Polytropic Models of White Dwarfs

UNC PHYS 331 Project

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What are polytropes?

Solutions to...

The Lane-Emden Equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n(\xi)$$

A dimensionless, 2nd order nonlinear differential equation relating the pressure of a spherically-symmetric gas distribution to the radius.

Why polytropes?

- Provide simplified stellar models - simple pressure/density relation
- Easier to solve than full equations of stellar structure
- Require less computational effort - some analytic solutions even exist!

Definitions

Definition

Polytropic fluid - Fluid with an equation of state that depends on only one variable

Definition

Polytropic index - Constant that relates pressure of a polytropic fluid to its volume (density). It may be any real number.

Definition

Poisson's equation Relates a force density function to a potential field

$$\nabla^2\Phi = f$$

Derivation 1: Poisson Equation

Can be derived multiple ways. From laws of mass conservation and hydrostatic equilibrium:

$$\begin{aligned} dM(r) &= 4\pi r^2\rho(r)dr \rightarrow \frac{dM(r)}{dr} = 4\pi r^2\rho(r) \\ \frac{dP(r)}{dr} &= -\frac{\rho(r)GM(r)}{r^2} \end{aligned}$$

These equations are related by multiplying the hydrostatic equation by r^2/ρ and differentiating:

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -G\frac{dM(r)}{dr}$$

Yielding Poisson's equation for gravity:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -4\pi G\rho(r)$$

Derivation 2: Working towards a dimensionless form

Define a polytropic state equation:

$$P = K\rho^{\frac{n+1}{n}}$$

Make it dimensionless:

$$\theta^n \equiv \frac{\rho}{\rho_c}$$

$$P(r) = K\rho_c^{\frac{n+1}{n}}\theta^{n+1}(r) = P_c\theta^{n+1}(r)$$

Substitute into Poisson and simplify:

$$\frac{(n+1)P_c}{4\pi G\rho_c^2}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta(r)}{dr}\right) = -\theta^n(r)$$

Derivation 3: More simplification

Define a new variable:

$$\alpha^2 \equiv \frac{(n+1)P_c}{4\pi G\rho_c^2}$$

Use it to define a dimensionless radius:

$$\xi \equiv \frac{r}{\alpha}$$

Substitute into the simplified Poisson:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta(\xi)}{d\xi}\right) = -\theta^n(\xi)$$

Why White Dwarfs?

- They are very dense
- So dense that they are completely degenerate
- We'll see why that's important shortly

Definitions

Definition

Degeneracy - In quantum mechanics, when 2 or more energy states correspond to the same measured energy

Definition

Degenerate Matter - Quantum version of an ideal gas. Appear under extremely high density or extremely low temperatures.

Degeneracy in White Dwarfs

High density results in complete degeneracy of electrons.
Pauli exclusion principle prevents more than 2 electrons in each energy state.
Density of electrons in with a range of momentum $[p, p + dp]$:

$$n_e(p, p + dp) \leq \frac{8\pi p^2 dp}{h^3}$$

When $n_e \ll \frac{8\pi p^2 dp}{h^3}$, behaves as an ideal gas
As $n_e \to \frac{8\pi p^2 dp}{h^3}$, degeneracy increases and equation of state becomes:

$$P = K\rho^{\gamma}$$

As the density increases further, the electrons become

In the non-relativistic case, $\gamma = 5/3$. In the relativistic case

Rearrange the Lane-Emden Equation

$$\frac{d^2\theta}{d\xi^2} = -\frac{2}{\xi}\frac{d\theta}{d\xi} - \theta^n(\xi)$$

Translating to a system of 1st order equations

$$\begin{cases} \phi & = \frac{d\theta}{d\xi} \\ \frac{d\phi}{d\xi} & = -\frac{2}{\xi}\phi - \theta^n \end{cases}$$

Boundary Values

Obtained from central density and hydrostatic equation

$$\begin{aligned} \xi &= 0 \\ \theta &= 1 \\ \frac{d\theta}{d\xi} &= 0 \end{aligned}$$

Runge-Kutta Solution

Used a 4th degree Runge-Kutta solver
Did not know the integration range beforehand:
To find the surface with arbitrary precision, backed up a step and halved the step size if $\theta < 0$ until desired precision reached

Problem!

