Probability and Random Processes

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Q)Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables each having probability density function

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

For $n \geq 1$, let $Y_n = |X_{2n} - X_{2n-1}|$. If $\overline{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$ for $n \geq 1$ and $\{\sqrt{n}(e^{-\overline{Y}_n} - e^{-1})\}_{n\geq 1}$ converges in distribution to a normal random variable with mean 0 and variance σ^2, then σ^2 (rounded off to two decimal places) equals (GATE ST 2023)

Solution:

1) Let $X, Y \sim \exp(1)$ and $Z = X - Y$

$$p_X(x) = e^{-x}u(x) \tag{1}$$
$$M_X(s) = E\left(e^{-sX}\right) \tag{2}$$
$$= \int_0^\infty e^{-sx}e^{-x}\,dx \tag{3}$$
$$= \frac{1}{s+1} \tag{4}$$

ROC for $M_X(s)$: $\text{Re}(s) > -1$
Similarly,

$$M_Y(s) = \frac{1}{s+1} \tag{5}$$
$$M_Y(-s) = \frac{1}{-s+1} \tag{6}$$

ROC for $M_Y(-s)$: $\text{Re}(s) < 1$

$$M_Z(s) = E\left(e^{-sZ}\right) \tag{7}$$

Using,

$$Z = X - Y \tag{8}$$
$$\implies M_Z(s) = E\left(e^{-s(X-Y)}\right) \tag{9}$$
$$= E\left(e^{-sX}\right)E\left(e^{sY}\right) \tag{10}$$
$$= M_X(s)M_Y(-s) \tag{11}$$
$$= \frac{1}{s+1} \times \frac{1}{-s+1} \tag{12}$$
$$M_Z(s) = \frac{1}{1-s^2} \tag{13}$$

The ROC for the laplace transform : $|\text{Re}(s)| < 1$

$$M_Z(s) = \frac{1}{2}\left(\frac{1}{1-s} + \frac{1}{1+s}\right) \tag{14}$$

Using Inverse Laplace transform,

$$P_Z(x) = \frac{1}{2}\left(e^{x}u\left(-x\right) + e^{-x}u\left(x\right)\right) \tag{15}$$
$$p_Z(x) = \frac{1}{2}e^{-|x|} \tag{16}$$
$$\implies Z \sim \text{Lap}\left(0, 1\right) \tag{17}$$

2) Let $T = |Z|$

$$p_Z(x) = \frac{1}{2}e^{-|x|} \tag{18}$$
$$F_Z(x) = \int_{-\infty}^{x} \frac{1}{2}e^{-|t|}\,dt \tag{19}$$
$$= \frac{1}{2} + \frac{1}{2}e^{-x} \tag{20}$$
$$F_T(x) = \Pr\left(T \leq x\right) \tag{21}$$
$$= \Pr\left(|Z| \leq x\right) \tag{22}$$
$$= \Pr\left(-x \leq Z \leq x\right) \tag{23}$$
$$F_T(x) = \frac{1}{2} - \frac{1}{2}e^{-x} - \left(-\frac{1}{2} + \frac{1}{2}e^{-x}\right) \tag{24}$$
$$F_T(x) = 1 - e^{-x} \text{ for } x > 0 \tag{25}$$
$$T \sim \exp\left(1\right) \tag{26}$$
$$\implies |Z| \sim \exp\left(1\right) \tag{27}$$

Using equations (8) and (27), we get:

$$|X_{2n} - X_{2n-1}| \sim \exp(1) \tag{28}$$
$$\implies Y_n \sim \exp(1) \tag{29}$$
$$M_{Y_n}(s) = \frac{1}{1+s} \tag{30}$$
$$\text{E}\left(Y_n\right) = \mu_1 \tag{31}$$
$$\mu_1 = -\frac{dM_{Y_n}(s)}{ds} \tag{32}$$
$$= -\frac{d}{ds}\left(\frac{1}{s+1}\right)\bigg|_{s=0} \tag{33}$$

