1 Numerical Recipes:

1.1 Standard BVP problem

Desire a solution to set of N coupled 1st order ODES satisfying n_1 boundary conditions at starting point x_1 and a remaining set of $n_2 = N - n_1$ conditions at final point x_2. (Recall all DEs of order higher than 1 can be written as coupled sets of 1st order)

Equations are:

$$\frac{dy_i(x)}{dx} = g_i(x, y_1, y_2, \ldots, y_n) \quad i = 1, 2, \ldots, N \tag{1}$$

For a *free boundary problem*: Only one boundary abcissa x_1 is specified, while other boundary x_2 is TBD so system has a solution satisfying total of $N+1$ conditions. Add an extra constant dependent variable:

$$y_{N+1} \equiv x_2 - x_1 \tag{2}$$

And another equation:

$$\frac{dy_{n+1}}{dx} = 0 \tag{3}$$

And also define new independent variable t:

$$t_{y_{N+1}} \equiv x - x_1, \quad 0 \leq t \leq 1 \tag{4}$$

This is now a system of $N+1$ differential equations for dy_i/dt in standard form with t varying between known limits 0 and 1.

$$\frac{dy_i}{dt} = g_i(t, x, y_1, y_2, \ldots, y_n), \quad i = 1, 2, \ldots, N+1 \tag{5}$$

1.2 How does this apply to our problem?

We do not know what the radius ξ_2 of the star is, only that θ must go to 0 at some point. We therefore have a free-boundary problem with an unknown ξ_2 and a known condition at that point.

Our original differential equation, rearranged into homogenous form:

$$\frac{d^2\theta(\xi)}{d\xi^2} + \frac{2}{\xi}\frac{d\theta(\xi)}{d\xi} + \theta^n(\xi) = 0 \tag{6}$$

Rearranged into a system of 1st-order equations:

$$\begin{cases} \dfrac{d\theta}{d\xi} = z \\ \dfrac{dz}{d\xi} = -\dfrac{2}{\xi}z - \theta^n \end{cases} \tag{7}$$

So I guess we add this new differential equation in?

$$\begin{cases} x = \xi \\ y_1 = \theta \\ y_2 = z \\ y_3 = x_2 - x_1 \\ t = x - x_1 \end{cases}$$

$$\begin{cases} \dfrac{dy_1}{dt} = y_2 \dfrac{dx}{dt} \\ \dfrac{dy_2}{dt} = -2xy_2 \dfrac{dx}{dt} - ? \\ \dfrac{dy_3}{dt} = 0 \end{cases} \tag{8}$$

2 Sheffield online lecture notes

2.1 Boundary values

Lane-Emden in form:

$$\left(\frac{1}{\xi^2}\right) \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n \tag{9}$$

To solve, need 2 boundary conditions. At center ($\xi = 0$), $\rho = \rho_c$ and hence $\theta = 1$. 2nd condition follows from equation of hydrostatic support in which $\frac{M}{r^2} \to 0$ as $r \to 0$. This means $\frac{dP}{dr} = 0$ at $r = 0$, and from the polytropic equation of state, $\frac{d\theta}{d\xi} = 0$ at $\xi = 0$.

2.2 How to solve with these conditions?

This looks like Euler's method, not the most accurate, but we can probably easily extend this to RK4.

Express Lane-Emden in form

$$\frac{d^2\theta}{d\xi^2} = -\left(\frac{2}{\xi}\frac{d\theta}{d\xi}\right) - \theta^n \tag{10}$$

Step outward from radius from the center and evaluate density at each radius. Value of the density θ_{i+1} given by value of density at the previous radius θ_i plus the change in density at each step.

$$\theta_{i+1} = \theta_i + \Delta\xi \left(\frac{d\theta}{d\xi}\right)_{i+1} \tag{11}$$

The rate of change of density with radius $\frac{d\theta}{d\xi}$ is an unknown in the above equation. To determine *its* value, use the same technique:

$$\left(\frac{d\theta}{d\xi}\right)_{i+1} = \left(\frac{d\theta}{d\xi}\right)_i + \Delta\xi \frac{d^2\theta}{d\xi^2} \tag{12}$$

The $\dfrac{d^2\theta}{d\xi^2}$ term is given by eq. 10:

$$\left(\frac{d\theta}{d\xi}\right)_{i+1} = \left(\frac{d\theta}{d\xi}\right)_i - \left(\frac{2}{\xi_i}\left(\frac{d\theta}{d\xi}\right)_i + \theta_i^n\right)\Delta\xi \tag{13}$$

Numerical integration for a particular polytropic index n can proceed as follows:

Starting at the center, at which values of ξ, $\dfrac{d\theta}{d\xi}$, andθ are known because of the boundary conditions given above, determine $\left(\dfrac{d\theta}{d\xi}\right)_{i+1}$. This value is used to determine θ_{i+1}. The radius is incremented by adding $\Delta\xi$ toξ and the process repeated until the surface is reached (when θ becomes negative).

In the online notes $\Delta\xi = 0.001$ and $\xi_0 = 10 \times 10^{-5}$ to avoid singularity at origin.

3 Mass-Radius Relation

$$\begin{aligned} M_* &= \int_0^{R_*} 4\pi r^2 \rho(r) dr \\ &= 4\pi\alpha^3\rho_c \int_0^{\xi_0} \xi^2\theta^n(\xi) d\xi \end{aligned}$$

We'll need to know α and ρ_c. We'll also need to use a numerical integrator for this part as well, since we only have a numerical solution.

Theorem 1. *Simpson's rule:*

Uses quadratic interpolation to numerically integrate over a region

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$\begin{aligned} \alpha^2 &= \frac{(n+1)P_c}{4\pi G\rho_c^2} \\ P_c &= K\rho_c^{\frac{n+1}{n}} \end{aligned}$$

Actually we can't use Simpson's rule without using interpolation because it depends on having a continuous function (uses a f(xavg) in the integral, which we of course wouldn't have in our vector). We could set that up, or we could just use matlab's builtin trapezoidal integrator.