21-128 and 15-151 problem sheet 3

Solutions to the following seven exercises and optional bonus problem are to be submitted through gradescope by 11PM on

Wednesday 21st September 2022.

Problem 1

Consider a function $f : A \to A$. Prove that if $f \circ f$ is injective, then f is injective.

Solution. Let $f : A \to A$ and assume that $f \circ f$ is an injection. Let $x, y \in A$ such that $f(x) = f(y)$. Applying f to both sides of the equation we see that $f(f(x)) = f(f(y))$, and by the injectivity of $f \circ f$ we conclude that $x = y$. Thus, f is an injection.

Problem 2

Consider functions $f : A \to B$ and $g : B \to A$. Prove that

(a) If $f \circ g$ is the identity function on B, then f is surjective.

(b) If $g \circ f$ is the identity function on A, then f is injective.

To remind you: given a set X, the identity function on X is the function $\mathrm{id}_X : X \to X$ defined by $\mathrm{id}_X(x) = x$ for all $x \in X$.

Solution.

(a) Suppose $f \circ g = \mathrm{id}_B$, and let $y \in B$. Let $x = g(y)$. Since $f \circ g = \mathrm{id}_B$, we have $f(x) = f(g(y)) = y$. Hence f is surjective.

(b) Suppose $g \circ f = \mathrm{id}_A$. Let $x, x' \in A$ with $f(x) = f(x')$. Then

$$x = g(f(x)) = g(f(x')) = x'$$

so f is injective.

Problem 3

Let $a, b, c, d \in \mathbb{R}$ be constants with $a \neq 0$ and $c \neq 0$. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ with $f(x) = ax + b$ and $g(x) = cx + d$ for all $x \in \mathbb{R}$. Prove that f and g are injective and surjective, but that the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h = g \circ f - f \circ g$ is neither injective nor surjective.

Solution. We will prove that f is injective and surjective; the proof for g is identical after relabelling.

- **Injectivity.** Fix $x, y \in \mathbb{R}$ and suppose $f(x) = f(y)$. Then $ax + b = ay + b$. Subtracting b from both sides gives $ax = ay$, and dividing through by a gives $x = y$. The last step is possible since $a \neq 0$. Hence f is injective.
- **Surjectivity.** Fix $y \in \mathbb{R}$, and define $x = \frac{y-b}{a}$. This is possible since $a \neq 0$. Then

  $$f(x) = a\left(\frac{y-b}{a}\right) + b = (y - b) + b = y$$

  So f is surjective.

Given $x \in \mathbb{R}$ we have

$$h(x) = (c(ax + b) + d) - (a(cx + d) + b) = bc + d - ad - b$$

Thus h is constant. Hence h is not injective, since for example $h(0) = h(1)$; and h is not surjective, since for example there is no $x \in \mathbb{R}$ with $h(x) = bc + cd - ad - ab + 1$.

Problem 4

Verify that the function $f : (0, 1) \to \mathbb{R}$ defined by

$$f(x) = \frac{2x - 1}{2x(1 - x)} \quad \text{for all } x \in (0, 1)$$

is a bijection.

Solution. First note that, given $x, y \in \mathbb{R}$ with $y \neq 0$, we have

$$\begin{aligned} f(x) = y \quad &\Leftrightarrow \quad (2y)x^2 + 2(1 - y)x - 1 = 0 \\ &\Leftrightarrow \quad x \in \left\{ \frac{y - 1 + \sqrt{y^2 + 1}}{2y}, \; \frac{y - 1 - \sqrt{y^2 + 1}}{2y} \right\} \end{aligned}$$

Call this result $(*)$.

We prove that f is a bijection by finding an inverse for f.

Define a function $g : \mathbb{R} \to (0,1)$ by letting, for $y \in \mathbb{R}$,

$$g(y) = \begin{cases} \dfrac{y - 1 + \sqrt{y^2+1}}{2y} & \text{if } y \neq 0 \\ \dfrac{1}{2} & \text{if } y = 0 \end{cases}$$

We verify that g is a well-defined function, and that g is a left inverse and a right inverse for f.

- g **is well-defined.** First note that the cases in the definition of g are mutually exclusive and cover all possibilities for $y \in \mathbb{R}$.