$$= \left. \frac{1}{(s+1)^2} \right|_{s=0} \tag{34}$$
$$\mathrm{E}\left(Y_n\right) = 1 \tag{35}$$
$$\mathrm{E}\left({Y_n}^2\right) = \mu_2 \tag{36}$$
$$\mu_2 = \frac{d^2 Y_n(s)}{ds^2} \tag{37}$$
$$= \left. \frac{d^2}{ds^2}\left(\frac{-1}{(s+1)^2}\right) \right|_{s=0} \tag{38}$$
$$= \left. \frac{2}{(s+1)^3} \right|_{s=0} \tag{39}$$
$$\mathrm{E}\left({Y_n}^2\right) = 2 \tag{40}$$

$$\mathrm{Var}(Y_n) = \mathrm{E}((Y_n - \mathrm{E}(Y_n))^2) \tag{41}$$
$$= \mathrm{E}((Y_n - 1)^2) \tag{42}$$
$$= \mathrm{E}({Y_n}^2) - 2\mathrm{E}(Y_n) + 1 = 1 \tag{43}$$

3) We know,

$$\overline{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i \tag{44}$$
$$\mathrm{E}\left(\overline{Y}_n\right) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}\left(Y_i\right) \tag{45}$$
$$= \frac{1}{n}\cdot(n) = 1 \tag{46}$$
$$\mathrm{E}\left(\overline{Y}_n\right) = 1 \tag{47}$$

$$\mathrm{var}\left(\overline{Y}_n\right) = \mathrm{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)^2\right] - \left(\mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right]\right)^2 \tag{48}$$
$$= \frac{1}{n^2}\left\{\mathrm{E}\left[\left(\sum_{i=1}^{n} Y_i\right)^2\right] - \left(\mathrm{E}\left[\sum_{i=1}^{n} Y_i\right]\right)^2\right\} \tag{49}$$

But

$$\mathrm{E}\left[\left(\sum_{i=1}^{n} Y_i\right)^2\right] = \mathrm{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} Y_i Y_j\right] \tag{50}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{E}\left[Y_i Y_j\right] \tag{51}$$

and

$$\left(\mathrm{E}\left[\sum_{i=1}^{n} Y_i\right]\right)^2 = \left(\sum_{i=1}^{n} \mathrm{E}\left[Y_i\right]\right)^2 \tag{52}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{E}\left[Y_i\right]\mathrm{E}\left[Y_j\right] \tag{53}$$

Putting (51) and (53) in (49), and using the definition of covariance,

$$\mathrm{var}\left(\overline{Y}_n\right) = \frac{1}{n^2}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\mathrm{E}\left[Y_i Y_j\right] - \mathrm{E}\left[Y_i\right]\mathrm{E}\left[Y_j\right]\right)\right\} \tag{54}$$
$$= \frac{1}{n^2}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{cov}\left(Y_i, Y_j\right)\right\} \tag{55}$$

As all the variables are i.i.d's and are thus uncorrelated,

$$\mathrm{cov}\left(Y_i, Y_j\right) = \begin{cases} 0 & \text{if } i \neq j \\ \mathrm{var}\left(Y_i\right) & \text{if } i = j \end{cases} \tag{56}$$

Putting (56) in (55),

$$\mathrm{var}\left(\overline{Y}_n\right) = \frac{1}{n^2}\left(\sum_{i=1}^{n}\mathrm{cov}\left(Y_i, Y_i\right)\right) \tag{57}$$
$$= \frac{1}{n^2}\left(\sum_{i=1}^{n}\mathrm{var}\left(Y_i\right)\right) \tag{58}$$
$$= \frac{1}{n^2}\cdot n = \frac{1}{n} \tag{59}$$
$$\mathrm{var}\left(\overline{Y}_n\right) = \frac{1}{n} \tag{60}$$
$$\implies \mathrm{E}\left(\overline{Y}_n\right) = 1 \text{ and } \mathrm{Var}\left(\overline{Y}_n\right) = \frac{1}{n} \tag{61}$$

By the Central Limit Theorem, $n \to \infty \implies \sqrt{n}(Y_n - \mu) \to \mathcal{N}(0,1)$

$$\frac{\overline{Y_n} - 1}{\frac{1}{\sqrt{n}}} \sim \mathcal{N}(0,1) \tag{62}$$
$$\sqrt{n}(\overline{Y_n} - 1) \sim \mathcal{N}(0,1) \tag{63}$$

