NCERT-12.9.Ex.14

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Question: In a bank, principal increases continuously at the rate of 5% per year. In how many years Rs 1000 double itself?
Theoretical Solution: Let P be the principle at any time t. According to the given problem,

$$\frac{dp}{dt} = \left(\frac{5}{100}\right) \times P \tag{1}$$
$$\frac{dp}{dt} = \frac{P}{20} \tag{2}$$

separating the variables in 2, we get

$$\frac{dp}{P} = \frac{dt}{20} \tag{3}$$

Integrating on both sides of 3, we get

$$\log P = \frac{t}{20} + C_1 \tag{4}$$
$$P = e^{\frac{t}{20}} \times e^{C_1} \tag{5}$$
$$P = Ce^{\frac{t}{20}} \tag{6}$$

Now, when $t = 0$, $P = 1000$.
Substituting the values of P and t in 6, we get $C = 1000$. Therefore, 6 gives

$$P = 1000e^{\frac{t}{20}} \tag{7}$$

Let t years be the time required to double the principal. Then

$$2000 = 1000e^{\frac{t}{20}} \tag{8}$$
$$t = 20 \ln 2 \approx 13.86 \tag{9}$$

Solution Using Trapezoid Rule: Formula of Trapezoidal rule over an interval (a, b) divided into n equal subintervals is:

$$\int_a^b f(x)\, dx \approx \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right] \tag{10}$$

If we do it for one subinterval:

$$\int_{a+nh}^{a+(n+1)h} f(x)\, dx \approx \frac{h}{2}\left[f(a+nh) + f(a+(n+1)h)\right] \tag{11}$$

To obtain difference equation using Trapezoidal rule, we need to integrate 3 for one subinterval.

$$\int_{P(t_n)}^{P(t_{n+1})} dp = \int_{t_n}^{t_{n+1}} \frac{Pdt}{20} \tag{12}$$

from 11

$$\left[P(t_{n+1}) - P(t_n)\right] = \frac{h}{2}\left[\frac{P(t_n) + P(t_{n+1})}{20}\right] \tag{13}$$

Here, $h = t_{n+1} - t_n$, Which gives

$$[P(t_{n+1}) - P(t_n)] = \frac{t_{n+1} - t_n}{2}\left[\frac{P(t_n) + P(t_{n+1})}{20}\right] \tag{14}$$

$$P_{n+1} = P_n\left[\frac{40 + t_{n+1} - t_n}{40 - t_{n+1} + t_n}\right] \tag{15}$$

$$P_{n+1} = P_n\left[\frac{40 + h}{40 - h}\right] \tag{16}$$

Depending upon the length of subinterval we choose, the accuracy of the result is determined. By considering one subinterval has 1 year time gap, and $P_0 = 1000$, $P_n = 2000$, $t_0 = 0$, and using 15 in recursion, we can find n, which indicates the time required to double the principal amount. The smaller the value of h, the more accurate the estimated time will be.
Choosing $h = 1yr$,

$$P_{n+1} = P_n\left[\frac{41}{39}\right] \tag{17}$$

$$P_1 = P_0\left[\frac{41}{39}\right] \tag{18}$$

$$P_2 = P_1\left[\frac{41}{39}\right] \tag{19}$$

$$\vdots \tag{20}$$

$$P_n = P_0\left[\frac{41}{39}\right]^n \tag{21}$$

Substituting $P_0 = 1000$ and $P_n = 2000$ in 21, we get $n \approx 13.86$,which means $P_{13} < 2000 < P_{14}$. **Usage of Bilinear transform:** For the equation 2 take the Laplace of the RHS to be $X(s)$.

$$\frac{dp}{dt} = h(t) \tag{22}$$

$$h(t) = \frac{p}{20} \tag{23}$$

Applying Laplace Transform on both sides and using Laplace properties, we get

$$sY(s) = X(s) \tag{24}$$

Substituting in the Transfer Function,

$$H(s) = \frac{Y(s)}{X(s)} \tag{25}$$

$$H(s) = \frac{1}{s} \tag{26}$$

Now, we have to apply Bilinear Transform on both sides of 26, which is conversion of s-domain to z-domain.

$$s = \frac{2}{h}\frac{1-\frac{1}{z}}{1+\frac{1}{z}} \tag{27}$$

$$H(z) = \frac{h}{2}\frac{1+\frac{1}{z}}{1-\frac{1}{z}} \tag{28}$$

$$\frac{Y(z)}{X(z)} = \frac{h}{2}\frac{1+\frac{1}{z}}{1-\frac{1}{z}} \tag{29}$$

$$\left(1-\frac{1}{z}\right)Y(z) = \frac{h}{2}\left(1+\frac{1}{z}\right)X(z) \tag{30}$$

Apply Inverse z-transform on both sides of 30,

$$P_{n+1} - P_n = \frac{h}{2}\left(h(t_n) + h(t_{n+1})\right) \tag{31}$$

$$P_{n+1} = P_n + \frac{h}{2}\left(h(t_n) + h(t_{n+1})\right) \tag{32}$$

$$P_{n+1} = P_n + \frac{h}{2}\left(\frac{P(t_n) + P(t_{n+1})}{20}\right) \tag{33}$$

which is same as 13.