

## 5.8.4

ADUDOTLA SRIVIDYA -EE25BTECH11006

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# Question

The centre of the circle passing through  $(0, 0)$  and  $(1, 0)$  and touching the circle  $x^2 + y^2 = 9$  is

# Solution

Let the required circle be represented as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0, \quad (1)$$

where  $\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$ .

**Condition 1: Passing through  $\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  and  $\mathbf{Q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$**

$$\mathbf{P}^T \mathbf{V} \mathbf{P} + 2\mathbf{u}^T \mathbf{P} + f = 0, \quad (2)$$

$$\mathbf{Q}^T \mathbf{V} \mathbf{Q} + 2\mathbf{u}^T \mathbf{Q} + f = 0. \quad (3)$$

Subtracting (2) from (3),

$$\left(\mathbf{Q}^T \mathbf{V} \mathbf{Q} - \mathbf{P}^T \mathbf{V} \mathbf{P}\right) + 2\mathbf{u}^T (\mathbf{Q} - \mathbf{P}) = 0. \quad (4)$$

# Solution

## Condition 2: Touching the given circle

The given circle is

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}_1^T \mathbf{x} + f_1 = 0, \quad (5)$$

where

$$\mathbf{u}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f_1 = -9. \quad (6)$$

Hence,

$$\mathbf{c}_1 = -\mathbf{u}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad r_1 = \sqrt{\mathbf{u}_1^T \mathbf{u}_1 - f_1} = 3. \quad (7)$$

For the required circle,

$$\mathbf{c}_2 = -\mathbf{u}, \quad r_2 = \sqrt{\mathbf{u}^T \mathbf{u} - f}. \quad (8)$$

# Solution

Since the circles touch each other internally,

$$\|\mathbf{c}_1 - \mathbf{c}_2\| = r_1 - r_2. \quad (9)$$

Substitute (8) into (9):

$$\|\mathbf{u}\| = 3 - \sqrt{\mathbf{u}^T \mathbf{u} - f}. \quad (10)$$

**Substitute condition from points**

From (2):

$$\mathbf{P}^T \mathbf{V} \mathbf{P} + 2\mathbf{u}^T \mathbf{P} + f = 0 \implies f = -2\mathbf{u}^T \mathbf{P} - \mathbf{P}^T \mathbf{V} \mathbf{P}. \quad (11)$$

# Solution

Substitute (11) into (10):

$$\|\mathbf{u}\| = 3 - \sqrt{\mathbf{u}^T \mathbf{u} + 2\mathbf{u}^T \mathbf{P} + \mathbf{P}^T \mathbf{V} \mathbf{P}}. \quad (12)$$

From (4), we also have

$$2\mathbf{u}^T (\mathbf{Q} - \mathbf{P}) = -(\mathbf{Q}^T \mathbf{V} \mathbf{Q} - \mathbf{P}^T \mathbf{V} \mathbf{P}). \quad (13)$$

**Now substitute specific vectors**

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (14)$$

# Solution

From (13),

$$2\mathbf{u}^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -(1 - 0) \implies \mathbf{u}^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{2}. \quad (15)$$

Let  $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ . Then

$$u_1 = -\frac{1}{2}. \quad (16)$$

From (12), using  $\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ,

$$\|\mathbf{u}\| = 3 - \sqrt{\mathbf{u}^T \mathbf{u}}. \quad (17)$$



# Solution

Let  $\sqrt{\mathbf{u}^T \mathbf{u}} = k$ .

$$k = 3 - k \implies 2k = 3 \implies k = \frac{3}{2}. \quad (18)$$

$$\mathbf{u}^T \mathbf{u} = \frac{9}{4}. \quad (19)$$

Hence,

$$u_1^2 + u_2^2 = \frac{9}{4}. \quad (20)$$

# Solution

Substitute  $u_1 = -\frac{1}{2}$ :

$$\frac{1}{4} + u_2^2 = \frac{9}{4} \implies u_2^2 = 2. \quad (21)$$

$$u_2 = \pm \sqrt{2}. \quad (22)$$

**Centre of the required circle:**

$$\mathbf{c}_2 = -\mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ -u_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \mp \sqrt{2} \end{pmatrix}. \quad (23)$$

**Hence, the two possible centres are:**

$$\mathbf{c}_2 = \begin{pmatrix} \frac{1}{2} \\ \sqrt{2} \end{pmatrix} \quad \text{and} \quad \mathbf{c}_2 = \begin{pmatrix} \frac{1}{2} \\ -\sqrt{2} \end{pmatrix} \quad (24)$$

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