Astronomy 501/701 Project - due 4/25

Ground Rules:

Get an early start, so that you can ask me questions. You may consult any source, including classmates, for help, but you must work alone when actually constructing your solution (i.e. no direct copying of code or papers). If you are uncertain what level of collaboration is allowed, ask me.

1. White dwarfs are supported by the pressure exerted by degenerate electrons. It's not a bad assumption to assume complete degeneracy, which implies equations of state of the form

$$P = K\rho^{\gamma}, \tag{1}$$

where $\gamma = 5/3$ for a non-relativistic gas and $4/3$ for a completely relativistic gas. The first step is to figure out what K is for each value of γ. For complete degeneracy, all electron orbitals are filled up to the fermi energy. The corresponding Fermi momentum is

$$p_F = \left(\frac{3n_e}{8\pi}\right)^{1/3} h \tag{2}$$

where n_e is the electron density. You can use the pressure integral (which I derived back when we were in Ch. 3) to figure out the pressure under the assumption that the electrons are either non-relativistic or extremely relativistic. Show that for these two cases, the pressure can be written as:

$$P_{non-rel} = K_1\rho^{5/3} = \frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{\rho}{m_H\mu_e}\right)^{5/3} \tag{3}$$

$$P_{rel} = K_2\rho^{4/3} = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3}\left(\frac{\rho}{m_H\mu_e}\right)^{4/3}. \tag{4}$$

For a white dwarf made of carbon and oxygen, $\mu_e \approx 2$.

2. For the non-relativistic case, we have an n=1.5 polytrope and n=3 for the relativistic case. Now it's time to solve the Lane-Emden equation. There are a number of ways to do this (including dropping it into Mathematica and hoping for the best), but in any event, you should be able to verify (or get close to) the following:

TABLE 7.1. Some parameters for $n = 1.5$, 2, and 3 polytropes

Index n	ξ_1	$\theta'(\xi_1)$	$\rho_c/\langle\rho\rangle$
1.5	3.6538	−0.20330	5.991
2.0	4.3529	−0.12725	11.402
3.0	6.8969	−0.04243	54.183

3. Plot the radius (in units of the solar radius) as a function of mass (also in solar units) for n=1.5. Interestingly, the radius decreases as the mass increases. For n=3, you'll notice that the mass is fixed - apparently there is a limiting mass for hydrostatic equilibrium with n=3, which we'll call the Chandrasekhar mass, M_{Ch}. What is this mass in units of solar mass?

4. A real white dwarf has a mixed equation of state - relativistic in the interior where the density is high, transitioning to non-relativistic towards the surface. This equation of state is complicated, but a good approximation is to write the pressure as

$$\frac{1}{P^2} = \frac{1}{P_1^2} + \frac{1}{P_2^2} \tag{5}$$

where the subscripts 1 and 2 refer to non-relativistic and relativistic, respectively. Decent approximations for the total pressure and densities are

$$P \approx \frac{GM^2}{R^4}, \ \rho \approx \frac{M}{R^3} \tag{6}$$

and so eqn. (5) can be solved for radius as a function of mass, which should be good up to some constant of order unity. Look at what happens for $M << M_{Ch}$ and $M = M_{Ch}$. The former situation should reduce to the non-relativistic mass-radius relation and so you can improve on your result by inserting the exact form for this. What's going on when $M = M_{Ch}$?

5. Now it's time to see how well your model predicts the observations. The table below lists masses and radii for white dwarfs observed in binary systems. How well does your mass-radius relation from part 4 do (plot this out)?

TABLE I: Observed masses and radii for white dwarfs

Name	Mass	Radius
	$M_\odot$	$R_\odot$
Sirius B	1.000 ± 0.016	0.0084 ± 0.0002
Procyon B	0.604 ± 0.018	0.0096 ± 0.0004
40 Eri B	0.501 ± 0.011	0.0136 ± 0.0002
CD-38 10980	0.74 ± 0.04	0.01245 ± 0.0004
W485A	0.59 ± 0.04	0.015 ± 0.001
L268-92	0.70 ± 0.12	0.0149 ± 0.001
L481-60	0.53 ± 0.05	0.1200 ± 0.0004
G154-B5B	0.46 ± 0.08	0.13 ± 0.002
G181-B5B	0.50 ± 0.05	0.011 ± 0.001
G156-64	0.59 ± 0.06	0.011 ± 0.001