Polytropes and Models of White Dwarf Stars

Erin Conn, Matthew Hurley

University of North Carolina at Chapel Hill

Abstract

I. Introduction

Understanding stellar mechanics requires the use of mathematical models of the internal structure of stars. We understand stars to be nearly spherical collections of hot gas held together by self-gravitation and holding themselves up against gravitational collapse by fluid and radiation pressure. Equations from Newtonian gravitational and fluid mechanics allow us to create models for the how the pressure, density, and temperature of stars varies with their mass and size, but they are complicated and highly coupled.[2] Additionally, as mass and density increase, quantum and relativistic effects become important.

Prior to the development of advanced electronic computers, solutions to these equations were difficult to impossible to compute. In 1870, American physicist Jonathan Homer Lane proposed a simplified model[3] by assuming the gas pressure depends only on the density of the gas (a *polytropic fluid*), eliminating any explicit dependence on temperature and decoupling the equations for pressure, temperature, and density.

$$P(r) = K\rho^{1+1/n} \tag{1}$$

Later Swiss physicist Robert Emden formalized the model in the dimensionless differential equation that bears their names, the Lane-Emden equation, whose derivation can be found in the appendix.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{2}$$

where ξ is a dimensionless function of the radius, θ is a dimensionless function relating density and pressure, and n is the *polytropic index* of the fluid. Solutions to this equation are called *polytropes*.

While Lane's polytropic simplification seemed unrealistic for most situations conceivable at the time, the vast simplification of its solutions over more realistic models proved a generous return on the trade with results that were still close enough to observed data to be very useful. Closed-form solutions[4] can be found readily found for polytropic indices $n = 0$ and $n = 1$, and with some algebraic substitution another one can be found for $n = 5$ (which has an infinite radius and will not be discussed further), and numerical solutions can be found using techniques known at the time of Lane's original publication.

In the 20th century new utility was found for this model with the development of quantum mechanics and the discovery of *white dwarfs*, stellar remnants composed of extremely dense gas. The electron density in white dwarfs approaches the density of available quantum energy states; and since the Pauli exclusion principle disallows multiple electrons from occupying the same quantum state[1, p.216], this results in what is called *electron degeneracy pressure* which is the main force opposing the dwarf's own gravity since white dwarfs are no longer actively undergoing fusion.[2, pp.163–166] White dwarfs are therefore composed of nearly fully *degenerate matter*. Serendipitously, fully degenerate matter has the following equation of state:

$$P(r) = K\rho(r)^{\gamma} \tag{3}$$

identical in form to eq. 1, *i.e.*, it is a polytropic gas with $\gamma \equiv 1 + \frac{1}{n}$[1]!

White dwarfs are therefore ideally suited to modeling with polytropes. The value of the polytropic index is found from fluid and quantum mechanical relations. In fact, there are two polytropic indices that apply for degenerate gases; in lower energy states the electrons have non-relativistic momenta and have a γ index of 5/3, which corresponds to a polytropic index of $n = 1.5$. As density increases, more of the electrons occupy higher energy states with higher momenta, and relativistic effects prevail, resulting in a γ index of 4/3, corresponding to a polytropic index of $n = 3$.

We used numerical integration methods to solve the Lane-Emden equation for these polytropic indices, found the relationship between the mass and radius of the resulting polytropes, and compared results with observed data for several known white dwarfs.

II. Methods

The Lane-Emden equation is a 2nd-order non-linear (for values of n other than 0 or 1) differential equation in one variable (ξ). It is not analytically solvable in most cases, but solutions to initial and boundary value problems for this equation can be found using numerical integration.

The major challenge in solving the equation lay in identifying the boundary conditions.

III. Results

We obtained the following parameters for non-relativistic ($n = 1.5$) and relativistic ($n = 3$) polytropes using a Runge-Kutta scheme in Matlab:

Table 1: *Solutions obtained with Runge-Kutta*

n	ξ_f	$\theta'(\xi_f)$	ρ_c
1.5	3.6838	-0.2033	5.9907
3	6.8968	-0.0424	54.1825

Compared with the parameters outlined in *Stellar Interiors*[2] any discrepancies can be attributed to rounding differences.

