Polytropes and Models of White Dwarf Stars

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Abstract

I. Introduction

Understanding stellar mechanics requires the use of mathematical models of the internal structure of stars. We understand stars to be nearly spherical collections of hot gas held together by self-gravitation and holding themselves up against gravitational collapse by fluid and radiation pressure. Equations from Newtonian gravitational and fluid mechanics allow us to create models for the how the pressure, density, and temperature of stars varies with their mass and size, but they are complicated and highly coupled.[3] Additionally, as mass and density increase, quantum and relativistic effects become important.

Prior to the development of advanced electronic computers, solutions to these equations were difficult to impossible to compute. In 1870, American physicist Jonathan Homer Lane proposed a simplified model[4] by assuming the gas pressure depends only on the density of the gas (a *polytropic fluid*), eliminating any explicit dependence on temperature and decoupling the equations for pressure, temperature, and density.

$$P(r) = K\rho^{1+1/n} \tag{1}$$

Later Swiss physicist Robert Emden formalized the model in the dimensionless differential equation that bears their names, the Lane-Emden equation, whose derivation can be found in Appendix A.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{2}$$

where ξ is a dimensionless function of the radius, θ is a dimensionless function related to the density[1], and n is the *polytropic index* of the fluid. Solutions to this equation are called *polytropes*.

While Lane's polytropic simplification seemed unrealistic for most situations conceivable at the time, the vast simplification of its solutions over more realistic models proved a generous return on the trade with results that were still close enough to observed data to be very useful. Closed-form solutions[5] can be found readily found for polytropic indices $n = 0$ and $n = 1$, and with some algebraic substitution another one can be found for $n = 5$ (which has an infinite radius and will not be discussed further), and numerical solutions can be found using techniques known at the time of Lane's original publication.

In the 20th century new utility was found for this model with the development of quantum mechanics and the discovery of *white dwarfs*, stellar remnants composed of extremely dense gas. The electron density in white dwarfs approaches the density of available quantum energy states; and since the Pauli exclusion principle disallows multiple electrons from occupying the same quantum state[2, p.216], this results in what is called *electron degeneracy pressure* which is the main force opposing the dwarf's own gravity since white dwarfs are no longer actively undergoing fusion.[3, pp.163–166] White dwarfs are therefore composed of nearly fully *degenerate matter*. Serendipitously, fully degenerate matter has the following equation of

[1] $\theta^n = \frac{\rho}{\rho_c}$, where ρ_c is the central density of the star

state:

$$P(r) = K\rho(r)^{\gamma} \tag{3}$$

identical in form to eq. 1, *i.e.*, it is a polytropic gas with $\gamma \equiv 1 + \frac{1}{n}$[2]!

White dwarfs are therefore ideally suited to modeling with polytropes. The value of the polytropic index is found from fluid and quantum mechanical relations. In fact, there are two polytropic indices that apply for degenerate gases; in lower energy states the electrons have non-relativistic momenta and have a γ index of 5/3, which corresponds to a polytropic index of $n = 1.5$. As density increases, more of the electrons occupy higher energy states with higher momenta, and relativistic effects prevail, resulting in a γ index of 4/3, corresponding to a polytropic index of $n = 3$.

We used numerical integration methods to solve the Lane-Emden equation for these polytropic indices, found the relationship between the mass and radius of the resulting polytropes, and compared results with observed data for several known white dwarfs.

II. Methods

The Lane-Emden equation is a 2nd-order nonlinear (for values of n other than 0 or 1) differential equation in one variable (ξ). It is not analytically solvable in most cases, but solutions to initial and boundary value problems for this equation can be found using numerical integration.

The major challenge in solving the equation lay in identifying the boundary conditions.

We started by rearranging the Lane-Emden equation to isolate the 2nd derivative:

$$\frac{d^2\theta}{d\xi^2} = -\frac{2}{\xi}\frac{d\theta}{d\xi} - \theta^n(\xi) \tag{4}$$

Then translating it into a system of first-order differential equations by defining a new variable $\phi \equiv \frac{d\theta}{d\xi}$:

$$\begin{cases} \phi = \frac{d\theta}{d\xi} \\ \frac{d\phi}{d\xi} = -\frac{2}{\xi}\phi - \theta^n \end{cases} \tag{5}$$

[2]not to be confused with the relativistic γ–factor

This rearrangement allows us to use numerical integrators such as Euler's method or Runge-Kutta schemes, as well as making it clearer where we want to have the boundaries defined.

