151 **E**xcellent Exam 3 Review

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Midterm 3

1 Things to Know

1.1 Counting

- Basic vocabulary
  - **Selection**: picking k things out of n, order does not matter; formally, a k-element subset of $[n]$
    - The number of selections of k things from n: $\binom{n}{k}$
    - Examples: picking committees out of n people (the order in which you pick people doesn't matter)
  - **Arrangement**: selections, except order does matter
    - Number of arrangments of k things from n: $\frac{n!}{(n-k)!} = \binom{n}{k}k!$ (verify this yourself)
    - Examples: picking committees and assigning each committee member a position, lining up a group of students for a picture, ice cream scoops
  - **Permutation**: special case of arrangement; arranging all n items
    - Number of permutations of n things: $\frac{n!}{0!} = n!$
    - Examples: arranging all n people in a row
  - **Independent Multiplication Principle**: states that in an m-step process, if all the steps are independent of each other, you can compute the total number of possible tuples generated by this process by multiplying the number of choices at each step
    - When using this principle, make sure to confirm that your process is injective in that each tuple is generated by a unique set of choices
  - **Dependent Multiplication Principle**: relaxes the IMP; the steps don't necessarily have to be independent of each other - as long as the *number of choices at each step* is the same regardless of the choices at the previous steps, you can still multiply all the choices together to get the total number of possible tuples
  - **Addition Principle**: Given a set S and $X = \{S_1, ..., S_n\}$ that is a ***partition*** of S, $|S| = \sum_{i=1}^{n} |S_i|$. Informally, if you have a bunch of cases, you can add up all the possibilities for your cases if they are mutually exclusive and exhaustive
    - Make sure to justify that your partition is valid!!
      - Show that $S = \bigcup_{i=1}^{n} S_i$; often this requires showing that each element of S falls in to one of the S_i' (usually $S_i \subseteq S$ so the other direction is trivial); this is the same as showing that your cases are *exhaustive*
      - Show that for any $i \neq j$, $S_i \cap S_j = \varnothing$; show that your partitioning sets are disjoint. This is the same as showing that your cases are *mutually exclusive*
- Common techniques
  - Casework
    - Split the problem into different cases and count each case; usually there aren't that many (and if you have a lot, it's probably worth considering using a different technique or redefining what you're casing on)

    * Useful for "at least n" problems with $n > 1$
    * Make sure your cases form a valid partition and cite the addition principle
  - Complementary counting
    * Count the total number of elements and subtract the ones that don't work
    * Useful for "at least one" problems, lattice path problems, etc.
  - Stars and Bars
    * Number of ways to split up n things among k people or solve $x_1+...+x_k = n$ with nonnegative x_i's is $\binom{n+k-1}{k-1}$
    * Think of it like adding in $k-1$ slots to the n you already have and choosing $k-1$ of the total number of slots to divide your group into k blocks
    * Common variation: ways to split up n things such that each person gets at least one - give each person one then use stars and bars to split up the remaining $n-k$ among the k people
  - Probability: you won't have any super complicated probability questions on the exam
    * Turn the problem into a counting problem: to find the probability of some event happening, calculate the size of the entire sample space (i.e. number of possible sequences of coin flips) and the size of the set of outcomes where that event happens (i.e. all the sequences with at least one head), and divide
    * Usually problems will involve flipping coins or counting card hands, pretty much the same examples we use for counting
  - Other tips
    * Look over the common types of counting i.e. words with repeated letters, counting card hands, lining up people (you've gotten a lot of practice already!)
    * When using a multistep process, if you're unsure whether it is injective (i.e. if it is overcounting), try to break it. Come up with an element in your set and try to change the order in which you pick things so that you can follow your process twice and get the same element.
    * If you are using cases, again try to break them by seeing if you can come up with a valid element of your set that does not fit in any of your cases or that fits in more than one case

