

## Midterm 1 Solutions

EECS 245, Fall 2025 at the University of Michigan

Name: \_\_\_\_\_

username: \_\_\_\_\_

UMID: \_\_\_\_\_

Room: ☐ 1013 DOW    ☐ 2725 BBB

### Instructions

- This exam consists of 8 problems, worth a total of 100 points, spread across 12 pages (6 sheets of paper).
- You have 80 minutes to complete this exam, unless you have extended-time accommodations through SSD.
- Write your username in the top right corner of each page.
- For free response problems, you must show all of your work (unless otherwise specified), and circle your final answer. We will not grade work that appears elsewhere, and you may lose points if your work is not shown.
- For multiple choice problems, completely fill in bubbles and square boxes; if we cannot tell which option(s) you selected, you may lose points.
  - ☐ A bubble means that you should only select one choice.
  - ☐ A square box means you should select all that apply.
- You may refer to a single two-sided handwritten notes sheet. Other than that, you may not refer to any other resources or technology during the exam (no phones, watches, or calculators).

You are to abide by the University of Michigan/Engineering Honor Code. To receive a grade, please sign below to signify that you have kept the Honor Code pledge.

*I have neither given nor received aid on this exam, nor have I concealed any violations of the Honor Code.*

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**Problem 1: Consider the Following... (15 pts)**

Consider the following dataset of  $n = 9$  values.

$y_1$	$y_2$	$y_3$	$y_4$	$y_5$	$y_6$	$y_7$	$y_8$	$y_9$
7	8	10	10	11	13	14	17	27

Suppose we'd like to find the optimal parameter,  $w^*$ , for the constant model  $h(x_i) = w$ , given this dataset of 9 values.

In parts a) through f), choose the empirical risk function  $R(w)$  that the given value of  $w^*$  is the minimizer of, for this particular dataset. If you believe the given value of  $w^*$  does not minimize any of the five options, select N/A.

- **Option 1:**  $R(w) = \frac{1}{n} \sum_{i=1}^n (y_i - w)^2$
- **Option 2:**  $R(w) = \frac{1}{n} \sum_{i=1}^n (27y_i - 13w)^2$
- **Option 3:**  $R(w) = \frac{1}{n} \sum_{i=1}^n 13|y_i - w|$
- **Option 4:**  $R(w) = \frac{1}{n} \sum_{i=1}^n \begin{cases} 13 & \text{if } y_i = w \\ 27 & \text{if } y_i \neq w \end{cases}$
- **Option 5:**  $R(w) = \lim_{p \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n |y_i - w|^p$

- a) (2.5 pts) 10 is the value of  $w$  that minimizes...  
☐ Option 1   ☐ Option 2   ☐ Option 3   ☒ Option 4   ☐ Option 5   ☐ N/A
- b) (2.5 pts) 11 is the value of  $w$  that minimizes...  
☐ Option 1   ☐ Option 2   ☒ Option 3   ☐ Option 4   ☐ Option 5   ☐ N/A
- c) (2.5 pts) 12 is the value of  $w$  that minimizes...  
☐ Option 1   ☐ Option 2   ☐ Option 3   ☐ Option 4   ☐ Option 5   ☒ N/A
- d) (2.5 pts) 13 is the value of  $w$  that minimizes...  
☒ Option 1   ☐ Option 2   ☐ Option 3   ☐ Option 4   ☐ Option 5   ☐ N/A
- e) (2.5 pts) 17 is the value of  $w$  that minimizes...  
☐ Option 1   ☐ Option 2   ☐ Option 3   ☐ Option 4   ☒ Option 5   ☐ N/A
- f) (2.5 pts) 27 is the value of  $w$  that minimizes...  
☐ Option 1   ☒ Option 2   ☐ Option 3   ☐ Option 4   ☐ Option 5   ☐ N/A

## Problem 2: Absolute Madness (17 pts)

Consider a dataset of  $n = 8$  values, where

$$y_1 = 1, \quad y_2 = y_3 = 4, \quad y_4 = y_5 = y_6 = \alpha, \quad y_7 = y_8 = 20$$

and  $4 < \alpha < 20$ .