We initially approached this as a boundary value problem with a free boundary[6, p.756] (since we do not know the range of ξ beforehand, only that $\theta(\xi)$ goes to zero at some point for $n < 5$). Difficulties with implementing a shooting method[1, pp.474–482] by varying the initial value of $\phi_0 = \frac{d\theta}{d\xi}|_{\xi=0}$ with such a boundary led us to re-evaluate the physics and arrive at a set of conditions to define an initial value problem that can be solved with a fairly straightforward fourth-degree Runge-Kutta scheme[1, pp.411–415].

The initial value of $\frac{d\theta}{d\xi}$ at the center of the polytrope must be zero, as there cannot physically be a cusp in the density function at the origin. However, this leads to a new issue: the value of $\frac{d^2\theta}{d\xi^2}$ in eq. 4 is now indeterminate at $\xi = 0$. To work around this problem, one could make a Taylor expansion about $\xi = 0$[3, p.339]:

$$\theta(\xi) = 1 - \frac{1}{3!}\xi^2 + \frac{n}{5!}\xi^4 - \frac{n(8n-5)}{3*7!}\xi^6 + \cdots \tag{6}$$

Taking the limit as $\xi \to 0$, $\theta'(\xi) \to -1/3$.

However, the Lane-Emden equation is very sensitive to perturbations near the origin[3, p.340], so a more stable way to treat the indeterminate origin is simply to offset the initial conditions by some small amount, starting the integration at $0 < \xi \ll 1$. We chose to begin iterations at $\xi = 0.0001$.

$\xi = 0$ is defined as the center of the polytrope, and the dimensionless θ ranges from 1 at the center to 0 at the surface, so the initial conditions for the polytrope solutions are:

$$\xi = 0$$
$$\theta = 1$$
$$\frac{d\theta}{d\xi} = 0$$

We started integration with a small arbitrary step size, but included a routine to modify the step size near $\theta(\xi)$ to provide arbitrary precision for the location of the surface of the polytrope (defined as the radius at which the density (and therefore θ goes to zero). If we detected that $\theta(\xi) < 0$, we backed up a step and halved the step size, continuing until the desired tolerance for the zero was reached[3].

After obtaining a solution to the Lane-Emden equation, we used the degenerate equations of state to calculate the relationship between the mass and the radius of the star for both the completely non-relativistic polytrope and the completely relativistic (see Appendix). The relativistic solution should have a nearly constant solution at a limiting mass of approximately $1.44\mathrm{M}_{\odot}$, known as the Chandrasekhar mass; the maximum mass that can be supported by electron degeneracy pressure. Beyond this mass, the star collapses into a neutron star.

Finding the mass-radius relation proved more challenging than anticipated as it involved formulae from fluid mechanics and quantum mechanics, which none of the research team have yet studied formally. Turning the dimensionless radius and density functions into actual radii and masses hinged on the calculation of the α factor[4]

$$\alpha = \frac{r}{\xi} = \sqrt{\frac{(n+1)P_c}{4\pi G\rho_c^2}} \tag{7}$$

where P_c is the central pressure, ρ_c is the central density, and $G = 6.67 \times 10^{-11}\,\frac{\mathrm{Nm}^2}{\mathrm{kg}^2}$ is Newton's gravitational constant. Unfortunately, the quantities P_c and ρ_c are unknown.

Instead, we focused on obtaining K in the polytropic equation of state (eq. 1, eq. 3) from the result of the pressure integrals for degenerate fluids, presented without derivation:

$$P_{nr} = K_1\rho^{5/3} \quad = \frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{\rho}{m_H\mu_e}\right)^{5/3} \tag{8}$$

$$P_r = K_1\rho^{4/3} \quad = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3}\left(\frac{\rho}{m_H\mu_e}\right)^{4/3} \tag{9}$$

where h is Planck's constant, m_e is the mass of an electron, m_H is the mass of a hydrogen atom, and μ_e is a constant related to the composition of the matter. μ_e for a white dwarf primarily composed of carbon and oxygen is given to be approximately 2.