We know,

$$\sqrt{n}(Y_n - k) \sim \mathcal{N}(0, \sigma^2) \tag{64}$$

Let us write the taylor expansion of $g\left(Y_n\right)$

around k

$$g(Y_n) = g(k) + g'(k)(Y_n - k) + \frac{1}{2}g''(k)(Y_n - k)^2 + . \tag{65}$$

Apply the Central Limit Theorem (CLT) to the standardized variable Z_n

$$Z_n = \frac{\sqrt{n}g'(k)(Y_n - k)}{\sigma\sqrt{n}} \tag{66}$$

$$n \to \infty \implies Z_n \to \mathcal{N}(0, 1) \tag{67}$$

Compare with standardised variable we get,

$$\implies \sqrt{n}(g(Y_n) - g(k)) \sim \mathcal{N}(0, \sigma^2[g'(k)]^2) \tag{68}$$

$$g(x) = e^{-x} \implies g'(x) = -e^{-x} \tag{69}$$

Using equation (68), we get:

$$\sqrt{n}(e^{-\overline{Y}_n} - e^{-1}) \sim \mathcal{N}(0, e^{-2}) \tag{70}$$

$$\implies \sigma^2 = e^{-2} = 0.14 \tag{71}$$

Steps for Simulation:

1) rand() / (double)RAND MAX:
   This generates a random variable between 0 and RAND MAX and divides it by RAND MAX to obtain a uniform distribution between 0 and 1
2) -log(rand() / (double)RAND MAX) :
   This transforms the uniform distribution between 0 and 1 into an exponential distribution by making the values vary from 0 to ∞.
3) Generate '2n' samples of Random Variable X from the given probability density function.
4) Now generate 'n' samples of $Y = |X_{2n} - X_{2n-1}|$.
5) Now find $\overline{Y}$ which is mean of 'n' samples of Y.
6) Now calculate $\sqrt{n}(e^{-\overline{Y}_n} - e^{-1})$ as result.
7) Now repeat the process for 'm' simulations so that we get m results.
8) Calculate the mean and the variance of the 'm' results obtained earlier using basic mean and variance formula.

