NCERT-10.4.3.3.1

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Question: Find the roots of the following Equation:

$$x - \frac{1}{x} = 3, x \neq 0 \tag{1}$$

Theoretical Solution: To eliminate the fraction, multiply 1 through by x (valid because $x \neq 0$):

$$x^2 - 1 = 3x. \tag{2}$$

Rearranging terms gives the quadratic equation:

$$f(x) = x^2 - 3x - 1 = 0. \tag{3}$$

Comparing this with the following equation:

$$ax^2 + bx + c = 0 \tag{4}$$

We get $a = 1, b = -3, c = -1$.
We solve this quadratic equation using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{5}$$

Substituting the values:

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2(1)}. \tag{6}$$

Simplify the expression:

$$x = \frac{3 \pm \sqrt{9+4}}{2} \tag{7}$$

$$x = \frac{3 \pm \sqrt{13}}{2}. \tag{8}$$

Thus, the solutions are:

$$\alpha = \frac{3 + \sqrt{13}}{2} \approx 3.3028 \text{and} \quad \beta = \frac{3 - \sqrt{13}}{2} \approx -0.3028. \tag{9}$$

Since $x \neq 0$, both solutions are valid.
Solution by using Newton-Raphson method: To apply the Newton-Raphson method, we use the iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{10}$$

where $f'(x)$ is the derivative of $f(x)$. The derivative of $f(x)$ is:

$$f'(x) = 2x - 3. \tag{11}$$

Substituting in 10, The iterative formula becomes:

$$x_{n+1} = x_n - \frac{{x_n}^2 - 3x_n - 1}{2x_n - 3}. \tag{12}$$

For the iterative process, choose an initial guess x_0. Then substitute it in 12 to obtain x_1, x_2, and so on until the value of ϵ is greater than that of $|x_{n+1} - x_n|$. Here, ϵ should be allotted a smaller value for better precision of the roots.

The Newton-Raphson method provides an efficient way to numerically approximate the solution. Iterative computation can be performed using a program to find the root up to the desired precision. For the graph obtained, initial guess are 4 and 0.5, and $\epsilon = 0.000001$.Roots obtained through iteration are $\alpha = 3.3027756377328092, \beta = -0.30277563792072887$.

Fixed Point Iteration: The given equation is 1

To apply the Fixed-Point Iteration Method, we rewrite the equation in the form:

$$x = g(x). \tag{13}$$

Possible rearrangements are:

$$x = \frac{1}{x} + 3. \tag{14}$$

$$x = \frac{x^2 - 1}{3} \tag{15}$$

The iterative formulas are:

$$x_{(n+1)1} = \frac{1}{x_n} + 3 \tag{16}$$

$$x_{(n+1)2} = \frac{x_n^2 - 1}{3}. \tag{17}$$

Choose an initial guess x_0. Continue the process until the difference between consecutive approximations $|x_{n+1} - x_n|$ is smaller than the desired tolerance ϵ. The convergence of the Fixed-Point Iteration depends on the choice of $g(x)$.

Steffensen's Method (Quadratic Convergence): Steffensen's method accelerates the convergence of fixed-point iteration. If $g(x)$ is the iteration function derived from the fixed-point form of $f(x)$, Steffensen's iterative formula is:

$$x_{n+1} = x_n - \frac{g(x_n)}{\frac{g(x_n + g(x_n))}{g(x_n)} - 1} \tag{18}$$

This eliminates the need to compute derivatives and achieves quadratic convergence.

Rewrite the equation in the fixed-point form, i.e., 13.

For this problem,they are:

$$g_1(x_n) = \frac{1}{x_n} + 3. \tag{19}$$

$$g_2(x_n) = \frac{x_n^2 - 1}{3}. \tag{20}$$

Substitute in 18

$$x_{(n+1)1} = x_n^2 + 4x_n + 1 \tag{21}$$

$$x_{(n+1)2} = x_n - \frac{\left(x_n^2 - 1\right)^2}{(x_n^2 + 3x_n - 1)^2 - 9x_n^2} \tag{22}$$

Continue iterating until $|x_{n+1} - x_n| < \epsilon$, where ϵ is the desired accuracy. Steffensen's method converges quadratically under suitable conditions, making it significantly faster than standard fixed-point iteration. **Solution using QR Algorithm:** A quadratic equation is given by:

$$ax^2 + bx + c = 0 \tag{23}$$

To construct the companion matrix, normalize the equation ($a \neq 0$) by dividing through by a:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \tag{24}$$

