Relations Review Solutions

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1. For each example below, determine whether the given relation R is an equivalence relation on the given set S:

   (a) (MIT 6.042) $S = P$, where P is the set of all people in the world today; $(x, y) \in R$ if and only if x is at least as tall as y.

   (b) (BYU) $S = \mathbb{Z}$; $(x, y) \in R$ if and only if $2x + 5y \equiv 0 \pmod 7$.

   (c) (For those of you in Matrices) $S = \mathbb{R}^n$ for some $2 \leq n \in \mathbb{N}$; $(\vec{x}, \vec{y}) \in R$ if and only if $\vec{x}$ and $\vec{y}$ are linearly independent.

   Solution.

   (a) R is not an equivalence relation since it is not symmetric. In particular, if x is strictly taller than y, then y cannot be strictly taller than x. (The fact that P is the set of all people in the world today implies that this strict inequality exists.)

   (b) R is an equivalence relation. Note that

   $$2x + 5y \equiv 0 \pmod 7 \quad \Leftrightarrow \quad 2(x - y) \equiv 0 \pmod 7 \quad \Leftrightarrow \quad x \equiv y \pmod 7.$$

   Since the latter is an equivalence relation, so is the former.

   (c) R is not an equivalence relation; in particular, it is not transitive. Consider $\vec{e_1} = (1, 0, \ldots, 0), \vec{e_2} = (0, 1, 0, \ldots, 0)$, and $2\vec{e_1} = (2, 0, \ldots, 0)$. Then, $\vec{e_1}$ and $\vec{e_2}$ are linearly independent and $\vec{e_2}$ and $2\vec{e_1}$ are linearly independent, but clearly $\vec{e_1}$ and $2\vec{e_1}$ are not linearly independent!

2. Suppose R is a relation on the set X that is both an equivalence relation and a partial order relation. Prove that R is the equality relation on X.

   Solution. Suppose there exist $x \neq y$ in X such that xRy. Since R is an equivalence relation, R is symmetric, so yRx. However, since R is a partial order relation, it is also antisymmetric, and so xRy and yRx implies $x = y$, contradiction. Thus, no two unequal elements in X are related under R. Combining this with the fact that R is reflexive gives that R must be the equality relation on X.

3. Let R be a relation on $\mathcal{P}([2n])$ defined by $(A, B) \in R$ if and only if $A \cap [n] = B \cap [n]$. Is R an equivalence relation? What happens if instead of using $[n], [2n]$, we generalize to C, S where $C \subseteq S$?

   Solution. R is an equivalence relation. The proof is the same as the proof that equality is an equivalence relation, with all instances of X replaced by $X \cap [n]$. This doesn't depend on the underlying sets being $[n], [2n]$ so it works for general C, S as well.

4. (Tripos 2011) Write down an equivalence relation on the positive integers that has exactly four equivalence classes, of which two are infinite and two are finite.

   Solution. Many examples are possible. For instance, one can say that $x \sim y$ if and only if $x = y = 0$, $x = y = 1$, $x \geq 2$ and $y \geq 2$ are both even, or $x \geq 3$ and $y \geq 3$ are both odd. It is not hard (albeit mildly tedious) to check that this is an equivalence relation.