Signals and Systems

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Part I.

Frequency Domain

1. Sinusoids and LSI Systems

In the first module, we have been looking at signals and systems in what is called their natural domain. By natural domain, we mean that the independent variable of the signal is the same as it was when the signal was recorded or observed or instituted. An example of a signal in its natural domain is a speech signal expressed as a function of time, as time is the naturally associated independent variable with the speech signal.
For processing signals, we need appropriate systems. Now, what processing needs to be done by the system can be a tricky thing to specify in the natural domain.
For example, say that we want to separate the voices of male and female singers from a chorus. Although it is qualitatively easier to understand, one can't give the description of such a system in the time domain. We need to have a broader view of systems, and more importantly, of signals, to be able to tackle this problem.
So in this module we are going to take the first step towards a change of paradigm, a change of world-view in describing signals and systems. We begin by understanding which signals are special from the point of view of the signals and systems. Let's visit the ideal voltage generator for that purpose.

1.1. The Ideal Voltage generator

The ideal voltage generator consists of a circular conducting coil rotating in a constant magnetic field.

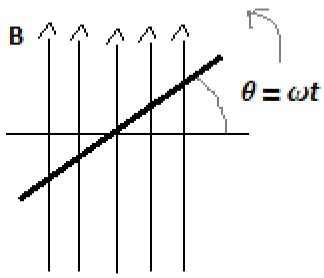


Figure 1.1.: Voltage generator (side view)

Let the area enclosed by the coil be A, the angular frequency of rotation be ω such that $\theta = \omega t$, and the value of the constant magnetic field be B. Now, by Faraday's law, the electric field $\mathcal{E}$ is defined as

$$\mathcal{E} = -\frac{\partial \Phi_B}{\partial t}$$

where the flux Φ_B in this case is given by

$$\Phi_B = BA\cos(\theta) = BA\cos(\omega t)$$

Hence the electric field will be

$$\mathcal{E} = -BA\omega\sin(\omega t)$$

This is the principle of generation of the voltage supply that we receive at our residences. This is one important reason why the sinusoidal voltage is highly favoured and deeply studied. And of course there are many other reasons why. Let us have a look at some other interesting properties of sinusoids.

1.2. Properties of sinusoids

The first interesting thing about sine waves is that when you add two sinusoids of the same frequency, it gives you back another sinusoid of the *same* frequency. Let's prove this formally. Say we have two sinusoids, $x_1(t)$ and $x_2(t)$, of the same frequency Ω, given by

$$x_1(t) = A_1 \cos(\Omega t + \phi_1)$$

$$x_2(t) = A_2 \cos(\Omega t + \phi_2)$$

Now consider a linear combination of the two sinusoids,

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$

Note that we can write x_1 and x_2 as

$$x_1(t) = A_1\{\cos(\Omega t)\cos(\phi_1) - \sin(\Omega t)\sin(\phi_1)\}$$

$$x_2(t) = A_2\{\cos(\Omega t)\cos(\phi_2) - \sin(\Omega t)\sin(\phi_2)\}$$

Hence we can write $x(t)$ as

$$x(t) = [\alpha_1 A_1 \cos(\phi_1) + \alpha_2 A_2 \cos(\phi_2)]\cos(\Omega t) + [-\alpha_1 A_1 \sin(\phi_1) - \alpha_2 A_2 \sin(\phi_2)]\sin(\Omega t)$$

Writing the two time-independent coefficients as P and Q, we get

$$x(t) = P\cos(\Omega t) + Q\sin(\Omega t) = \sqrt{P^2+Q^2}\left\{\frac{P}{\sqrt{P^2+Q^2}}\cos(\Omega t) + \frac{Q}{\sqrt{P^2+Q^2}}\sin(\Omega t)\right\}$$

This can be easily seen to be

$$x(t) = \sqrt{P^2+Q^2}\cos(\Omega t + \Phi)$$

with

$$\Phi = -\tan^{-1}\left(\frac{Q}{P}\right)$$

Hence, linearly combining two sinusoids of the same frequency gives back a sinusoid with the same frequency.
Another interesting thing about sinusoids is that when we differentiate a sine wave, we get back a sinusoid of the same frequency.

$$\frac{d}{dt}\{A\cos(\Omega t + \phi)\} = -A\Omega\sin(\Omega t + \phi)$$

It doesn't matter whether it is cosine or sine since they differ only by a phase of $\pi/2$.

