Polytropic Models of White Dwarfs

UNC PHYS 331 Project

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What are polytropes?

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Solutions to...

The Lane-Emden Equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n(\xi)$$

A dimensionless, 2nd order nonlinear differential equation relating the pressure of a spherically-symmetric gas distribution to the radius.

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- Provide simplified stellar models - simple pressure/density relation
- Easier to solve than full equations of stellar structure
- Require less computational effort - some analytic solutions even exist!

Definitions

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Polytropic fluid - Fluid with an equation of state that depends on only one variable

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Poisson's equation Relates a force density function to a potential field

$$\nabla^2\Phi = f$$

Derivation 1: Poisson Equation

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$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -4\pi G \rho(r)$$

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$$\frac{(n+1)P_c}{4\pi G\rho_c^2}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta(r)}{dr}\right) = -\theta^n(r)$$

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$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta(\xi)}{d\xi}\right) = -\theta^n(\xi)$$

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Why White Dwarfs?

- They are very dense
- So dense that they are completely degenerate
- We'll see why that's important shortly

Definitions

Definition

Degeneracy - In quantum mechanics, when 2 or more energy states correspond to the same measured energy

Definition

Degenerate Matter - Quantum version of an ideal gas. Appear under extremely high density or extremely low temperatures.

Degeneracy in White Dwarfs

High density results in complete degeneracy of electrons.
energy state.

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Pauli exclusion principle prevents more than 2 electrons in each energy state.

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Density of electrons in with a range of momentum $[p, p+dp]$:

$$n_e(p, p+dp) \leq \frac{8\pi p^2 dp}{h^3}$$

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When $n_e \ll \frac{8\pi p^2 dp}{h^3}$, behaves as an ideal gas

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As $n_e \rightarrow \frac{8\pi p^2 dp}{h^3}$, degeneracy increases and equation of state becomes:

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As the density increases further, the electrons become relativistic.

Degeneracy in White Dwarfs

High density results in complete degeneracy of electrons.
energy state.

As $n_e \to \frac{8\pi p^2 dp}{h^3}$, degeneracy increases and equation of state becomes:

$$P = K\rho^\gamma$$

In the non-relativistic case, $\gamma = 5/3$. In the relativistic case, $\gamma = 4/3$.

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Rearrange the Lane-Emden Equation

$$\frac{d^2\theta}{d\xi^2} = -\frac{2}{\xi}\frac{d\theta}{d\xi} - \theta^n(\xi)$$

Translating to a system of 1st order equations

$$\begin{cases} \phi & = \frac{d\theta}{d\xi} \\ \frac{d\phi}{d\xi} & = -\frac{2}{\xi}\phi - \theta^n \end{cases}$$

Boundary Values

Obtained from central density and hydrostatic equation

$$\begin{aligned} \xi &= 0 \\ \theta &= 1 \\ \frac{d\theta}{d\xi} &= 0 \end{aligned}$$

Runge-Kutta Solution

Used a 4th degree Runge-Kutta solver
Did not know the integration range beforehand:
To find the surface with arbitrary precision, backed up a step and halved the step size if $\theta < 0$ until desired precision reached

Problem!

Singularity at $\xi_0 = 0$:

$$\phi' = -\left(\frac{2}{\xi_0}\phi\right) - \theta_0^n$$

Need to work around this somehow:

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Singularity at $\xi_0 = 0$:

$$\phi' = -\left(\frac{2}{\xi_0}\phi\right) - \theta_0^n$$

Need to work around this somehow:

- Taylor expand at $\xi = 0$ and take limit as $\xi \to 0$: $\phi' \to -\frac{1}{3}$
- Offset the starting point: $0 < \xi_0 \ll 1$

Getting something useful

Finding the density & pressure:

$$\frac{\rho}{\rho_c} = \frac{1}{3}\frac{\xi_f}{\theta(\xi_f)}$$

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Solutions for n=1.5 and n=3

Parameters for n=1.5, 3 polytropes[1]

n	ξ_f	$\theta'(\xi_f)$	$\rho_c/\langle\rho\rangle$
1.5	3.6538	-0.20330	5.991
3	6.8969	-0.04243	54.1825