  Also note that we may divide by y and take the square root in the first case, since $y \neq 0$ and $y^2 + 1 \geq 0$.

  Finally, note that $g(y) \in (0,1)$ for all $y \in \mathbb{R}$. Certainly, if $y = 0$ then $g(y) = \frac{1}{2} \in (0,1)$. If $y \neq 0$, then $y^2 > 0$, so that $\sqrt{y^2+1} > \sqrt{1} = 1$. Moreover

  $$y^2 + 1 < y^2 + 2|y| + 1 = (|y|+1)^2 \quad \text{and so} \quad \sqrt{y^2+1} < |y| + 1$$

  Hence,

  $$\begin{aligned} \left| \frac{1}{2} - g(y) \right| &= \left| \frac{y - (y - 1 + \sqrt{y^2+1})}{2y} \right| && \text{by combining fractions} \\ &= \left| \frac{\sqrt{y^2+1} - 1}{2y} \right| && \text{by simplifying} \\ &= \frac{\sqrt{y^2+1} - 1}{2|y|} && \text{since } \sqrt{y^2+1} - 1 > 0 \\ &< \frac{|y|}{2|y|} && \text{since } \sqrt{y^2+1} < |y| + 1 \\ &= \frac{1}{2} && \text{by cancellation} \end{aligned}$$

  Since $\left| \frac{1}{2} - g(y) \right| < \frac{1}{2}$, it follows that $0 < g(y) < 1$, as required.

- g **is a right inverse for** f**.** To see this, fix $y \in \mathbb{R}$, and let $x = g(y)$. We need to show that $f(x) = y$. If $y = 0$ then $x = \frac{1}{2}$, so that $f(x) = 0 = y$ as required. If $y \neq 0$, then $x = \frac{y-1+\sqrt{y^2+1}}{2y}$, so that $f(x) = y$ by $(*)$ above.

- g **is a left inverse for** f**.** To see this, fix $x \in \mathbb{R}$, and let $y = f(x)$. We need to show that $g(y) = x$.

  If $x = \frac{1}{2}$ then $y = 0$, so that $g(y) = \frac{1}{2} = x$, as required.

  If $x \neq \frac{1}{2}$ then $y \neq 0$, so that $x = \frac{y-1+\sqrt{y^2+1}}{2y}$ or $x = \frac{y-1-\sqrt{y^2+1}}{2y}$ by $(*)$ above. If x takes the first of these values, then $x = g(y)$, so we're done. Hence, it remains to show that $x \neq \frac{y-1-\sqrt{y^2+1}}{2y}$. To see this, note that if $x = \frac{y-1-\sqrt{y^2+1}}{2y}$, then

$$\begin{aligned}
\left|\frac{1}{2} - x\right| &= \left|\frac{y-(y-1-\sqrt{y^2+1})}{2y}\right| && \text{by combining fractions} \\
&= \left|\frac{1+\sqrt{y^2+1}}{2y}\right| && \text{by simplifying} \\
&\geq \frac{1+|y|}{2|y|} && \text{since } \sqrt{y^2+1} > |y| \\
&= \frac{1}{2|y|} + \frac{1}{2} && \text{by separating fractions} \\
&> \frac{1}{2} && \text{since } \frac{1}{2|y|} > 0
\end{aligned}$$

  so that $|\frac{1}{2} - x| > \frac{1}{2}$ and $x \notin (0,1)$.

Hence f is a bijection and $g = f^{-1}$.

Problem 5

(a) Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ where f and g are surjective and let $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ s.t. $\forall\,(x,y) \in \mathbb{R} \times \mathbb{R},\ h(x,y) = f(x,y) + g(x,y)$. Determine, with proof, whether or not h is necessarily surjective.

(b) For all positive integers n, define S_n to be the set of bijections on $[n]$. Let

$$\begin{aligned}
A &= \{\, f \in S_n \mid \forall\, x \in [n]\, (f(x) \neq x) \,\} \\
B &= \{\, f \in S_n \mid \exists\, g \in S_n\, \forall\, x \in [n]\, (f(g(x)) \neq g(x)) \,\}
\end{aligned}$$

Show that $A = B$.