$$\cos(\Omega t + \phi) = \sin(\Omega t + \phi + \pi/2)$$

Now let's see how we can describe a change in amplitude of a sinusoid. Suppose the original sinusoid is given by

$$x_1(t) = A\cos(\Omega t + \phi)$$

and let the changed sinusoid is given by

$$x_2(t) = B\cos(\Omega t + \phi)$$

where A and B are positive constants. Then,

$$x_2(t) = \frac{B}{A}x_1(t)$$

So the change of amplitude in a sinusoid is simply described by a multiplying constant. But this is not true for a phase change. Suppose the original sinusoid is given by

$$x_1(t) = A\cos(\Omega t + \phi_1)$$

and the changed sinusoid is given by

$$x_2(t) = A\cos(\Omega t + \phi_2)$$

Then, unlike the amplitude case, $x_2(t)/x_1(t)$ doesn't turn out to be a constant independent of time. This will cause a problem while dealing with inductors or capacitors. In inductors, for example, the voltage drop is proportional to the derivative of the current passing through it. Hence, if a sinusoidal current is passing through it, the voltage drop across it will also be a sinusoid, but will have a phase difference of 90^0 with the current. Hence, we cannot establish a relationship like the resistor ($V = RI$) in case of the inductor since V/I won't be a constant independent of time.
To deal with this problem, we will have to use the rotating complex number representation of the sinusoid.

1.3. Sinusoids as Rotating Complex Numbers

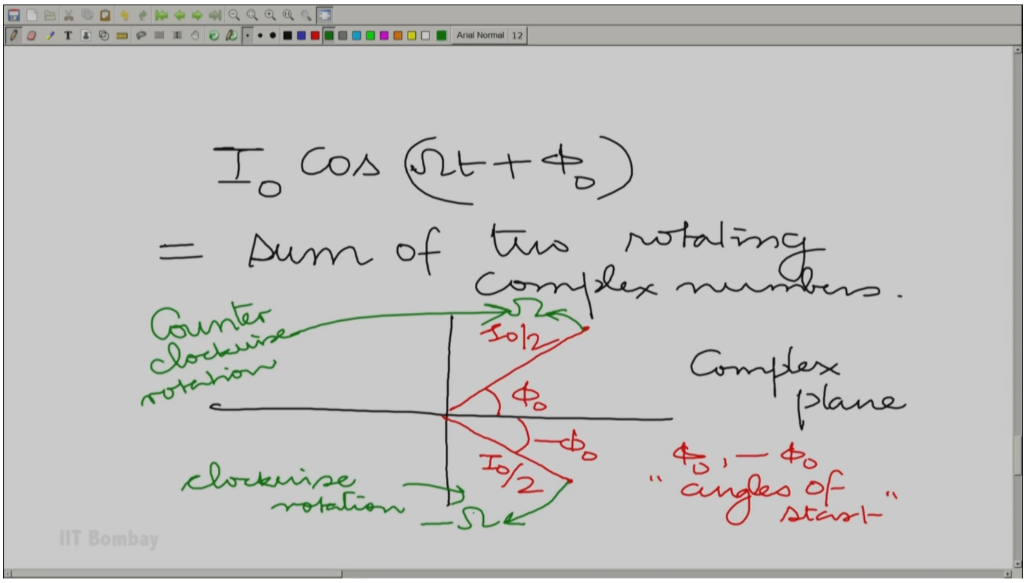
We can think of a sinusoid $I_0\cos(\Omega t + \phi_0)$ as a combination of two rotating complex numbers. Let the first begin at time $t = 0$ with an angle of ϕ_0. Let it have a magnitude of $I_0/2$ and let the other begin with the same magnitude $I_0/2$ but the opposite starting angle $(-\phi_0)$. Let them both rotate with an angular velocity of Ω, but the first one in counter clockwise and the second one in clockwise direction.
We can express these two complex numbers in the polar form as

$$c_1 = \frac{I_0}{2}e^{j\{\Omega t+\phi_0\}}$$

$$c_2 = \frac{I_0}{2}e^{-j\{\Omega t+\phi_0\}}$$

where $j = \sqrt{-1}$. One can see that $c_1 + c_2$ gives back the original sinusoid.
It seems like a silly thing to do to describe a sinusoid as a combination of two rotating complex numbers. But the reason for doing this is clear when we look at the inductor once again.
Let's assume for now that we provide to the inductor, an input which is just one of those rotating complex numbers. So