Singularity at $\xi_0 = 0$:

$$\phi' = -\left(\frac{2}{\xi_0}\phi\right) - \theta_0^n$$

Need to work around this somehow:

- Taylor expand at $\xi = 0$ and take limit as $\xi \to 0$: $\phi' \to -\frac{1}{3}$
- Offset the starting point: $0 < \xi_0 \ll 1$

Getting something useful

Finding the density & pressure:

$$\frac{\rho}{\rho_c} = \frac{1}{3}\frac{\xi_f}{\theta(\xi_f)}$$

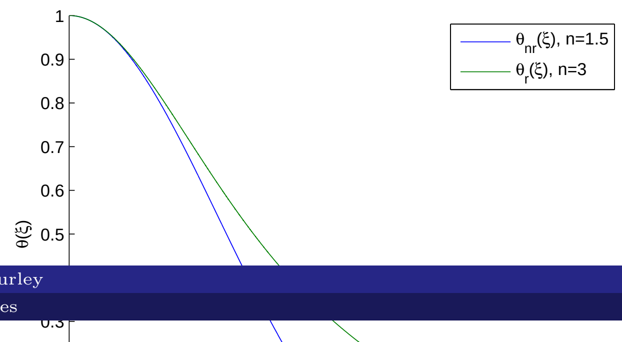
Solutions for n=1.5 and n=3

Parameters for n=1.5, 3 polytropes[1]

n	ξ_f	$\theta'(\xi_f)$	$\rho_c/\langle\rho\rangle$
1.5	3.6538	-0.20330	5.991
3	6.8969	-0.04243	54.1825

Our calculated values:

n	ξ_f	$\theta'(\xi_f)$	$\rho_c/\langle\rho\rangle$
1.5	3.6538	-0.2033	5.9907
3	6.8968	-0.0424	54.1825



Density Profile

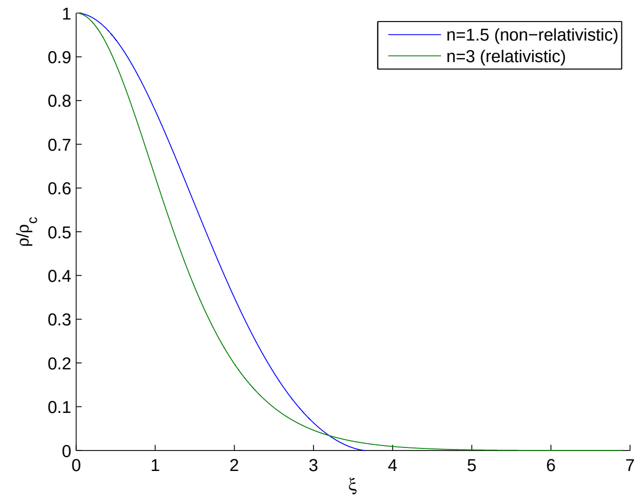
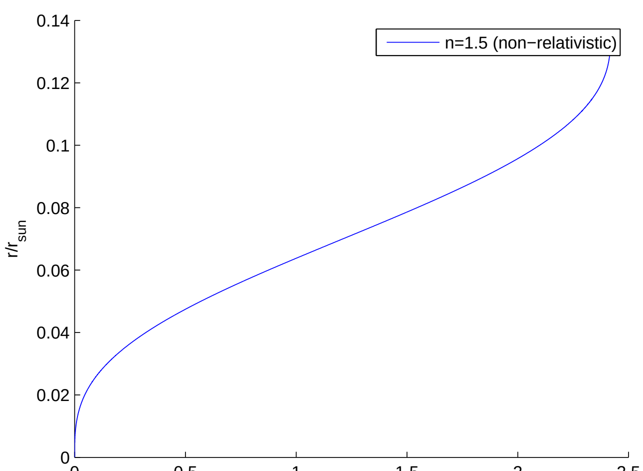


Image created by presenters in Matlab

Mass - Radius Relationship

We did not get this working correctly yet. We believe we're having trouble with unit conversions



Challenges

- Major difficulty was in correctly framing the problem.
- Initially tried to use a shooting method - but with free boundary it became prohibitively difficult.
- Rearranged & used known physics to turn into initial value problem.

- Computing the mass-radius relationship
- Sources differed on derivation
- UNITS

Where to go from here?

Fix our calculation of the mass-radius relationship
Real white dwarfs have a mixed equation of state;
non-relativistic near surface and highly relativistic near core.
Approximate this state equation to find behavior near
Chandrasekhar mass

Questions?

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