Matgeo Presentation - Problem 4.13.60

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Problem Statement

A line through $A(5,4)$ meets the lines $x+3y+2=0$, $2x+y+4=0$ and $x-y-5=0$ at B, C, D respectively. If

$$\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2,$$

find the equation of the line.

Data

Line	Value
$\mathbf{x_1}$	$\begin{pmatrix}\frac{1}{3} & 1\end{pmatrix}\mathbf{x_1} = -\frac{2}{3}$
$\mathbf{x_2}$	$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x_2} = -4$
$\mathbf{x_3}$	$\begin{pmatrix}-1 & 1\end{pmatrix}\mathbf{x_3} = -5$
$\mathbf{x_4}$	$\mathbf{x_4} = \begin{pmatrix}5\\4\end{pmatrix} + k\begin{pmatrix}1\\m\end{pmatrix}$

Table : Lines

Solution

Let the required line be

$$\mathbf{x_4} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} + k \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{0.1}$$

Hence the points B, C, D can be written as

$$\mathbf{B} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} + k_1 \begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} 5 + k_1 \\ 4 + k_1 m \end{pmatrix} \tag{0.2}$$

$$\mathbf{C} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} 5 + k_2 \\ 4 + k_2 m \end{pmatrix} \tag{0.3}$$

$$\mathbf{D} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} + k_3 \begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} 5 + k_3 \\ 4 + k_3 m \end{pmatrix} \tag{0.4}$$

Solution

Find k_1, k_2, k_3
Since $\mathbf{B}$ lies on $\mathbf{x_1}$

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix} \mathbf{B} = -\frac{2}{3} \tag{0.5}$$

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 5 + k_1 \\ 4 + k_1 m \end{pmatrix} = -\frac{2}{3} \tag{0.6}$$

$$\frac{17}{3} + \left(m + \frac{1}{3}\right) k_1 = -\frac{2}{3} \tag{0.7}$$

$$\left(m + \frac{1}{3}\right) k_1 = -\frac{19}{3} \tag{0.8}$$

$$k_1 = \frac{-19}{3m + 1} \tag{0.9}$$

Solution

Since $\mathbf{C}$ lies on $\mathbf{x_2}$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{C} = -4 \tag{0.10}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 5 + k_2 \\ 4 + k_2 m \end{pmatrix} = -4 \tag{0.11}$$

$$(2 + m)k_2 + 14 = -4 \tag{0.12}$$

$$(2 + m)k_2 = -18 \tag{0.13}$$

$$k_2 = \frac{-18}{m + 2} \tag{0.14}$$

Since $\mathbf{D}$ lies on $\mathbf{x_3}$

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{D} = -5 \tag{0.15}$$

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 5 + k_3 \\ 4 + k_3 m \end{pmatrix} = -5 \tag{0.16}$$

$$(m - 1)k_3 - 1 = -5 \tag{0.17}$$

$$k_3 = \frac{-4}{m - 1} \tag{0.18}$$

Solution

Find distances

$$\|\mathbf{B}-\mathbf{A}\| = |k_1|\sqrt{1+m^2} \tag{0.19}$$

$$\|\mathbf{C}-\mathbf{A}\| = |k_2|\sqrt{1+m^2} \tag{0.20}$$

$$\|\mathbf{D}-\mathbf{A}\| = |k_3|\sqrt{1+m^2} \tag{0.21}$$

Use the given equation

$$\left(\frac{15}{\|\mathbf{B}-\mathbf{A}\|}\right)^2 + \left(\frac{10}{\|\mathbf{C}-\mathbf{A}\|}\right)^2 = \left(\frac{6}{\|\mathbf{D}-\mathbf{A}\|}\right)^2 \tag{0.22}$$

Substitute distances:

$$\frac{225}{k_1^2(1+m^2)} + \frac{100}{k_2^2(1+m^2)} = \frac{36}{k_3^2(1+m^2)} \tag{0.23}$$

