

NCERT-12.6.6.12

S. Sai Akshita - EE24BTECH11054

Question: A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.

Theoretical Solution: Assume that θ be the angle between the hypotenuse and the height of the triangle, and x and y represent the portions of the hypotenuse corresponding to the lengths a and b projected along the hypotenuse.

The total length of the hypotenuse c can then be written as:

$$c = x + y. \quad (1)$$

The relationships between a , b , x , y , and θ are:

$$x = a \sec \theta, \quad y = b \csc \theta. \quad (2)$$

Thus, by substituting 2 in 1, the hypotenuse becomes:

$$c = a \sec \theta + b \csc \theta. \quad (3)$$

To minimize c , we differentiate it with respect to θ . Differentiating c gives:

$$\frac{dc}{d\theta} = a \sec \theta \tan \theta - b \csc \theta \cot \theta. \quad (4)$$

Setting $\frac{dc}{d\theta} = 0$ for critical points, we have:

$$a \sec \theta \tan \theta = b \csc \theta \cot \theta. \quad (5)$$

Using trigonometric identities, this simplifies to:

$$\frac{a \sin \theta}{\cos^2 \theta} = \frac{b \cos \theta}{\sin^2 \theta}. \quad (6)$$

Cross-multiplying:

$$a \sin^3 \theta = b \cos^3 \theta. \quad (7)$$

Dividing through by $\cos^3 \theta \sin^3 \theta$ gives:

$$\frac{\tan^3 \theta}{1} = \frac{b}{a}. \quad (8)$$

Thus:

$$\tan \theta = \left(\frac{b}{a} \right)^{\frac{1}{3}}. \quad (9)$$

Using $\tan \theta = \left(\frac{b}{a} \right)^{\frac{1}{3}}$, we calculate $\sin \theta$ and $\cos \theta$:

$$\sin \theta = \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}} = \frac{\left(\frac{b}{a} \right)^{\frac{1}{3}}}{\sqrt{1 + \left(\frac{b}{a} \right)^{\frac{2}{3}}}}, \quad (10)$$

$$\cos \theta = \frac{1}{\sqrt{1 + \tan^2 \theta}} = \frac{1}{\sqrt{1 + \left(\frac{b}{a}\right)^2}}. \quad (11)$$

Substituting these into 3, we get:

$$c = a \sqrt{1 + \left(\frac{b}{a}\right)^2} + b \sqrt{1 + \left(\frac{a}{b}\right)^2}. \quad (12)$$

Factoring and simplifying further, the result becomes:

$$c = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}. \quad (13)$$

Thus, the minimum length of the hypotenuse is:

$$c_{\min} = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}. \quad (14)$$

Gradient Descent Method for finding local minima:

To find the value of θ for which the value of c will be the lowest, Gradient Descent Method can be used iteratively. For that, we need to choose a random value for θ , say θ_0 , and apply the method on repeat until we can be as close as we can to the local minima. The process is as follows:

We treat the length of the hypotenuse as a function of the angle θ . 3

$$c = a \sec \theta + b \csc \theta. \quad (15)$$

By differentiating 3 and substituting the value of θ_0 , we get to know the slope of the tangent at that point. By moving along the downward side of tangent, and by using the following Iteration,

$$\theta_{n+1} = \theta_n - \eta \times \left. \frac{dc}{d\theta} \right|_{\theta=\theta_n} \quad (16)$$

By substitution, we get:

$$\theta_{n+1} = \theta_n - \eta \left(\frac{a \sin \theta_n}{\cos^2 \theta_n} - \frac{b \cos \theta_n}{\sin^2 \theta_n} \right) \quad (17)$$

Performing iterations,

$$\theta_1 = \theta_0 - \eta \times \left(\frac{a \sin \theta_0}{\cos^2 \theta_0} - \frac{b \cos \theta_0}{\sin^2 \theta_0} \right) \quad (18)$$

$$\theta_2 = \theta_1 - \eta \times \left(\frac{a \sin \theta_1}{\cos^2 \theta_1} - \frac{b \cos \theta_1}{\sin^2 \theta_1} \right) \quad (19)$$

$$\theta_3 = \theta_2 - \eta \times \left(\frac{a \sin \theta_2}{\cos^2 \theta_2} - \frac{b \cos \theta_2}{\sin^2 \theta_2} \right) \quad (20)$$

And so on

By opting certain value of ϵ such that

$$\left| \left(\frac{a \sin \theta_n}{\cos^2 \theta_n} - \frac{b \cos \theta_n}{\sin^2 \theta_n} \right) \right| < \epsilon \quad (21)$$

We can obtain the minimizing value of θ , i.e., θ^* such that $c(\theta^*)$ is minimum. For this iteration, the smaller the values of η and ϵ are, the greater the accuracy of the result will be.

Theory vs Simulation plot assuming certain values: In order to plot theory vs sim plot, I'm assuming the values of a and b to be 2 and 3 respectively. Theoretically, the value of θ^* is 0.844 radians. For simulation, I chose $\theta_0 = 0.1030$, $\epsilon = 0.000001$, and $\eta = 0.0001$. This resulted in θ^* to be approximately 0.851, which is almost the same value as the theoretical one. This shows the high efficiency of **Gradient Descent Method** in finding local minimas and maximas.

