21-128 and 15-151 problem sheet 2

Solutions to the following seven exercises and optional bonus problem are to be submitted through gradescope by 11PM on

Wednesday 14th September 2022.

Problem 1

Determine which are tautologies where A, B and C are non-empty subsets of $\mathbb{Z}$.

(a) $(A \cap B \cap C) \subset (A \cup B)$

(b) $(A \setminus B) \cap (B \setminus A) = \varnothing$

(c) $(A \cap B \neq \varnothing) \implies ((A \setminus B) \subset A)$

Solution.

(a) This statement is false. Consider $A = B = C$. Then

$$
\begin{aligned}
& (A \cap B \cap C) \subset (A \cup B) \\
\implies & A \subset (A \cup B) && \text{(since } A = B = C\text{)} \\
\implies & A \subset (A \cup A) && \text{(since } A = B\text{)} \\
\implies & A \subset A && \text{(since } A = B\text{)}
\end{aligned}
$$

We know this is false since $A = A$. Therefore $(A \cap B \cap C) \subset (A \cup B)$ is false.

(b) This statement is true. $\varnothing \subseteq (A \setminus B) \cap (B \setminus A)$ since $\varnothing$ is a subset of any set, so it remains to show that $x \in (A \setminus B) \cap (B \setminus A) \implies x \in \varnothing$ and thus

$$
\begin{aligned}
& x \in (A \setminus B) \cap (B \setminus A) \\
\implies & (x \in A \setminus B) \wedge (x \in B \setminus A) && \text{(def. of } \cap\text{)} \\
\implies & x \in A \wedge x \notin B \wedge x \in B \wedge x \notin A && \text{(def. of } \setminus\text{)} \\
\implies & \bot && \text{(since } (\top \wedge \bot) \equiv \bot\text{)}
\end{aligned}
$$

Since there are no such elements x, the implication holds. Therefore $(A \setminus B) \cap (B \setminus A) = \varnothing$ and the statement is true.

(c) This statement is true. Since $A \cap B \neq \varnothing$, $\exists x \in A \cap B \implies (x \in A) \wedge (x \in B)$. Also, since $(A \setminus B) = \{a \mid (a \in A) \wedge (a \notin B)\} \subseteq A$. Note that $x \notin (A \setminus B)$ and $x \in A$ means that $A \neq (A \setminus B)$. Therefore, since $A \setminus B$ is a subset of A but not equal, $(A \setminus B) \subset A$ (it is a proper subset). So $(A \cap B \neq \varnothing) \implies ((A \setminus B) \subset A)$.

Problem 2

Prove that
$$\{x \in \mathbb{Z} : 5 \mid x\} = \{x \in \mathbb{Z} : 5 \mid (10 - 4x)\}.$$

Solution. We need to prove that each set is a subset of the other. That is, we need to prove that, given $x \in \mathbb{Z}$, we have

(a) If $5 \mid x$ then $4 \mid (10 - 4x)$; and

(b) If $5 \mid (10 - 4x)$ then $5 \mid x$.

We will prove each statement separately.

(a) Suppose $5 \mid x$. Then $x = 5y$ for some integer y, so $10 - 4x = 10 - 20y = 5(2 - 4y)$ and $5 \mid (10 - 4x)$.

(b) Suppose $5 \mid (10 - 4x)$. Then $10 - 4x = 5y$ for some integer y, so $10 + x = 5y + 5x \implies x = 5y + 5x - 10 = 5(y + x - 2)$ and $5 \mid x$.

Thus, the two sets are equal.

Problem 3

Let $A, B \subseteq \mathbb{N}$ be finite and let $C = \{a \in A \mid \exists\, b \in B \text{ s.t. } (a + b) \in A\}$.
For what sets A, B does $A = C$?

Solution. $A = C$ only when $A = \varnothing$ or when $0 \in B$.

By definition, $C \subseteq A$, so to show $A = C$, we only need to show $A \subseteq C$.

If $A = \varnothing$, $A = \varnothing \subseteq C$ since the empty set is a subset of all sets. So $A = C$.

Then either $0 \in B$ or $0 \notin B$. If $0 \in B$, let $a \in A$ and $b = 0 \in B$. Then $\forall\, a \in A$, $(a+b) = a+0 = a$, and since $(a + b) \in A$, $a \in C$. So $A \subseteq C$ and $A = C$.

