21128/15151 Counting Things

Emily Zhu

1 Counting Numerically

Counting is generally really difficult; there are many sets which cannot be counted in any nice way, and even if a set is nice to count, there are still many ways to accidentally undercount/overcount and relatively few ways to count correctly. Even so, that doesn't stop us from trying!

1.1 Fundamental Principles

Proposition 1 (Addition Principle)**.** If a finite set S is partitioned into $S_1, \ldots, S_k$, i.e. $\bigcup_{i=1}^k S_i = S$ and $S_i \cap S_j = \varnothing$ for $i \neq j$, then $|S| = \sum_{i=1}^k |S_i|$.

Informally, this states that if the thing we're trying to count can be completely split into some k nonoverlapping cases, then we can add the results we get for the cases to get what we're trying to count. For this, it is important to explicitly check that your cases actually cover everything and that they do not overlap. Often, these are pretty obvious, but nonzero work should be shown, especially in claiming that everything is attained. Technically, we should also be showing that our cases give us elements in the desired set, but we normally define the cases to do exactly that.

Proposition 2 (Multiplication Principle (informal))**.** If a set of n-tuples is created by picking coordinates in a process such that there are a_1 choices for the first coordinate, and given any choice of values of the first i coordinates, there are a_{i+1} choices for the $i+1^{st}$ coordinate, then the number of such n-tuples is $\prod_{i=1}^n a_i$.

Even more informally, this states that if the thing we're trying to count is found by some process in which each step always has the same number of choices (regardless of everything else), then we can multiply these numbers of choices to get what we're trying to count. It is extremely important to note that generates a tuple; often, we want to count the number of sets rather than tuples. We can work around this by treating these sets as having a specific ordering (which we get to decide), and then counting the number of tuples with that precise ordering.

Proposition 3 (Complementary Counting)**.** Given a finite set X and a subset $U \subseteq X$. Then $|U| = |X| - |X \setminus U|$.

Some times it is easier to count the complement than what you originally wanted—this is often the case when your original set is complicated or filled with many cases. A keyword which may suggest complementary counting is something like "at least" or "at most," where the complement has very few cases when looking at exact quantities. Of course, if the original "at least" only requires 2 exact cases when the complement requires like 4, attempting to use the complement is just much more work.

1.2 Checking Your Answer

After writing down a process, it can still be unclear if your process actually produces the set you wanted—perhaps it may have missed some elements or repeated some elements. (Formally, we want a bijection from the tuples found to whatever set we desire; typically we won't make any such bijection explicit.) It may be helpful to think about what the process produces and consider a sample element that we want to get from the set. Given this sample object, can you attain it somehow in your process? Can you attain it multiple ways? The latter is very commonly a problem which emerges when trying to count sets. Steps which say “pick anything” are prone to overcounting, especially if there is some other step with overlapping choices. Attempting to run your process may also help you to determine how to specify your process in your writeup.

Another common way to check your answer is to attempt to count your set in a different way—perhaps you can change the order in which you pick things, perhaps you can use complementary counting, or perhaps you can interpret your set differently (i.e. change the scenario you are in; I for one, like thinking about lattice paths).

Yet another way to check your answer (or actually to approach the problem) is to try a much smaller case of the same problem or a similar problem, so you can explicitly list out cases. This is often tedious and prone to other errors (maybe you made a different problem or you made a mistake listing out cases), but it can still be helpful in attempting to make your problem more concrete/approachable.

1.3 Lots of Problems

Example 1 (Warmup)**.** How many ways are there to label the 4 corners of a square with letters (i.e. A to Z) such that adjacent corners get different letters?

Thoughts. We begin labeling the top left corner—26 choices. We label the top right and bottom left—25 choices each. Now when we get to labeling the bottom right corner, we need it to be different from the top right and bottom left. At this point you may be tempted to say there are 24 choices left but if the top right and bottom left corner are the same, then there are actually 25 choices! We fix our argument to case on if corners are the same and then proceed to write it up...

Example 2 (Two Suits)**.** How many hands of 4 cards from a standard 52-card deck contain cards of exactly 2 suits?