As usual, let  $R_{\text{abs}}(w)$  represent the mean absolute error of a constant prediction  $w$  on this dataset of 8 values.

a) (3 pts) Is the value of  $w^*$ , the minimizer of  $R_{\text{abs}}(w)$ , unique? Select and fill out one option below.

☒ The value of  $w^*$  is unique, and is equal to

$\alpha$

☐ The value of  $w^*$  is not unique; any value between

and

is a minimizer.

b) (6 pts) Find the value of  $R_{\text{abs}}(\alpha)$ , for any valid choice of  $\alpha$ . Show your work, and circle your final answer, which should be an expression involving  $\alpha$  and other constants, but no other variables, and no summation notation.

**Solution:**  $R_{\text{abs}}(\alpha) = \frac{\alpha+31}{8}$ .

Let's start with the definition of  $R_{\text{abs}}(w)$  and plug in  $w = \alpha$ .

$$\begin{aligned} R_{\text{abs}}(\alpha) &= \frac{1}{8} \sum_{i=1}^8 |y_i - \alpha| \\ &= \frac{|1 - \alpha| + |4 - \alpha| + |4 - \alpha| + |\alpha - \alpha| + |\alpha - \alpha| + |\alpha - \alpha| + |20 - \alpha| + |20 - \alpha|}{8} \\ &= \frac{|1 - \alpha| + 2|4 - \alpha| + 2|20 - \alpha|}{8} \end{aligned}$$

Since  $\alpha > 1$  and  $\alpha > 4$ , we know that  $|1 - \alpha| = \alpha - 1$  and  $|4 - \alpha| = \alpha - 4$ . Similarly, since  $\alpha < 20$ , we have  $|20 - \alpha| = 20 - \alpha$ .

$$\begin{aligned} R_{\text{abs}}(\alpha) &= \frac{|1 - \alpha| + 2|4 - \alpha| + 2|20 - \alpha|}{8} \\ &= \frac{\alpha - 1 + 2(\alpha - 4) + 2(20 - \alpha)}{8} \\ &= \frac{\alpha - 1 + 2\alpha - 8 + 40 - 2\alpha}{8} \\ &= \frac{\alpha + 31}{8} \end{aligned}$$

Recall,

$$y_1 = 1, \quad y_2 = y_3 = 4, \quad y_4 = y_5 = y_6 = \alpha, \quad y_7 = y_8 = 20$$

where  $4 < \alpha < 20$ .

c) (8 pts) Let the minimum possible value of  $R_{\text{abs}}(w)$  be  $M$ . Given that

$$R_{\text{abs}}(20) - M = \frac{9}{2}$$

find the value of  $\alpha$ . Show your work, and circle your final answer, which should be a number with no variables.

*Hint: It's possible to answer this without using your answer from the previous part.*

**Solution:**  $\alpha = 11$ .

Since  $\alpha$  minimizes  $R_{\text{abs}}(w)$ , we know that  $R_{\text{abs}}(\alpha) = M$ . In the previous part, we found an expression for  $R_{\text{abs}}(\alpha)$ . One common solution was to find another expression for  $R_{\text{abs}}(20)$  (which is also a function of  $\alpha$ ), and then to solve for the  $\alpha$  such that

$$R_{\text{abs}}(20) - R_{\text{abs}}(\alpha) = \frac{9}{2}$$

Here's another solution. Since  $(\alpha, M)$  is the vertex of  $R_{\text{abs}}(w)$ , we know that the slope to the left of it is negative and the slope to the right of it is positive.

