12.56

EE25BTECH11043 - Nishid Khandagre

Question: The area enclosed between the curves $y^2 = 4x$ and $x^2 = 4y$ is

Solution:
The equation of a parabola in Matrix form is

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{0.1}$$

For $y^2 = 4x$:

$$\mathbf{V_1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.2}$$

$$\mathbf{u_1} = -2\mathbf{e_1} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{0.3}$$

$$f_1 = 0 \tag{0.4}$$

For $x^2 = 4y$:

$$\mathbf{V_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{0.5}$$

$$\mathbf{u_2} = -2\mathbf{e_2} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \tag{0.6}$$

$$f_2 = 0 \tag{0.7}$$

The intersection of two conics with parameters $\mathbf{V_i}$, $\mathbf{u_i}$, f_i, $i = 1, 2$ is defined as

$$\mathbf{X}^T \left(\mathbf{V}_1 + \mu \mathbf{V}_2\right)\mathbf{X} + 2(\mathbf{u_1} + \mu \mathbf{u_2})^T \mathbf{X} \ + \ (f_1 + \mu f_2) \ = \ 0 \tag{0.8}$$

$$\implies \begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{0.9}$$

$$\implies \begin{vmatrix} \mu & 0 & -2 \\ 0 & 1 & -2\mu \\ -2 & -2\mu & 0 \end{vmatrix} = 0 \tag{0.10}$$

$$\implies \begin{vmatrix} \mu & 0 & -2 \\ 0 & 1 & -2\mu \\ -2 & -2\mu & 0 \end{vmatrix} \xleftrightarrow{R_3 \leftrightarrow R_3 + \frac{2}{\mu} \times R_1} \begin{vmatrix} \mu & 0 & -2 \\ 0 & 1 & -2\mu \\ 0 & -2\mu & -\frac{4}{\mu} \end{vmatrix} \tag{0.11}$$

$$\xleftrightarrow{R_3 \leftrightarrow R_3 + 2\mu \times R_2} \begin{vmatrix} \mu & 0 & -2 \\ 0 & 1 & -2\mu \\ 0 & 0 & -(\frac{4}{\mu} + 4\mu^2) \end{vmatrix} = 0 \tag{0.12}$$

$$\implies -(4 + 4\mu^3) = 0 \tag{0.13}$$

$$\implies \mu = -1 \tag{0.14}$$

Substituting the value of $\mu = -1$ in (0.8) we get points of intersection as

$$\mathbf{x_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{0.15}$$

$$\mathbf{x_2} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{0.16}$$

Area of the desired region is given by

$$A = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx \tag{0.17}$$

$$A = \left[\frac{4}{3}x^{3/2} - \frac{x^3}{12}\right]_0^4 \tag{0.18}$$

$$A = \left(\frac{4}{3}(4)^{3/2} - \frac{(4)^3}{12}\right) - (0 - 0) \tag{0.19}$$

$$A = \left(\frac{4}{3}(8) - \frac{64}{12}\right) \tag{0.20}$$

$$A = \left(\frac{32}{3} - \frac{16}{3}\right) \tag{0.21}$$

$$A = \frac{16}{3} \tag{0.22}$$

Thus, the area enclosed between the curves is $\frac{16}{3}$.

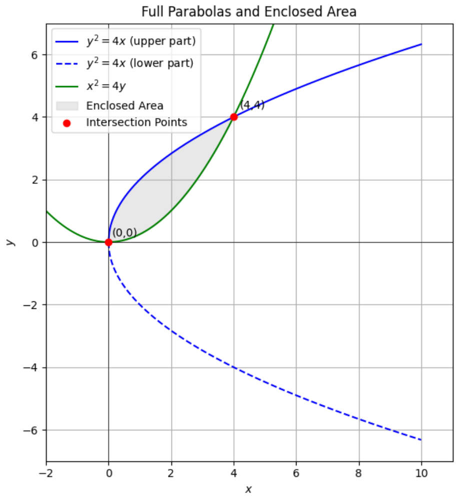
Full Parabolas and Enclosed Area
$y^2 = 4x$ (upper part)
$y^2 = 4x$ (lower part)
$x^2 = 4y$
Enclosed Area
Intersection Points
(4,4)
(0,0)
y
x
6
4
2
0
−2
−4
−6
−2
0
2
4
6
8
10

Fig. 0.1