Thoughts. I actually screwed up many times when trying to count this. The main idea is that we want to pick which 2 suits appear and then pick the 4 cards from these suits. This suggest $\binom{4}{2}\binom{26}{4}$—but this does not work since it also includes cases where only 1 suit appears. You may then think that you just need to remove the number of ways to get 1 suit, but it turns out that this actually counts the number of ways to get cards of a single suit multiple times! Thus, it may be better to explicitly case on how many cards of each suit appear. We can also write an alternative proof using inclusion/exclusion and complementary counting which basically does this.

Example 3. How many ways are there for me to distribute some amount of my 10 (essentially indistinguishable) Sinosaurus plushies to 4 distinguishable people? Note: I may want to keep some.

Remark 4. We can rephrase this as how many solutions are there to $x_1+x_2+x_3+x_4 \leq 10$, where $x_i \in \mathbb{N}$ for $i \in [4]$?

Remark 5. The technique used to count the number of solutions to $\sum_{i=1}^{k} x_i = n$ is often called stars and bars (or balls and urns). This is since we are trying to arrange n indistinguishable stars and $k-1$ indistinguishable dividing bars in $n+k-1$ slots. It's also important to remember that the space before the first and after the last divider also account for x_1 and x_n respectively, so we only need $k-1$ dividers.

Example 4. Given 2 red dice and 3 green dice, how many ways are there such that exactly 3 dice rolls have the same value and the remaining two have different values? Note: I cannot tell the red dice apart, so getting a 5 and a 3 on the red dice is the same as getting a 3 and a 5.

Definition 6 (Lattice Path)**.** A lattice path is some sequence of moves each going one unit right or up; often we consider it from $(0,0)$ to (m,n) for $m,n \in \mathbb{N}$.

Example 5. How many lattice paths are there from $(0,0)$ to (m,n)?

Example 6. How many lattice paths from $(0,0)$ to $(5,3)$ avoid $(1,1),(1,2),(2,1),(2,2)$?

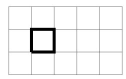
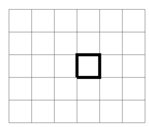


Figure 1: Avoid the darkened square

Remark **7.** While in this case, the path avoiding the removed square was easy to directly case on, this is not generally the case. Consider running through the argument for:



Example 7. Find the number of lattice paths from $(0,0)$ to (m,n) where you go left exactly once and the whole path remains within the rectangle formed by $(0,0),(m,0),(m,n),(0,n)$.

Example 8. How many functions are there from $[5] \to [8]$ which are not injective? What about from $[k] \to [n]$ in general, where $k \leq n$? (What happens if $k > n$?)

Thoughts. Please don't actually list them out. Explicitly trying to figure out where outputs match is also really difficult, but the "not" in the example should be quite suggestive...

Example 9. How many "words" can be formed from rearranging the letters of COMMITTEE?

In counting, don't be afraid to try different approaches if your first approach seems tedious or seems to count the wrong thing. Do be very careful in constructing your processes though! Also remember that upon seeing solutions, they may seem very simple, but this does not mean that they are easy to find.

2 Counting in Two Ways

The main idea of counting in two ways is to show that two expressions are equal by exhibiting a set and showing that both sides count the size of this set. Note that by the uniqueness of the cardinality of a set, we can conclude that the two expressions are thus equal. It is also possible to have the expressions count two sets and then find a bijection between the sets—this may result in more work since you then need to define a function and prove that it is well-defined and bijective.

Perhaps the hardest part of counting in two ways is constructing the set; this is where you need to determine what your expressions can actually count and how they interact. Once you determine your set, you have likely figured out how your expressions count the set. In some sense, this reverses counting numerically, since you start with (abstract) sizes and try to build something of that size.

*Important: Please do **not** do algebraic manipulations to an expression when we ask you to prove it by counting in 2 ways! Also, "simple" manipulations should be fairly easy to reason about.*

2.1 Fundamental Expressions

When counting in two ways, it's very useful to have a sense of what the following can count/have prototypical examples of how to think about various expressions. This list is certainly nonexhaustive, and as you get practice, you should gain intuition on which of these may be easiest to work with. Note that in counting in two ways, you will often have a combination of these expressions, so it is good to understand how these sets interact.