The slope on the line segment between  $(\alpha, M)$  and  $(20, M + \frac{9}{2})$  is

$$\text{slope} = \frac{\# \text{left} - \# \text{right}}{n} = \frac{6 - 2}{8} = \frac{1}{2}$$

So, now we know that on the line segment between  $(\alpha, M)$  and  $(20, M + \frac{9}{2})$ , the slope is  $\frac{1}{2}$ . This is all we need to solve for  $\alpha$ . Since the slope of a line segment is its change in  $y$  over its change in  $x$ , we have:

$$\frac{M + \frac{9}{2} - M}{20 - \alpha} = \frac{1}{2}$$

Solving for  $\alpha$ , we get:

$$\alpha = 11$$

### Problem 3: Spreading Your Wings (12 pts)

Consider a dataset of  $n$  points,  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ , where

- the means of  $x_1, x_2, \dots, x_n$  and  $y_1, y_2, \dots, y_n$  are 15 and 5, respectively
- the variances of  $x_1, x_2, \dots, x_n$  and  $y_1, y_2, \dots, y_n$  are  $\sigma_x^2$  and  $\sigma_y^2$ , respectively
- the correlation coefficient between  $x_1, x_2, \dots, x_n$  and  $y_1, y_2, \dots, y_n$  is  $r$

We define a new set of values,  $z_1, z_2, \dots, z_n$ , as follows:

$$z_i = 3x_i - y_i, \quad i = 1, 2, \dots, n$$

- a) (4 pts) Suppose we fit a simple linear regression line to the dataset  $(x_1, z_1), (x_2, z_2), \dots, (x_n, z_n)$  by minimizing mean squared error. Note that  $z$  is the variable being predicted, not  $y$ . Let  $h(x_i)$  represent the corresponding line.

What is the value of  $h(15)$ ? Your answer should be a number with no variables.

$$h(15) = \boxed{40}$$

**Solution:**  $h(15) = 40$ .

The key fact being assessed here is that the line that minimizes mean squared error always passes through

(mean of input variable, mean of output variable)

Normally this is stated as the line passing through the point  $(\bar{x}, \bar{y})$ , but here the output variable is  $z$ , not  $y$ .

The mean of  $z$  is  $3\bar{x} - \bar{y}$ , as we explored in a homework problem, and this is

$$3\bar{x} - \bar{y} = 3(15) - 5 = 40$$

- b) (8 pts)  $\sigma_z^2$ , the variance of  $z_1, z_2, \dots, z_n$ , can be written in the form  $\sigma_z^2 = 9\sigma_x^2 + \sigma_y^2 + C$ .

(i) What is the value of  $C$ ?

- ☐  $-6\sigma_x\sigma_y$    
 ☐  $6\sigma_x\sigma_y$    
 ☒  $-6r\sigma_x\sigma_y$    
 ☐  $6r\sigma_x\sigma_y$    
 ☐  $-6nr\sigma_x\sigma_y$    
 ☐  $6nr\sigma_x\sigma_y$

(ii) Show your work in the box below. English explanations are not enough.

**Solution:** We'll find the answer by expanding out the definition of  $\sigma_z^2$  and simplifying.

$$\begin{aligned}
\sigma_z^2 &= \frac{1}{n} \sum_{i=1}^n (z_i - \bar{z})^2 \\
&= \frac{1}{n} \sum_{i=1}^n (3x_i - y_i - (3\bar{x} - \bar{y}))^2 \\
&= \frac{1}{n} \sum_{i=1}^n \underbrace{(3x_i - 3\bar{x} - y_i + \bar{y})^2}_{\text{distributed the negative sign and rearranged}} \\
&= \frac{1}{n} \sum_{i=1}^n \underbrace{(3(x_i - \bar{x}) - (y_i - \bar{y}))^2}_{\text{treat this as } (a - b)^2} \\
&= \frac{1}{n} \sum_{i=1}^n (9(x_i - \bar{x})^2 - 6(x_i - \bar{x})(y_i - \bar{y}) + (y_i - \bar{y})^2) \\
&= 9 \left( \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \right) - 6 \left( \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) \right) + \left( \frac{1}{n} \sum_{i=1}^n (y_i - \bar{y})^2 \right) \\
&= 9\sigma_x^2 + \sigma_y^2 - 6 \left( \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) \right)
\end{aligned}$$