$$I = \frac{I_0}{2}e^{j\{\Omega t+\phi_0\}}$$



Now, the voltage across an inductor is given by

$$V = L\frac{dI}{dt} = L\frac{I_0}{2}(j\Omega)e^{j\{\Omega t+\phi_0\}}$$

Hence we can see that

$$V/I = j\Omega L$$

Hence, now the ratio of the voltage and current for an inductor is a complex *constant independent of time*!
Similarly, for the other rotating complex number, we will get V/I again to be a constant, but with a minus sign $(-j\Omega L)$. So in some sense the actual angular frequency, whether positive or negative, is reflected. It is left as an exercise to do a similar analysis for capacitances and verify that for capacitors,

$$V/I = \frac{1}{j\Omega C}$$

This makes the analysis of RLC circuits (circuits consisting of Resistor(s), Inductor(s) and Capacitors(s)) much easier as we can interpret these constants as generalised resistances, or *impedances*. In that sense, the resistor has an impedance of R, an inductor has an inductance of $j\Omega L$ and a capacitor has an impedance of $1/j\Omega C$.
Notice that c_1 and c_2 defined above are complex conjugates of each other. So whatever happens to c_1 is mirrored in c_2. For example, the V/I for c_2 $(-j\Omega L)$ is the complex conjugate of the V/I for c_1 $(j\Omega L)$.
The analysis of sinusoids using rotating complex numbers is known as 'phasor' analysis. As the rotating complex numbers are constant in magnitude, but change their phase, they are called 'phasors'.
This is the reason why we earlier demanded that our systems should accept a complex signal in general.
In a nutshell, dealing with amplitudes is easy but the dealing with phases is a problem in sinusoids and that problem is overcame when you go to the phasor instead of the corresponding sinusoid.

1.4. Understanding Phase Change In Sinusoids and Phasors

If we consider a sinusoid as a combination of two rotating complex numbers, which we will henceforth call phasors, moving in opposite direction,

$$2Acos(\Omega t+\phi)=Ae^{j(\Omega t+\phi)}+Ae^{-j(\Omega t+\phi)}$$

Let us focus our attention on one of the phasors for now. Let us look at what happens when we introduce a phase change.

$$Ae^{j(\Omega t+\phi)}\xrightarrow{Phase\ Change}Ae^{j(\Omega t+\phi+\Delta\phi)}$$
$$Ae^{j(\Omega t+\phi+\Delta\phi)}=Ae^{j(\Omega t+\phi)}e^{j\Delta\phi}$$

We hence find that a change of phase results just in a multiplying factor which is a constant independent of time.
Now we try to do a similar calculation for the complex conjugate (the phasor moving in the opposite direction).

$$Ae^{-j(\Omega t+\phi)}\xrightarrow{Phase\ Change}Ae^{-j(\Omega t+\phi)}e^{-j\Delta\phi}$$

Notice that it is the complex conjugate of the previous multiplying factor we found.

$$2Acos(\Omega t+\phi)\xrightarrow{Phase\ Change}Ae^{j(\Omega t+\phi)}e^{j\Delta\phi}+Ae^{-j(\Omega t+\phi)}e^{-j\Delta\phi}$$

We hence understand that using only one phasor is enough to fully determine the behaviour of the sinusoid. This explains why we used complex signals in our previous analysis, due to their ease in mathematical understanding.
We will now see the result of inputting a sinusoidal signal to a stable linear shift invariant system, with impulse response h(t) (Assume that the impulse response is real).
We deal with stable LSI because what we can say from the stability of the system is that y(t) is going to be bounded.

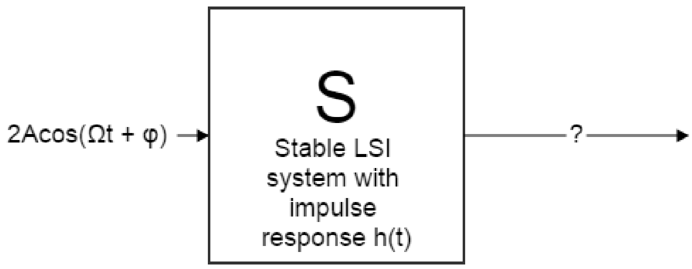


Figure 1.2.: Sinusoidal Input To Stable LSI System

The direct solution via convolution seems very difficult. In fact if you try to calculate via the convolution integral, we get the following expression.

$$y(t)=\int_{-\infty}^{+\infty}h(\lambda)2Acos(\Omega(t-\lambda)+\phi)d\lambda$$

We will now see how to solve it using phasors, making the analysis mathematically convenient.

