2.10.54

EE25BTECH11025 - Ganachari Vishwambhar

Question:
Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be unit vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$. Which of the following are correct?

1) $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} = \mathbf{0}$
2) $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} \neq \mathbf{0}$
3) $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{c} \neq \mathbf{0}$
4) $\mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}$ are mutually perpendicular.

Solution:

Given:

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0 \tag{1}$$

$$\mathbf{c} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \tag{2}$$

$$\tag{3}$$

This $\mathbf{c}$ lies in span of $\mathbf{a}, \mathbf{b}$.

Since $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are all in 2D space, if all three are non-zero unit vectors satisfying this relation, they must be linearly dependent.

Therefore, the 2×2 matrix $\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix}$ cannot be invertible.

$$\left| \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \right| = 0 \tag{4}$$

So the matrix is singular.
In 2D, norm is defined by the determinant:

$$\|\mathbf{a} \times \mathbf{b}\| = \left| \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \right| \tag{5}$$

So if $\left| \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \right| = 0$, then

$$\mathbf{a} \times \mathbf{b} = 0 \tag{6}$$

Similarly, we can show the same for the vectors $\mathbf{a}$ and $\mathbf{b}$.
Thus, the correct option is (1):

$$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} = \mathbf{0} \tag{7}$$

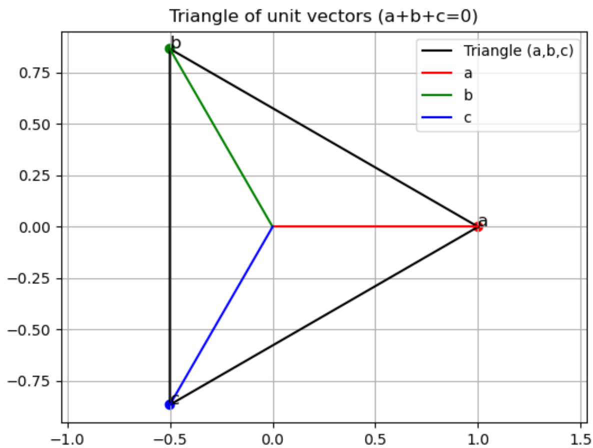

Fig. 1: Plot of the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$