2.10.41

EE25BTECH11012-BEERAM MADHURI

Question:
Let the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ be such that $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{0}$. Let A and B be planes determined by the pairs of vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}, \mathbf{d}$ respectively. Then the angle between A and B is

a) 0 b) $\frac{\pi}{4}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{2}$

Solution:
given,

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = 0 \tag{0.1}$$

$$A : \text{span of } \mathbf{a}, \mathbf{b}$$
$$B : \text{span of } \mathbf{c}, \mathbf{d}$$

Cross product of 2 vectors can be written using a skew-symmetric matrix:

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b} \quad \text{where} \quad [\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{0.2}$$

Thus,

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a} \times \mathbf{b}]_\times (\mathbf{c} \times \mathbf{d}) = 0 \tag{0.3}$$
$$[\mathbf{a} \times \mathbf{b}]_\times (\mathbf{c} \times \mathbf{d}) = 0 \iff (\mathbf{c} \times \mathbf{d}) \parallel (\mathbf{a} \times \mathbf{b}) \tag{0.4}$$
$$(\mathbf{a} \times \mathbf{b}) = \lambda (\mathbf{c} \times \mathbf{d}) \tag{0.5}$$

normals to planes A and B:

$$n_A = \mathbf{a} \times \mathbf{b} \tag{0.6}$$
$$n_B = \mathbf{c} \times \mathbf{d} \tag{0.7}$$
$$n_A = \lambda n_B \tag{0.8}$$

Angle between Planes A and B = Angle between Normals n_A and n_B

Angle between planes A and B:

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_A^\top \mathbf{n}_B}{\|\mathbf{n}_A\| \, \|\mathbf{n}_B\|}\right) \tag{0.9}$$

$$= \cos^{-1}\left(\frac{\lambda\|\mathbf{n}_B\|^2}{|\lambda|\|\mathbf{n}_B\|^2}\right) \tag{0.10}$$

$$= \cos^{-1}(\pm 1) \tag{0.11}$$

$$\tag{0.12}$$

Considering acute angle,

$$\theta = 0 \tag{0.13}$$

$$\therefore n_A \parallel n_B \tag{0.14}$$

$$\therefore planeA \parallel planeB \tag{0.15}$$

Hence, Angle between the planes is 0.
option (a).

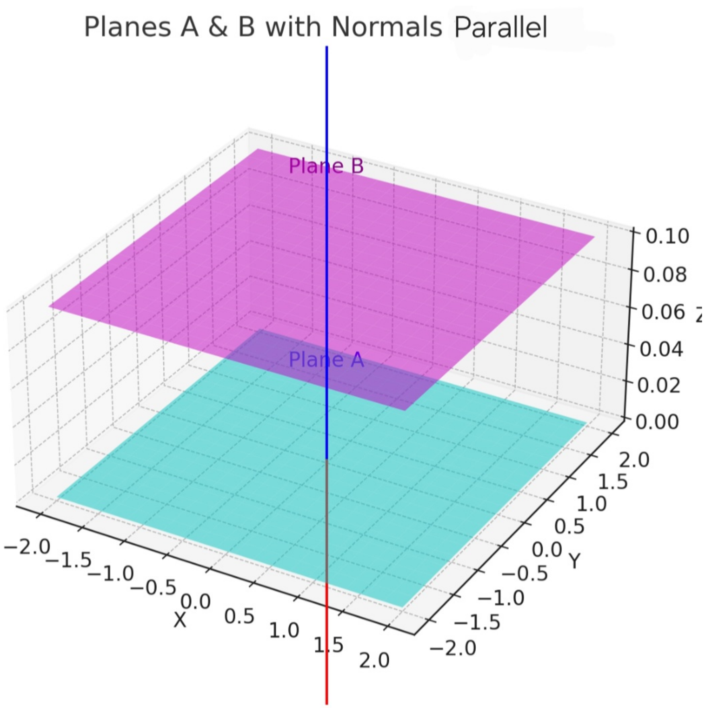

Fig. 0.1: Planes A and B