12.576

EE25BTECH11043 - Nishid Khandagre

Question: If the characteristic polynomial and minimal polynomial of a square matrix $\mathbf{A}$ are $(\lambda-1)(\lambda+1)^4(\lambda-2)^5$ and $(\lambda-1)(\lambda+1)(\lambda-2)$, respectively, then the rank of the matrix $\mathbf{A}+\mathbf{I}$ is?

Solution: Given:

$$\chi_A(\lambda) = (\lambda-1)(\lambda+1)^4(\lambda-2)^5 \tag{0.1}$$

$$m_A(\lambda) = (\lambda-1)(\lambda+1)(\lambda-2) \tag{0.2}$$

Size of $\mathbf{A}$=degree of χ_A

$$\deg \chi_A = 1+4+5 = 10 \tag{0.3}$$

Thus, $\mathbf{A}$ is a 10×10 matrix.

The minimal polynomial $m_A(\lambda)$ has simple roots
(all linear factors with exponent 1).

$$m_A(\lambda) = (\lambda-1)(\lambda+1)(\lambda-2) \tag{0.4}$$

Since all roots are distinct, the matrix $\mathbf{A}$ is diagonalizable.

Eigenvalues of $\mathbf{A}+\mathbf{I}$ and the zero-eigenspace:

If λ is an eigenvalue of $\mathbf{A}$, then $\lambda+1$ is an eigenvalue of .

The eigenvalue 0 of $\mathbf{A}+\mathbf{I}$ corresponds to the eigenvalue -1 of $\mathbf{A}$. From $\chi_A(\lambda)$, the algebraic multiplicity of $\lambda=-1$ is 4.
Since $\mathbf{A}$ is diagonalizable, the geometric multiplicity of $\lambda=-1$ is equal to its algebraic multiplicity, which is 4.
Therefore, the geometric multiplicity of 0 for $\mathbf{A}+\mathbf{I}$ is 4.

$$\text{nullity}(\mathbf{A}+\mathbf{I}) = \dim\ker(\mathbf{A}+\mathbf{I}) = 4 \tag{0.5}$$

Rank-nullity theorem:

$$\text{rank}(\mathbf{A}+\mathbf{I}) + \text{nullity}(\mathbf{A}+\mathbf{I}) = n \tag{0.6}$$

Here, $n=10$ and $\text{nullity}(\mathbf{A}+\mathbf{I}) = 4$.

$$\text{rank}(\mathbf{A}+\mathbf{I}) = 10-4 \tag{0.7}$$

$$= 6 \tag{0.8}$$

Thus, the rank of the matrix $\mathbf{A}+\mathbf{I}$ is 6.