- Counting in 2 ways
  - Most common terms and their meanings
    * a^n: n things with a choices each, length n base-a strings
    * $\binom{n}{k}$: picking k things out of n
    * $n!$: ordering n things
    * $\sum_{k=i}^{j}$: this defines a partition; pay attention to the bounds
    * n: picking 1 thing out of n
  - Other tips
    * Any time you see a +, you are defining a partition, so make sure to justify what the partition is
    * Start with the simpler side and figure out what it's counting and try to break down the RHS into cases or a multistep process to count the set again
    * Often times the LHS and RHS are the same process with the steps reversed (i.e. counting committees examples)
    * When trying to figure out the partition, it can help to draw pictures - I usually default to a number line to see how k divides up my group of n; this can be especially useful when you have unusual bounds on the summation (ex: the strange number line problem from the counting review session is almost trivial once you put it on a number line)
    * Another tip for figuring out the partition: expand the summation by writing out the first and last few terms and seeing how the summation counts the set with small cases
    * Don't forget that when terms are multiplied together, it's a multistep process

1.2 Inequalities

- Basic Inequalities
  - Know the whole chain of inequalities: $\max(x, y) \geq \mathrm{QM}(x, y) \geq \mathrm{AM}(x, y) \geq \mathrm{GM}(x, y) \geq \mathrm{HM}(x, y) \geq \min(x, y)$
  - $\mathrm{QM}(x, y) = \sqrt{\frac{x^2+y^2}{2}}$
  - $\mathrm{AM}(x, y) = \frac{x+y}{2}$
  - $\mathrm{GM}(x, y) = \sqrt{xy}$
  - $\mathrm{HM}(x, y) = \frac{2}{\frac{1}{x}+\frac{1}{y}}$
- Cauchy-Schwarz
  - Two forms: given $\vec{x} =< u_1, \dots, u_n >$ and $\vec{y} =< v_1, \dots, v_n$, we know:
    * $|\vec{x} \cdot \vec{y}| \leq ||\vec{x}|| \cdot ||\vec{y}||$ (squaring both sides can also be useful)
    * $(u_1v_1 + ...u_nv_n)^2 \leq (u_1^2 + ... + u_n^2)(v_1^2 + ... + v_n^2)$
  - When coming up with vectors, a good rule of thumb is that the number of distinct variables in the inequality given is the length of the vectors that you're using
  - Often it is easier to use the second form than the first when trying to figure out what vectors you have
  - Don't forget that the magnitude of a vector is the same regardless of the order of the components; so if the dot product is not working out, try rearranging the elements in your vector(s)
- Triangle Inequality: given a triangle with side lengths a, b, c, we have the following:
  - $a + b \geq c$
  - $a + c \geq b$
  - $b + c \geq a$

1.3 Other

- Chinese Remainder Theorem
  - Formal statement: Let m, n be moduli and let $a, b \in \mathbb{Z}$. If m and n are coprime, then there exists an integer solution x to the system of congruences $x \equiv a \bmod m$ and $x \equiv b \bmod n$. If x and y are two such solutions, then $x \equiv y \bmod mn$
  - Informally, this states that you can solve a system of congruences if the moduli are coprime
  - What you need to know for the exam is how to solve a system of congruences; the method is the same as what is used in the proof of CRT - this is also what you did in HW 7 q3)
  - The general process is to use the definition of mods with one of the system of congruences to write x in terms of that modulus and some integer k, then plug that in to the second congruence and write an expression for k in terms of mn and some other integer j. Then, you can rewrite x as congruent to something modulo mn. Refer to the review questions and Clive for a worked out example
- Finiteness
  - Closure properties:
    * Union of finite sets is finite
    * Subset of a finite set is finite
    * Finite cartesian product of finite sets is finite
    * Intersection of finite sets is finite
    * Set difference of finite sets is finite

    * Power set of a finite set is finite
  - Remember that any finite subset of the naturals has a largest element
  - Any subset of the naturals with a largest element is finite

- Bijections
  - Recall how to prove injectivity and surjectivity
    * Injectivity: Show for some a, b in the domain that $f(a) = f(b) \implies a = b$
    * Surjectivity: Given y in the codomain, find x in the domain such that $f(x) = y$
  - Recall how to find and prove a two-sided inverse
  - Cantor-Schroder-Bernstein: If there are injections $f : X \to Y$ and $g : Y \to X$, then $\exists h : X \to Y$ that is bijective. (it's ok if you don't understand the formal proof of this theorem, but the intuition is good to know)

2 Problems

We've put together a bunch of problems for you to use for review. I'd recommend sitting down and solving them without looking at the solutions. Remember, you can be terser and provide less justification (but not *no* justification) on the midterm than you would on a typical homework. Also in general, these problems are **not** necessarily representative of exam problems, but rather meant to test your conceptual understanding of a wide range of topics. Good luck!