So,

$$C = -6 \left( \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) \right)$$

But, recall that

$$r = \frac{1}{n} \sum_{i=1}^n \left( \frac{x_i - \bar{x}}{\sigma_x} \right) \left( \frac{y_i - \bar{y}}{\sigma_y} \right)$$

which means that

$$r\sigma_x\sigma_y = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$

So,

$$C = -6 \left( \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) \right) = -6r\sigma_x\sigma_y$$



#### Problem 4: Mission Impossible (12 pts)

- a) (6 pts) Suppose  $\vec{u}, \vec{v} \in \mathbb{R}^n$  are **non-zero** vectors, and suppose that

$$|\vec{u} \cdot \vec{v}| = \|\vec{u}\| \|\vec{v}\|$$

For each statement below, determine whether it is impossible, possible, or guaranteed to be true, given the above assumptions. **Select exactly one option from each row.** The first statement has been done for you as an example.

	statement	impossible?	possible?	guaranteed?
(i)	$\ \vec{u}\  = 5$	<input type="radio"/>	<input checked="" type="radio"/>	<input type="radio"/>
(ii)	$\vec{u}$ and $\vec{v}$ are orthogonal	<input checked="" type="radio"/>	<input type="radio"/>	<input type="radio"/>
(iii)	$\ \vec{u} - \vec{v}\  = 0$	<input type="radio"/>	<input checked="" type="radio"/>	<input type="radio"/>
(iv)	$\vec{u}$ and $\vec{v}$ span a 1-dimensional subspace of $\mathbb{R}^n$	<input type="radio"/>	<input type="radio"/>	<input checked="" type="radio"/>
(v)	$\vec{u}$ and $\vec{v}$ span a 2-dimensional subspace of $\mathbb{R}^n$	<input checked="" type="radio"/>	<input type="radio"/>	<input type="radio"/>
(vi)	$\ \vec{u} + \vec{v}\  = \ \vec{u}\  + \ \vec{v}\ $	<input type="radio"/>	<input checked="" type="radio"/>	<input type="radio"/>

**Solution:** Remember that for **any** two vectors  $\vec{u}$  and  $\vec{v}$ ,

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$$

The fact that we're told that

$$|\vec{u} \cdot \vec{v}| = \|\vec{u}\| \|\vec{v}\|$$

tells us that  $\cos \theta = 1$  or  $\cos \theta = -1$ , which means that the angle between  $\vec{u}$  and  $\vec{v}$  is  $0^\circ$  or  $180^\circ$ , which means that  $\vec{u}$  and  $\vec{v}$  are scalar multiples of each other. (They may point in the same or opposite directions.) This is the key insight to assessing each of the statements.

- b) (6 pts) Suppose  $\vec{w}, \vec{z} \in \mathbb{R}^n$ . Given that  $\|\vec{w}\| = \|\vec{z}\| = \|\vec{w} - \vec{z}\| = 1$ , find  $\|\vec{w} + \vec{z}\|$ . Show your work, and circle your final answer, which should be a number with no variables.

**Solution:**  $\|\vec{w} + \vec{z}\| = \sqrt{3}$ .

We're asked to find  $\|\vec{w} + \vec{z}\|$ . To do so, let's expand out  $\|\vec{w} + \vec{z}\|^2$  as we've done in the past, and see how to utilize what we were given.

$$\begin{aligned}\|\vec{w} + \vec{z}\|^2 &= (\vec{w} + \vec{z}) \cdot (\vec{w} + \vec{z}) \\ &= \vec{w} \cdot \vec{w} + 2\vec{w} \cdot \vec{z} + \vec{z} \cdot \vec{z} \\ &= \|\vec{w}\|^2 + 2\vec{w} \cdot \vec{z} + \|\vec{z}\|^2 \\ &= 1 + 2\vec{w} \cdot \vec{z} + 1 \\ &= 2 + 2\vec{w} \cdot \vec{z}\end{aligned}$$

Above, we've plugged in  $\|\vec{w}\|^2 = 1$  and  $\|\vec{z}\|^2 = 1$ . We need to know  $\vec{w} \cdot \vec{z}$ , which we don't yet know.

