# 2.10.28

AI25BTECH11039-Harichandana Varanasi

## Question

**Q 2.10.28.** For non-zero vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, the relation

$$\left|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\right| = \|\mathbf{a}\| \, \|\mathbf{b}\| \, \|\mathbf{c}\|$$

holds if and only if

1) $\mathbf{a} \cdot \mathbf{b} = 0, \; \mathbf{b} \cdot \mathbf{c} = 0$
2) $\mathbf{b} \cdot \mathbf{c} = 0, \; \mathbf{c} \cdot \mathbf{a} = 0$
3) $\mathbf{c} \cdot \mathbf{a} = 0, \; \mathbf{a} \cdot \mathbf{b} = 0$
4) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 0$

**Solution:** Let

$$A = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix}, \qquad G = A^\top A = \begin{pmatrix} \mathbf{a}^\top \mathbf{a} & \mathbf{a}^\top \mathbf{b} & \mathbf{a}^\top \mathbf{c} \\ \mathbf{b}^\top \mathbf{a} & \mathbf{b}^\top \mathbf{b} & \mathbf{b}^\top \mathbf{c} \\ \mathbf{c}^\top \mathbf{a} & \mathbf{c}^\top \mathbf{b} & \mathbf{c}^\top \mathbf{c} \end{pmatrix}. \tag{4.1}$$

The scalar triple product equals the determinant of the column matrix,

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \det \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix} = \det A. \tag{4.2}$$

Hence

$$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|^2 = (\det A)^2 = \det(A^\top A) = \det G. \tag{4.3}$$

By Hadamardâs inequality for the positive semidefinite Gram matrix $G$,

$$\det G \leq (\mathbf{a}^\top \mathbf{a}) \, (\mathbf{b}^\top \mathbf{b}) \, (\mathbf{c}^\top \mathbf{c}) = \|\mathbf{a}\|^2 \, \|\mathbf{b}\|^2 \, \|\mathbf{c}\|^2, \tag{4.4}$$

with equality *iff* the columns of $A$ are pairwise orthogonal, i.e.,

$$\mathbf{a} \cdot \mathbf{b} = 0, \qquad \mathbf{b} \cdot \mathbf{c} = 0, \qquad \mathbf{c} \cdot \mathbf{a} = 0. \tag{4.5}$$

Taking square roots in (4.3) and (4.4) yields

$$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = \|\mathbf{a}\| \, \|\mathbf{b}\| \, \|\mathbf{c}\| \iff (4.5) holds. \tag{4.6}$$

Thus the correct option is $\boxed{\text{(d)}}$.

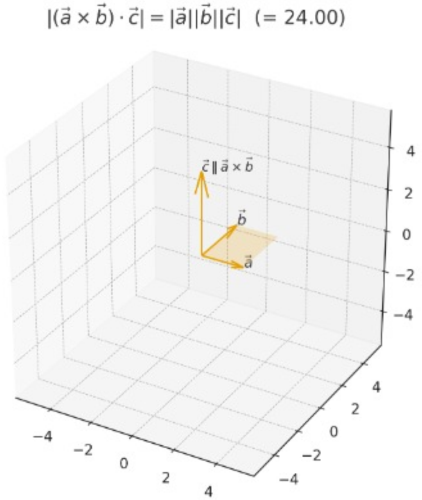


Fig. 4.1: Illustration of $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |\mathbf{a}|\,|\mathbf{b}|\,|\mathbf{c}|$ with $\mathbf{a} \perp \mathbf{b}$ and $\mathbf{c} \parallel (\mathbf{a} \times \mathbf{b})$.