Sets

A set S is a collection of unique elements with no set order

Set Operations

$\subseteq$: Subset

$\subsetneq$: Proper Subset

$\cup$: Union

$\cap$: Intersection

$/$: Set minus

A^c : Complement

* Make sure you know

1. Distribution
2. De Morgan's Laws
3. Converting to propositional logic definitions

Double Containment

Two sets A & B are equal iff $A \subseteq B$ and $B \subseteq A$

1. Show $a \in A \Rightarrow a \in B$
2. Show $b \in B \Rightarrow b \in A$

Examples

Let $A = \{a : a \equiv 1 \bmod 2\}$ and $B = \{b : b = 2k+1,\ k \in \mathbb{Z}\}$

1. $A \subseteq B$

def of mod

$a \in A \Rightarrow a \equiv 1 \bmod 2 \Rightarrow 2 \mid (a-1) \Rightarrow \exists k \in \mathbb{Z},\ 2k = a-1 \Rightarrow a = 2k+1 \Rightarrow a \in B$

2. $B \subseteq A$

Same as 1 with the implications reversed

* In this case you can do biimplications, but be careful when doing so! *

Functions

A function f from a set $\overset{\text{domain}}{X}$ to set $\overset{\text{codomain}}{Y}$ such that:

$$\underbrace{\forall x \in X}_{\text{totality}},\ \underbrace{\exists}_{\text{existence}}\overset{!}{\ }_{\text{uniqueness}}\, y \in Y,\ y = f(x)$$

A function is **well-defined** if it satisfies these properties:

1. Totality: all $x \in X$ have a corresponding output
2. Existence: an $f(x)$ exists for each $x \in X$
3. Uniqueness: the $f(x)$ for each x is unique

Types of Functions

1. A function $f: X \to Y$ is an **injection** if:

   $\forall x_1, x_2 \in X,\ f(x_1) = f(x_2) \implies x_1 = x_2$ OR f has a left inverse

2. A function $f: X \to Y$ is a **surjection** if:

   $\forall y \in Y,\ \exists x \in X,\ f(x) = y$ OR f has a right inverse

3. A function $f: X \to Y$ is a **bijection** if:

   f is injective & surjective OR f has a two-sided inverse

Examples

Prove that $f: \mathbb{R} \to \mathbb{R}$ via $f(x) = \begin{cases} -x^3 & x \le 1 \\ -\ln x - 1 & x > 1 \end{cases}$ is a bijection

Define $f^{-1}: \mathbb{R} \to \mathbb{R}$ via $f^{-1}(x) = \begin{cases} \sqrt[3]{-x} & x \ge -1 \\ e^{-x-1} & x < -1 \end{cases}$

Well-defined

1. Total ✓ (Cases are exhaustive)
2. Existence ✓ ($\sqrt[3]{x}$ & e^x are defined over $\mathbb{R}$)
3. Uniqueness ✓ (by algebra)

Left Inverse

Case 1: $x \leq 1$

$$f^{-1}(f(x)) = f^{-1}(-x^3)$$
$$= \sqrt[3]{-(-x)^3}$$
$$= x$$

$x \leq 1$
$\Rightarrow x^3 \leq 1$
$\Rightarrow -x^3 \geq -1$

Case 2: $x > 1$

$$f^{-1}(f(x)) = f^{-1}(-\ln x - 1)$$
$$= e^{-(-\ln x - 1) - 1}$$
$$= e^{\ln x + 1 - 1}$$
$$= x$$

$x > 1$
$\Rightarrow \ln x > \ln(1)$
$\Rightarrow \ln x > 0$
$\Rightarrow \ln x + 1 > 1$
$\Rightarrow -\ln x - 1 < -1$

Right Inverse

Case 1: $x \geq -1$

$$f(f^{-1}(x)) = f(\sqrt[3]{-x})$$
$$= -(\sqrt[3]{-x})^3$$
$$= x$$

$x \geq -1$
$\Rightarrow -x \leq 1$
$\Rightarrow \sqrt[3]{-x} \leq 1$

Case 2: $x < -1$

$$f(f^{-1}(x)) = f(e^{-x-1})$$
$$= -\ln(e^{-x-1}) - 1$$
$$= -(-x-1)\ln e - 1$$
$$= x + 1 - 1$$
$$= x$$

$x < -1$
$\Rightarrow -x > 1$
$\Rightarrow -x - 1 > 0$
$\Rightarrow e^{-x-1} > 1$