Initially we attempted to follow a procedure given by Dhillon[**?**] to find the mass as a function of radius

$$M = -4\pi\alpha^3\rho_c\xi^2\frac{d\theta}{d\xi} \tag{10}$$

however, we still do not know ρ_c for this relation or for the calculation of α, though at least we now know K for calculating the central pressure once we have this value.

Dhillon uses a formula for the mean density of the Sun, which we used with the value of $\rho_c/\langle\rho_c\rangle = -1/3(\xi/\theta')$, which we *can* obtain from the polytrope. However, this resulted in wildly inaccurate results as this model is accurate for main-sequence stars, not white dwarfs.

We then went back to *Stellar Interiors*[3, p.336] which gives a relation between K, M, and R:

$$K = \left[\frac{4\pi}{\xi^{n+1}(-\theta')^{n-1}}\right]_{\xi_f}^{1/n}\frac{G}{n+1}M^{1-1/n}R^{-1+3/n} \tag{11}$$

and since we know K, we can solve this for R vs. M. Note that for $n = 3$, the relativistic case, M does not depend on R and is instead a constant, which should be the Chandrasekhar mass.

Actual white dwarfs have a mixed equation of state, approaching a completely relativistic degenerate gas at the core, where densities are high, and completely non-relativistic at the surface where densities approach zero. We attempted a purely mathematical model of this mixed state equation using a bisection algorithm to find a combination constant that results in the radius going to zero at the Chandrasekhar mass.

III. Results

We obtained the following parameters for non-relativistic ($n = 1.5$) and relativistic ($n =$

[3]This is not the same as a typical adaptive-step integrator, which has a pre-defined range of integration and adjusts the step size based on the estimated error of each step

[4]See the derivation of the Lane-Emden equation in Appendix A

3) polytropes using a Runge-Kutta scheme in Matlab:

Table 1: *Solutions obtained with Runge-Kutta*

n	ξ_f	$\theta'(\xi_f)$	$\rho_c/\langle\rho\rangle$
1.5	3.6838	-0.2033	5.9907
3	6.8968	-0.0424	54.1825

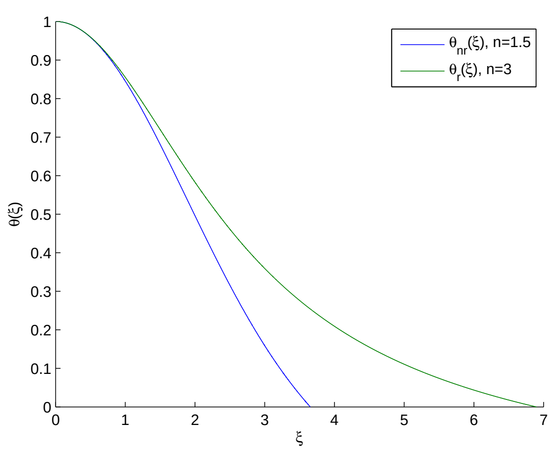
Compared with the parameters outlined in *Stellar Interiors*[3] any discrepancies are attributable to differences in significant figures.

Table 2: *Parameters for $n = 1.5$ and $n = 3$ polytropes[3, p.340]*

n	ξ_f	$\theta'(\xi_f)$	$\rho_c/\langle\rho\rangle$
1.5	3.6538	-0.20330	5.991
3	6.8969	-0.04243	54.1825

Plots of the solutions have the expected shape matching those found in the literature[3][5][**?**]:

