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EE25BTECH11043 - Nishid Khandagre

Question: If $\mathbf{A}$ is square symmetric real valued matrix of dimension 2n, the eigenvalues of $\mathbf{A}$ are

- a) 2n distinct real values numbers
- b) 2n real values, not necessarily distinct
- c) n distinct pairs of complex conjugate numbers
- d) n pairs of complex conjugate numbers, not necessarily distinct

Solution: Let $\mathbf{A}$ be a real symmetric matrix of size $2n \times 2n$.

$$\mathbf{A} = \mathbf{A}^T = \bar{A}^\top = \mathbf{A}^* \tag{0.1}$$

Let $\mathbf{v}$ be an eigenvector and λ its eigenvalue:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{0.2}$$

Take Hermitian inner product of both sides with $\mathbf{v}$:

$$\mathbf{v}^*\mathbf{A}\mathbf{v} = \lambda\mathbf{v}^*\mathbf{v} \tag{0.3}$$

Since $\mathbf{A} = \mathbf{A}^*$:

$$\mathbf{v}^*\mathbf{A}\mathbf{v} = (\mathbf{A}\mathbf{v})^*\mathbf{v} \tag{0.4}$$

$$= (\lambda\mathbf{v})^*\mathbf{v} \tag{0.5}$$

$$= \overline{\lambda}\mathbf{v}^*\mathbf{v} \tag{0.6}$$

Equating both expressions for $\mathbf{v}^*\mathbf{A}\mathbf{v}$:

$$\lambda\mathbf{v}^*\mathbf{v} = \overline{\lambda}\mathbf{v}^*\mathbf{v} \tag{0.7}$$

Since $\mathbf{v}^*\mathbf{v} > 0$ (as $\mathbf{v}$ is an eigenvector, it must be non-zero):

$$\lambda = \overline{\lambda} \tag{0.8}$$

$$\implies \lambda \text{ is real.} \tag{0.9}$$

Therefore, All eigenvalues are real.
$\mathbf{A}$ has 2n eigenvalues since its dimension is $2n \times 2n$. The eigenvalues may be repeated (not necessarily distinct).
Therefore, the correct option is (b).