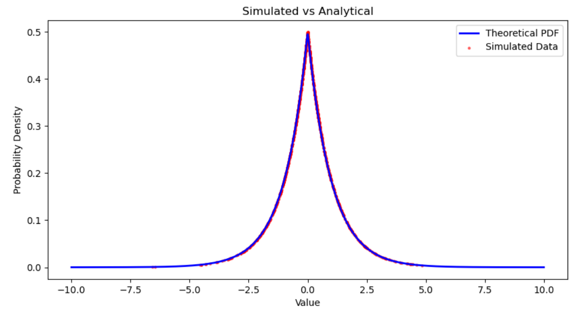

Fig. 1: pdf of the laplacian

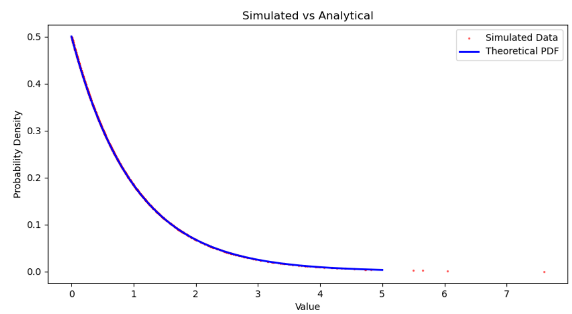

Fig. 2: pdf of absolute of the laplacian

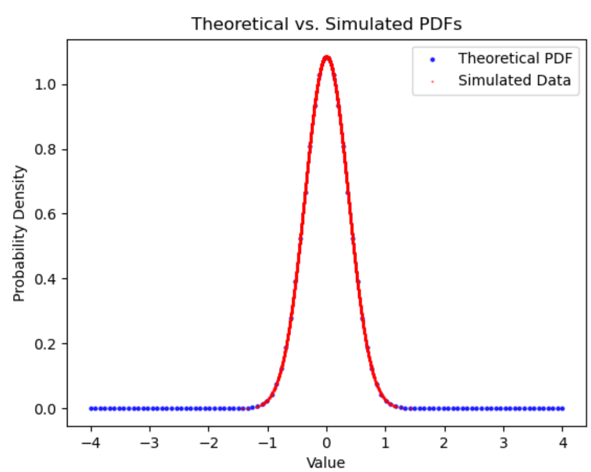

Fig. 3: Gaussian pdf

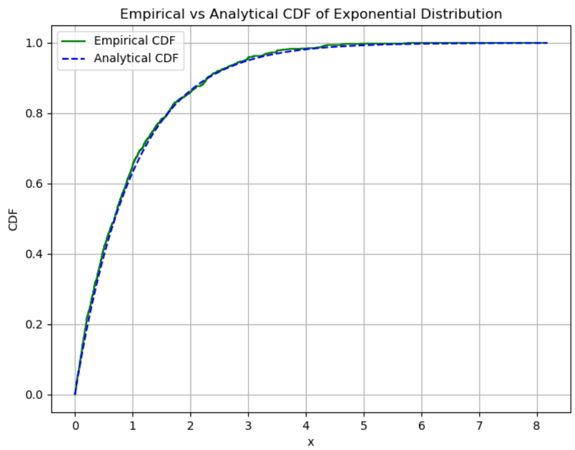

Fig. 4: Cdf of X

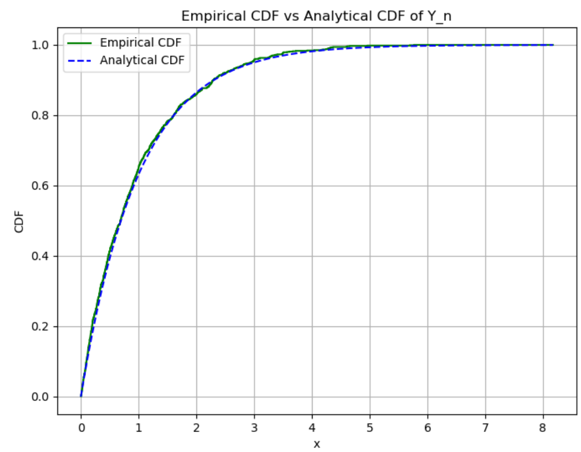

Fig. 7: Cdf of Y_n

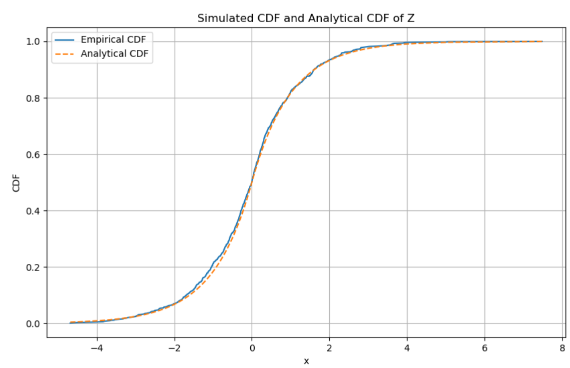

Fig. 5: Cdf of Z

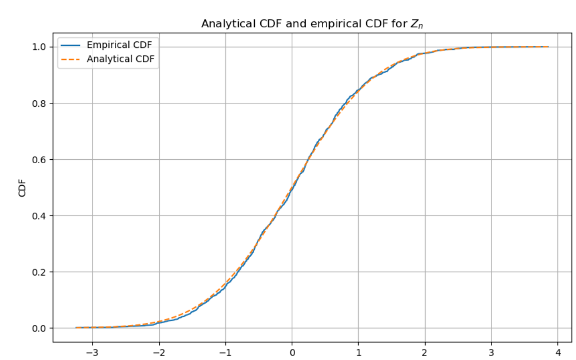

Fig. 8: Cdf of Z_n

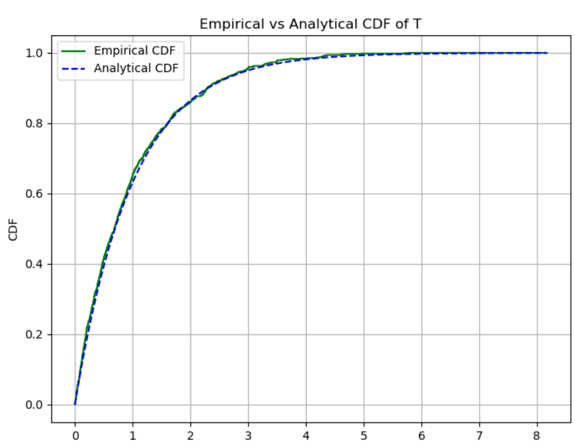

Fig. 6: Cdf of T