12.784

EE25BTECH11043 - Nishid Khandagre

Question: Let A = (a_{ij}) be a 3×3 real matrix such that A$\begin{pmatrix}1\\2\\1\end{pmatrix} = 2\begin{pmatrix}1\\2\\1\end{pmatrix}$, A$\begin{pmatrix}0\\1\\1\end{pmatrix} = 2\begin{pmatrix}0\\1\\1\end{pmatrix}$ and A$\begin{pmatrix}-1\\1\\0\end{pmatrix} = 4\begin{pmatrix}-1\\1\\0\end{pmatrix}$. If m is the degree of the minimal polynomial of A, then $a_{11}+a_{21}+a_{31}+m$ equals

Solution: Given eigen relations:

$$\mathbf{A}\begin{pmatrix}1\\2\\1\end{pmatrix} = 2\begin{pmatrix}1\\2\\1\end{pmatrix} \tag{0.1}$$

$$\mathbf{A}\begin{pmatrix}0\\1\\1\end{pmatrix} = 2\begin{pmatrix}0\\1\\1\end{pmatrix} \tag{0.2}$$

$$\mathbf{A}\begin{pmatrix}-1\\1\\0\end{pmatrix} = 4\begin{pmatrix}-1\\1\\0\end{pmatrix} \tag{0.3}$$

The eigenvectors are :

$$\mathbf{v_1} = \begin{pmatrix}1\\2\\1\end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix}0\\1\\1\end{pmatrix}, \quad \mathbf{v_3} = \begin{pmatrix}-1\\1\\0\end{pmatrix} \tag{0.4}$$

as $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independent

Form the matrix $\mathbf{P}$ with these eigenvectors as columns:

$$\mathbf{P} = \begin{pmatrix}1 & 0 & -1\\2 & 1 & 1\\1 & 1 & 0\end{pmatrix} \tag{0.5}$$

The inverse of $\mathbf{P}$ is:

$$\mathbf{P}^{-1} = \begin{pmatrix}\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\\-\frac{1}{2} & -\frac{1}{2} & \frac{3}{2}\\-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\end{pmatrix} \tag{0.6}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} \end{pmatrix} \tag{0.7}$$

$$\mathbf{AP} = \begin{pmatrix} \mathbf{Av_1} & \mathbf{Av_2} & \mathbf{Av_3} \end{pmatrix} \tag{0.8}$$

$$\mathbf{AP} = \begin{pmatrix} 2\mathbf{v_1} & 2\mathbf{v_2} & 4\mathbf{v_3} \end{pmatrix} \tag{0.9}$$

$$\mathbf{AP} = \mathbf{PD} \tag{0.10}$$

$$\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{0.11}$$

$$\mathbf{PD} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{0.12}$$

$$= \begin{pmatrix} 2 & 0 & -4 \\ 4 & 2 & 4 \\ 2 & 2 & 0 \end{pmatrix} \tag{0.13}$$

Now, compute $\mathbf{A} = \mathbf{PDP}^{-1}$:

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & -4 \\ 4 & 2 & 4 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \tag{0.14}$$

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 0 & 0 & 2 \end{pmatrix} \tag{0.15}$$

The sum of the first column elements is:

$$a_{11} + a_{21} + a_{31} = 3 + (-1) + 0 = 2 \tag{0.16}$$

The eigenvalues of $\mathbf{A}$ are 2 (with algebraic multiplicity 2) and 4 (with algebraic multiplicity 1).
Since there are three linearly independent eigenvectors, the matrix $\mathbf{A}$ is diagonalizable. The minimal polynomial is the product of distinct linear factors corresponding to the eigenvalues.

$$m_A(x) = (x-2)(x-4) \tag{0.17}$$

The degree of the minimal polynomial, m is 2.

$$a_{11} + a_{21} + a_{31} + m = 2 + 2 = 4 \tag{0.18}$$