Question

Problem

Given points $A(1, -2)$, $B(2, 3)$, $C(a, 2)$ and $D(-4, -3)$ which form a parallelogram.

- Find the value of a.
- Find the height of the parallelogram taking AB as base.

Step 1: Represent the Vectors

Represent the points as vectors:

$$\mathbf{A} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} a \\ 2 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}. \tag{1}$$

Condition for parallelogram (diagonals bisect):

$$\mathbf{A} + \mathbf{C} = \mathbf{B} + \mathbf{D}. \tag{2}$$

Step 2: Solve for a

Substituting:

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} a \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \end{pmatrix}. \tag{3}$$

Simplify RHS:

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}. \tag{4}$$

Thus:

$$\begin{pmatrix} 1+a \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}. \tag{5}$$

So,

$$1 + a = -2 \implies a = -3. \tag{6}$$

Hence,

$$\mathbf{C} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}. \tag{7}$$

Step 3: Base and Side Vectors

Base vector:

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}. \tag{8}$$

Side vector:

$$\mathbf{D} - \mathbf{A} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}. \tag{9}$$

Step 4:

The projection of $\mathbf{D} - \mathbf{A}$ on $\mathbf{B} - \mathbf{A}$

The projection of $\mathbf{D} - \mathbf{A}$ on $\mathbf{B} - \mathbf{A}$ is

$$\mathbf{P} - \mathbf{A} = \frac{(\mathbf{B} - \mathbf{A})^\top (\mathbf{D} - \mathbf{A})}{\|\mathbf{B} - \mathbf{A}\|^2} (\mathbf{B} - \mathbf{A}) \tag{10}$$

Compute inner products:

$$(\mathbf{B} - \mathbf{A})^\top (\mathbf{D} - \mathbf{A}) = 1(-5) + 5(-1) = -10 \tag{11}$$

$$\|\mathbf{B} - \mathbf{A}\|^2 = 1^2 + 5^2 = 26. \tag{12}$$

So,

$$\mathbf{P} - \mathbf{A} = \frac{-10}{26} \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -\frac{5}{13} \\ -\frac{25}{13} \end{pmatrix}. \tag{13}$$

Step 5: Perpendicular Component

Perpendicular component:

$$\mathbf{r} = (\mathbf{D} - \mathbf{A}) - (\mathbf{P} - \mathbf{A}) = \begin{pmatrix} -5 \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{5}{13} \\ -\frac{25}{13} \end{pmatrix} = \begin{pmatrix} -\frac{60}{13} \\ \frac{12}{13} \end{pmatrix}. \tag{14}$$

Height:

$$h = \|\mathbf{r}\| = \sqrt{\left(-\frac{60}{13}\right)^2 + \left(\frac{12}{13}\right)^2}. \tag{15}$$

Final Answer

$$h = \frac{\sqrt{3744}}{13} = \frac{12\sqrt{26}}{13} = \frac{24}{\sqrt{26}}$$

$$\boxed{a = -3}, \qquad \boxed{h = \frac{12\sqrt{26}}{13}}$$

C Code (1/2)

```c
#include <stdio.h>
#include <math.h>

double norm(double v[2]) {
    return sqrt(v[0]*v[0] + v[1]*v[1]);
}

int main() {
    double A[2] = {1, -2};
    double B[2] = {2, 3};
    double D[2] = {-4, -3};
    double C[2];

    // Find 'a' using parallelogram condition: A + C = B + D
    C[0] = (B[0] + D[0]) - A[0];
    C[1] = 2; // given y-coordinate
    printf("a = %lf\n", C[0]);
```

C Code (2/2)

```c
    // Vectors
    double u[2] = {B[0]-A[0], B[1]-A[1]};
    double v[2] = {C[0]-A[0], C[1]-A[1]};

    // Projection of v on u
    double dot_uv = u[0]*v[0] + u[1]*v[1];
    double dot_uu = u[0]*u[0] + u[1]*u[1];
    double proj[2] = { (dot_uv/dot_uu)*u[0], (dot_uv/dot_uu)*u[1]
        };

    // Perpendicular component
    double w[2] = { v[0]-proj[0], v[1]-proj[1] };
    double h = norm(w);

    printf("Height = %lf\n", h);
    return 0;
}
```

Python Code (1/2)

```python
import numpy as np
import matplotlib.pyplot as plt

def norm(v):
    return np.sqrt(np.dot(v, v))

# Points
A = np.array([1, -2], dtype=float)
B = np.array([2, 3], dtype=float)
C = np.array([None, 2], dtype=float)
D = np.array([-4, -3], dtype=float)

# Find a using parallelogram condition
a_val = (B[0] + D[0]) - A[0]
C[0] = a_val
print("a =", a_val)
```

```python
# Base and AC vectors
u = B - A
v = C - A

# Projection
proj = (np.dot(v,u)/np.dot(u,u)) * u
w = v - proj
h = norm(w)
print("Height =", h)

# Plot
fig, ax = plt.subplots()
x_vals = [A[0],B[0],C[0],D[0],A[0]]
y_vals = [A[1],B[1],C[1],D[1],A[1]]
ax.plot(x_vals, y_vals, 'k-')
ax.plot([C[0], (A+proj)[0]], [C[1], (A+proj)[1]], 'r-')
plt.show()
```

Parallelogram Plot