2.1 Counting

You've been given plenty of counting problems to practice, so we didn't write more problems for this section. In particular, I'd recommend Josh A's "Counting Review Problems".

2.2 Inequalities

1. Let $a, b, c, d > 0$ be reals. Show that $(a+b+c+d)\left(1/a + 1/b + 1/c + 1/d\right) \geq 16$.

2. Say $a_1, ...a_n, b_1, ...b_n$ are reals. Show that
$(a_1 b_1 + ... + a_n b_n)^2 \leq \left(a_1^2 + 2a_2^2 + ... + na_n^2\right)\left(b_1^2 + b_2^2/2 + ... + b_n^2/n\right)$.

3. Use Cauchy-Schwartz to prove QM-AM for an arbitrary number of variables. In particular, show that $\sqrt{\frac{1}{n}\sum_{i=1}^n x_i^2} \geq \frac{1}{n}\left(\sum_{i=1}^n x_i\right)$.

4. Find the maximum volume of a rectangular prism with surface area 6.

2.3 Others

5. Say $x \in \mathbb{N}$ satisfies $x \equiv 3 \mod 13$ and $x \equiv 12 \mod 25$. What is the smallest possible value for x?

6. Find an explicit bijection between $(0, 1)$ and $\mathbb{R}^+$.

3 Solutions

Note: *These are the complete solutions to the problems. Once again, we don't expect this much detail on the exam.*

3.1 Counting

Uh... we don't have any problems, so ... I don't think we're gonna have solutions in this section any time soon.

3.2 Inequalities

Problem 1 *Let $a, b, c, d > 0$ be reals. Show that $(a+b+c+d)\left(1/a + 1/b + 1/c + 1/d\right) \geq 16$.*

Solution:
We note that from AM-HM, we have that:
$\frac{a+b+c+d}{4} \geq \frac{4}{1/a+1/b+1/c+1/d}$.
Cross-multiplying, we get:
$(a+b+c+d)(1/a + 1/b + 1/c + 1/d) \geq 16$ ■

Problem 2 *Say $a_1, ...a_n, b_1, ...b_n$ are reals. Show that*
$(a_1b_1 + ... + a_nb_n)^2 \leq \left(a_1^2 + 2a_2^2 + ... + na_n^2\right)\left(b_1^2 + b_2^2/2 + ... + b_n^2/n\right)$.

Solution:
Let $x = \left(a_1, \sqrt{2}a_2, ...\sqrt{n}a_n\right)$ and $y = (b_1, b_2/\sqrt{2}, ...b_n/\sqrt{n})$.
We have that $||x||^2 = (a_1^2 + 2a_2^2 + ... + na_n^2)$ and $||y||^2 = (b_1^2 + b_2^2/2 + ... + b_n^2/n)$. Also, note that $x \cdot y = a_1b_1 + ... + a_nb_n$.
Then, the desired result follows directly from Cauchy-Schwartz, which states that $(x \cdot y)^2 \leq ||x||^2 \cdot ||y||^2$. ■

Problem 3 *Use Cauchy-Schwartz to prove QM-AM for an arbitrary number of variables. In particular, show that $\sqrt{\frac{1}{n}\sum_{i=1}^n x_i^2} \geq \frac{1}{n}\left(\sum_{i=1}^n x_i\right)$.*