But, we have enough information to find it, if we expand out  $\|\vec{w} - \vec{z}\|^2$ , which we were told is equal to 1.

$$\begin{aligned}\|\vec{w} - \vec{z}\|^2 &= (\vec{w} - \vec{z}) \cdot (\vec{w} - \vec{z}) \\ 1 &= \vec{w} \cdot \vec{w} - 2\vec{w} \cdot \vec{z} + \vec{z} \cdot \vec{z} \\ 1 &= \|\vec{w}\|^2 - 2\vec{w} \cdot \vec{z} + \|\vec{z}\|^2 \\ 1 &= 1 - 2\vec{w} \cdot \vec{z} + 1 \\ 1 &= 2 - 2\vec{w} \cdot \vec{z}\end{aligned}$$

Solving the above gives us  $\vec{w} \cdot \vec{z} = \frac{1}{2}$ . This gives

$$\|\vec{w} + \vec{z}\|^2 = 2 + 2\vec{w} \cdot \vec{z} = 2 + 2 \cdot \frac{1}{2} = 3$$

And so,

$$\|\vec{w} + \vec{z}\| = \sqrt{3}$$

**Problem 5: Back to Normal (12 pts)**

Consider the orthogonal vectors  $\vec{u}_1 = \begin{bmatrix} 13 \\ -3 \\ 2 \end{bmatrix}$ ,  $\vec{u}_2 = \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix}$ , and  $\vec{u}_3 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ .

- a) (4 pts) Find the equation of the plane spanned by  $\vec{u}_2$  and  $\vec{u}_3$  in standard form, i.e.  $ax + by + cz + d = 0$ . Circle your final answer.

**Solution:** Plane:  $13x - 6y + 4z = 0$  (or any scalar multiple of this equation).

Most students took the cross product of  $\vec{u}_2$  and  $\vec{u}_3$  to find a vector that is orthogonal to the plane spanned by  $\vec{u}_2$  and  $\vec{u}_3$ , and then used that vector to define the plane.

But, we were already told that all three vectors are orthogonal to each other, which means that the vector orthogonal to the plane spanned by  $\vec{u}_2$  and  $\vec{u}_3$  is  $\vec{u}_1$ . So, we can use  $\vec{u}_1$  to define the plane.

$$\vec{u}_1 \cdot (x, y, z) = 0 \implies 13x - 3y + 2z = 0$$

So, the equation of the plane spanned by  $\vec{u}_2$  and  $\vec{u}_3$  is  $13x - 6y + 4z = 0$  (or any scalar multiple of this equation).

- b) (8 pts) There is one value of  $k$  such that the projection of  $\vec{x} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$  onto  $\vec{u}_k$  is just  $\vec{u}_k$  itself.

(i) What is the value of  $k$ ? ☐ 1 ☐ 2 ☒ 3

(ii) Show your work in the box below. English explanations are not enough.

**Solution:** We're told that for one of the three provided vectors —  $\vec{u}_1$ ,  $\vec{u}_2$ , or  $\vec{u}_3$  — the projection of  $\vec{x}$  onto that vector is just that vector itself.