1.5. Phasor Input to LSI System

Consider the output when a phasor is input into a Stable LSI system.

$$Ae^{j(\Omega t+\phi)} \xrightarrow{LSI} \int_{\infty}^{\infty} Ae^{j(\Omega(t-\lambda+\phi)}h(\lambda)d\lambda$$

$$= Ae^{j(\Omega t+\phi)} \int_{\infty}^{\infty} e^{-j\Omega\lambda}h(\lambda)d\lambda$$

So, we see that we obtain the input phasor, with a multiplying factor $\int_{-\infty}^{\infty} e^{-j\Omega\lambda}h(\lambda)d\lambda$, which is dependent only on Ω and h (it is constant with respect to time).
We will now come back to the importance of why we chose our system to be stable. We will do this by trying to find a bound to the absolute value of the multiplying factor we just found.

$$|\int_{-\infty}^{\infty} e^{-j\Omega\lambda}h(\lambda)d\lambda| \leq \int_{-\infty}^{\infty} |e^{-j\Omega\lambda}||h(\lambda)|d\lambda$$

$$\int_{-\infty}^{\infty} |e^{-j\Omega\lambda}||h(\lambda)|d\lambda = \int_{-\infty}^{\infty} |h(\lambda)|d\lambda$$

And as per the condition of stability the integral of the absolute value of the impulse response is bounded. Hence the absolute value of the multiplying factor is also bounded. In fact we have obtained a concrete bound to this integral, namely the absolute integral of the impulse response. We can do similar mathematical analysis for the conjugate as well.
This multiplying factor is called the *Frequency Response* of the LSI system at angular frequency Ω.

$$Frequency\ Response\ H(\Omega) = \int_{-\infty}^{\infty} e^{-j\Omega\lambda}h(\lambda)d\lambda$$

1.6. Sinusoid Input to LSI System

We will now come back to our original problem of inputting a sinusoidal signal into a stable linear shift invariant system. We'll begin by simplifying the outputs of the two phasors.

$$H(\Omega)Ae^{j(\Omega t+\phi)} = |H(\Omega)|Ae^{j(\Omega t+\phi+\angle H(\Omega))}$$

The complex conjugate of the frequency response is given by

$$\bar{H}(\Omega) = \int_{-\infty}^{\infty} e^{j\Omega\lambda}h(\lambda)d\lambda = H(-\Omega)$$

Hence,

$$H(-\Omega)Ae^{-j(\Omega t+\phi)} = |H(\Omega)|Ae^{-j(\Omega t+\phi+\angle H(\Omega))}$$

Hence by the formula for summing conjugate complex numbers, the output for inputting a sinusoid given by $2Acos(\Omega t+\phi)$ is given by:

$$2Acos(\Omega t+\phi) \xrightarrow{Stable\ LSI} 2A|H(\Omega)|cos(\Omega t+\phi+\angle H(\Omega))$$

This indicates that on passing through a stable LSI system, a sinusoid goes through an amplitude and phase shift independent of time, and only dependent on the impulse response and angular frequency of the sinusoid.
Also, if we view an input to a stable LSI system as the sum of multiple sinusoids, the

output can be visualized as the sum of the corresponding outputs of the sinusoids through the stable LSI system. The stability of the LSI systems guarantees the existence of the frequency response of the system. But if it is unstable it may or may not have a frequency response.

1.7. Periodic Input to Shift Invariant Systems

Periodic inputs can be represented by a linear combination (which can be countably infinite) of sinusoids having frequencies which are multiples of the frequency of the input. This property is referred to as a Fourier Series expansion of the input.
The Fourier Series expansion and our understanding from the previous section gives us a simple method to compute the output of a periodic input in an LSI system as the linear combination of the output of these sinusoids.
First consider the following lemma.
Lemma: A periodic input to a shift invariant system produces a periodic output.
Proof:

Let input x(t) with period T

$$x(t+T) = x(t) \quad \forall t$$

$$x(t) \xrightarrow{SI\ system} y(t)$$

Now by shift invariance

$$x(t+T) \xrightarrow{SI\ system} y(t+T)$$

$$y(t) = y(t+T) \quad \forall t$$

Hence the output is also periodic, infact with the same period T.