Figure 1: *Plot of $n = 1.5$ and $n = 3$ polytropes*

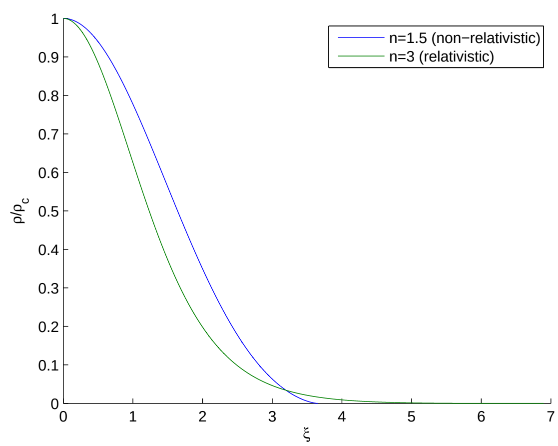


The first physically meaningful relationship that is most readily found from the polytrope is the density profile of the star; the relationship between density and radius. Obtaining this relationship in dimensionless form is just a matter of using the definition of θ:

$$\theta^n = \frac{\rho}{\rho_c} \tag{12}$$

then plotting θ^n vs. ξ:

Figure 2: *Density relation*

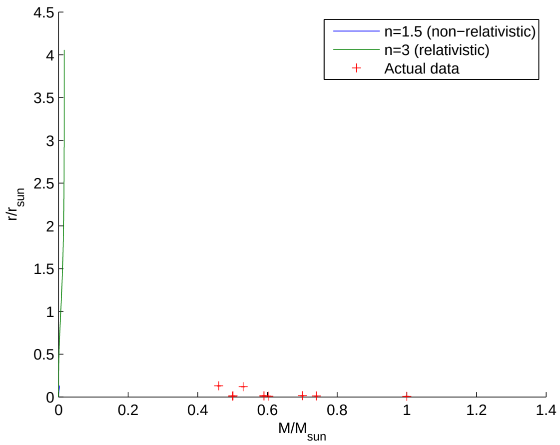


Here we see that for a completely relativistic degenerate gas, the density initially falls off more quickly with increasing radius than for a non-relativistic gas, but this dropoff slows considerably near the stellar surface resulting in a radius almost twice as large as the non-relativistic case.

A plot of our initial attempt at a mass/radius relation compared with actual data obtained from astronomical surveys[5] shows the severe problems with the model:

[5]see Appendix B

Figure 3: *Incorrect mass/radius relation*



We do not see the expected constant value for the relativistic mass, and the shapes of the relations do not match expectations (we should be seeing the radius *decrease* with increasing mass).

After fixing our procedure, we find the expected relation. The relativistic polytrope results in a constant mass of $1.4373\mathrm{M}_\odot$, which rounds to the expected value of $M_C = 1.44\mathrm{M}_\odot$.

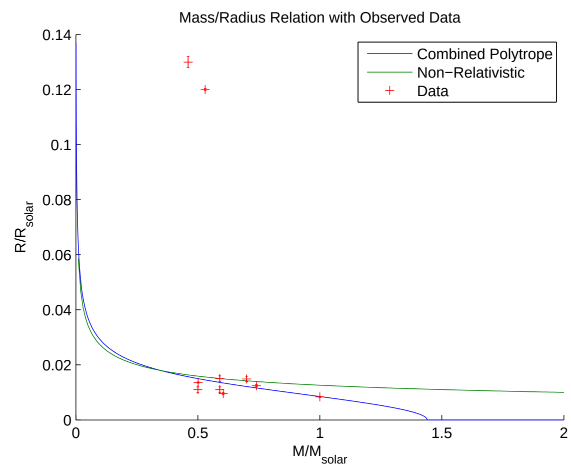
We then combined the relativistic and non-relativistic models by forcing the radius to go to zero at the Chandrasekhar mass (values in this relation are from the non-relativistic and relativistic mass/radius relations, with w as a combination constant:

$$R = \sqrt{w\left(\frac{1}{G^2 M_C^4}\right) - \frac{5.4707 \times 10^{12}}{M_C^{(}8/3)/4.7338 \times 10^{16} M_C^{10/3}}}$$

We used a bisection algorithm to find the value of w that results in a root at $M = M_C$.

Plotting the combined model against observed data, we find that it fits better than the non-relativistic model at larger masses:

Figure 4: *Mass/radius relation for non-relativistic & combined pressures with data*



IV. Discussion

References

[1] A. Gilat and V. Subramaniam. *Numerical Methods for Engineers and Scientists*. John Wiley & Sons, Inc., 111 River Street, Hoboken, NJ 07030-5774, 3rd edition, 2014.