Remember that the projection of  $\vec{x}$  onto  $\vec{u}_k$  is given by

$$\text{proj}_{\vec{u}_k} \vec{x} = \frac{\vec{x} \cdot \vec{u}_k}{\vec{u}_k \cdot \vec{u}_k} \vec{u}_k$$

So, we need to find the vector  $\vec{u}_k$  such that the scalar  $\frac{\vec{x} \cdot \vec{u}_k}{\vec{u}_k \cdot \vec{u}_k}$  is equal to 1, or equivalently,  $\vec{x} \cdot \vec{u}_k = \vec{u}_k \cdot \vec{u}_k$ . We can check this equality for each of the three provided vectors.

$$1. \quad \vec{x} \cdot \vec{u}_1 = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 13 \\ -3 \\ 2 \end{bmatrix} = 7 \cdot 13 + 3 \cdot (-3) + 1 \cdot 2 = 84$$

$$\vec{u}_1 \cdot \vec{u}_1 = 13^2 + (-3)^2 + 2^2 = 180$$

$84 \neq 180$ , so  $\vec{u}_1$  is not the vector we're looking for.

$$2. \quad \vec{x} \cdot \vec{u}_2 = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix} = 7 \cdot 0 + 3 \cdot 4 + 1 \cdot 6 = 18$$

$$\vec{u}_2 \cdot \vec{u}_2 = 0^2 + 4^2 + 6^2 = 52$$

$18 \neq 52$ , so  $\vec{u}_2$  is not the vector we're looking for.

$$3. \quad \vec{x} \cdot \vec{u}_3 = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = 7 \cdot 1 + 3 \cdot 3 + 1 \cdot (-2) = 14$$

$$\vec{u}_3 \cdot \vec{u}_3 = 1^2 + 3^2 + (-2)^2 = 14$$

$14 = 14$ , so  $\vec{u}_3$  is the vector we're looking for.

**Problem 6: Needed Me (11 pts)**

Suppose  $\vec{x} = \begin{bmatrix} c \\ 1 \\ 0 \end{bmatrix}$ ,  $\vec{y} = \begin{bmatrix} 1 \\ c \\ 1 \end{bmatrix}$ , and  $\vec{z} = \begin{bmatrix} 0 \\ 1 \\ c \end{bmatrix}$ , where  $c \in \mathbb{R}$  is a constant.

- a) (8 pts) Find a **positive value** of  $c$  such that  $\vec{x}$ ,  $\vec{y}$ , and  $\vec{z}$  are linearly **dependent**. Show your work, and circle your final answer, which should be a positive number with no variables.

**Solution:**  $c = \sqrt{2}$ .

For  $\vec{x}$ ,  $\vec{y}$ , and  $\vec{z}$  to be linearly dependent, there must exist scalars  $a$ ,  $b$ , and  $c$  such that

$$a\vec{x} + b\vec{y} + \vec{z} = \vec{0}$$

(or equivalently,  $a\vec{x} + b\vec{y} + d\vec{z} = \vec{0}$ , but the former approach involves one fewer variable to solve for).

Substituting in the given vectors, we have

$$a \begin{bmatrix} c \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ c \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ c \end{bmatrix}$$

As a system of equations, we have

$$ca + b = 0$$

$$a + cb = 1$$

$$b = c$$

The third equation gives us  $b = c$ , and the second gives us  $a = 1 - cb = 1 - c^2$ . Substituting these into the first equation gives us

$$c(1 - c^2) + c = 0 \implies c - c^3 + c = 0 \implies c(2 - c^2) = 0$$

This equation has three solutions for  $c$ :  $c = 0$ ,  $c = \sqrt{2}$ , and  $c = -\sqrt{2}$ . We're asked to find a **positive** value of  $c$ , so  $c = \sqrt{2}$  for this part, and either 0 or  $-\sqrt{2}$  for the next part.

