4.13.67

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Question

The area of the triangle formed by the intersection of line parallel to X axis and passing through $\mathbf{p}$(h,k) with the lines y=x and x+y=2 is $4h^2$.Find the locus of point $\mathbf{p}$

Solution:

line parallel to X axis is of the form

$$\mathbf{n}^T\mathbf{x} = c. \tag{1}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = c. \tag{2}$$

As the above line passes through $\mathbf{p}$(h,k)

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = c. \implies c = k. \tag{3}$$

The three lines are as follows

$$y = k \implies \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = k. \tag{4}$$

$$-x + y = 0 \implies \begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0. \tag{5}$$

$$x + y = 2 \implies \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2. \tag{6}$$

Let $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$ be the point of intersection of above 3 lines

On solving equation (4) and (5)

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k \\ 0 \end{pmatrix}. \tag{7}$$

$$\implies \mathbf{A} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k \\ k \end{pmatrix} \tag{8}$$

On solving equation (5) and (6)

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_1 + R_2} \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \tag{9}$$

$$\implies \mathbf{B} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{10}$$

On solving equation (4) and (6)

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k \\ 2 \end{pmatrix}. \tag{11}$$

$$\implies \mathbf{C} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2-k \\ k \end{pmatrix} \tag{12}$$

$$\text{area of } \triangle \text{ABc} = \frac{1}{2}\left\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{B} - \mathbf{C})\right\| \tag{13}$$

$$= \frac{1}{2}\left\|\begin{pmatrix} k-1 \\ k-1 \end{pmatrix} \times \begin{pmatrix} k-1 \\ 1-k \end{pmatrix}\right\| \tag{14}$$

$$= \frac{1}{2}(2(k-1)^2) = (k-1)^2. \tag{15}$$

Given area of the triangle formed by the intersection of above 3 lines is $4h^2$.

$$\implies (k-1)^2 = 4h^2. \tag{16}$$

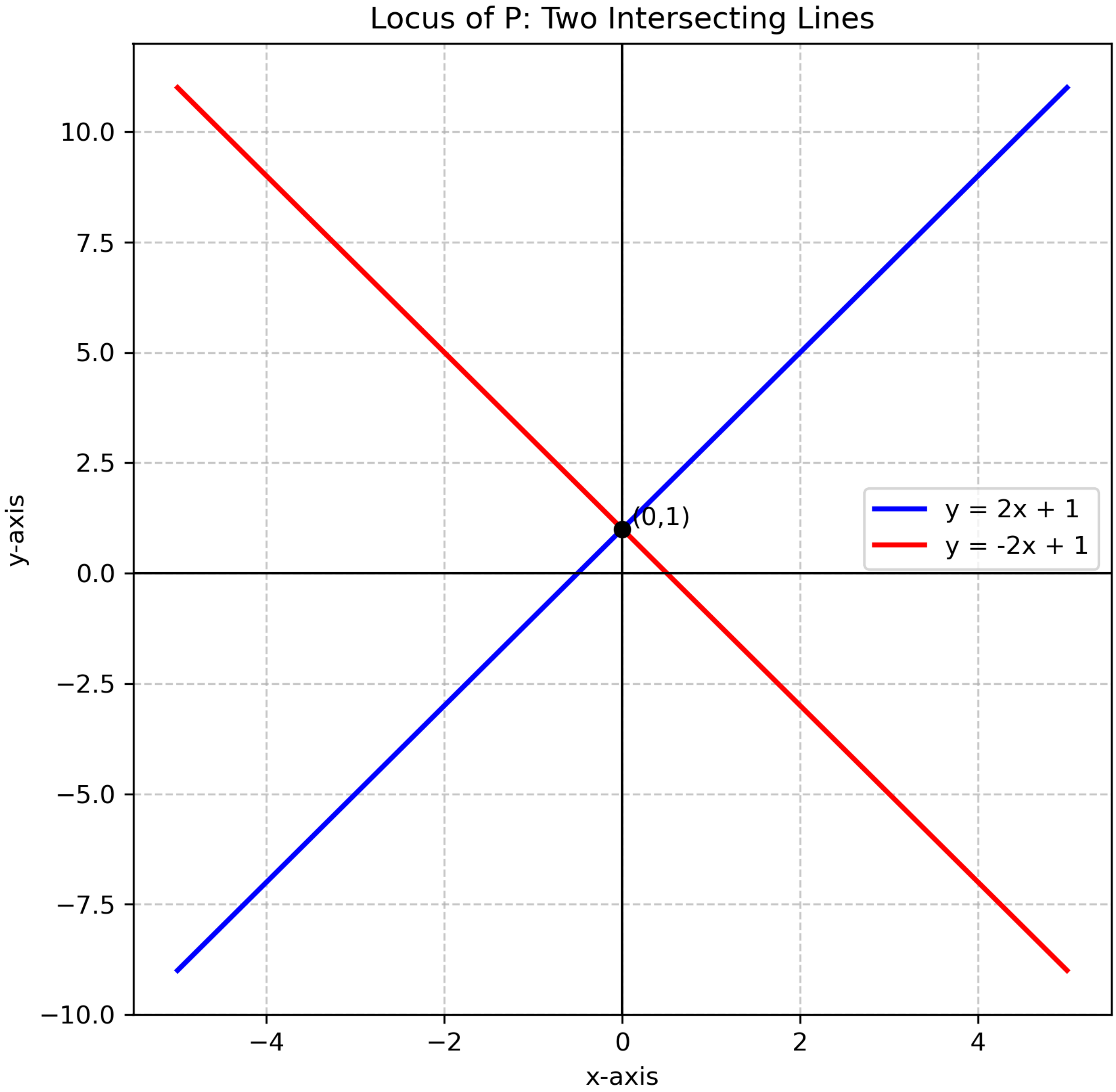
$$\implies (y-1)^2 = 4x^2 \tag{17}$$

$$\implies (y-1-2x)(y-1+2x) = 0 \tag{18}$$

$\implies$ The locus of $\mathbf{p}$ is pair of straight lines

$$y - 1 - 2x = 0. \implies \begin{pmatrix} -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1. \tag{19}$$

$$y - 1 + 2x = 0. \implies \begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1. \tag{20}$$


Locus of P: Two Intersecting Lines
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
y-axis
−4
−2
0
2
4
x-axis
(0,1)
y = 2x + 1
y = -2x + 1