The companion matrix for the quadratic equation is:

$$C = \begin{bmatrix} 0 & -\frac{c}{a} \\ 1 & -\frac{b}{a} \end{bmatrix} \tag{25}$$

For 3, it is:

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \tag{26}$$

The eigenvalues of the companion matrix C are the roots of the quadratic equation. The goal is to use the QR algorithm to approximate the eigenvalues of C. The steps are as follows:

1) Start with the matrix C:

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}. \tag{27}$$

2) Perform QR decomposition: Factor C into:

$$C = QR, \tag{28}$$

   where:
   - Q is an orthogonal matrix ($Q^T Q = I$),
   - R is an upper triangular matrix.
3) Update the matrix: Compute:

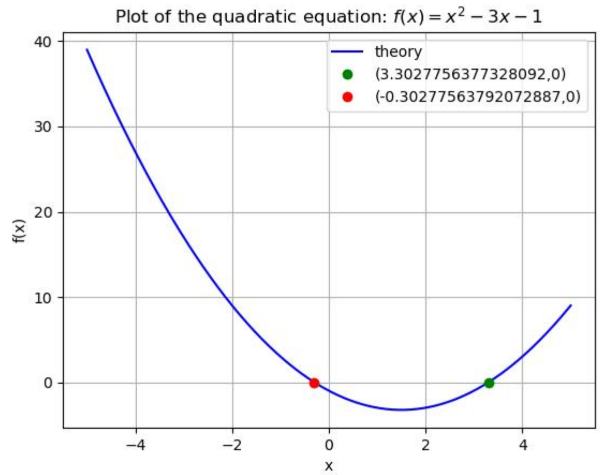
$$C_{k+1} = RQ. \tag{29}$$

4) Iterate: Repeat the QR decomposition and update step until C_k converges to an upper triangular matrix. The diagonal entries of the resulting matrix are the eigenvalues of C.

By using the QR Decomposition, the eigenvalues,i.e., the roots obtained are

$$\alpha = 3.30277564 \tag{30}$$

$$\beta = -0.30277564 \tag{31}$$


Plot of the quadratic equation: $f(x) = x^2 - 3x - 1$
theory
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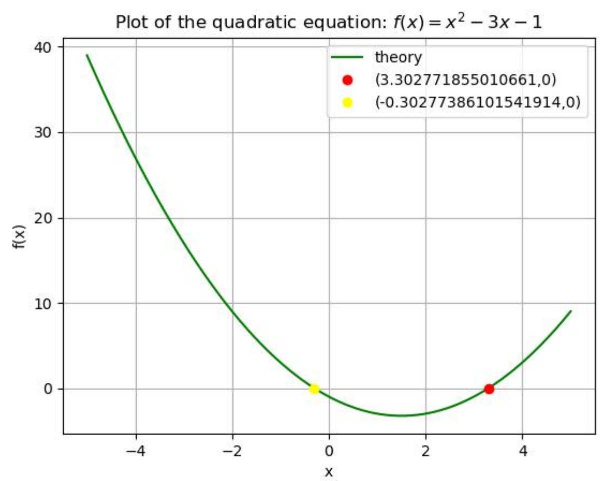
(-0.30277563792072887,0)
f(x)
x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theory
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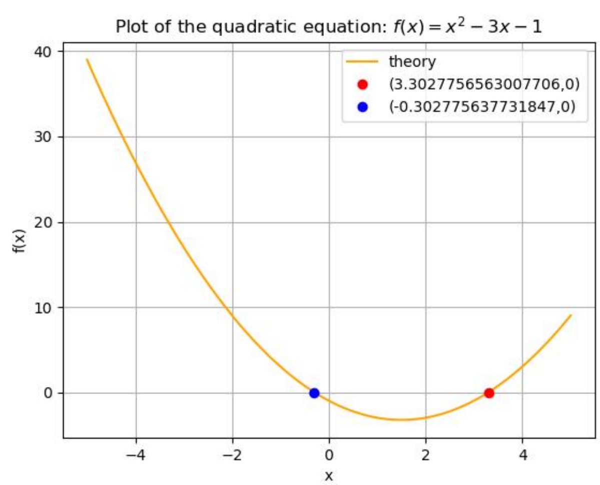
(-0.30277386101541914,0)
f(x)
x


Plot of the quadratic equation: $f(x) = x^2 - 3x - 1$
theory
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f(x)
x