Solutions for n=1.5 and n=3

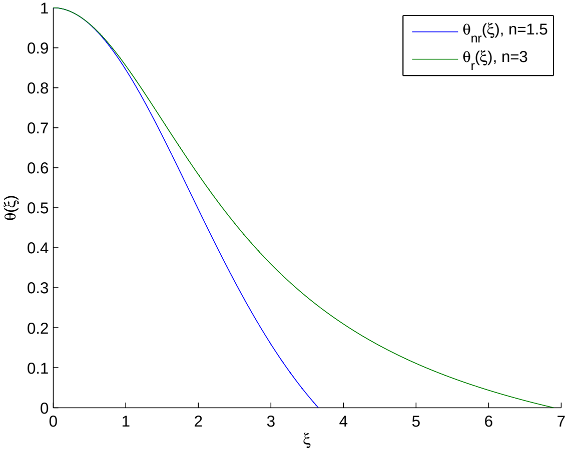


Image created by presenters in Matlab

Density Profile

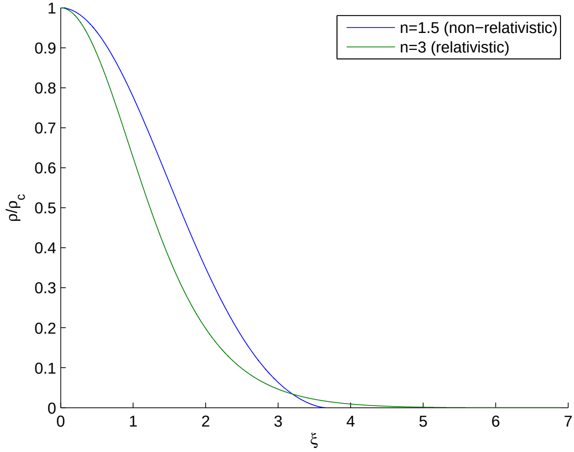
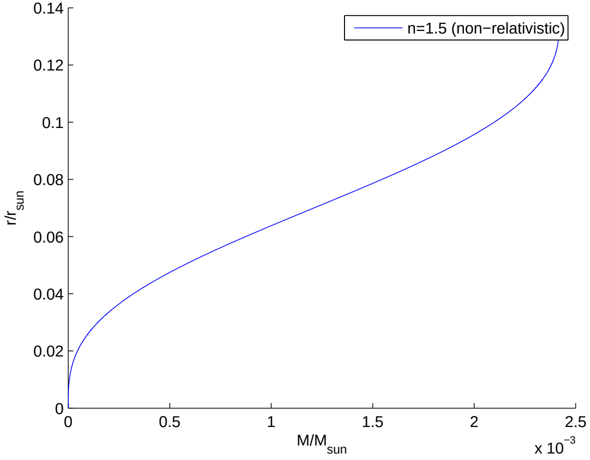


Image created by presenters in Matlab

Mass - Radius Relationship

We did not get this working correctly yet. We believe we're having trouble with unit conversions

Mass - Radius Relationship



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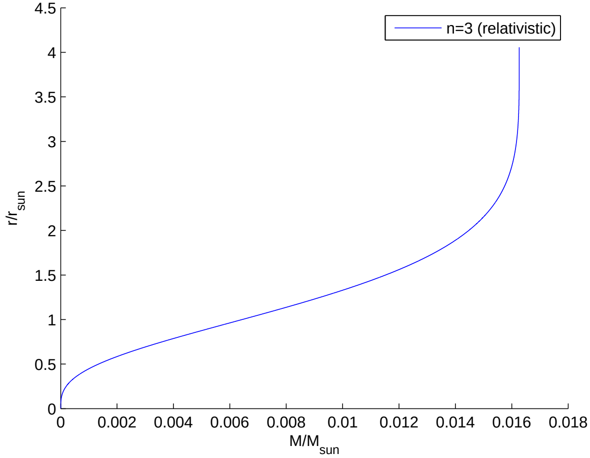

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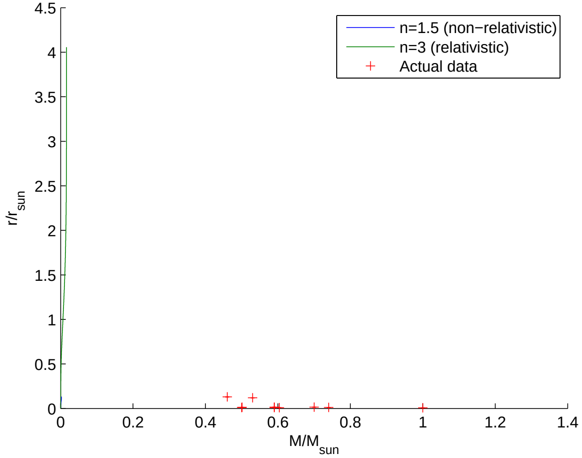

Image create by presenters in Matlab, Data from astronomical surveys of white dwarfs in binary systems

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Challenges

- Major difficulty was in correctly framing the problem.
- Initially tried to use a shooting method - but with free boundary it became prohibitively difficult.
- Rearranged & used known physics to turn into initial value problem.

Challenges

- Computing the mass-radius relationship
- Sources differed on derivation
- UNITS

Where to go from here?

Fix our calculation of the mass-radius relationship
Real white dwarfs have a mixed equation of state;
non-relativistic near surface and highly relativistic near core.
Approximate this state equation to find behavior near Chandrasekhar mass

Questions?

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