- b) (3 pts) Provide one **other** value of  $c$  (that is, not your answer from the previous part) such that  $\vec{x}$ ,  $\vec{y}$ , and  $\vec{z}$  are linearly **dependent**. Your answer should be a number with no variables.

other value of  $c =$

**0 or  $-\sqrt{2}$**

**Problem 7: High Definition (12 pts)**

Suppose  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{12}$  are 12 non-zero vectors in  $\mathbb{R}^7$ . Furthermore, suppose:

- $\vec{x}_1, \vec{x}_2$ , and  $\vec{x}_3$  span a 2-dimensional subspace of  $\mathbb{R}^7$ .
- $\vec{x}_4, \vec{x}_5$ , and  $\vec{x}_6$  span **the same** 2-dimensional subspace of  $\mathbb{R}^7$  as  $\vec{x}_1, \vec{x}_2$ , and  $\vec{x}_3$ , i.e.

$$\text{span}(\{\vec{x}_4, \vec{x}_5, \vec{x}_6\}) = \text{span}(\{\vec{x}_1, \vec{x}_2, \vec{x}_3\})$$

- a) (4 pts) Let  $r$  be the dimension of the subspace of  $\mathbb{R}^7$  spanned by  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{12}$ . What are the smallest and largest possible values of  $r$ ? Your answers should be integers with no variables.

smallest possible value of  $r =$

2

largest possible value of  $r =$

7

- b) (4 pts) Which of the following **could** form a basis for  $\mathbb{R}^7$ ? Select all that apply. Blank answers will receive no credit.

- ☐  $\{\vec{x}_7, \vec{x}_8, \vec{x}_9, \vec{x}_{10}, \vec{x}_{11}, \vec{x}_{12}\}$   
☒  $\{\vec{x}_6, \vec{x}_7, \vec{x}_8, \vec{x}_9, \vec{x}_{10}, \vec{x}_{11}, \vec{x}_{12}\}$   
☒  $\{\vec{x}_1, \vec{x}_5, \vec{x}_8, \vec{x}_9, \vec{x}_{10}, \vec{x}_{11}, \vec{x}_{12}\}$   
☐  $\{\vec{x}_1, \vec{x}_2, \vec{x}_5, \vec{x}_9, \vec{x}_{10}, \vec{x}_{11}, \vec{x}_{12}\}$   
☒  $\{\vec{x}_1, \vec{x}_2, \vec{x}_8, \vec{x}_9, \vec{x}_{10}, \vec{x}_{11}, \vec{x}_{12}\}$

**Solution:** The first choice only includes 6 vectors, but since the span of  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{12}$  is 7-dimensional, it must include at least 7 vectors. So, the first choice is not a valid basis.

The fourth choice includes 7 vectors, but we know that  $\vec{x}_1, \vec{x}_2, \vec{x}_5$  are a linearly **dependent** set since they all lie on the same 2-dimensional subspace of  $\mathbb{R}^7$  (and you only need 2 vectors to uniquely define a 2-dimensional subspace), so the fourth choice is not a valid basis.

The other options all include 7 vectors that *could* be linearly independent, and so they could form a basis for  $\mathbb{R}^7$ .

- c) (4 pts) Suppose the intersection of  $\text{span}(\{\vec{x}_1, \vec{x}_2\})$  and  $\text{span}(\{\vec{x}_4, \vec{x}_5\})$  is a line (i.e. a 1-dimensional subspace) in  $\mathbb{R}^7$ . Which of the following **must** be true? Select all that apply. Blank answers will receive no credit.

*Hint: Don't forget the assumptions introduced at the start of the problem.*

- ☐  $\vec{x}_2, \vec{x}_4$ , and  $\vec{x}_5$  can all be written as scalar multiples of  $\vec{x}_1$ .  
☐ The set  $\{\vec{x}_2, \vec{x}_4\}$  is linearly independent.  
☐ The set  $\{\vec{x}_3, \vec{x}_4\}$  is linearly independent.  
☐ The set  $\{\vec{x}_3, \vec{x}_6\}$  is linearly independent.  
☐ None of the above.

