2.10.33

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Question

Let α, β, γ be distinct real numbers. The points with position vectors $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$, $\beta\hat{i} + \gamma\hat{j} + \alpha\hat{k}$, $\gamma\hat{i} + \alpha\hat{j} + \beta\hat{k}$

1. are collinear
2. form an equilateral triangle
3. form a scalene triangle
4. form a right angled triangle

Solution

Let $\mathbf{A}$ be $\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$, $\mathbf{B}$ be $\begin{pmatrix}\beta\\ \gamma\\ \alpha\end{pmatrix}$, and $\mathbf{C}$ be $\begin{pmatrix}\gamma\\ \alpha\\ \beta\end{pmatrix}$.

First, we need to check when the three points are collinear. We can do this using the collinearity matrix:

$$\begin{pmatrix}\mathbf{C}-\mathbf{A} & \mathbf{B}-\mathbf{A}\end{pmatrix}^T \tag{1}$$

If the rank of the matrix is 1, then the points are collinear.

$$\begin{pmatrix}\gamma-\alpha & \alpha-\beta & \beta-\gamma\\ \beta-\alpha & \gamma-\beta & \alpha-\gamma\end{pmatrix} \tag{2}$$

The rank of this matrix will be 1 only when all the elements in the bottom row of the matrix are equal to 0. This occurs only when $\alpha = \beta = \gamma$, which contradicts the fact that α, β, γ are distinct.

Therefore the points must be non-collinear and form a triangle.

The sides of the triangle are $\mathbf{A} - \mathbf{B}, \mathbf{B} - \mathbf{C}, \mathbf{C} - \mathbf{A}$.

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} \alpha - \beta \\ \beta - \gamma \\ \gamma - \alpha \end{pmatrix} \tag{3}$$

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} \beta - \gamma \\ \gamma - \alpha \\ \alpha - \beta \end{pmatrix} \tag{4}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} \gamma - \alpha \\ \alpha - \beta \\ \beta - \gamma \end{pmatrix} \tag{5}$$

$||\mathbf{A} - \mathbf{B}|| = ||\mathbf{B} - \mathbf{C}|| = ||\mathbf{C} - \mathbf{A}|| = \sqrt{(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2}$

The three points therefore form an **equilateral triangle**, so option (2) is correct.

For example, let us take $\alpha = 2, \beta = 1, \gamma = 3$. We get an equilateral triangle as shown below:

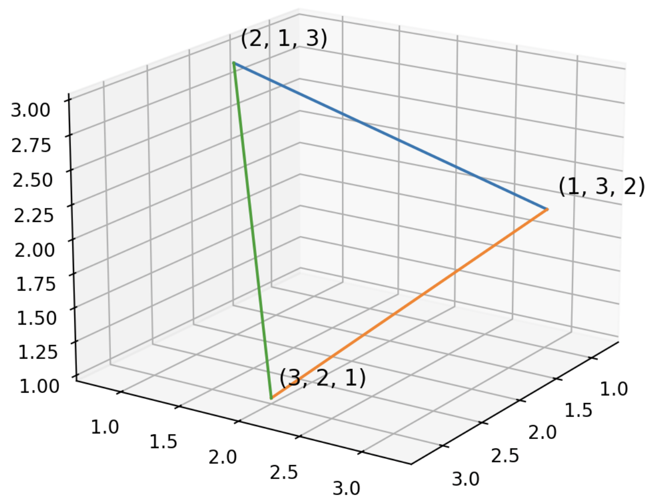

Figure 1: Equilateral Triangle