Problem (4.4.28) : The x-coordinate of a point on the line joining the points $\mathbf{P}(2,2,1)$ and $\mathbf{Q}(5,1,-2)$ is 4. Find its z-coordinate.

Solution:

Input variable	Value
$\mathbf{P}$	$\begin{pmatrix}2\\2\\1\end{pmatrix}$
$\mathbf{Q}$	$\begin{pmatrix}5\\1\\-2\end{pmatrix}$
$\mathbf{R}$	$\begin{pmatrix}4\\y\\z\end{pmatrix}$

Table 1

Form the column vectors $\mathbf{Q}-\mathbf{P}$ and $\mathbf{R}-\mathbf{P}$ and the matrix $\mathbf{M}$ whose columns are these vectors:

$$\mathbf{Q}-\mathbf{P} = \begin{pmatrix}3\\-1\\-3\end{pmatrix}, \qquad \mathbf{R}-\mathbf{P} = \begin{pmatrix}2\\y-2\\z-1\end{pmatrix}, \tag{1}$$

$$\mathbf{M} = \begin{pmatrix}3 & 2\\-1 & y-2\\-3 & z-1\end{pmatrix}. \tag{2}$$

Take the transpose $\mathbf{M}^{\mathbf{T}}$:

$$\mathbf{M}^{\mathbf{T}} = \begin{pmatrix}3 & -1 & -3\\2 & y-2 & z-1\end{pmatrix}. \tag{3}$$

Perform the row operation $R_2 \leftarrow R_2 - \frac{2}{3}R_1$. Writing the entries explicitly gives

$$R_1 = \begin{pmatrix}3 & -1 & -3\end{pmatrix}, \tag{4}$$

$$R_2 = \begin{pmatrix}2 & y-2 & z-1\end{pmatrix} - \tfrac{2}{3}\begin{pmatrix}3 & -1 & -3\end{pmatrix} = \begin{pmatrix}2-\frac{2}{3}\cdot 3 & (y-2)-\frac{2}{3}\cdot(-1) & (z-1)-\frac{2}{3}\cdot(-3)\end{pmatrix}. \tag{5}$$

Thus after the row operation we have

$$\mathbf{M^T}_{\text{after}} = \begin{pmatrix}3 & -1 & -3\\2-\frac{2}{3}\cdot 3 & (y-2)-\frac{2}{3}\cdot(-1) & (z-1)-\frac{2}{3}\cdot(-3)\end{pmatrix} \tag{6}$$

Carry out the indicated multiplications to simplify the entries of the second row:

$$2 - \tfrac{2}{3} \cdot 3 = 2 - 2 = 0, \tag{7}$$

$$(y - 2) - \tfrac{2}{3} \cdot (-1) = y - 2 + \tfrac{2}{3} = y - \tfrac{4}{3}, \tag{8}$$

$$(z - 1) - \tfrac{2}{3} \cdot (-3) = z - 1 + 2 = z + 1. \tag{9}$$

So the fully simplified matrix after the row operation is

$$\mathbf{M^T}_{\text{after}} = \begin{pmatrix} 3 & -1 & -3 \\ 0 & y - \tfrac{4}{3} & z + 1 \end{pmatrix} \tag{10}$$

For the columns of M to be linearly dependent (equivalently for P, Q, R to be collinear) we require rank$(M) = 1$. Since rank$(\mathbf{M})$ = rank$(\mathbf{M^T})$, and $\mathbf{M^T}_{\text{after}}$ has two rows, rank $= 1$ means the second row must be the zero row. Hence

$$y - \tfrac{4}{3} = 0, \tag{11}$$

$$z + 1 = 0. \tag{12}$$

Therefore

$$y = \tfrac{4}{3}, \qquad z = -1. \tag{13}$$

$$\boxed{z = -1} \tag{14}$$

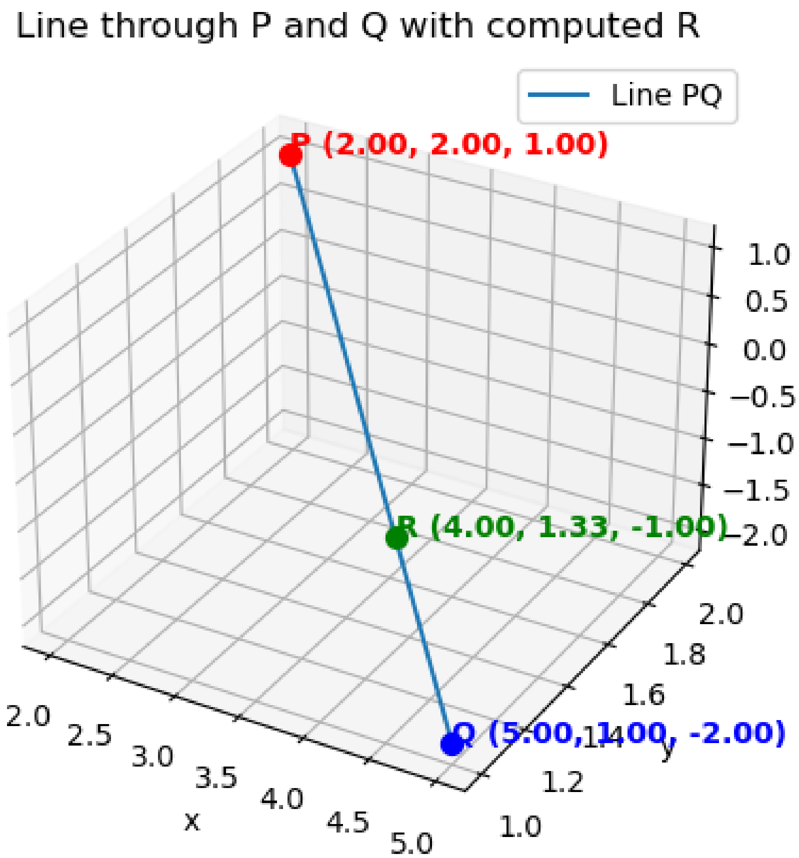
Line through P and Q with computed R
Line PQ
P (2.00, 2.00, 1.00)
R (4.00, 1.33, -1.00)
Q (5.00, 1.00, -2.00)
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
2.0
1.8
1.6
1.4
1.2
1.0
y
2.0
2.5
3.0
3.5
4.0
4.5
5.0
x

Figure 1