

# POLYTROPIC MODELS OF WHITE DWARFS

## UNC PHYS 331 PROJECT

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# WHAT ARE POLYTROPES?

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Solutions to...

## The Lane-Emden Equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n(\xi)$$

A dimensionless, 2nd order nonlinear differential equation relating the pressure of a spherically-symmetric gas distribution to the radius.

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- ▶ Provide simplified stellar models - simple pressure/density relation
- ▶ Easier to solve than full equations of stellar structure
- ▶ Require less computational effort - some analytic solutions even exist!



# DEFINITIONS

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**Poisson's equation** Relates a force density function to a potential field

$$\nabla^2\Phi = f$$

# DERIVATION 1: POISSON EQUATION

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Yielding Poisson's equation for gravity:

$$\frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2}{\rho(r)} \frac{dP(r)}{dr} \right) = -4\pi G \rho(r)$$

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$$\frac{(n+1)P_c}{4\pi G \rho_c^2} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\theta(r)}{dr} \right) = -\theta^n(r)$$



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# WHY WHITE DWARFS?

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- ▶ They are very dense
- ▶ So dense that they are completely degenerate
- ▶ We'll see why that's important shortly

# DEFINITIONS

## DEFINITION

**Degeneracy** - In quantum mechanics, when 2 or more energy states correspond to the same measured energy

## DEFINITION

**Degenerate Matter** - Quantum version of an ideal gas. Appear under extremely high density or extremely low temperatures.

# DEGENERACY IN WHITE DWARFS

High density results in complete degeneracy of electrons.  
energy state.

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High density results in complete degeneracy of electrons.  
Pauli exclusion principle prevents more than 2 electrons in each energy state.

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Density of electrons in with a range of momentum  $[p, p + dp]$ :

$$n_e(p, p + dp) \leq \frac{8\pi p^2 dp}{h^3}$$

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When  $n_e \ll \frac{8\pi p^2 dp}{h^3}$ , behaves as an ideal gas

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$$P = K\rho^\gamma$$



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As the density increases further, the electrons become relativistic.

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As  $n_e \rightarrow \frac{8\pi p^2 dp}{h^3}$ , degeneracy increases and equation of state becomes:

$$P = K\rho^\gamma$$

In the non-relativistic case,  $\gamma = 5/3$ . In the relativistic case,  
 $\gamma = 4/3$ .

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# REARRANGE THE LANE-EMDEN EQUATION

$$\frac{d^2\theta}{d\xi^2} = -\frac{2}{\xi} \frac{d\theta}{d\xi} - \theta^n(\xi)$$

# TRANSLATING TO A SYSTEM OF 1ST ORDER EQUATIONS

$$\begin{cases} \phi &= \frac{d\theta}{d\xi} \\ \frac{d\phi}{d\xi} &= -\frac{2}{\xi}\phi - \theta^n \end{cases}$$

# BOUNDARY VALUES

Obtained from central density and hydrostatic equation

$$\xi = 0$$

$$\theta = 1$$

$$\frac{d\theta}{d\xi} = 0$$

# RUNGE-KUTTA SOLUTION

Used a 4th degree Runge-Kutta solver

Did not know the integration range beforehand:

To find the surface with arbitrary precision, backed up a step and halved the step size if  $\theta < 0$  until desired precision reached

# PROBLEM!

Singularity at  $\xi_0 = 0$ :

$$\phi' = - \left( \frac{2}{\xi_0} \phi \right) - \theta_0^n$$

Need to work around this somehow:



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- Taylor expand at  $\xi = 0$  and take limit as  $\xi \rightarrow 0$ :  $\phi' \rightarrow -\frac{1}{3}$

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Singularity at  $\xi_0 = 0$ :

$$\phi' = - \left( \frac{2}{\xi_0} \phi \right) - \theta_0^n$$

Need to work around this somehow:

- ▶ Taylor expand at  $\xi = 0$  and take limit as  $\xi \rightarrow 0$ :  $\phi' \rightarrow -\frac{1}{3}$
- ▶ Offset the starting point:  $0 < \xi_0 \ll 1$

# GETTING SOMETHING USEFUL

Finding the density & pressure:

$$\frac{\rho}{\rho_c} = \frac{1}{3} \frac{\xi_f}{\theta(\xi_f)}$$

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# SOLUTIONS FOR N=1.5 AND N=3

Parameters for n=1.5, 3 polytropes[1]

| $n$ | $\xi_f$ | $\theta'(\xi_f)$ | $\rho_c/\langle\rho\rangle$ |
|-----|---------|------------------|-----------------------------|
| 1.5 | 3.6538  | -0.20330         | 5.991                       |
| 3   | 6.8969  | -0.04243         | 54.1825                     |

|                        | $n$ | $\xi_f$ | $\theta'(\xi_f)$ | $\rho_c/\langle\rho\rangle$ |
|------------------------|-----|---------|------------------|-----------------------------|
| Our calculated values: | 1.5 | 3.6538  | -0.2033          | 5.9907                      |
|                        | 3   | 6.8968  | -0.0424          | 54.1825                     |