Solution.

(a) The following counter example demonstrates that h does not have to be surjective:

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined such that $\forall\,(x, y) \in \mathbb{R} \times \mathbb{R},\ f(x, y) = x$.
Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined such that $\forall\,(x, y) \in \mathbb{R} \times \mathbb{R},\ g(x, y) = -x$.

f and g are well-defined: they are total because they are clearly defined for all $(x, y) \in \mathbb{R}$. Existence is satisfied because $\forall\,(x, y) \in \mathbb{R} \times \mathbb{R} \quad x \in \mathbb{R} \implies f(x, y) \in \mathbb{R}$ and $-x \in \mathbb{R} \implies f(x, y) \in \mathbb{R}$, and uniqueness is satisfied because the outputs of x and $-x$ can only take on one value per input.

We now show f and g are surjective. Fix $y \in \mathbb{R}$. We see that $f(y, 0) = y$ and $g(-y, 0) = y$. We can verify easily that $(y, 0) \in \mathbb{R} \times \mathbb{R}$ and $(-y, 0) \in \mathbb{R} \times \mathbb{R}$, since $y \in \mathbb{R} \implies -y \in \mathbb{R}$. Thus, we have shown surjectivity of both f and g.

We see that $\forall\,(x, y) \in \mathbb{R} \times \mathbb{R}\ \ h(x, y) = f(x, y) + g(x, y) = x + -x = 0$. The negation of the definition of surjectivity for h is true: $\exists\, y \in \mathbb{R}\ \forall\,(a, b) \in \mathbb{R} \times \mathbb{R}\ \ h(a, b) \neq y$. We can pick $y = 1$, and we see the statement is true since $\forall\,(x, y) \in \mathbb{R} \times \mathbb{R}\ \ h(x, y) = 0 \neq 1$.

(b) We show the result via double containment:

$A \subseteq B$: Fix an arbitrary $f \in A$, and choose g to be the identity function, we see that g is a well defined bijection, and furthermore:

$$\begin{aligned} & f(x) \neq x && (f \in A) \\ \implies & f(g(x)) \neq g(x) && (g(x) = x) \\ \implies & f \in B \end{aligned}$$

$B \subseteq A$: For an arbitrary $f \in B$, we know there is a bijection g such that $\forall\, x \in [n]\ \ f(g(x)) \neq g(x)$, and we need to show $\forall\, x \in [n]\ \ f(x) \neq x$ to show $f \in A$, so fix $x \in [n]$:

$$\begin{aligned} & g \text{ bijective} \\ \implies & g \text{ surjective} && (\text{by def. of bijective}) \\ \implies & \exists\, y \in [n]\ \ g(y) = x && (g \text{ is surjective}) \\ \implies & f(g(y)) \neq g(y) && (f \in B) \\ \implies & f(x) \neq x && (g(y) = x) \\ \implies & f \in A \end{aligned}$$

Therefore, since $A \subseteq B$ and $B \subseteq A$, $A = B$.

Problem 6

(a) Find a function $f : \mathbb{N} \to \mathbb{N}$ that is surjective and not injective, and show that it has at least two right-inverses.

(b) Suppose that $f : A \to B$ is surjective but not injective. Prove that f has at least two right-inverses.

Solution.

(a) We can take $f : \mathbb{N} \to \mathbb{N}$ by defining $f(0) = 0$, and $f(x) = x - 1$ for all $x \geq 1$.

First we show that it is a well-defined function.

Totality: Note that the naturals can be exhaustively defined as either 0, or greater than or equal to 1, so our function is total.

Existence: Note that for all naturals n greater than 1, $n-1$ is also natural, so for all $n > 1$, $f(n)$ exists in our codomain. at 0, note we return 0, which is natural, so the function exists everywhere. This function is not injective, since for $x_1 = 0, x_2 = 1, x_1 \neq x_2 \wedge f(0) = f(1) = 0$.

Uniqueness: Note that we partition our set into 0, and $x \geq 1$. There are no elements in both sets, so we just need to show that for each set separately, we map every x to one unique y, which we clearly do.

