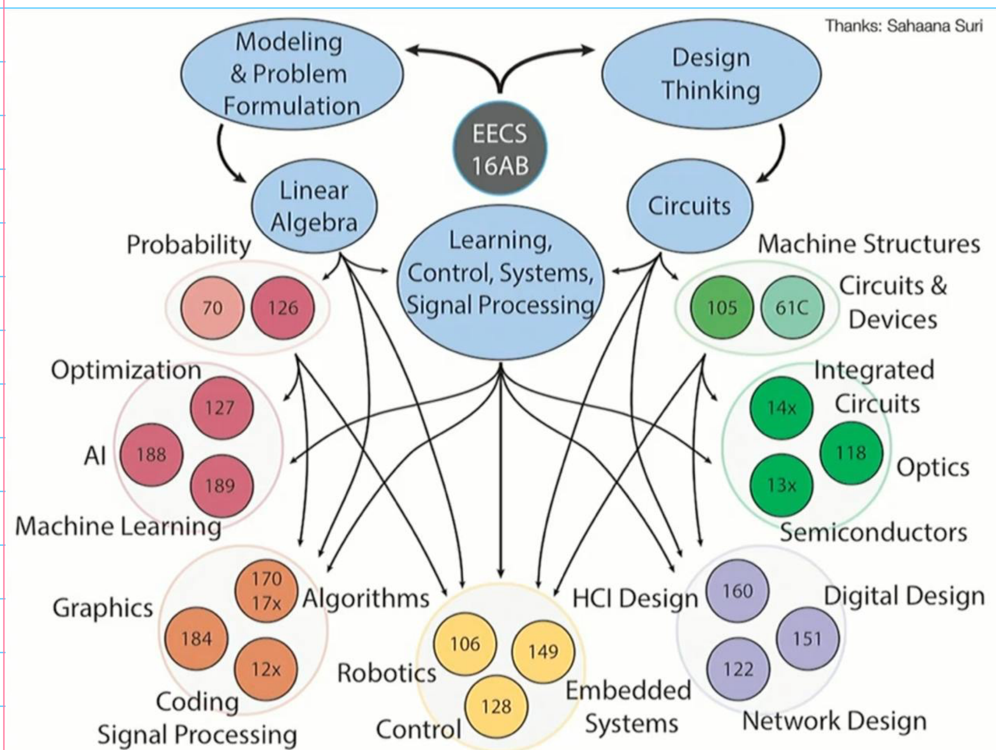


Can fix this by using a higher order/degree model

This is just the beginning

1) What you accomplished this semester
   - Built a camera
   - Built two types of touchscreen
   - Built a GPS system
2) Berkeley courses, your degree, and your career are a marathon, not a sprint
3) It's about your slope, not your intercept

EECS 16A has opened up the course ecosystem for you:



Go Forth and do Great Things!