

February 9, 2024

Exercise 9.5

25) Find the sum of the following series up to n terms and obtain the Z-transform:

$$\dots + 0 + \frac{1^3}{1} + \frac{1^3 + 2^3}{1 + 3} + \frac{1^3 + 2^3 + 3^3}{1 + 3 + 5} + \dots$$

Solution:

| Symbol | Description |
|--------|----------------------------|
| $x(n)$ | general term of the series |
| $s(n)$ | sum of terms until x(n) |
| $X(z)$ | Z-transform of x(n) |
| $S(z)$ | Z-transform of s(n) |

Table 0: Parameters

$$x_1(n) = \sum_{i=0}^n (i+1)^3 \quad (3)$$

$$= (n+1)^3 * u(n) \quad (4)$$

$$x_1(n) \xrightarrow{Z} X_1(z) \quad (5)$$

$$X_1(z) = \left(\frac{1 + 4z^{-1} + z^{-2}}{(1 - z^{-1})^4} \right) \left(\frac{1}{1 - z^{-1}} \right) \quad (6)$$

$$= \frac{1}{(1 - z^{-1})^5} + \frac{4z^{-1}}{(1 - z^{-1})^5} + \frac{z^{-2}}{(1 - z^{-1})^5} \quad (7)$$

$$x_1(n) = \frac{(n+4)(n+3)(n+2)(n+1)}{24} u(n+4) \quad (8)$$

$$+ \frac{(n+3)(n+2)(n+1)n}{6} u(n+3)$$

$$+ \frac{(n+2)(n+1)n(n-1)}{24} u(n+2)$$

$$= \frac{(n+2)^2(n+1)^2}{4} u(n) \quad (9)$$

$$x(n) = \frac{\sum_{i=0}^n (i+1)^3}{\sum_{i=0}^n (2i+1)} u(n) \quad (1)$$

$$= \frac{x_1(n)}{x_2(n)} u(n) \quad (2)$$

$$x_2(n) = \sum_{i=0}^n (2i+1) \quad (10)$$

$$= (2n+1) * u(n) \quad (11)$$

$$x_2(n) \xleftrightarrow{Z} X_2(z) \quad (12)$$

$$X_2(z) = \left(\frac{1+z^{-1}}{(1-z^{-1})^2} \right) \left(\frac{1}{1-z^{-1}} \right) \quad (13)$$

$$= \frac{1}{(1-z^{-1})^3} + \frac{z^{-1}}{(1-z^{-1})^3} \quad (14)$$

$$x_2(n) = \frac{(n+2)(n+1)}{2} u(n+2) \quad (15)$$

$$+ \frac{(n+1)(n)}{2} u(n+1) \\ = (n+1)^2 u(n) \quad (16)$$

using (??),

$$x(n) = \frac{(n+2)^2}{4} u(n) \quad (17)$$

$$s(n) = \sum_{r=-\infty}^n x(r) \quad (18)$$

using (??),

$$s(n) = \sum_{r=-\infty}^n \frac{(r+2)^2}{4} u(r) \quad (19)$$

$$= \frac{(n+2)^2}{4} * u(n) \quad (20)$$

$$s(n) \xleftrightarrow{Z} S(z) \quad (21)$$

$$S(z) = \left(\frac{4-3z^{-1}+z^{-2}}{4(1-z^{-1})^3} \right) \left(\frac{1}{1-z^{-1}} \right) \quad (22)$$

$$= \frac{1}{(1-z^{-1})^4} - \frac{3z^{-1}}{4(1-z^{-1})^4} \quad (23)$$

$$+ \frac{z^{-2}}{4(1-z^{-1})^4} \quad (24)$$

$$s(n) = \frac{(n+3)(n+2)(n+1)}{6} u(n+3) \\ (25)$$

$$- \frac{(n+2)(n+1)(n)}{8} u(n+2) \\ + \frac{(n+1)(n)(n-1)}{24} u(n+1) \quad (26)$$

$$= \left(1 + \frac{37n}{24} + \frac{5n^2}{8} + \frac{n^3}{12} \right) u(n) \quad (27)$$

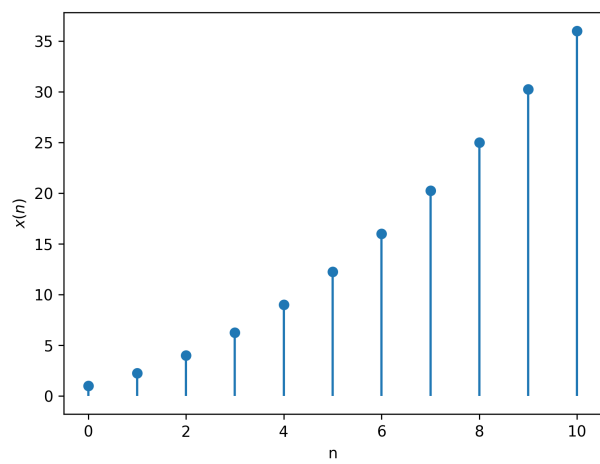


Figure 0: Plot of $x(n)$ vs n