Solution:
Let $x = (x_1, x_2, ...x_n)$ and $y = (1/n, 1/n, ...1/n)$. Then $x \cdot y = \sum_{i=1}^n x_i/n = \frac{1}{n}\left(\sum_{i=1}^n x_i\right)$. Note that this is positive.
Also, $||x|| = \sqrt{x_1^2 + ... + x_n^2}$ and $||y|| = \sqrt{n \cdot 1/n^2} = \sqrt{1/n}$
Then $||x|| \cdot ||y|| \geq |x \cdot y|$.
Then $\sqrt{x_1^2 + ... + x_n^2}\sqrt{1/n} \geq \frac{1}{n}\left(\sum_{i=1}^n x_i\right)$. Rearranging the left side, we get $\sqrt{\frac{1}{n}\sum_{i=1}^n x_i^2} \geq \frac{1}{n}\left(\sum_{i=1}^n x_i\right)$. ■

Problem 4 *Find the maximum volume of a rectangular prism with surface area 6.*

Solution:
Say our rectangular prism has side lengths x, y, z. We have that $2(xy + yz + xz) = 6$ (i.e. $xy + yz + xz = 3$. We want to find the maximum value of xyz.
Note that by AM-GM, we have that $1 = \frac{xy+yz+xz}{3} \geq \sqrt[3]{(xy)(yz)(xz)} = \sqrt[3]{(xyz)^2}$
Then $(xyz)^{2/3} \leq 1 \implies xyz \leq 1$. Note that it does indeed equal 1 when $x = y = z$. Then, the maximum volume is 1. ■

3.3 Others

Problem 5 *Say $x \in \mathbb{N}$ satisfies $x \equiv 3$ mod 13 and $x \equiv 12$ mod 25. What is the smallest possible value for x?*

Solution:
We know that if $\gcd(m, n) = 1$ then $x \equiv a \bmod m$ and $x \equiv b \bmod n$ is equivalent to $x \equiv am\overline{m}+bn\overline{n} \bmod mn$, where $\overline{m} \equiv m^{-1} \bmod n$ and $\overline{n} \equiv n^{-1} \bmod m$.
In this case we have that $a = 3, b = 12, m = 13$ and $n = 25$. We then see that $\overline{m} = 2$ and $\overline{n} = 12$ (to find these would require you to use the Euclidean algorithm). Then $x \equiv 12 \cdot 13 \cdot 2 + 3 \cdot 25 \cdot 12 \bmod \ \equiv 312 + 900 \equiv 1212 \equiv 237 \bmod (13 \cdot 25)$. Then note that the smallest value of x that works is 237. ■

Problem 6 *Find an explicit bijection between $(0, 1)$ and $\mathbb{R}^+$.*

Solution:
Define $f : (0, 1) \to \mathbb{R}^+$ as $f(x) = {}^1\!/\!_x - 1$.
First note that this is well-defined. Totality (note $x \neq 0$) and uniqueness are clear. We show existence:
We want to show that $x \in (0, 1] \implies {}^1\!/\!_x - 1$. Note that $0 < x < 1 \implies {}^1\!/\!_x > 1 \implies {}^1\!/\!_x - 1 > 0$. So, the output is indeed in $\mathbb{R}^+$.
Now we show bijectivity by demonstrating a two-sided inverse $g : \mathbb{R}^+ \to (0, 1)$. In particular, define $g(y) = \frac{1}{y+1}$. Again, totality and uniqueness are clear. Existence is again true since $y > 0 \implies y + 1 > 1 \implies {}^1\!/\!_{y+1} < 1$. It is also clearly positive.
We now show it is indeed a left and right inverse.
Left inverse:

$$\begin{aligned} g(f(x)) &= g({}^1\!/\!_x - 1) \\ &= \frac{1}{({}^1\!/\!_x - 1) + 1} \\ &= \frac{1}{{}^1\!/\!_x} \\ &= x \end{aligned}$$

Right inverse:

$$\begin{aligned} f(g(y)) &= f\left(\frac{1}{y+1}\right) \\ &= \frac{1}{{}^1\!/\!_{y+1}} - 1 \\ &= y + 1 - 1 \\ &= y \end{aligned}$$

Hence, g is a two-sided inverse for f.
Hence f is bijective. ■