Solving differential equation NCERT-12.9.ex.12

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Question:

$$xdy = \left(2x^2 + 1\right) dx$$

Solution:

The aim is to find the difference equation using the trapezoidal law using the following initial conditions $x_0 = 1$ and $y_0 = 1$

$$\frac{dy}{dx} = 2x + \frac{1}{x} \tag{1}$$

1. Start with the Differential Equation

The given differential equation is:

$$\frac{dy}{dx} = f(x, y). \tag{2}$$

Integrate this over the interval :

$$\int_{y_n}^{y_{n+1}} dy = \int_{x_n}^{x_{n+1}} f(x, y) \, dx \tag{3}$$

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x, y) \, dx. \tag{4}$$

3. Generalize Over Multiple Intervals

Now, consider the integral over multiple intervals from to :

$$\int_{x_0}^{x_n} f(x) \, dx. \tag{5}$$

Using the trapezoidal rule, we approximate this integral as:

$$\int_{x_0}^{x_n} f(x) \, dx \approx \frac{h}{2} \left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right]. \tag{6}$$

This can be expressed as:

$$\int_{x_0}^{x_n} f(x) \, dx \approx h \left[\frac{1}{2} f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2} f(x_n) \right]. \tag{7}$$

The trapezoidal rule approximates an integral over an interval as:

$$\int_{x_n}^{x_{n+1}} f(x) \, dx \approx \frac{h}{2} \left[f(x_n) + f(x_{n+1}) \right], \tag{8}$$

4. Substitute Back into the Differential Equation

Returning to the differential equation, , the difference equation becomes:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]. \tag{9}$$

The function is:

$$f(x, y) = 2x + \frac{1}{x}. \tag{10}$$

Substitute and into the general trapezoidal formula:

$$y_{n+1} = y_n + \frac{h}{2}\left[2x_n + \frac{1}{x_n} + 2x_{n+1} + \frac{1}{x_{n+1}}\right]. \tag{11}$$

Express x_{n+1} Explicitly
since

$$x_{n+1} = x_n + h, \tag{12}$$

substitute this into the second term:

$$2x_{n+1} + \frac{1}{x_{n+1}} = 2(x_n + h) + \frac{1}{x_n + h}. \tag{13}$$

Now rewrite the formula:

$$y_{n+1} = y_n + \frac{h}{2}\left[2x_n + \frac{1}{x_n} + 2(x_n + h) + \frac{1}{x_n + h}\right]. y_{n+1} = y_n + \frac{h}{2}\left(4x_n + 2h + \frac{1}{x_n} + \frac{1}{x_n + h}\right) \tag{14}$$

theoretical solution:

$$\frac{dy}{dx} = 2x + \frac{1}{x} \tag{15}$$

$$dy = \left(2x + \frac{1}{x}\right)dx \tag{16}$$

$$\int dy = \int \left(2x + \frac{1}{x}\right)dx \tag{17}$$

$$y = x^2 + \log|x| + C \tag{18}$$

given it passes through $(1, 1)$ C = 0 Therefore:

$$y = x^2 + \log|x| \tag{19}$$