- $\binom{n}{k}$: picking k things from n things
  - number of subsets of $[n]$ with size k
  - number of binary strings with k 1's (or k 0's)
  - number of lattice paths from $(0,0)$ to $(k, n-k)$ (see examples above)
- 2^n: making some binary choice n times (see a^n as well)
  - number of subsets of $[n]$
  - number of binary strings of length n
- a^n: deciding from a choices n times
  - number of functions $f : [n] \to [a]$
  - number of ways to color n distinguishable objects with a colors
  - number of ways to divide n things into a (possibly empty) groups
  - number of a-ary strings of length n (i.e. made of a symbols)
- $n!$: ordering n things
  - number of permutations of $[n]$ (or any other set)
- $\sum_{k=0}^{n}$: partitioning based on some k
  - k can represent the size of the sets in the partition
  - k can represent a special element in the sets (eg. the largest element; often you will pick fewer elements on one side for this case)
  - k can represent the number of elements we pick from some part of a partition
- n: pick 1 thing from n things
  - it can be helpful to interpret n as $\binom{n}{1}$ sometimes

There are other important identities (some of which we've alluded to in **Counting Numerically** such as $\binom{n+k-1}{k-1}$) which may also help you interpret the expressions in a more natural way. Perhaps one of the most important of those identities would be that $\binom{n}{k} = \binom{n}{n-k}$ so you can think about the things with that property (e.g. being in the set, being a 1) or without it. Also remember that order does not matter in multiplication, so rearranging the terms in the product may also help. Often expressions will be written in a way that looks aesthetically pleasing without regards to the process used to make them.

A good thing to think about is how many things you're picking and how many things you're picking from. From there, you also want to think about how these things interact—are they distinguishable? is it a single set? is a tuple of sets? a set of tuples? Or if you're working with a specific scenario, how exactly do these committees/cases interact? Do you have special elements? A lot of this intuition will come from practice, and the rest will come from just trying a lot of scenarios and attempting many matchings of "variables" to amounts of stuff getting picked.

2.2 Lots of Problems

(these are mostly sourced from *Mathematical Thinking: Problem-Solving and Proofs* by John. P. D'Angelo and Douglas B. West)
(I've also gotten lazy on showing intuition so you'll have to be at the session to hear about it!)

Example 10. By counting in 2 ways, show that $\sum_{k=1}^{n} 2^{k-1} = 2^n - 1$ for $n \in \mathbb{N}$.

Example 11. By counting in 2 ways, show that $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$ for $n \in \mathbb{N}$. (This is also equal to $n\sum_{k=1}^{n} \binom{n-1}{k-1}$)

Example 12. By counting in 2 ways, show that $\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$ for $m, n \in \mathbb{N}$.

This is also Example 6.2.44 in the Book.

Example 13. By counting in 2 ways, show that $\sum_{k=-m}^{n} \binom{m+k}{r}\binom{n-k}{s} = \binom{m+n+1}{r+s+1}$ for $m, n, r, s \in \mathbb{N}$.

Remark 8. Note that for $S = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$, $|S| = b - a + 1$ and not $b - a$. To convince yourself, define a bijection $f : [b - a + 1] \to S$ via $f(x) = a - 1 + x$. What happens when we plug in $f(b - a + 1)$? (Check that it's actually a bijection)

Example 14. By counting in 2 ways, show that $\binom{n}{k}\binom{n-k}{m} = \binom{n-m}{k}\binom{n}{m}$ for $n, k, m \in \mathbb{N}$.

Example 15. By counting in 2 ways, show that $\sum_{k=0}^{m} \binom{n}{k}\binom{m}{m-k}2^{n-k} = \sum_{l=0}^{n} \binom{n}{l}\binom{n+m-l}{m}$ for $m, n \in \mathbb{N}$. *Hint*[1]

[1]Hint: pick 2 subsets