1.8. Conclusion

In this lecture we understood the analysis of inputting sinusoidal input to a stable LSI system. We saw how we could simplify our analysis by the use of phasors. In the upcoming lecture we will learn about the use of inner product and vector analysis in understanding signals.

1.9. Introduction

In the previous lecture, we have learnt how a periodic input given to a Linear Shift Invariant system results in a periodic output with the same period. Now in this lecture we will see some new concepts by which every input can be expressed as sinusoidal inputs. Henceforth, it will be simpler to analyse their outputs.

1.10. Relations between Signals and Vectors

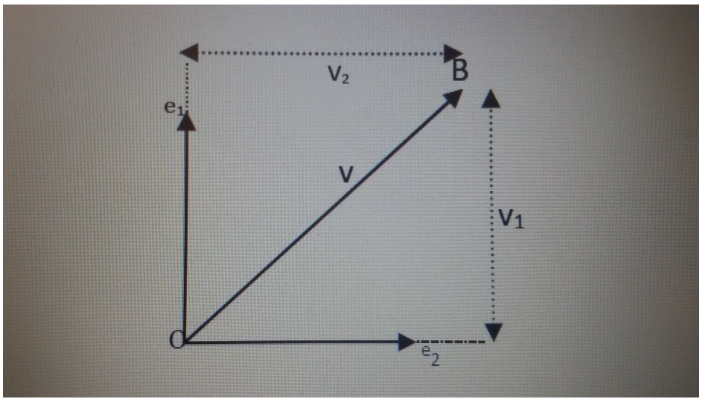
Consider a 2-dimensional vector, we can decompose this vector into two perpendicular components with e_1 and e_2 as the unit vectors along those directions. To find these components, we need to use Dot Product.

1.10.1. Dot Product

Dot Product of two vectors u and v is defined as the magnitude of vector u multiplied by the magnitude of v multiplied by cosine of the angle between these two vectors.

$$u.v = |u||v|cos\theta$$

where θ is the angle between vectors u and v. If u is a unit vector, then the dot product of vectors v and u gives the component of vector v in the direction of u, with the value |v|cosθ.



Consider the above figure, here OB is a vector denoted by v which is expressed as

$$v = v_1e_1 + v_2e_2$$

where v_1 and v_2 are the components in e_1 and e_2 directions respectively. By this we can say that vectors e_1 and e_2 spans the space in 2D i.e. we can express any vector as a linear combination of e1 and e2. A collection of vectors span a space. Suppose if any vector can be expressed uniquely as a linear combination of $\{u_1,u_2,u_3 \; .... \; u_n\}$, where $\{u_1,u_2,u_3...u_n\}$ are linearly independent, then $\{u_1,u_2...u_n\}$ is said to form a basis.
Also, if $\langle u_i,u_j \rangle = 0$ for i,j belonging from 1 to n,i.e. dot product of any two vectors is zero, then the basis is said to be an *orthogonal basis*. Finally, for a n-dimensional space, a collection of n linearly independent vectors forms the basis.
The dot product of two vectors u and v is equal to the sum of the products of the corresponding perpendicular components. Suppose

$$u = u_1e_1 + u_2e_2$$

$$v = v_1e_1 + v_2e_2$$

therefore,

$$u.v = u_1v_1 + u_2v_2$$

1.10.2. Dot Product of Discrete Sequences

Now, consider a discrete sequence with 2 non-zero points. This sequence can be compared to a 2-dimensional vector v with v_1 and v_2 as its perpendicular components such that it is equal to the values at the 2 non zero points of the sequence. Similarly for a sequence with n non-zero points can be considered as a vector in n dimensional space. Also, the concept of dot product is similarly applied to the discrete sequences.

