5.13.59

EE25BTECH11033 - Kavin

Question:
Let $\mathbf{P} = \left(a_{ij}\right)$ be a 3×3 matrix and let $\mathbf{Q} = \left(b_{ij}\right)$, where $b_{ij} = 2^{i+j} a_{ij}$ for $1 \le i, j \le 3$. If the determinant of $\mathbf{P}$ is 2, then the determinant of the matrix $\mathbf{Q}$ is

1) 2^{10}
2) 2^{11}
3) 2^{12}
4) 2^{13}

Solution:
Let the matrix $\mathbf{P}$ be ,

$$\mathbf{P} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{1}$$

also,

$$\left|\mathbf{P}\right| = \left|a_{ij}\right| = 2 \tag{2}$$

and the matrix $\mathbf{Q}$ be ,

$$\mathbf{Q} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \tag{3}$$

Given that,

$$b_{ij} = 2^{i+j} a_{ij} \tag{4}$$

The determinant of the matrix $\mathbf{Q}$ is given by:

$$\left|\mathbf{Q}\right| = \left|b_{ij}\right| = \left|2^{i+j} a_{ij}\right| \tag{5}$$

Split the exponent using the property $2^{i+j} = 2^i \cdot 2^j$:

$$\left|\mathbf{Q}\right| = \left|2^i \cdot 2^j \cdot a_{ij}\right| \tag{6}$$

First, for each row i (from $i = 1$ to 3), factor out the common term 2^i:

$$\left|\mathbf{Q}\right| = (2^1)(2^2)(2^3) \cdot \left|2^j a_{ij}\right| \tag{7}$$

The product of these factors is:

$$\prod_{i=1}^{3} 2^i = 2^{\sum_{i=1}^{3} i} = 2^{\frac{3(3+1)}{2}} = 2^6 \tag{8}$$

This simplifies the expression for the determinant to:

$$\left|\mathbf{Q}\right| = 2^6 \left|2^j a_{ij}\right| \tag{9}$$

Now, look at the remaining determinant, $\left|2^j a_{ij}\right|$. For each column j (from $j = 1$ to 3), factor out the common term 2^j:

$$\left|2^j a_{ij}\right| = (2^1)(2^2)(2^3) \cdot \left|a_{ij}\right| = 2^6 \left|\mathbf{P}\right| \tag{10}$$

Substituting this back into our expression for $\left|\mathbf{Q}\right|$:

$$\left|\mathbf{Q}\right| = 2^6 \cdot \left(2^6 \left|\mathbf{P}\right|\right) \tag{11}$$

$$\left|\mathbf{Q}\right| = 2^{12} \left|\mathbf{P}\right| \tag{12}$$

$$\implies \left|\mathbf{Q}\right| = 2^{12} \cdot 2 \tag{13}$$

$$\implies \left|\mathbf{Q}\right| = 2^{13} \tag{14}$$