7.4.13

EE25BTECH11006 - ADUDOTLA SRIVIDYA

Question:
The centre of the circle passing through $(0,0)$ and $(1,0)$ and touching the circle $x^2+y^2=9$ is
Solution:
Let the required circle be represented as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0, \tag{1}$$

where $\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$.

Condition 1: Passing through $\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\mathbf{P}^T\mathbf{V}\mathbf{P} + 2\mathbf{u}^T\mathbf{P} + f = 0, \tag{2}$$
$$\mathbf{Q}^T\mathbf{V}\mathbf{Q} + 2\mathbf{u}^T\mathbf{Q} + f = 0. \tag{3}$$

Subtracting (2) from (3),

$$\left(\mathbf{Q}^T\mathbf{V}\mathbf{Q} - \mathbf{P}^T\mathbf{V}\mathbf{P}\right) + 2\mathbf{u}^T\left(\mathbf{Q} - \mathbf{P}\right) = 0. \tag{4}$$

Condition 2: Touching the given circle
The given circle is

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u_1}^T\mathbf{x} + f_1 = 0, \tag{5}$$

where

$$\mathbf{u_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f_1 = -9. \tag{6}$$

Hence,

$$\mathbf{c_1} = -\mathbf{u_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad r_1 = \sqrt{\mathbf{u_1}^T\mathbf{u_1} - f_1} = 3. \tag{7}$$

For the required circle,

$$\mathbf{c_2} = -\mathbf{u}, \quad r_2 = \sqrt{\mathbf{u}^T\mathbf{u} - f}. \tag{8}$$

Since the circles touch each other internally,

$$\|\mathbf{c_1} - \mathbf{c_2}\| = r_1 - r_2. \tag{9}$$

Substitute (8) into (9):

$$\|\mathbf{u}\| = 3 - \sqrt{\mathbf{u}^T\mathbf{u} - f}. \tag{10}$$

Substitute condition from points

From (2):

$$\mathbf{P}^T\mathbf{V}\mathbf{P} + 2\mathbf{u}^T\mathbf{P} + f = 0 \implies f = -2\mathbf{u}^T\mathbf{P} - \mathbf{P}^T\mathbf{V}\mathbf{P}. \tag{11}$$

Substitute (11) into (10):

$$\|\mathbf{u}\| = 3 - \sqrt{\mathbf{u}^T\mathbf{u} + 2\mathbf{u}^T\mathbf{P} + \mathbf{P}^T\mathbf{V}\mathbf{P}}. \tag{12}$$

From (4), we also have

$$2\mathbf{u}^T(\mathbf{Q} - \mathbf{P}) = -\left(\mathbf{Q}^T\mathbf{V}\mathbf{Q} - \mathbf{P}^T\mathbf{V}\mathbf{P}\right). \tag{13}$$

Now substitute specific vectors

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{14}$$

From (13),

$$2\mathbf{u}^T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -(1 - 0) \implies \mathbf{u}^T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{2}. \tag{15}$$

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Then

$$u_1 = -\frac{1}{2}. \tag{16}$$

From (12), using $\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

$$\|\mathbf{u}\| = 3 - \sqrt{\mathbf{u}^T\mathbf{u}}. \tag{17}$$

Let $\sqrt{\mathbf{u}^T\mathbf{u}} = k$.

$$k = 3 - k \implies 2k = 3 \implies k = \frac{3}{2}. \tag{18}$$

$$\mathbf{u}^T\mathbf{u} = \frac{9}{4}. \tag{19}$$

Hence,

$$u_1^2 + u_2^2 = \frac{9}{4}. \tag{20}$$

Substitute $u_1 = -\frac{1}{2}$:

$$\frac{1}{4} + u_2^2 = \frac{9}{4} \implies u_2^2 = 2. \tag{21}$$

$$u_2 = \pm\sqrt{2}. \tag{22}$$

Centre of the required circle:

$$\mathbf{c_2} = -\mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ -u_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \mp\sqrt{2} \end{pmatrix}. \tag{23}$$

Hence, the two possible centres are:

$$\mathbf{c_2} = \begin{pmatrix} \frac{1}{2} \\ \sqrt{2} \end{pmatrix} \quad \text{and} \quad \mathbf{c_2} = \begin{pmatrix} \frac{1}{2} \\ -\sqrt{2} \end{pmatrix} \tag{24}$$