**Solution:** The intended answer to the problem was options 1 and 3. The scenario we had in mind was that  $\text{span}(\{\vec{x}_1, \vec{x}_2\}) = \text{span}(\{\vec{x}_4, \vec{x}_5\}) =$  the same line. The two spans can't both be different planes that happen to intersect in a line, since we're told that  $\vec{x}_1, \vec{x}_2, \vec{x}_3$  span a 2-dimensional subspace of  $\mathbb{R}^7$  and  $\vec{x}_4, \vec{x}_5, \vec{x}_6$  span the same 2-dimensional subspace of  $\mathbb{R}^7$ . So, if the two spans are planes, they're the same plane, and they would intersect at a plane. Since the two spans intersect at a line, **we thought** they'd both have to be lines. If that was the case, then  $\vec{x}_2, \vec{x}_4$ , and  $\vec{x}_5$  would all be scalar multiples of  $\vec{x}_1$ , and so  $\vec{x}_3$  would have to not be on that line (for  $\vec{x}_1, \vec{x}_2, \vec{x}_3$  to span a 2-dimensional subspace), which is why Options 1 and 3 were our originally intended answers.

But after releasing exam scores, a student brought up a possibility we hadn't considered: it's possible that  $\text{span}(\{\vec{x}_1, \vec{x}_2\})$  is a plane, and  $\text{span}(\{\vec{x}_4, \vec{x}_5\})$  is a line that is contained on that plane. That setup would satisfy all of the assumptions provided in the problem statement, but it would imply that none of the options are true. So, retroactively, we gave full credit to everyone for this part.

**Problem 8: Worst-Case Scenario (8 pts)**

Suppose  $a, b, c, d, e$  are positive real numbers. Find the **largest** real number  $T$  such that it's guaranteed that

$$(a + b + c + d + e) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} \right) \geq T$$

Think of  $T$  as the "best possible lower bound". For instance, we know that the expression on the left-hand side above must be greater than or equal to 0, since  $a, b, c, d, e$  are all positive, but  $T = 0$  is not the answer, since there's a larger value of  $T$  that also guarantees the inequality holds.

Show your work, and circle your final answer, which should be a number with no variables.

*Hint: Use the Cauchy-Schwarz inequality.*



**Solution:**  $T = 25$ .

Recall, the Cauchy-Schwarz inequality states that for any two vectors  $\vec{u}$  and  $\vec{v}$ ,

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$$

Let's define two vectors  $\vec{u}$  and  $\vec{v}$  and then apply the Cauchy-Schwarz inequality to them.

$$\vec{u} = \begin{bmatrix} \sqrt{a} \\ \sqrt{b} \\ \sqrt{c} \\ \sqrt{d} \\ \sqrt{e} \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{b}} \\ \frac{1}{\sqrt{c}} \\ \frac{1}{\sqrt{d}} \\ \frac{1}{\sqrt{e}} \end{bmatrix}$$

Let's compute the three quantities involved in the inequality.

- $\|\vec{u}\| = \sqrt{a + b + c + d + e}$
- $\|\vec{v}\| = \sqrt{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}}$
- $|\vec{u} \cdot \vec{v}| = |\sqrt{a} \cdot \frac{1}{\sqrt{a}} + \sqrt{b} \cdot \frac{1}{\sqrt{b}} + \sqrt{c} \cdot \frac{1}{\sqrt{c}} + \sqrt{d} \cdot \frac{1}{\sqrt{d}} + \sqrt{e} \cdot \frac{1}{\sqrt{e}}| = 5$

So, we have that

$$5 \leq \sqrt{a + b + c + d + e} \cdot \sqrt{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}}$$

Squaring both sides of the inequality gives us

$$25 \leq (a + b + c + d + e) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} \right)$$

This means that for any positive values of  $a, b, c, d, e$ , it's impossible for  $(a + b + c + d + e) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} \right)$  to be less than 25. Finding a value equal to 25 is doable if we set  $a = b = c = d = e = 1$ . So,  $T = 25$  is the largest possible value of  $T$  that guarantees the inequality holds.

(1 pt) Congrats on finishing Midterm 1! Here's a free point.

Feel free to draw us a picture about EECS 245 in the box below.