If $0 \notin B$, consider the maximum element of A ($\max(A)$) (since A finite, this element exists). Then $\forall\, b \in B$, $\max(A) + b > \max(A)$ since $b \in \mathbb{N} \setminus \{0\}$(positive elements only). Therefore $\forall\, b \max(A) + b \notin A$, so $\max(A) \notin C$. Thus $A \not\subseteq C$ and $A \neq C$.

Therefore, $A = C$ only when $A = \varnothing$ or when $0 \in B$.

Problem 4

Let $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$. Prove

$$(A \triangle B) \setminus (B \triangle C) \subseteq ((B \cap C) \setminus A) \cup (A \setminus (B \cup C)).$$

Solution. Let $x \in (A \triangle B) \setminus (B \triangle C)$. Then

$$\begin{aligned}
& x \in (A \triangle B) \setminus (B \triangle C) \\
\implies & x \in ((A \setminus B) \cup (B \setminus A)) \setminus ((B \setminus C) \cup (C \setminus B)) && \text{(def. of } \triangle) \\
\implies & x \in (A \setminus B) \cup (B \setminus A) \wedge x \notin ((B \setminus C) \cup (C \setminus B)) && \text{(def. of } \setminus) \\
\implies & x \in (A \setminus B) \cup (B \setminus A) \wedge \neg(x \in (B \setminus C) \cup (C \setminus B)) && \text{(def. of } \notin) \\
\implies & x \in (A \setminus B) \cup (B \setminus A) \wedge \neg(x \in (B \setminus C) \vee x \in (C \setminus B)) && \text{(def. of } \cup) \\
\implies & x \in (A \setminus B) \cup (B \setminus A) \wedge \neg((x \in B \wedge x \notin C) \vee (x \in C \wedge x \notin B)) && \text{(def. of } \setminus) \\
\implies & x \in (A \setminus B) \cup (B \setminus A) \wedge \neg(x \in B \wedge x \notin C) \wedge \neg(x \in C \wedge x \notin B)) && \text{(DeMorgan's)} \\
\implies & x \in (A \setminus B) \cup (B \setminus A) \wedge (x \notin B \vee x \in C) \wedge (x \notin C \wedge x \in B)) && (*, \text{DeMorgan's})
\end{aligned}$$

Case 1. $x \in (A \setminus B)$. Then $x \in A \wedge x \notin B$ (def of $\setminus$). Since $(x \notin C \vee x \in B)$ by $*$ and $x \notin B$, then $x \notin C$. So

$$\begin{aligned}
& x \in A \wedge x \notin B \wedge x \notin C \\
\implies & x \in A \wedge \neg(x \in B) \wedge \neg(x \in C) && \text{(def. of } \notin) \\
\implies & x \in A \wedge \neg(x \in B \vee x \in C) && \text{(DeMorgan's)} \\
\implies & x \in A \wedge \neg(x \in B \cup C) && \text{(def. of } \cup) \\
\implies & x \in A \wedge x \notin (B \cup C) && \text{(def. of } \notin) \\
\implies & x \in (A \setminus (B \cup C) && \text{(def. of } \setminus) \\
\implies & x \in ((B \cap C) \setminus A) \cup (A \setminus (B \cup C)) && \text{(def. of } \cup)
\end{aligned}$$

Case 2. $x \in (B \setminus A)$. Then $x \in B \wedge x \notin A$ (def of $\setminus$). Since $(x \notin B \vee x \in C)$ by $*$ and $x \in B$, then $x \in C$. So

$$\begin{aligned}
& x \in B \wedge x \in C \wedge x \notin A \\
\implies & x \in (B \cap C) \wedge x \notin A && \text{(def. of } \cap) \\
\implies & x \in ((B \cap C) \setminus A) && \text{(def. of } \setminus) \\
\implies & x \in ((B \cap C) \setminus A) \cup (A \setminus (B \cup C)) && \text{(def. of } \cup)
\end{aligned}$$

[Note: The sets are actually equal!]

Problem 5

Find a non-empty set A such that

1. $\forall S \in A, S \subseteq \mathbb{N}$
2. $\forall S \in A, \exists S' \in A, (S \neq S') \wedge (S \cup S') = S$

Solution. Consider the set $A = \{\{x \in \mathbb{N} \mid x > i\} \mid i \in \mathbb{N}\}$.
This set looks like $\{\{1, 2, 3, 4, \cdots\}, \{2, 3, 4, 5, \cdots\}, \cdots\}$. For an arbitrary $S \in A$, by definition we have $S = \{x \in \mathbb{N} \mid x > i\}$ for some $i \in \mathbb{N}$. Also by definition, the set $\{x \in \mathbb{N} \mid x > i + 1\}$ is in A. Note that this set is not equal to S because $i + 1 \in S$, but $i + 1 \notin \{x \in \mathbb{N} \mid x > i + 1\}$. We proceed to show that $S \cup \{x \in \mathbb{N} \mid x > i + 1\} = S$ via double containment.

