10.7.91

EE25BTECH11043 - Nishid Khandagre

Question: The point of intersection of the tangents at the ends of the latus rectum of the parabola $y^2 = 4x$ is

Solution: The general form of a conic section is

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{0.1}$$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, f = 0 \tag{0.2}$$

Latus rectum endpoints are $\begin{pmatrix} \text{a} \\ \pm 2\text{a} \end{pmatrix}$

For $y^2 = 4x$, $a = 1$.So the endpoints of the latus rectum are: $\mathbf{q_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$,$\mathbf{q_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

The tangent at a point $\mathbf{q}$ on the conic is given by the formula:

$$\left(\mathbf{V}\mathbf{q} + \mathbf{u}\right)^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0 \tag{0.3}$$

For $\mathbf{q_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$:

$$\mathbf{V}\mathbf{q_1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{0.4}$$

$$\mathbf{V}\mathbf{q_1} + \mathbf{u} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \tag{0.5}$$

$$\mathbf{u}^\top \mathbf{q_1} = \begin{pmatrix} -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -2 \tag{0.6}$$

$$f = 0 \tag{0.7}$$

The tangent at $\mathbf{q_1}$ is:

$$\begin{pmatrix} -2 \\ 2 \end{pmatrix}^\top \mathbf{x} - 2 = 0 \tag{0.8}$$

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}^\top \mathbf{x} = 1 \tag{0.9}$$

For $\mathbf{q_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$:

$$\mathbf{V}\mathbf{q_2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \tag{0.10}$$

$$\mathbf{V}\mathbf{q_2} + \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \tag{0.11}$$

$$\mathbf{u}^\top \mathbf{q_2} = \begin{pmatrix} -2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = -2 \tag{0.12}$$

$$f = 0 \tag{0.13}$$

The tangent at $\mathbf{q_2}$ is:

$$\begin{pmatrix} -2 \\ -2 \end{pmatrix}^\top \mathbf{x} - 2 = 0 \tag{0.14}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}^\top \mathbf{x} = -1 \tag{0.15}$$

Intersection of the two tangents:

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}^\top \mathbf{x} = 1 \tag{0.16}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}^\top \mathbf{x} = -1 \tag{0.17}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{0.18}$$

$$\left(\begin{array}{cc|c} -1 & 1 & 1 \\ 1 & 1 & -1 \end{array}\right) \tag{0.19}$$

$R_1 \to -R_1$

$$\left(\begin{array}{cc|c} 1 & -1 & -1 \\ 1 & 1 & -1 \end{array}\right) \tag{0.20}$$

$R_2 \to R_2 - R_1$

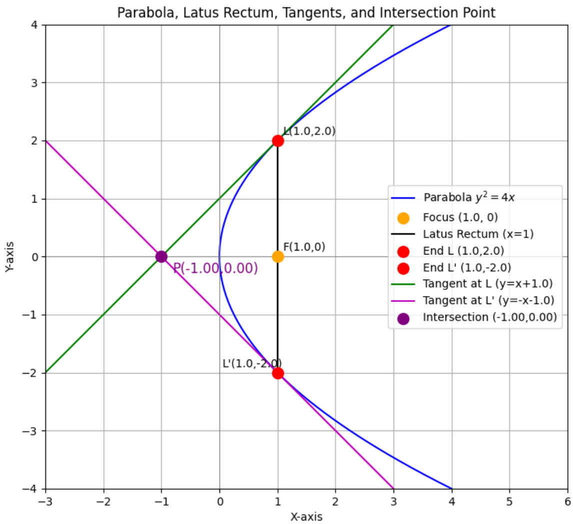
$$\left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 2 & 0 \end{array}\right) \tag{0.21}$$

$R_2 \to R_2/2,\ R_1 \to R_1 + R_2$

$$\left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right) \tag{0.22}$$

$$\mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{0.23}$$

Therefore, the point of intersection is $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.



Parabola, Latus Rectum, Tangents, and Intersection Point
L(1.0,2.0)
F(1.0,0)
P(-1.00,0.00)
L'(1.0,-2.0)
Parabola $y^2 = 4x$
Focus (1.0, 0)
Latus Rectum (x=1)
End L (1.0,2.0)
End L' (1.0,-2.0)
Tangent at L (y=x+1.0)
Tangent at L' (y=-x-1.0)
Intersection (-1.00,0.00)
X-axis
Y-axis

Fig. 0.1