2.9.22

AI25BTECH11006

Question: Let $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ be three vectors such that $|\overrightarrow{a}| = 1$, $|\overrightarrow{b}| = 2$, and $|\overrightarrow{c}| = 3$. If the projection of $\overrightarrow{b}$ along $\overrightarrow{a}$ is equal to the projection of $\overrightarrow{c}$ along $\overrightarrow{a}$, and $\overrightarrow{b}$ and $\overrightarrow{c}$ are perpendicular to each other, then find $|3\overrightarrow{a} - 2\overrightarrow{b} + 2\overrightarrow{c}|$.

Solution using Gram Matrix:

Let us define the vector:

$$\mathbf{v} = 3\mathbf{a} - 2\mathbf{b} + 2\mathbf{c} \tag{1}$$

$$\|\mathbf{v}\|^2 = \mathbf{v}^T\mathbf{v} \tag{2}$$

Let the Gram matrix be defined as:

$$G = \begin{bmatrix} \mathbf{a}^T\mathbf{a} & \mathbf{a}^T\mathbf{b} & \mathbf{a}^T\mathbf{c} \\ \mathbf{b}^T\mathbf{a} & \mathbf{b}^T\mathbf{b} & \mathbf{b}^T\mathbf{c} \\ \mathbf{c}^T\mathbf{a} & \mathbf{c}^T\mathbf{b} & \mathbf{c}^T\mathbf{c} \end{bmatrix} \tag{3}$$

Let the coefficient vector be:

$$\mathbf{k} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} \tag{4}$$

Then,

$$\|\mathbf{v}\|^2 = \mathbf{k}^T G \mathbf{k} \tag{5}$$

$$\mathbf{a}^T\mathbf{a} = 1, \quad \mathbf{b}^T\mathbf{b} = 4, \quad \mathbf{c}^T\mathbf{c} = 9 \tag{6}$$

$$\mathbf{b}^T\mathbf{c} = 0, \quad \mathbf{a}^T\mathbf{b} = d, \quad \mathbf{a}^T\mathbf{c} = d \tag{7}$$

$$\frac{\mathbf{b}^T\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{c}^T\mathbf{a}}{\|\mathbf{a}\|} \quad \Rightarrow \quad \mathbf{b}^T\mathbf{a} = \mathbf{c}^T\mathbf{a} \tag{8}$$

Thus, we can set $\mathbf{a}^T\mathbf{b} = \mathbf{a}^T\mathbf{c} = d$.

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$$G = \begin{bmatrix} 1 & d & d \\ d & 4 & 0 \\ d & 0 & 9 \end{bmatrix} \tag{9}$$

Computing $\|\mathbf{v}\|^2$:

$$\|\mathbf{v}\|^2 = \begin{bmatrix} 3 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & d & d \\ d & 4 & 0 \\ d & 0 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} \tag{10}$$

Multiply step-by-step:

$$= 3(3\times 1+(-2)\times d+2\times d)+(-2)(3\times d+(-2)\times 4+2\times 0)+2(3\times d+(-2)\times 0+2\times 9) \tag{11}$$

Simplifying:

$$= 3(3 + (-2d) + 2d) + (-2)(3d - 8) + 2(3d + 18) \tag{12}$$

$$= 3(3) + (-2)(3d - 8) + 2(3d + 18) \tag{13}$$

$$= 9 + (-6d + 16) + (6d + 36) \tag{14}$$

$$= 9 + 16 + 36 \tag{15}$$

$$= 61 \tag{16}$$

Thus,

$$\|\mathbf{v}\| = \sqrt{61} \tag{17}$$

$$\boxed{\|3\mathbf{a} - 2\mathbf{b} + 2\mathbf{c}\| = \sqrt{61}} \tag{18}$$

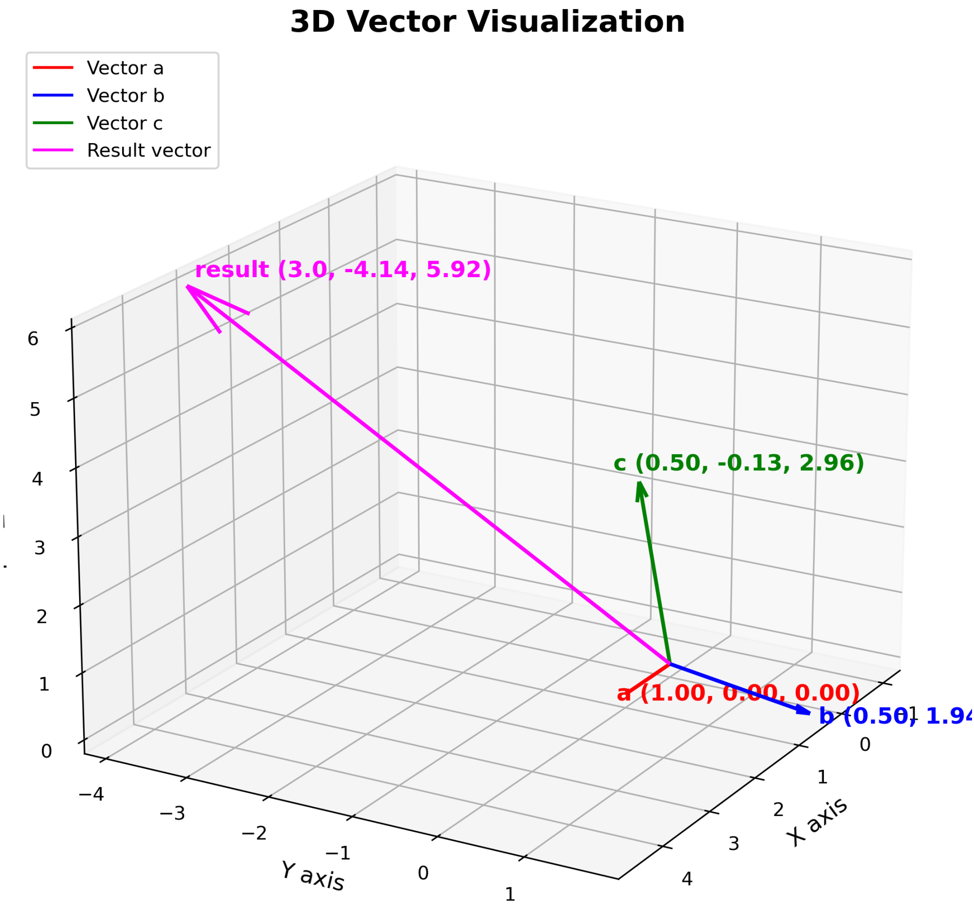
3D Vector Visualization
Vector a
Vector b
Vector c
Result vector
result (3.0, -4.14, 5.92)
c (0.50, -0.13, 2.96)
a (1.00, 0.00, 0.00)
b (0.50, 1.94
Y axis
X axis
−4
−3
−2
−1
0
1
0
1
2
3
4
5
6
0
1
2
3
4

Fig. 1