For example : Consider x[n] = $(1/2)^n$ u[n];
i.e.

$$x[n] = (1/2)^n \quad for \;\; n \geq 0$$

$$0 \quad for \;\; n < 0$$

$$y[n] = (1/3)^n u[n]$$

i.e.

$$y[n] = (1/3)^n \quad for \;\; n \geq 0$$

$$0 \quad for \;\; n < 0$$

Lets calculate the dot product of x[n] and y[n],i.e. summing the product of corresponding components. We have,

$$\langle x[n], y[n] \rangle = \sum_{n=-\infty}^{\infty} \; x[n]y[n]$$

$$\langle x[n], y[n] \rangle = \sum_{n=0}^{\infty} \; x[n]y[n]$$

$$\langle x[n], y[n] \rangle = \sum_{n=0}^{\infty} \; (1/2)^n * (1/3)^n$$

$$\langle x[n], y[n] \rangle = \sum_{n=0}^{\infty} \; (1/6)^n$$

$$\langle x[n], y[n] \rangle = \frac{1}{(1-(1/6))}$$

$$\langle x[n], y[n] \rangle = \frac{6}{5}$$

By this we compute the dot product of two discrete sequences. This dot product is also called as *Inner Product*. Inner product of two vectors u and v is represented by $\langle u, v \rangle$.

1.10.3. Inner Product of Continuous Signals

In a discrete sequence, a unit vector can be expressed as $\delta[n-N]$ for all integers Z. A n-dimensional vector v can be expressed as

$$v = v_1 e_1 + v_2 e_2 + ... + v_n e_n$$

Now writing the above equation in terms of sequences, we have

$$x[n] = \sum_{n=-\infty}^{\infty} \; x[N]\delta[n-N]$$

where x[n] is the component along dimension N.

Similarly applying the same concept for continuous time functions, we need to replace summation by integral, which gives,

$$x(t) = \int_{-\infty}^{\infty} x(\lambda)\delta[t-\lambda] \; \mathrm{d}\lambda$$

where x(λ) is the component in direction λ and $\delta[t-\lambda]$ is continuous impulse at λ similar to unit vector.So inner product of x(t) and y(t) is given by

$$\langle x(t), y(t)\rangle = \int_{-\infty}^{\infty} x(t)y(t)\ \mathrm{d}t$$

1.11. Sinusoids with same period

Let us see how we can express any signal as sinusoidal functions. So, first we need to find sinusoidal signals which are perpendicular. Consider a signal x(t) which is periodic with period T,i.e.

$$x(t) = x(t+T)$$

. Assume that it can be expressed as a sum of sinusoidal functions. We should take a sinusoidal function which is periodic with period T. So it is of the form $A_k \cos(\frac{2\pi}{T}kt+\phi_k)$. We have,

$$x(t) = \sum_{k=-\infty}^{\infty} A_k \cos(\frac{2\pi}{T}kt+\phi_k)$$

Let's take two different k .

$$x_1(t) = A_1 \cos(\frac{2\pi}{T}k_1 t+\phi_1)$$

and

$$x_2(t) = A_2 \cos(\frac{2\pi}{T}k_2 t+\phi_2)$$

Consider the inner product of $x_1(t)$ and $x_2(t)$ with the interval going from 0 to T. We are here restricting our interval to T because the integral might diverge when we integrate from 0 to ∞. We have,

$$\langle x_1(t), x_2(t)\rangle = \int_0^T x_1(t)x_2(t)\ \mathrm{d}t$$

$$\langle x_1(t), x_2(t)\rangle = \int_0^T \cos(\frac{2\pi}{T}k_1 t+\phi_1)\cos(\frac{2\pi}{T}k_2 t+\phi_2)\ \mathrm{d}t$$

Using

$$2cosAcosB = cos(\frac{A+B}{2}) + cos(\frac{A-B}{2})$$

We have,

$$\langle x_1(t), x_2(t)\rangle = \int_0^T \left\{\cos(\frac{2\pi}{T}\frac{(k_1+k_2)}{2}t+\frac{\phi_1+\phi_2}{2}) + \cos(\frac{2\pi}{T}\frac{(k_1-k_2)}{2}t+\frac{\phi_1-\phi_2}{2})\right\}\ \mathrm{d}t$$

If (k_1 + k_2) or (k_1 - k_2) are not zero, that means we have a finite number of cycles of the sinusoids, implying the integral is zero. However if k_1=k_2, the second integral becomes T times $\cos(\theta_1-\theta_2)$. Thus we have,

$$\langle x_1(t), x_2(t)\rangle \neq 0 \quad for\ \ k_1 = k_2$$

$$\langle x_1(t), x_2(t)\rangle = 0 \quad for\ \ k_1 \neq k_2$$

Hence, we have proved that two sinusoids with same time period are perpendicular if they don't have same angular frequency and vice-versa. Thus using this important concept, we will be able to write any signal as sum of sinusoidal signals and hence analysis of these signals will be simpler.