2.8.31

EE25BTECH11044 - Sai Hasini Pappula

Question. Given $A(1, -2)$, $B(2, 3)$, $C(a, 2)$, and $D(-4, -3)$ which form a parallelogram. Find the value of a and the height of the parallelogram when AB is taken as the base. Use only vector projections and norms.

Solution. Represent the points as vectors:

$$\mathbf{A} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} a \\ 2 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}. \tag{0.1}$$

Since the diagonals of a parallelogram bisect each other,

$$\mathbf{A} + \mathbf{C} = \mathbf{B} + \mathbf{D}. \tag{0.2}$$

That is,

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} a \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}. \tag{0.3}$$

Equating components gives

$$1 + a = -2 \quad \implies \quad a = -3. \tag{0.4}$$

Thus

$$\mathbf{C} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}. \tag{0.5}$$

Now define the base and side vectors:

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \qquad \mathbf{D} - \mathbf{A} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}. \tag{0.6}$$

The projection of $\mathbf{D} - \mathbf{A}$ on $\mathbf{B} - \mathbf{A}$ is

$$\mathbf{P} - \mathbf{A} = \frac{(\mathbf{B} - \mathbf{A})^\top(\mathbf{D} - \mathbf{A})}{\|\mathbf{B} - \mathbf{A}\|^2}(\mathbf{B} - \mathbf{A}) \tag{0.7}$$

Compute:

$$(\mathbf{B} - \mathbf{A})^\top(\mathbf{D} - \mathbf{A}) = 1(-5) + 5(-1) = -10, \qquad \|\mathbf{B} - \mathbf{A}\|^2 = 1^2 + 5^2 = 26. \tag{0.8}$$

So

$$\mathbf{P} - \mathbf{A} = \frac{\mathbf{-10}}{\mathbf{26}}\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -\frac{5}{13} \\ -\frac{25}{13} \end{pmatrix}. \tag{0.9}$$

Subtracting, the perpendicular component is

$$\mathbf{r} = (\mathbf{D} - \mathbf{A}) - (\mathbf{P} - \mathbf{A}) = \begin{pmatrix} -5 \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{5}{13} \\ -\frac{25}{13} \end{pmatrix} = \begin{pmatrix} -\frac{60}{13} \\ \frac{12}{13} \end{pmatrix}. \tag{0.10}$$

The required height is the norm of $\mathbf{r}$:

$$h = \|\mathbf{r}\| = \sqrt{\left(-\tfrac{60}{13}\right)^2 + \left(\tfrac{12}{13}\right)^2}. \tag{0.11}$$

$$h = \frac{\sqrt{3744}}{13} = \frac{12\sqrt{26}}{13} = \frac{24}{\sqrt{26}}. \tag{0.12}$$

Final Answer:

$$\boxed{a = -3\,, \qquad h = \tfrac{12\sqrt{26}}{13} = \tfrac{24}{\sqrt{26}}} \tag{0.13}$$