[2] D. J. Griffiths. *Introduction to Quantum Mechanics*. Pearson Education Inc., Upper Saddle River, NJ 07458, USA, 2nd edition, 2005.

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Appendices

A. Derivation of the Lane-Emden Equation[5, pp.176–179]

The Lane-Emden equation can be derived multiple ways; one way is from the equations for hydrostatic equilibrium and mass conservation:

$$\frac{dP(r)}{dr} = -\frac{\rho(r)GM(r)}{r^2} \tag{13}$$

$$dM(r) = 4\pi r^2 \rho(r) dr \rightarrow \frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \tag{14}$$

where $P(r)$ is the gas pressure as a function of radial distance from the center of the distribution, $\rho(r)$ is the gas density, G is Newton's gravitational constant, and $M(r)$ is the mass enclosed within a sphere of radius r.

These equations can be related by multiplying eq. 13 by r^2/ρ:

$$\frac{r^2}{\rho(r)}\frac{dP(r)}{dr} = -\frac{r^2}{\rho(r)}\frac{\rho(r)GM(R)}{r^2}$$

then differentiating with respect to r:

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -G\frac{dM(r)}{dr}$$

Substituting in eq. 14 we obtain Poisson's equation for gravitational potential:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -4\pi G\rho(r) \tag{15}$$

Now, using the polytropic state equation:

$$P = K\rho^{\frac{n+1}{n}} \tag{16}$$

where n is called the *polytropic index* and K is a constant, and defining a dimensionless function $\theta(r)$:

$$\rho(r) = \rho_c \theta^n r \tag{17}$$

where ρ_c is the central density of the star, we can rewrite the pressure as a function of $\theta(r)$:

$$P(r) = K\rho_c^{\frac{n+1}{n}}\theta^{n+1}(r) = P_c\theta^{n+1}(r)$$

where $P_c = K\rho_c^{\frac{n+1}{n}}$ is the central pressure of the star. Substituting this into eq. 15:

$$K\rho_c^{\frac{n+1}{n}}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho_c\theta^n(r)}\frac{d\theta^{n+1}(r)}{dr}\right) = -4\pi G\rho_c\theta^n(r)$$

This can be simplified a bit by realizing that $\frac{d\theta^{n+1}(r)}{dr} = (n+1)\theta^n(r)\frac{d\theta(r)}{dr}$:

$$\frac{(n+1)P_c}{4\pi G\rho_c^2}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta(r)}{dr}\right) = -\theta^n(r) \tag{18}$$

Since we defined $\theta(r)$ as a dimensionless function, this equation requires that $\frac{(n+1)P_c}{4\pi G\rho_c^2}$ has the dimension of length squared. For further simplification, we can define a new constant factor α that depends on the polytropic index n:

$$\alpha^2 = \frac{(n+1)P_c}{4\pi G\rho_c^2} \tag{19}$$

and a new dimensionless radius ξ:

$$\xi = \frac{r}{\alpha} \tag{20}$$

Substituting this ξ into eq. 18 we finally obtain the Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta(\xi)}{d\xi}\right) = -\theta^n(\xi) \tag{21}$$

B. Data Tables

Table 3: *Observed masses and radii for white dwarfs*

Name	Mass $M_\odot$	Radius $R_\odot$
Sirius B	1.000 ± 0.016	0.0084 ± 0.0002
Procyon B	0.604 ± 0.018	0.0096 ± 0.0004
40 Eri B	0.501 ± 0.011	0.0136 ± 0.0002
CD-38 10980	0.74 ± 0.04	0.01245 ± 0.0004
W485A	0.59 ± 0.04	0.0015 ± 0.001
L268-92	0.70 ± 0.12	0.0149 ± 0.001
L481-60	0.53 ± 0.05	0.1200 ± 0.0004
G154-B5B	0.46 ± 0.08	0.13 ± 0.002
G181-B5B	0.50 ± 0.05	0.011 ± 0.001
G156-64	0.59 ± 0.06	0.011 ± 0.001