Solution for JEE problem from 2003

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If the two circles $(x-1)^2 + (y-3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$, intersects in two distinct points, then find the conditions on r.

Graph1

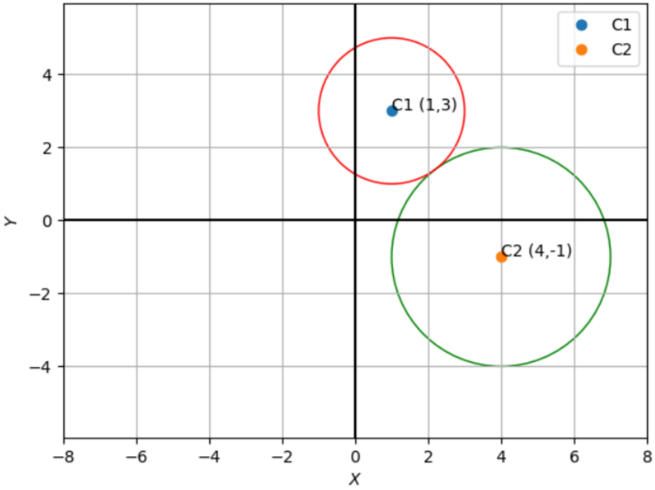

Figure: graph1

Graph2

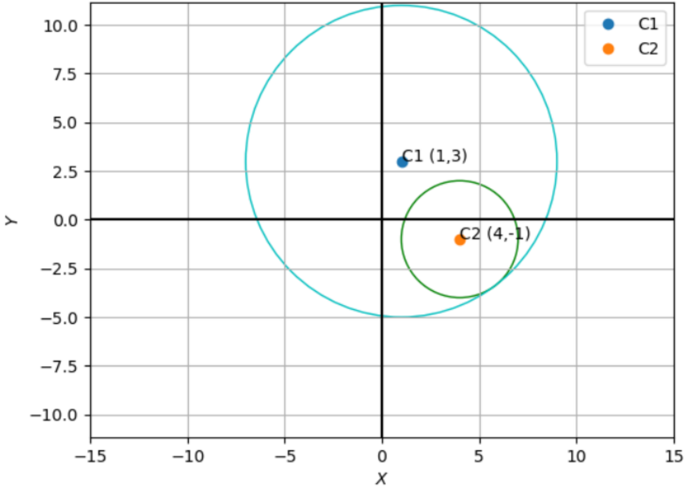

Figure: graph2

from the above graph it is clear that for given r_1 and r_2

$$|r_1 + r_2| > C_1C_2 > |r_1 - r_2|$$

Now,
General equation of a circle at a given center G and radius R is

$$C \equiv \mathbf{X}^{\mathbf{T}}\mathbf{X} - 2\mathbf{G}^T\mathbf{X} + \mathbf{G}^T\mathbf{G} - R^2 = 0$$

Given equation of fixed circle is $x^2 + y^2 - 8x + 2y + 8 = 0$
or in matrix form

$$C_1 \equiv \mathbf{X}^T\mathbf{X} - 2\begin{pmatrix} 4 & -1 \end{pmatrix}\mathbf{X} + 8 = 0$$

and the equation for variable circle is $(x-1)^2 + (y-3)^2 = r^2$
or in matrix form

$$C_2 \equiv \mathbf{X}^T\mathbf{X} - 2\begin{pmatrix} 1 & 3 \end{pmatrix}\mathbf{X} + 10 - r^2 = 0$$

solution

So radius of $C_1 is$

$$R_1 = \sqrt{\begin{pmatrix}4 & -1\end{pmatrix}^T \begin{pmatrix}4 & -1\end{pmatrix} - 8} = 3$$

and radius of $C_2 is$

$$R_2 = \sqrt{\begin{pmatrix}1 & 3\end{pmatrix}^T \begin{pmatrix}1 & 3\end{pmatrix} - 10 + r^2} = r$$

now the distance between any two points $\mathbf{P_1}$ and $\mathbf{P_2}$ is given by this relation

$$\mathbf{P_1 P_2} = ||\mathbf{P_1} - \mathbf{P_2}||$$

so distance between center of C_1 and C_2 is given by

$$C_1 C_2 = \left\|\begin{pmatrix}4 & -1\end{pmatrix} - \begin{pmatrix}1 & 3\end{pmatrix}\right\| = 5$$

So from the above discussed inequality which is

$$|r_1 + r_2| > C_1C_2 > |r_1 - r_2|$$

and substituting the values r_1, r_2, and C_1C_2 we get

$$|r + 3| > 5 > |r - 3|$$

therefore the condition on r is this

$$2 < r < 8$$

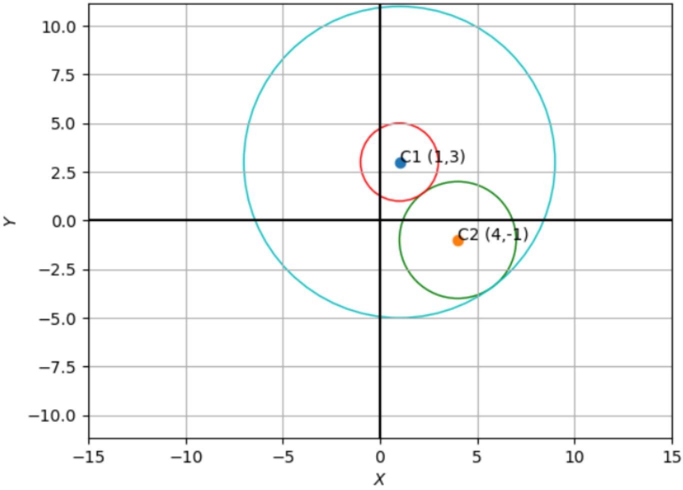

Figure: graph1