- To show that $S \subseteq S \cup \{x \in \mathbb{N} \mid x > i + 1\}$, fix an arbitrary $x \in S$. By the definition of set union, we have $x \in S \cup \{x \in \mathbb{N} \mid x > i + 1\}$, which suffices for this direction.
- To show that $S \cup \{x \in \mathbb{N} \mid x > i + 1\} \subseteq S$, fix an arbitrary $x \in S \cup \{x \in \mathbb{N} \mid x > i + 1\}$. We wish to show that $x \in S$. We have two consider based on the set union definition.
  - Case 1 is when the assumption is that $x \in S$, since this is one way to have x in the union above. Under this assumption, we immediately see that $x \in S$ as desired.
  - Case 2 is when x is in the union because x is in $\{x \in \mathbb{N} \mid x > i + 1\}$. By definition, this tells us that $x > i + 1$, but this must mean that $x > i$, since $i + 1 > i$. By the definition of S, we must have $x \in S$ as desired.

Problem 6

Let $\mathbb{N}^+$ denote the set of positive integers and consider the function $f : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{R}$ defined by

$$f(a, b) = \frac{(a + 1)(a + 2b)}{2}$$

(a) Show that the image of f is a subset of $\mathbb{N}^+$.

(b) Determine exactly which positive integers are elements of the image of f. (**Hint:** Formulate a hypothesis by trying values.)

Solution.

(a) Given $(a,b) \in \mathbb{N}^+ \times \mathbb{N}^+$, either a is even or a is odd. If a is even then $a = 2k$ for some $k \in \mathbb{N}^+$, so $a+2b = 2k+2b = 2(k+b)$, and so $f(a,b) = (a+1)(k+b) \in \mathbb{N}^+$. If a is odd then $a = 2k-1$ for some $k \in \mathbb{N}^+$, so $a+1 = 2k$, and so $f(a,b) = k(a+2b) \in \mathbb{N}^+$.

Hence $f(a,b) \in \mathbb{N}^+$ for all $(a,b) \in \mathbb{N}^+ \times \mathbb{N}^+$, and so the image of f is a subset of $\mathbb{N}^+$.

(b) Here is a table listing the value of $f(a,b)$ for the first few natural number values of a and b:

	1	2	3	4	5
1	3	5	7	9	11
2	6	9	12	15	18
3	10	14	18	22	26
4	15	20	25	30	35
5	21	27	33	39	45

Clearly every odd number greater than or equal to 3 appears. What about the other numbers? The even numbers that appear so far are 6, 10, 12, 14, 18, 20, 22, ... We're missing 1, 2, 4, 8, 16, ..., all of which are powers of 2. That is, they're the numbers which are *not* divisible by an odd number which is greater than 1.

Claim. The image of f consists of exactly those natural numbers which are divisible by an odd number greater than or equal to 3.

Proof. First note that every element of the image of f is divisible by some odd number greater than 1. Indeed, given $(a,b) \in \mathbb{N}^+ \times \mathbb{N}^+$, there are two cases:

- Case 1: a is even. In this case, we saw in (a) that $f(a,b) = (a+1)d$ for some $d \in \mathbb{N}^+$. This shows that $f(a,b)$ has an odd divisor $a+1$, which is greater than 1.
- Case 2: a is odd. In this case, $f(a,b) = k(a+2b)$ for some $k \in \mathbb{N}^+$. Since a is odd, it follows that $a+2b$ is odd, so again $f(a,b)$ has an odd divisor greater than 1.

Now need to show that if $n \in \mathbb{N}^+$ is divisible by an odd number greater than 1, then $n = f(a,b)$ for some $(a,b) \in \mathbb{N}^+ \; times \mathbb{N}^+$. So suppose n is divisible by an odd number greater than 1. In particular $n > 1$, so that

$$n = s(2t+1)$$

for some $s,t \in \mathbb{N}^+$.

- If $s > t$ then $s - t \in \mathbb{N}^+$, and

$$f(2t, s-t) = \frac{(2t+1)(2t+2(s-t))}{2} = (2t+1)(t+s-t) = s(2t+1) = n$$

so that n is in the image of f.

- If $s \le t$ then $t - s \ge 0$, so $t - s + 1 \in \mathbb{N}^+$. Then

  $$f(2s-1, t-s+1) = \frac{(2s-1+1)(2s-1+2(t-s+1))}{2} = s(2t+1) = n$$

  so that n is in the image of f.