Table 2: *Parameters for $n = 1.5$ and $n = 3$ polytropes[2]*

n	ξ_f	$\theta'(\xi_f)$	ρ_c
1.5	3.6538	-0.20330	5.991
3	6.8969	-0.04243	54.1825

IV. Discussion

References

[1] D. J. Griffiths. *Introduction to Quantum Mechanics.* Pearson Education Inc., Upper Saddle River, NJ 07458, USA, 2nd edition, 2005.

[2] C. J. Hansen, S. D. Kawaler, and V. Trimble. *Stellar interiors: physical principles, structure, and evolution.* Springer-Verlag, New York, 2nd edition, 2004.

[3] J. H. Lane. On the theoretical temperature of the sun under the hypothesis of a gaseous mass maintaining its volume by its internal heat and depending on the laws of gases known to terrestrial experiment. *The American Journal of Science and Arts*, 50:57–74, 1870.

[4] F. LeBlanc. *An introduction to stellar astrophysics.* John Wiley & Sons Ltd, The Atrium, Southerngate, Chichester, West Sussex, PO19 8SQ, United Kingdom, 1st edition, 2010.

[1] not to be confused with the relativistic γ–factor

Appendices

A. Derivation of the Lane-Emden Equation[4, pp.176–179]

The Lane-Emden equation can be derived multiple ways; one way is from the equations for hydrostatic equilibrium and mass conservation:

$$\frac{dP(r)}{dr} = -\frac{\rho(r)GM(r)}{r^2} \tag{4}$$

$$dM(r) = 4\pi r^2 \rho(r) dr \rightarrow \frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \tag{5}$$

where $P(r)$ is the gas pressure as a function of radial distance from the center of the distribution, $\rho(r)$ is the gas density, G is Newton's gravitational constant, and $M(r)$ is the mass enclosed within a sphere of radius r.

These equations can be related by multiplying eq. 4 by r^2/ρ:

$$\frac{r^2}{\rho(r)}\frac{dP(r)}{dr} = -\frac{r^2}{\rho(r)}\frac{\rho(r)GM(R)}{r^2}$$

then differentiating with respect to r:

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -G\frac{dM(r)}{dr}$$

Substituting in eq. 5 we obtain Poisson's equation for gravitational potential:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -4\pi G\rho(r) \tag{6}$$

Now, using the polytropic state equation:

$$P = K\rho^{\frac{n+1}{n}} \tag{7}$$

where n is called the *polytropic index* and K is a constant, and defining a dimensionless function $\theta(r)$:

$$\rho(r) = \rho_c \theta^n r \tag{8}$$

where ρ_c is the central density of the star, we can rewrite the pressure as a function of $\theta(r)$:

$$P(r) = K\rho_c^{\frac{n+1}{n}}\theta^{n+1}(r) = P_c\theta^{n+1}(r)$$

where $P_c = K\rho_c^{\frac{n+1}{n}}$ is the central pressure of the star. Substituting this into eq. 6:

$$K\rho_c^{\frac{n+1}{n}}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho_c\theta^n(r)}\frac{d\theta^{n+1}(r)}{dr}\right) = -4\pi G\rho_c\theta^n(r)$$

This can be simplified a bit by realizing that $\frac{d\theta^{n+1}(r)}{dr} = (n+1)\theta^n(r)\frac{d\theta(r)}{dr}$:

$$\frac{(n+1)P_c}{4\pi G\rho_c^2}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta(r)}{dr}\right) = -\theta^n(r) \tag{9}$$

Since we defined $\theta(r)$ as a dimensionless function, this equation requires that $\frac{(n+1)P_c}{4\pi G\rho_c^2}$ has the dimension of length squared. For further simplification, we can define a new variable α that depends on the polytropic index n:

$$\alpha^2 = \frac{(n+1)P_c}{4\pi G\rho_c^2} \tag{10}$$

and a new dimensionless radius ξ:

$$\xi = \frac{r}{\alpha} \tag{11}$$

Substituting this ξ into eq. 9 we finally obtain the Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta(\xi)}{d\xi}\right) = -\theta^n(\xi) \tag{12}$$