Now, we prove that it is surjective. to do so, we **define a right inverse as follows**: Pick some $n \in \mathbb{N}$. We then have that $f^{-1}(n) = n + 1$. Note totality holds since we've defined our function in the general case, existence holds since the naturals are closed under addition, and uniqueness holds since we've defined only one value per n. Next, note $f \circ f^{-1}(x) = f(x + 1)$. Note $x \in \mathbb{N} \implies x + 1 > 0$, so $f(x + 1) = x$. We have proven surjectivity. Note that the first right inverse is provided by our proof of surjectivity.

Now to find the second. We can define another right-inverse by changing the value at 0. If we set $h(0) = 0$ and $h(x) = x + 1$ for all $x \geq 1$, then h is also a right-inverse of f. This function exists everywhere because the naturals are closed under addition and 0 is in the naturals. This function total again because the sets are disjoint and it is unique everywhere since the sets $\{0\}$ and $x \geq 1$ are disjoint and cover all of the naturals, and in each case we define $f(x)$ as one value. Now to prove this is a valid right inverse: $f(h(0)) = f(0) = 0$, and for $x \geq 1$, $f(h(x)) = f(x + 1) = x + 1 - 1 = x$.

(b) Suppose $f : A \to B$ is surjective but not injective. First, we show that $A \neq \varnothing$.

Assume for the sake of contradiction that $A = \varnothing$. Then there is exactly one function $f : A \to B$, which does absolutely nothing. However, this is not surjective unless $B = \varnothing$ as well, but then it is injective, contradiction. Therefore, $A \neq \varnothing$. This also gives us that $B \neq \varnothing$.

Now we can talk about the elements of A and B. Since f is not injective, $\exists\, b \in B$ such that $\exists\, a_1, a_2 \in A$ with $a_1 \neq a_2$ and $f(a_1) = f(a_2) = b$. Therefore, the preimage of b in A has size at least 2. Now, for every $b' \in B$ with $b' \neq b$, since f is surjective, we know that $f^{-1}[\{b'\}] \neq \varnothing$. Then, we can choose (!) the minimum $a \in f^{-1}(b')$ and set $g(b') = a$. For b, we now have at least two choices for where the right inverse can send it. In one case, we can set $g(b) = a_1$, and in the other case, we can set $g(b) = a_2$, giving us at least two possible right-inverses. So our functions are:

$$\begin{cases} a & x \neq b \\ a_1 & x = b \end{cases}$$

$$\begin{cases} a & x \neq b \\ a_2 & x = b \end{cases}$$

Define g_1 and g_2 as being the right inverses that send $b \mapsto a_1$ and $b \mapsto a_2$ respectively. We check that these are actually right inverses. First note trivially that both functions satisfy existence, as they are defined to send elements to our original set A. Also, since we cannot have $x \neq b$ and $x = b$, and for each section of the piecewise we send all elements to one value, we have uniqueness. Finally, note that our set can be partitioned totally into $x = b$ or $x \neq b$, so totality also holds for both. Now to prove these are both right inverses.

$f(g_1(b)) = f(a_1) = b$ and for all other $b' \in B$, we have defined g_1 so that $g_1(b') \in f^{-1}[\{b'\}]$, so $f(g_1(b')) = b'$.

For the other function, the proof is essentially the same: $f(g_2(b)) = f(a_2) = b$ and for all other $b' \in B$, we have defined g_2 so that $g_2(b') \in f^{-1}[\{b'\}]$, so $f(g_2(b')) = b'$.

Problem 7

Let $g : \mathbb{R} \to \mathbb{R}$ and define $f : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ via $f(S) = \{g(x) \mid x \in S\}$.

a) Determine necessary and sufficient conditions on g to have

$$\forall\, A, B \subseteq \mathbb{R} \;\; f(A \cap B) = f(A) \cap f(B)$$

b) Show that if g is surjective, then f is surjective.

Solution.

a) I claim this holds if and only if g is injective.

First, if g is not injective, then there are some distinct $a, b \in \mathbb{R}$ such that $g(a) = g(b)$. But then $\{a\} \cap \{b\} = \varnothing$ so that $f(\{a\} \cap \{b\}) = \varnothing$, while $f(\{a\}) \cap f(\{b\}) = \{g(a)\} \cap \{g(b)\} = \{g(a)\}$.