Solution

Multiply throughout by $(1+m^2)$:

$$\frac{225}{k_1^2}+\frac{100}{k_2^2}=\frac{36}{k_3^2} \tag{0.24}$$

Substitute values of k_1, k_2, k_3:

$$\frac{225}{\left(\frac{-19}{3m+1}\right)^2}+\frac{100}{\left(\frac{-18}{m+2}\right)^2}=\frac{36}{\left(\frac{-4}{m-1}\right)^2} \tag{0.25}$$

Simplify:

$$\frac{225(3m+1)^2}{361}+\frac{100(m+2)^2}{324}=\frac{9(m-1)^2}{4} \tag{0.26}$$

$$\tag{0.27}$$

$$429031m^2+1108138m-45869=0 \tag{0.28}$$

Solution

This gives a quadratic in m. Solving by using the quadratic formula, we get

$$m = 0.04075, \quad m = -2.62364 \tag{0.29}$$

Answer :
Final equations of the line
For $m = 0.04075$:

$$\mathbf{x_4} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} + k \begin{pmatrix} 1 \\ 0.04075 \end{pmatrix} \tag{0.30}$$

For $m = -2.62364$:

$$\mathbf{x_4} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} + k \begin{pmatrix} 1 \\ -2.62364 \end{pmatrix} \tag{0.31}$$

Plot

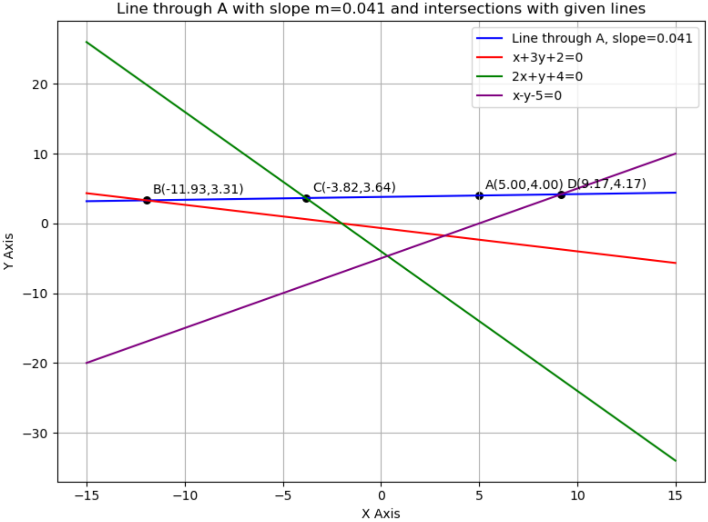

Fig : Lines 1

Plot

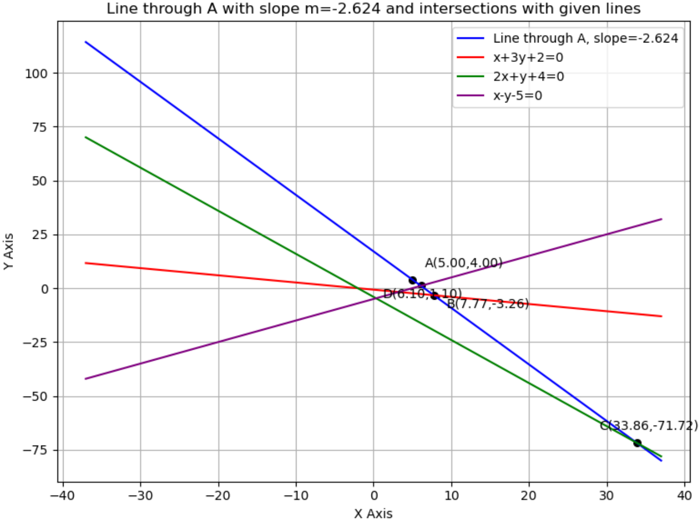

Fig : Lines 2