2.3.2

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Question

Q. Find the angle between unit vectors **a** and **b** such that $\sqrt{3}\,\mathbf{a}-\mathbf{b}$ is also a unit vector.

Solution

Given: $\|\mathbf{a}\| = \|\mathbf{b}\| = 1$ and $\left\| \sqrt{3}\,\mathbf{a} - \mathbf{b} \right\| = 1$.

Use the length definition $\|x\|^2 = x^\top x$ and the scalarâproduct relation $\mathbf{a}^\top \mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta$.

$$\begin{aligned}
1 &= \left\| \sqrt{3}\,\mathbf{a} - \mathbf{b} \right\|^2 \\
&= (\sqrt{3}\,\mathbf{a} - \mathbf{b})^\top (\sqrt{3}\,\mathbf{a} - \mathbf{b}) \\
&= 3\,\mathbf{a}^\top\mathbf{a} + \mathbf{b}^\top\mathbf{b} - 2\sqrt{3}\,\mathbf{a}^\top\mathbf{b} \\
&= 3 \cdot 1 + 1 \cdot 1 - 2\sqrt{3}\cos\theta \\
&= 4 - 2\sqrt{3}\cos\theta.
\end{aligned}$$

Hence $2\sqrt{3}\cos\theta = 3$, so $\cos\theta = \dfrac{\sqrt{3}}{2}$ and therefore $\boxed{\theta = 30^\circ}$.

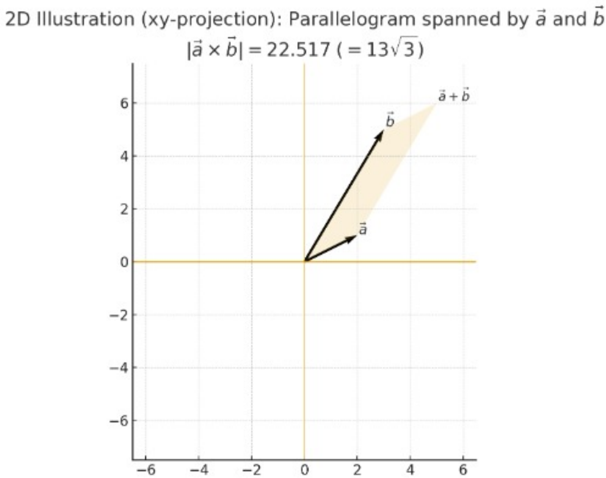

Fig. 0.1: xy-projection of **a** and **b**; $|\mathbf{a} \times \mathbf{b}| = 13\sqrt{3}$.