4.11.3

EE25BTECH11025 - Ganachari Vishwambhar

Question:
Find the equation of the line passing through (2,-1,2) and (5,3,4) and the equation of the plane passing through (2,0,3), (1,1,5), and (3,2,4). Also, find their point of intersection.
Solution:

Let:

$$\mathbf{P}_1 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}; \mathbf{P}_2 = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix} \tag{1}$$

$$\mathbf{A} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}; \mathbf{B} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}; \mathbf{C} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \tag{2}$$

Direction vector of the line:

$$\mathbf{m} = \mathbf{P}_2 - \mathbf{P}_1 = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \tag{3}$$

Vector form of the line can be written as:

$$\mathbf{x} = \mathbf{P}_1 + \kappa\mathbf{m} \tag{4}$$

Vector form of the line can be written as:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \mathbf{n} = \mathbf{1} \tag{5}$$

$$\begin{pmatrix} 2 & 0 & 3 \\ 1 & 1 & 5 \\ 3 & 2 & 4 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{6}$$

Augmented matrix can be written as:

$$\left(\begin{array}{ccc|c} 2 & 0 & 3 & 1 \\ 1 & 1 & 5 & 1 \\ 3 & 2 & 4 & 1 \end{array}\right) R_2 \leftrightarrow R_1 \left(\begin{array}{ccc|c} 1 & 1 & 5 & 1 \\ 2 & 0 & 3 & 1 \\ 3 & 2 & 4 & 1 \end{array}\right) \frac{R_2 \rightarrow R_2 - 2R_1}{R_3 \rightarrow R_3 - 3R_1} \tag{7}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 5 & 1 \\ 0 & -2 & -7 & -1 \\ 0 & -1 & -11 & -2 \end{array}\right) \frac{R_2 \leftrightarrow R_3}{R_2 \rightarrow -R_2} \left(\begin{array}{ccc|c} 1 & 1 & 5 & 1 \\ 0 & 1 & 11 & 2 \\ 0 & -2 & -7 & -1 \end{array}\right) \tag{8}$$

$$\frac{R_1 \rightarrow R_1 - R_2}{R_3 \rightarrow R_3 + 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -6 & -1 \\ 0 & 1 & 11 & 2 \\ 0 & 0 & 15 & 3 \end{array}\right) R_3 \rightarrow \frac{1}{15} R_3 \tag{9}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -6 & -1 \\ 0 & 1 & 11 & 2 \\ 0 & 0 & 1 & \frac{1}{5} \end{array}\right) \frac{R_1 \rightarrow R_1 + 6R_3}{R_2 \rightarrow R_2 - 11R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{5} \\ 0 & 1 & 0 & \frac{-1}{5} \\ 0 & 0 & 1 & \frac{1}{5} \end{array}\right) \tag{10}$$

Therefore, the plane equation is:

$$\begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \mathbf{x} = 5 \tag{11}$$

$$\mathbf{n}^\top \mathbf{x} = c \tag{12}$$

Substituting (4) in (11):

$$\mathbf{n}^\top \left(\mathbf{P}_1 + \kappa \mathbf{m} \right) = c \tag{13}$$

$$(\mathbf{n}^\top \mathbf{P}_1) + (\kappa \mathbf{n}^\top \mathbf{m}) = c \tag{14}$$

$$\kappa = \frac{c - (\mathbf{n}^\top \mathbf{P}_1)}{\mathbf{n}^\top \mathbf{m}} \tag{15}$$

The point of intersection is (from(4)):

$$\mathbf{x} = \mathbf{P}_1 + \left(\frac{c - (\mathbf{n}^\top \mathbf{P}_1)}{\mathbf{n}^\top \mathbf{m}} \right) \mathbf{m} \tag{16}$$

Substituting the values from (11), (1) and (3):

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \left(\frac{0}{-3} \right) \begin{pmatrix} 3 & 4 & 2 \end{pmatrix} \tag{17}$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{18}$$

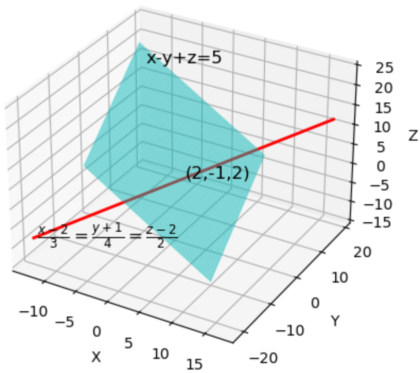

Fig. 1: Plot of the given plane and line