Matrices in Geometry 4.13.42

EE25BTECH11037 - Divyansh

Question: Let $0 < \alpha < \frac{\pi}{2}$ be a fixed angle. If $\mathbf{P} = (\cos\theta,\ \sin\theta)$ and $\mathbf{Q} = (\cos(\alpha-\theta),\ \sin(\alpha-\theta))$, then $\mathbf{Q}$ can be obtained from $\mathbf{P}$ by

(a) clockwise rotation around the origin through an angle α
(b) anticlockwise rotation around the origin through an angle α
(c) reflection in the line through origin with slope $\tan\alpha$
(d) reflection in the line through origin with slope $\tan\left(\frac{\alpha}{2}\right)$

Solution:

We know that $\mathbf{Q} = \begin{pmatrix}\cos(\alpha-\theta)\\ \sin(\alpha-\theta)\end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$

We also know that the rotation matrix $\mathbf{R} = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$, where α is anticlockwise.

We can obtain $\mathbf{Q}$ by

$$\mathbf{Q} = \mathbf{R}\mathbf{P} \implies \mathbf{Q} = \begin{pmatrix}\cos\theta\cos\alpha - \sin\theta\sin\alpha\\ \sin\theta\cos\alpha + \cos\theta\sin\alpha\end{pmatrix} \tag{1}$$

We know from trigonometric identities that

$$\cos(\alpha-\theta) = \cos\theta\cos\alpha + \sin\theta\sin\alpha \tag{2}$$

$$\sin(\alpha-\theta) = \cos\theta\sin\alpha - \sin\theta\cos\alpha \tag{3}$$

If we take α clockwise, that is, exchange it with $-\alpha$, we will get the rotation matrix as

$$\mathbf{R} = \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix} \tag{4}$$

Thus, the required rotation matrix is $\mathbf{R}$ using which $\mathbf{Q}$ can be obtained from $\mathbf{P}$ by clockwise rotation around the origin through an angle α, therefore the correct option is (a)

Let us plot a graph for $\theta = 45^\circ$ and $\alpha = 30^\circ$

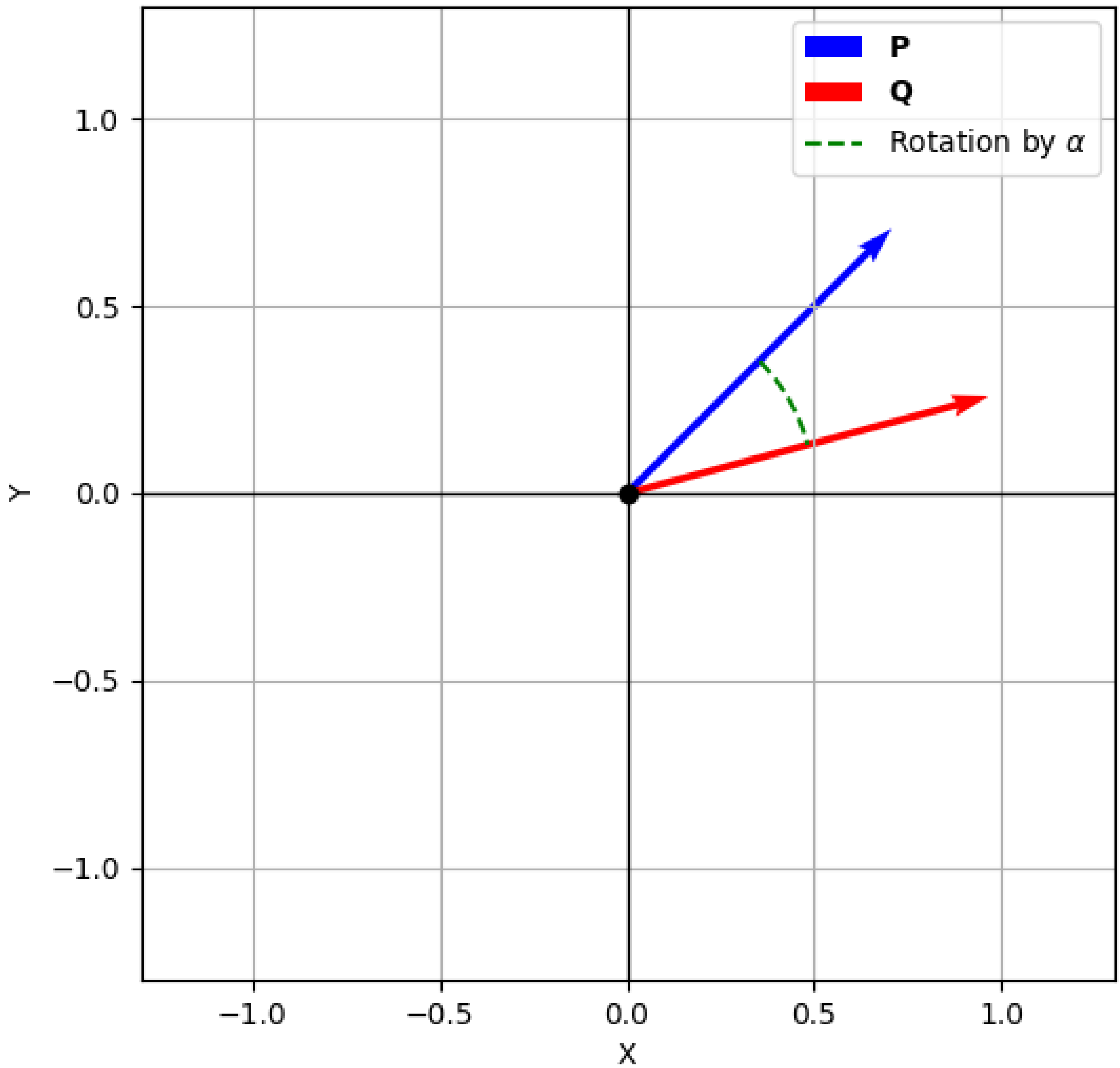

Fig. 1: Graph for 4.13.42