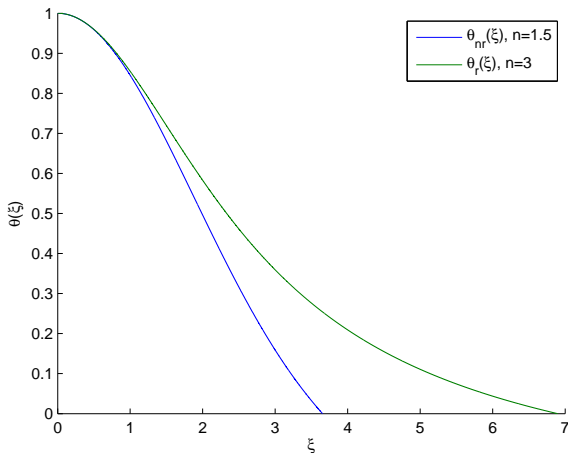
SOLUTIONS FOR  $N=1.5$  AND  $N=3$ 

Image created by presenters in Matlab

# DENSITY PROFILE

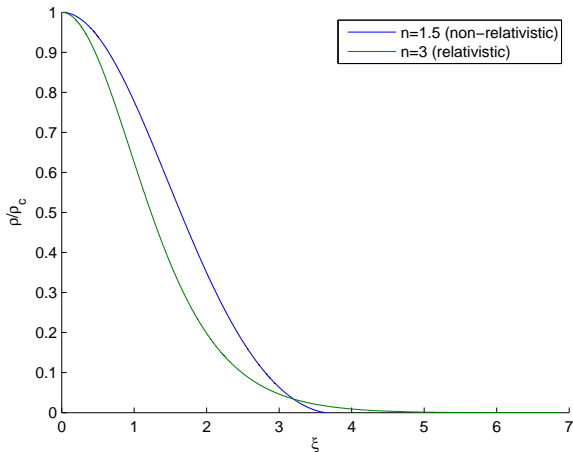


Image created by presenters in Matlab

# MASS - RADIUS RELATIONSHIP

We did not get this working correctly yet. We believe we're having trouble with unit conversions



# MASS - RADIUS RELATIONSHIP

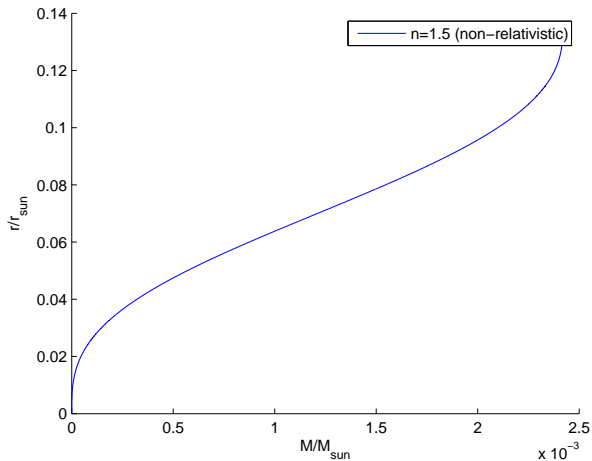


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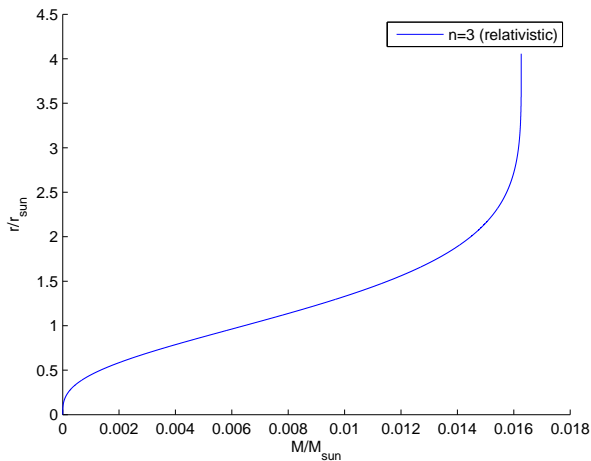


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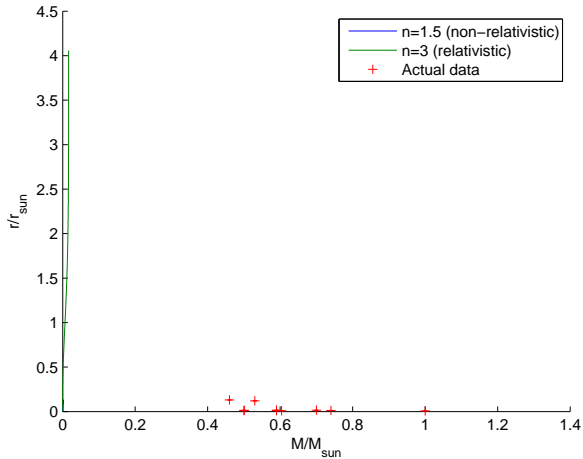


Image create by presenters in Matlab, Data from astronomical surveys of white dwarfs in binary systems

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# CHALLENGES

- ▶ Major difficulty was in correctly framing the problem.
- ▶ Initially tried to use a shooting method - but with free boundary it became prohibitively difficult.
- ▶ Rearranged & used known physics to turn into initial value problem.

# CHALLENGES

- ▶ Computing the mass-radius relationship
- ▶ Sources differed on derivation
- ▶ UNITS

# WHERE TO GO FROM HERE?

Fix our calculation of the mass-radius relationship  
Real white dwarfs have a mixed equation of state;  
non-relativistic near surface and highly relativistic near core.  
Approximate this state equation to find behavior near  
Chandrasekhar mass

# QUESTIONS?



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V. Dhillon.

Solving the Lane-Emden equation

*PHY 213 - The structure of main-sequence stars*

[http://www.vikdhillon.staff.shef.ac.uk/teaching/  
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