Problem 2.10.21 : Let $\mathbf{a}$ and $\mathbf{b}$ be two non-collinear unit vectors. If

$$\mathbf{u} = \mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\mathbf{b}, \quad \mathbf{v} = \mathbf{a} \times \mathbf{b}, \tag{1}$$

find $\|\mathbf{v}\|$.

(a) $\|\mathbf{u}\|$
(b) $\|\mathbf{u}\| + |\mathbf{u} \cdot \mathbf{a}|$
(c) $\|\mathbf{u}\| + |\mathbf{u} \cdot \mathbf{b}|$
(d) $\|\mathbf{u}\| + \mathbf{u} \cdot (\mathbf{a} + \mathbf{b})$

Solution

$$\begin{aligned}
\|\mathbf{u}\|^2 &= \mathbf{u}^T\mathbf{u} \\
&= \big(\mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\mathbf{b}\big)^T\big(\mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\mathbf{b}\big) \\
&= \mathbf{a}^T\mathbf{a} - 2(\mathbf{a} \cdot \mathbf{b})^2 + (\mathbf{a} \cdot \mathbf{b})^2\,\mathbf{b}^T\mathbf{b} \\
&= \|\mathbf{a}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \qquad (\text{since } \|\mathbf{a}\| = \|\mathbf{b}\| = 1) \\
&= 1 - (\mathbf{a} \cdot \mathbf{b})^2.
\end{aligned} \tag{2}$$

$$\begin{aligned}
\|\mathbf{v}\|^2 &= \|\mathbf{a} \times \mathbf{b}\|^2 \\
&= \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \qquad (\text{vector identity}) \\
&= 1 - (\mathbf{a} \cdot \mathbf{b})^2.
\end{aligned} \tag{3}$$

$$(2) \text{ and } (3) \implies \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 \implies \|\mathbf{v}\| = \|\mathbf{u}\|. \tag{4}$$

$$\boxed{\|\mathbf{v}\| = \|\mathbf{u}\|} \tag{5}$$

Option A is correct

Vectors from C Library

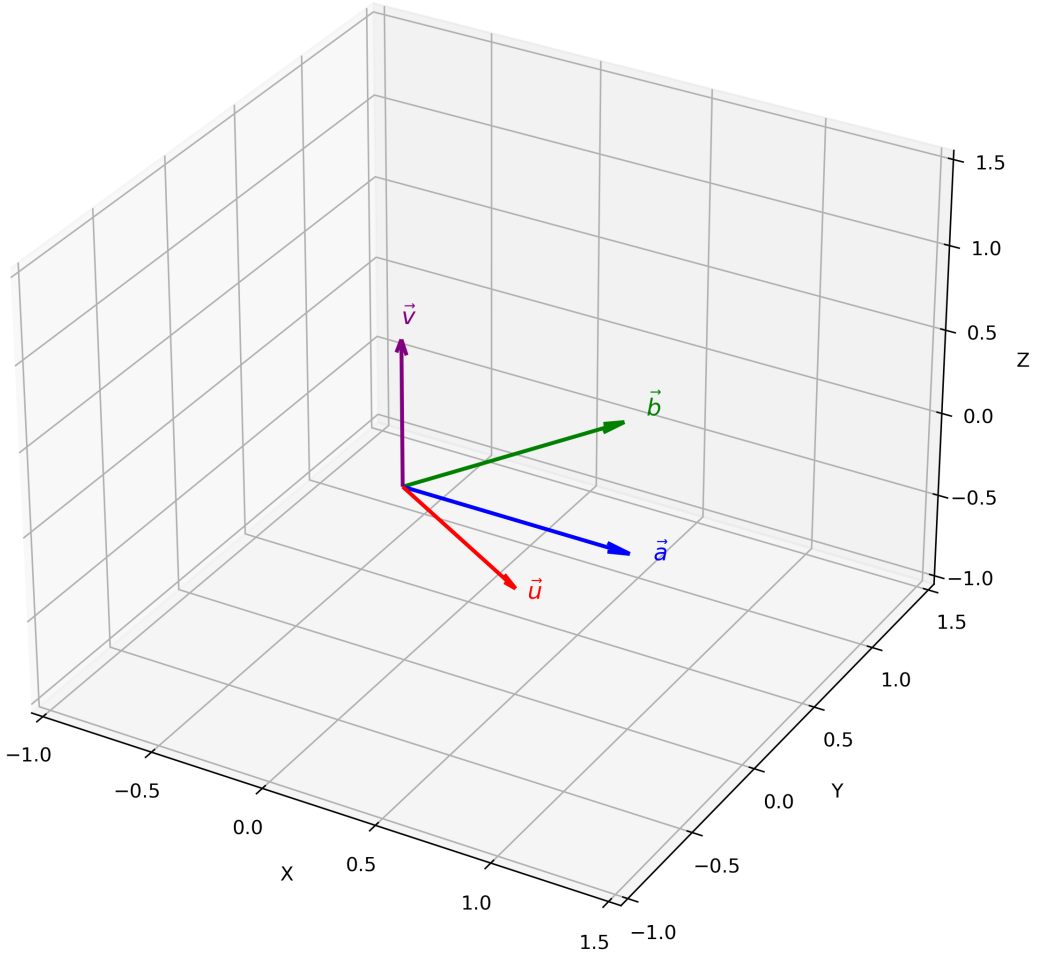

Figure 1