10.7.75

EE25BTECH11026-Harsha

Question:
Find the equations of tangents drawn from origin to the circle $x^2+y^2-2rx-2hy+h^2 = 0$,are

1) $x = 0$
2) $y = 0$
3) $\left(h^2 - r^2\right)x - 2rhy = 0$
4) $\left(h^2 - r^2\right)x + 2rhy = 0$

Solution:
Let us solve the given question theoretically and then verify the solution computationally.

Given the equation of circle,

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{4.1}$$

where, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$,$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,$\mathbf{u} = \begin{pmatrix} -r \\ -h \end{pmatrix}$ and $f = h^2$.

It is given that the tangents pass through the origin.

$$\therefore \mathbf{n}^\top\mathbf{x} = 0 \tag{4.2}$$

where $\mathbf{n}$ is the direction vector of the tangent.

It is known that for any conic , the condition of tangency is given by,

$$\mathbf{n}^\top\mathbf{\Sigma}\mathbf{n} = 0 \tag{4.3}$$

where,

$$\mathbf{n} = \begin{pmatrix} 1 \\ m \end{pmatrix} \text{(Direction vector of tangent)} \tag{4.4}$$

$$\mathbf{\Sigma} = (\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - S(\mathbf{h})\mathbf{V} \tag{4.5}$$

$\mathbf{h}$ is the point through which the tangent passes and $S(\mathbf{h}) = \mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f = 0$.
From (4.2), (4.5) reduces to,

$$\mathbf{\Sigma} = \mathbf{u}\mathbf{u}^\top - f\mathbf{V} \tag{4.6}$$

yielding,

$$\mathbf{n}^\top\left(\mathbf{u}\mathbf{u}^\top - f\mathbf{V}\right)\mathbf{n} = 0 \tag{4.7}$$

$$\implies \mathbf{n}^\top \mathbf{u}\mathbf{u}^\top \mathbf{n} - f\mathbf{n}^\top \mathbf{V}\mathbf{n} = 0 \tag{4.8}$$

$$\therefore \|\mathbf{u}^\top \mathbf{n}\|^2 = f\mathbf{n}^\top \mathbf{V}\mathbf{n} \tag{4.9}$$

Substituting $\mathbf{V}$ in (4.9),

$$\implies \|\mathbf{u}^\top \mathbf{n}\|^2 = f\|\mathbf{n}\|^2 \tag{4.10}$$

$$\implies (rm+h)^2 = h^2\left(1+m^2\right) \tag{4.11}$$

$$\therefore m\left(\left(r^2-h^2\right)m - 2rh\right) = 0 \tag{4.12}$$

$$\implies \mathbf{n} = \begin{pmatrix}1\\0\end{pmatrix} \qquad \mathbf{n} = \begin{pmatrix}h^2-r^2\\-2rh\end{pmatrix} \tag{4.13}$$

From the figure, it is clearly verified that the theoretical solution matches with the computational solution.