In any case, n is in the image of f.

Problem 7

For $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, show that (a) and (b) below have different meanings.

(a) $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon]$.

(b) $(\exists \delta > 0)(\forall \varepsilon > 0)(\forall x \in \mathbb{R})[|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon]$.

(**Hint:** Find a function $f : \mathbb{R} \to \mathbb{R}$ and an element $a \in \mathbb{R}$ for which (a) and (b) have different truth values.)

Solution. Define $f(x) = x$ for all $x \in \mathbb{R}$ and let $a = 0$. Then

- (a) is true. To see this, fix $\varepsilon > 0$ and let $\delta = \varepsilon$. Suppose $x \in \mathbb{R}$ and $|x| < \delta$. Then $|f(x) - f(0)| = |x| < \delta = \varepsilon$, so statement (a) is true.
- (b) is false. To see this, we will prove its negation, which is:

  $$(\forall \delta > 0)(\exists \varepsilon > 0)(\exists x \in \mathbb{R})[|x - a| < \delta \wedge |f(x) - f(a)| \ge \varepsilon]$$

  So fix $\delta > 0$, let $\varepsilon = \frac{\delta}{3}$, and let $x = \frac{\delta}{2}$. Then

  $$|x - 0| = \frac{\delta}{2} < \delta \quad \text{and} \quad |f(x) - f(0)| = |x| = \frac{\delta}{2} \ge \frac{\delta}{3} = \varepsilon$$

  So the negation of (b) is true, and hence (b) is false.

Bonus Problem (2 points)

A function $g : \mathbb{R} \to \mathbb{R}$ is *even* if $g(-x) = g(x)$ for all $x \in \mathbb{R}$, or *odd* if $h(-x) = -h(x)$ for all $x \in \mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$.

(a) Prove that there exists a unique pair of functions $g, h : \mathbb{R} \to \mathbb{R}$ such that g is even, h is odd, and $f = g + h$. (**Hint:** Express both $f(x)$ and $f(-x)$ in terms of $g(x)$ and $h(x)$, and solve the resulting system of equations.)

(b) When f is a polynomial function, express g and h as in (a) in terms of the coefficients of f.

Solution.

(a) For each $x \in \mathbb{R}$, define $g(x) = \frac{f(x)+f(-x)}{2}$ and $h(x) = \frac{f(x)-f(-x)}{2}$. Then for all $x \in \mathbb{R}$ we have

- $g(-x) = \frac{f(-x)+f(x)}{2} = \frac{f(x)+f(-x)}{2} = g(x)$, so g is even.
- $h(-x) = \frac{f(-x)-f(x)}{2} = -\frac{f(x)-f(-x)}{2} = -h(x)$, so h is odd.
- $g(x) + h(x) = \frac{(f(x)+f(-x))+(f(x)-f(-x))}{2} = \frac{2f(x)}{2} = f(x)$, so $f = g + h$.

Now if $g', h' : \mathbb{R} \to \mathbb{R}$ are functions which are even and odd, respectively, such that $f = g' + h'$, then we must have

$$\begin{cases} f(x) & = g'(x) + h'(x) \\ f(-x) & = g'(x) - h'(x) \end{cases}$$

for all $x \in \mathbb{R}$. Adding these equations yields $f(x) + f(-x) = 2g'(x)$, so that

$$g'(x) = \frac{f(x) + f(-x)}{2} = g(x)$$

for all $x \in \mathbb{R}$, and so $g' = g$. Likewise subtracting the second from the first yields $f(x) - f(-x) = 2h'(x)$, so that

$$h'(x) = \frac{f(x) - f(-x)}{2} = h(x)$$

for all $x \in \mathbb{R}$, and hence $h' = h$.

So the pair (g, h) is unique.

(b) Suppose f is a polynomial, so that there exist real numbers $a_0, a_1, \dots, a_n$ such that $f(x) = \sum_{i=0}^{n} a_i x^i$ for all $x \in \mathbb{R}$. Then

- $g(x) = \frac{1}{2} \sum_{i=0}^{n} (a_i + (-1)^i a_i) x^i = a_0 + a_2 x^2 + a_4 x^4 + \cdots$; and
- $h(x) = \frac{1}{2} \sum_{i=0}^{n} (a_i - (-1)^i a_i) x^i = a_1 x + a_3 x^3 + a_5 x^5 + \cdots$.

That is, g is the sum of the even-power terms and h is the sum of the odd-power terms.