Next, if g is injective, then we will show the condition is true via double containment. Let $A, B \subseteq \mathbb{R}$ be arbitrary. If either are empty then both sides of our equivalence are empty.

($\subseteq$) Fix an arbitrary $x \in f(A \cap B)$. We will show that $x \in f(A) \cap f(B)$. Note by definition of f that $x = g(x_0)$ for some $x_0 \in A \cap B$. Therefore $x_0 \in A$ and $x_0 \in B$. Thus, $g(x_0) \in f(A)$ and $g(x_0) \in f(B)$, so that $g(x_0) = x \in f(A) \cap f(B)$ as desired.

($\supseteq$) Fix an arbitrary $x \in f(A) \cap f(B)$. Then $x \in f(A)$ and $x \in f(B)$. In other words, there exists some $a \in A$ with $g(a) = x$ and some $b \in B$ with $g(b) = x$. By injectivity of g, $a = b$. But then $a \in A \cap B$ so that $f(a) = x \in f(A \cap B)$ as desired.

b) We construct a right inverse for f under the assumption that g is surjective. Since g is surjective, it has a right inverse. Namely, $g^{-1} : \mathbb{R} \to \mathbb{R}$ such that $\forall x \in \mathbb{R}\ g(g^{-1}(x)) = x$. We define a right inverse $f^{-1} : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ via $f^{-1}(S) = \bigcup_{x \in S}\{g^{-1}(x)\}$. Note that f^{-1} is well defined because we've defined it for all inputs and g^{-1} takes in a real number (totality), its output is unique for a given input since g^{-1} well-defined (uniqueness), and it satisfies existence because every member of its output is a real number, since $g^{-1}(x)$ is guaranteed to be a real number when x is a real number (well-definedness of g). See that we have $\forall S \subseteq \mathbb{R}\ f(f^{-1}(S)) = f(\{g^{-1}(x) \mid x \in S\}) = \{g(y) \mid y \in \{g^{-1}(x) \mid x \in S\}\} = \{g(g^{-1}(x)) \mid x \in S\} = S$ by right inversedness of g^{-1}.

Bonus Problem - 2 points

Let $f : A \to B$ be a function.

(a) Prove that there exists a set X and functions $p : A \to X$ and $i : X \to B$, with p surjective and i injective, such that $f = i \circ p$.

(b) Prove that there exists a set Y and functions $j : A \to Y$ and $q : Y \to B$, with j injective and q surjective, such that $f = q \circ j$.

Solution.

(a) Let $X = f[A]$. Define $p : A \to X$ by $p(a) = f(a)$ for all $a \in A$, and define $i : X \to B$ by $i(b) = b$ for all $b \in X$. Then p is well-defined since if $a \in A$ then $f(a) \in X$, and i is well-defined since $X \subseteq B$. Now

- If $a \in A$ then $i(p(a)) = i(f(a)) = f(a)$, so that $f = i \circ p$.
- Given $b \in X$ we have $b = f(a) = p(a)$ for some $a \in A$, so that p is surjective.
- Given $b, b' \in X$, if $i(b) = i(b')$ then $b = b'$ simply by definition of i, so that i is injective.

(b) If $A \neq \varnothing$ then let $Y = A \times B$, and define

$$j(a) = (a, f(a)) \quad \text{and} \quad q(a, b) = b$$

for all $a \in A$ and $b \in B$. Then

- j is injective. If $a, a' \in A$ and $j(a) = j(a')$, then $(a, f(a)) = (a', f(a'))$. Since the first components must be equal, we have $a = a'$.
- q is surjective. Given $b \in B$, let $a \in A$ be any element (*note: this is where we use the fact that A is non-empty*). Then $b = q(a, b)$.
- $f = q \circ j$, since if $a \in A$ then $q(j(a)) = q(a, f(a)) = f(a)$.

On the other hand, if $A = \varnothing$ then f is vacuously injective, and moreover the identity function $\mathrm{id}_B : B \to B$ is surjective, so we can define $j = f$ and $q = \mathrm{id}_B$.