

10.7.91

EE25BTECH11043 - Nishid Khandagre

Question: The point of intersection of the tangents at the ends of the latus rectum of the parabola $y^2 = 4x$ is

Solution: The general form of a conic section is

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (0.1)$$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, f = 0 \quad (0.2)$$

Latus rectum endpoints are $\begin{pmatrix} a \\ \pm 2a \end{pmatrix}$

For $y^2 = 4x$, $a = 1$. So the endpoints of the latus rectum are: $\mathbf{q}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{q}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

The tangent at a point \mathbf{q} on the conic is given by the formula:

$$(\mathbf{V} \mathbf{q} + \mathbf{u})^T \mathbf{x} + \mathbf{u}^T \mathbf{q} + f = 0 \quad (0.3)$$

For $\mathbf{q}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$:

$$\mathbf{V} \mathbf{q}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad (0.4)$$

$$\mathbf{V} \mathbf{q}_1 + \mathbf{u} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \quad (0.5)$$

$$\mathbf{u}^T \mathbf{q}_1 = (-2 \quad 0) \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -2 \quad (0.6)$$

$$f = 0 \quad (0.7)$$

The tangent at \mathbf{q}_1 is:

$$\begin{pmatrix} -2 \\ 2 \end{pmatrix}^T \mathbf{x} - 2 = 0 \quad (0.8)$$

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}^T \mathbf{x} = 1 \quad (0.9)$$

For $\mathbf{q}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$:

$$\mathbf{V}\mathbf{q}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \quad (0.10)$$

$$\mathbf{V}\mathbf{q}_2 + \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \quad (0.11)$$

$$\mathbf{u}^\top \mathbf{q}_2 = \begin{pmatrix} -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = -2 \quad (0.12)$$

$$f = 0 \quad (0.13)$$

The tangent at \mathbf{q}_2 is:

$$\begin{pmatrix} -2 \\ -2 \end{pmatrix}^\top \mathbf{x} - 2 = 0 \quad (0.14)$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}^\top \mathbf{x} = -1 \quad (0.15)$$

Intersection of the two tangents:

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}^\top \mathbf{x} = 1 \quad (0.16)$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}^\top \mathbf{x} = -1 \quad (0.17)$$

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (0.18)$$

$$\left(\begin{array}{cc|c} -1 & 1 & 1 \\ 1 & 1 & -1 \end{array} \right) \quad (0.19)$$

$$R_1 \rightarrow -R_1$$

$$\left(\begin{array}{cc|c} 1 & -1 & -1 \\ 1 & 1 & -1 \end{array} \right) \quad (0.20)$$

$$(0.21)$$

$$R_2 \rightarrow R_2 - R_1$$

$$\left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 2 & 0 \end{array} \right) \quad (0.22)$$

$$(0.23)$$

$$R_2 \rightarrow R_2/2, R_1 \rightarrow R_1 + R_2$$

$$\left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 0 \end{array} \right) \quad (0.24)$$

$$\mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad (0.25)$$

Therefore, the point of intersection is $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

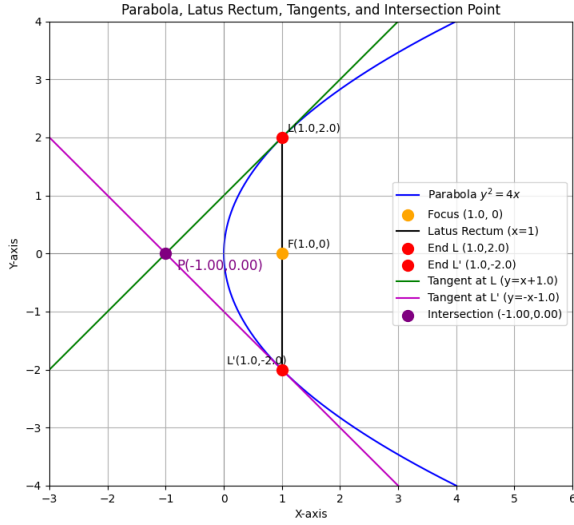


Fig. 0.1