Solving differential equation NCERT-12.9.ex.12

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Question:
$xdy = \left(2x^2 + 1\right) dx$
Solution:
The aim is to find the difference equation using the trapezoidal law using the following initial conditions $x_0 = 1$ and $y_0 = 1$

$$\frac{dy}{dx} = 2x + \frac{1}{x} \tag{1}$$

1. Start with the Differential Equation
The given differential equation is:

$$\frac{dy}{dx} = f(x, y). \tag{2}$$

Integrate this over the interval :

$$\int_{y_n}^{y_{n+1}} dy = \int_{x_n}^{x_{n+1}} f(x, y)\, dx \tag{3}$$

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x, y)\, dx. \tag{4}$$

3. Generalize Over Multiple Intervals
Now, consider the integral over multiple intervals from to :

$$\int_{x_0}^{x_n} f(x)\, dx. \tag{5}$$

Using the trapezoidal rule, we approximate this integral as:

$$\int_{x_0}^{x_n} f(x)\, dx \approx \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]. \tag{6}$$

This can be expressed as:

$$\int_{x_0}^{x_n} f(x)\, dx \approx h\left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right]. \tag{7}$$

The trapezoidal rule approximates an integral over an interval as:

$$\int_{x_n}^{x_{n+1}} f(x)\, dx \approx \frac{h}{2}\left[f(x_n) + f(x_{n+1})\right], \tag{8}$$

4. Substitute Back into the Differential Equation
Returning to the differential equation, , the difference equation becomes:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]. \tag{9}$$

The function is:

$$f(x, y) = \frac{2x+1}{x}. \tag{10}$$

Substitute and into the general trapezoidal formula:

$$y_{n+1} = y_n + \frac{h}{2}\left[\frac{2x_n+1}{x_n} + \frac{2x_{n+1}+1}{x_{n+1}}\right]. \tag{11}$$

Express x_{n+1} Explicitly
since

$$x_{n+1} = x_n + h, \tag{12}$$

substitute this into the second term:

$$\frac{2x_{n+1}+1}{x_{n+1}} = \frac{2(x_n+h)+1}{x_n+h}. \tag{13}$$

Now rewrite the formula:

$$y_{n+1} = y_n + \frac{h}{2}\left[\frac{2x_n+1}{x_n} + \frac{2(x_n+h)+1}{x_n+h}\right]. \tag{14}$$