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Question

If $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}, \mathbf{b} = 4\hat{i} + 3\hat{j} + 4\hat{k}$ and $\mathbf{c} = \hat{i} + \alpha\hat{j} + \beta\hat{k}$ are linearly dependant vectors and $|c| = \sqrt{3}$, then

1. $\alpha = 1, \beta = -1$
2. $\alpha = 1, \beta = \pm 1$
3. $\alpha = -1, \beta = -1$
4. $\alpha = \pm 1, \beta = 1$

Solution

Given three vectors in $\mathbb{R}^3$:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} \tag{1}$$

We are told:

- The vectors are linearly dependent.
- The magnitude of $\mathbf{c}$ is $\sqrt{3}$.

$$\|\mathbf{c}\|^2 = 1^2 + \alpha^2 + \beta^2 = 3 \quad \Rightarrow \quad \alpha^2 + \beta^2 = 2 \tag{1}$$

Place the vectors as rows of a matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 4 \\ 1 & \alpha & \beta \end{pmatrix} \tag{2}$$

Apply row operations:

$$R_2 \to R_2 - 4R_1 \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & \alpha & \beta \end{pmatrix} \tag{3}$$

$$R_3 \to R_3 - R_1 \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & \alpha - 1 & \beta - 1 \end{pmatrix} \tag{4}$$

Normalize second row:

$$R_2 \to \frac{1}{-2}R_2 \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & \alpha - 1 & \beta - 1 \end{pmatrix} \tag{5}$$

Eliminate second column:

$$R_1 \to R_1 - R_2 \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & \alpha - 1 & \beta - 1 \end{pmatrix} \tag{6}$$

$$R_3 \to R_3 - (\alpha - 1)R_2 \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & \beta - 1 \end{pmatrix} \tag{7}$$

Final RREF matrix:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \beta - 1 \end{pmatrix} \tag{8}$$

For the rows to be linearly dependent, the third row must be zero:

$$\beta - 1 = 0 \quad \Rightarrow \quad \beta = 1 \tag{2}$$

Substitute Equation (2) into Equation (1):

$$\alpha^2 + 1 = 2 \quad \Rightarrow \quad \alpha^2 = 1 \quad \Rightarrow \quad \alpha = \pm 1 \tag{9}$$

Final Answer

$$\boxed{\alpha = \pm 1, \quad \beta = 1} \tag{10}$$