# POLYTROPIC MODELS OF WHITE DWARFS UNC PHYS 331 PROJECT

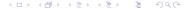
Erin Conn Matthew Hurley

April 13, 2014



#### THEORY

Theory



# Polytropes

# Table of Contents

THEORY

Polytropes



#### WHAT ARE POLYTROPES?

Solutions to...

The Lane-Emden Equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n(\xi)$$

A dimensionless, 2nd order nonlinear differential equation relating the pressure of a spherically-symmetric gas distribution to the radius.

# WHY POLYTROPES?

▶ Provide simplified stellar models - simple pressure/density relation

# WHY POLYTROPES?

- ▶ Provide simplified stellar models simple pressure/density relation
- ► Easier to solve than full equations of stellar structure

### WHY POLYTROPES?

- ▶ Provide simplified stellar models simple pressure/density relation
- ► Easier to solve than full equations of stellar structure
- ▶ Require less computational effort some analytic solutions even exist!

# **DEFINITIONS**

#### **DEFINITION**

Polytropic process - Thermodynamic process that obeys the relation

$$PV^n = C$$

Theory

#### DEFINITIONS

#### DEFINITION

Polytropic process - Thermodynamic process that obeys the relation

$$PV^n = C$$

#### DEFINITION

Polytropic index - Constant that relates pressure of a polytropic fluid to its volume (density). It may be any real number.

Theory

#### **DEFINITIONS**

#### **DEFINITION**

Polytropic process - Thermodynamic process that obeys the relation

$$PV^n = C$$

#### DEFINITION

Polytropic index - Constant that relates pressure of a polytropic fluid to its volume (density). It may be any real number.

#### DEFINITION

Poisson's equation Relates a force density function to a potential field

$$\nabla^2 \Phi = f$$

Can be derived multiple ways. From laws of mass conservation and hydrostatic equilibrium:

Can be derived multiple ways. From laws of mass conservation and hydrostatic equilibrium:

$$dM(r) = 4\pi r^2 \rho(r) dr \rightarrow \frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

# DERIVATION 1: POISSON EQUATION

Can be derived multiple ways. From laws of mass conservation and hydrostatic equilibrium:

$$\begin{array}{ll} dM(r) &= 4\pi r^2 \rho(r) dr \rightarrow \frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \\ \frac{dP(r)}{dr} &= -\frac{\rho(r)GM(r)}{r^2} \end{array}$$

Can be derived multiple ways. From laws of mass conservation and hydrostatic equilibrium:

$$\begin{array}{ll} dM(r) &= 4\pi r^2 \rho(r) dr \rightarrow \frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \\ \frac{dP(r)}{dr} &= -\frac{\rho(r)GM(r)}{r^2} \end{array}$$

These equations are related by multiplying the hydrostatic equation by  $r^2/\rho$  and differentiating:

Can be derived multiple ways. From laws of mass conservation and hydrostatic equilibrium:

$$\begin{array}{ll} dM(r) &= 4\pi r^2 \rho(r) dr \rightarrow \frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \\ \frac{dP(r)}{dr} &= -\frac{\rho(r)GM(r)}{r^2} \end{array}$$

These equations are related by multiplying the hydrostatic equation by  $r^2/\rho$  and differentiating:

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -G\frac{dM(r)}{dr}$$

Can be derived multiple ways. From laws of mass conservation and hydrostatic equilibrium:

$$\begin{array}{ll} dM(r) &= 4\pi r^2 \rho(r) dr \rightarrow \frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \\ \frac{dP(r)}{dr} &= -\frac{\rho(r)GM(r)}{r^2} \end{array}$$

These equations are related by multiplying the hydrostatic equation by  $r^2/\rho$  and differentiating:

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -G\frac{dM(r)}{dr}$$

Yielding Poisson's equation for gravity:



# Derivation 1: Poisson Equation

Can be derived multiple ways. From laws of mass conservation and hydrostatic equilibrium:

$$\begin{array}{ll} dM(r) &= 4\pi r^2 \rho(r) dr \rightarrow \frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \\ \frac{dP(r)}{dr} &= -\frac{\rho(r)GM(r)}{r^2} \end{array}$$

These equations are related by multiplying the hydrostatic equation by  $r^2/\rho$  and differentiating:

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -G\frac{dM(r)}{dr}$$

Yielding Poisson's equation for gravity:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -4\pi G\rho(r)$$



Theory

# DERIVATION 2: WORKING TOWARDS A DIMENSIONLESS FORM

Define a polytropic state equation:

