

1 Numerical Recipes:

1.1 Standard BVP problem

Desire a solution to set of N coupled 1st order ODES satisfying n_1 boundary conditions at starting point x_1 and a remaining set of $n_2 = N - n_1$ conditions at final point x_2 . (Recall all DEs of order higher than 1 can be written as coupled sets of 1st order)

Equations are:

$$\frac{dy_i(x)}{dx} = g_i(x, y_1, y_2, \dots, y_n) \quad i = 1, 2, \dots, N \quad (1)$$

For a *free boundary problem*: Only one boundary abscissa x_1 is specified, while other boundary x_2 is TBD so system has a solution satisfying total of $N + 1$ conditions. Add an extra constant dependent variable:

$$y_{N+1} \equiv x_2 - x_1 \quad (2)$$

And another equation:

$$\frac{dy_{N+1}}{dx} = 0 \quad (3)$$

And also define new independent variable t :

$$t_{y_{N+1}} \equiv x - x_1, \quad 0 \leq t \leq 1 \quad (4)$$

This is now a system of $N + 1$ differential equations for dy_i/dt in standard form with t varying between known limits 0 and 1.

$$\frac{dy_i}{dt} = g_i(t, x, y_1, y_2, \dots, y_n), \quad i = 1, 2, \dots, N + 1 \quad (5)$$

1.2 How does this apply to our problem?

We do not know what the radius ξ_2 of the star is, only that θ must go to 0 at some point. We therefore have a free-boundary problem with an unknown ξ_2 and a known condition at that point.

Our original differential equation, rearranged into homogenous form:

$$\frac{d^2\theta(\xi)}{d\xi^2} + \frac{2}{\xi} \frac{d\theta(\xi)}{d\xi} + \theta^n(\xi) = 0 \quad (6)$$

Rearranged into a system of 1st-order equations:

$$\begin{cases} \frac{d\theta}{d\xi} = z \\ \frac{dz}{d\xi} = -\frac{2}{\xi}z - \theta^n \end{cases} \quad (7)$$

So I guess we add this new differential equation in?

$$\left\{ \begin{array}{l} x = \xi \\ y_1 = \theta \\ y_2 = z \\ y_3 = x_2 - x_1 \\ t = x - x_1 \end{array} \right. \quad \left\{ \begin{array}{l} \frac{dy_1}{dt} = y_2 \frac{dx}{dt} \\ \frac{dy_2}{dt} = -2xy_2 \frac{dx}{dt} - ? \\ \frac{dy_3}{dt} = 0 \end{array} \right. \quad (8)$$

2 Sheffield online lecture notes

2.1 Boundary values

Lane-Emden in form:

$$\left(\frac{1}{\xi^2} \right) \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n \quad (9)$$

To solve, need 2 boundary conditions. At center ($\xi = 0$), $\rho = \rho_c$ and hence $\theta = 1$. 2nd condition follows from equation of hydrostatic support in which $\frac{M}{r^2} \rightarrow 0$ as $r \rightarrow 0$. This means $\frac{dP}{dr} = 0$ at $r = 0$, and from the polytropic equation of state, $\frac{d\theta}{d\xi} = 0$ at $\xi = 0$.

2.2 How to solve with these conditions?

This looks like Euler's method, not the most accurate, but we can probably easily extend this to RK4.

Express Lane-Emden in form

$$\frac{d^2\theta}{d\xi^2} = - \left(\frac{2}{\xi} \frac{d\theta}{d\xi} \right) - \theta^n \quad (10)$$

Step outward from radius from the center and evaluate density at each radius. Value of the density θ_{i+1} given by value of density at the previous radius θ_i plus the change in density at each step.

$$\theta_{i+1} = \theta_i + \Delta\xi \left(\frac{d\theta}{d\xi} \right)_{i+1} \quad (11)$$

The rate of change of density with radius $\frac{d\theta}{d\xi}$ is an unknown in the above equation. To determine its value, use the same technique:

$$\left(\frac{d\theta}{d\xi} \right)_{i+1} = \left(\frac{d\theta}{d\xi} \right)_i + \Delta\xi \frac{d^2\theta}{d\xi^2} \quad (12)$$

The $\frac{d^2\theta}{d\xi^2}$ term is given by eq. 10:

$$\left(\frac{d\theta}{d\xi}\right)_{i+1} = \left(\frac{d\theta}{d\xi}\right)_i - \left(\frac{2}{\xi_i} \left(\frac{d\theta}{d\xi}\right)_i + \theta_i^n\right) \Delta\xi \quad (13)$$

Numerical integration for a particular polytropic index n can proceed as follows:

Starting at the center, at which values of ξ , $\frac{d\theta}{d\xi}$, and θ are known because of the boundary conditions given above, determine $\left(\frac{d\theta}{d\xi}\right)_{i+1}$. This value is used to determine θ_{i+1} . The radius is incremented by adding $\Delta\xi$ to ξ and the process repeated until the surface is reached (when θ becomes negative).

In the online notes $\Delta\xi = 0.001$ and $\xi_0 = 10 \times 10^{-5}$ to avoid singularity at origin.

3 Mass-Radius Relation

$$\begin{aligned} M_* &= \int_0^{R_*} 4\pi r^2 \rho(r) dr \\ &= 4\pi \alpha^3 \rho_c \int_0^{\xi_0} \xi^2 \theta^n(\xi) d\xi \end{aligned}$$

We'll need to know α and ρ_c . We'll also need to use a numerical integrator for this part as well, since we only have a numerical solution.

Theorem 1. *Simpson's rule:*

Uses quadratic interpolation to numerically integrate over a region

$$\int_a^b f(x) dx \approx \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

$$\begin{aligned} \alpha^2 &= \frac{(n+1)P_c}{4\pi G \rho_c^2} \\ P_c &= K \rho_c^{\frac{n+1}{n}} \end{aligned}$$

Does this mean we need to know K as well?