Theory

# DERIVATION 2: WORKING TOWARDS A DIMENSIONLESS FORM

Define a polytropic state equation:

$$P = K\rho^{\frac{n+1}{n}}$$

# DERIVATION 2: WORKING TOWARDS A DIMENSIONLESS FORM

Define a polytropic state equation:

$$P = K\rho^{\frac{n+1}{n}}$$

Make it dimensionless:

$$\theta^n \equiv \frac{\rho}{\rho_c}$$

Theory

# DERIVATION 2: WORKING TOWARDS A DIMENSIONLESS FORM

Define a polytropic state equation:

$$P = K\rho^{\frac{n+1}{n}}$$

Make it dimensionless:

$$\theta^n \equiv \frac{\rho}{\rho_c}$$

$$P(r) = K \rho_c^{\frac{n+1}{n}} \theta^{n+1}(r) = P_c \theta^{n+1}(r)$$

# DERIVATION 2: WORKING TOWARDS A DIMENSIONLESS FORM

Define a polytropic state equation:

$$P = K\rho^{\frac{n+1}{n}}$$

Make it dimensionless:

$$\theta^n \equiv \frac{\rho}{\rho_c}$$

$$P(r) = K \rho_c^{\frac{n+1}{n}} \theta^{n+1}(r) = P_c \theta^{n+1}(r)$$

Substitute into Poisson and simplify:

# DERIVATION 2: WORKING TOWARDS A DIMENSIONLESS FORM

Define a polytropic state equation:

$$P = K\rho^{\frac{n+1}{n}}$$

Make it dimensionless:

$$\theta^n \equiv \frac{\rho}{\rho_c}$$

$$P(r) = K \rho_c^{\frac{n+1}{n}} \theta^{n+1}(r) = P_c \theta^{n+1}(r)$$

Substitute into Poisson and simplify:

$$\frac{(n+1)P_c}{4\pi G\rho_c^2} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\theta(r)}{dr} \right) = -\theta^n(r)$$

Define a new variable:



Define a new variable:

$$\alpha^2 \equiv \frac{(n+1)P_c}{4\pi G\rho_c^2}$$

Define a new variable:

$$\alpha^2 \equiv \frac{(n+1)P_c}{4\pi G\rho_c^2}$$

Use it to define a dimensionless radius:

Define a new variable:

$$\alpha^2 \equiv \frac{(n+1)P_c}{4\pi G\rho_c^2}$$

Use it to define a dimensionless radius:

$$\xi \equiv \frac{r}{\alpha}$$

Define a new variable:

$$\alpha^2 \equiv \frac{(n+1)P_c}{4\pi G\rho_c^2}$$

Use it to define a dimensionless radius:

$$\xi \equiv \frac{r}{\alpha}$$

Substitute into the simplified Poisson:

Define a new variable:

$$\alpha^2 \equiv \frac{(n+1)P_c}{4\pi G\rho_c^2}$$

Use it to define a dimensionless radius:

$$\xi \equiv \frac{r}{\alpha}$$

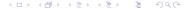
Substitute into the simplified Poisson:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta(\xi)}{d\xi} \right) = -\theta^n(\xi)$$

# Table of Contents

THEORY

White Dwarfs



Theory

### PLACEHOLDER

Something about degenerate matter and how polytropic models suit it?

Theory

# PLACEHOLDER

Relativistic vs. Non-Relativistic?



#### METHODS



$$\frac{d^2\theta}{d\xi^2} = -\frac{2}{\xi} \frac{d\theta}{d\xi} - \theta^n(\xi)$$

# Translating to a system of 1st order **EQUATIONS**

$$\begin{cases} \phi = \frac{d\theta}{d\xi} \\ \frac{d\phi}{d\xi} = -\frac{2}{\xi}\phi - \theta^n \end{cases}$$

# Obtained from central density and hydrostatic equation

$$\xi = 0$$

$$\theta = 1$$

$$\frac{d\theta}{d\xi} = 0$$

Singularity at  $\xi_0 = 0$ :

$$\phi' = -\left(\frac{2}{\xi_0}\phi\right) - \theta_0^n$$

Need to work around this somehow:

## PROBLEM!

Singularity at  $\xi_0 = 0$ :

$$\phi' = -\left(\frac{2}{\xi_0}\phi\right) - \theta_0^n$$

Need to work around this somehow:

▶ Taylor expand at  $\xi = 0$  and take limit as  $\xi \to 0$ :  $\phi' \to -\frac{1}{3}$ 

## PROBLEM!

Singularity at  $\xi_0 = 0$ :

$$\phi' = -\left(\frac{2}{\xi_0}\phi\right) - \theta_0^n$$

Need to work around this somehow:

- ▶ Taylor expand at  $\xi = 0$  and take limit as  $\xi \to 0$ :  $\phi' \to -\frac{1}{3}$
- Offset the starting point:  $0 < \xi_0 \ll 1$



# Table of Contents

THEORY

Polytropes

METHODS

RESULTS

DISCUSSION



# PLACEHOLDER

## Table of Contents

THEORY

Polytropes

White Dwarfs

METHODS

RESULTS

DISCUSSION

