1. Determine, with proof, whether each statement below is true.

(i) $(\forall x \in \mathbb{Q}^+)(\exists y \in \mathbb{N})(x^2 - y^2 = 0)$

Solution. The statement is false. We prove the negation by considering $x = \frac{1}{2} \in \mathbb{Q}^+$. Since the square of any natural number is an integer and $\frac{1}{4}$ is not an integer we have that $\frac{1}{4} - y^2 \neq 0$ for every natural number y. The proof is complete.

(ii) $(\forall x \in \mathbb{Q}^+)(\exists y \in \mathbb{Q}^+)(x^2 - y^2 = 0)$

Solution. The statement is true. Let $x \in \mathbb{Q}^+$. Consider $y = x \in \mathbb{Q}^+$. Then $x^2 - y^2 = x^2 - x^2 = 0$. The proof is complete.

(iii) $(\forall x \in \mathbb{Q}^+)(\exists! y \in \mathbb{R})(x^2 - y^2 = 0)$

Solution. The statement is false. We prove the negation by considering $x = 1 \in \mathbb{Q}^+$. For this choice of x there are two elements $y \in \mathbb{R}$ such that $x^2 - y^2 = 0$, namely $y = 1$ and $y = -1$. The proof is complete.

2. Prove that for arbitrary sets A, B, and C

$$(A \cap B) \cup C \subseteq (A \cup B \cup C) \setminus (A \setminus (B \cup C)).$$

Solution. Let $x \in (A \cap B) \cup C$.

Case 1) $x \in C$. Since $x \in C$, we know that $x \in A \cup B \cup C$. Since $x \in C$, we know that $x \in B \cup C$ and hence $x \notin A \setminus (B \cup C)$. Thus, $x \in A \cup B \cup C) \setminus (A \setminus (B \cup C)$, as desired.

Case 2) $x \in A \cap B$. Since $x \in A$, we know that $x \in A \cup B \cup C$. Moreover, since $x \in B$ we have $x \in B \cup C$, so $x \notin A \setminus (B \cup C)$. Thus, $x \in A \cup B \cup C) \setminus (A \setminus (B \cup C)$, as desired.

In either of the possible cases, the desired result holds.

3. Recall that an integer larger than 1 is prime if and only if it has only 1 and itself as positive integer factors. Prove that for any prime number $p \geq 3$, there exist **unique** positive integers x and y such that

$$p = x^2 - y^2.$$

Solution. Let p be a prime number greater than or equal to 3. Note, in particular, that this means that p is odd. As in class, we observe that $x = \frac{p+1}{2}$ and $y = \frac{p-1}{2}$ are positive integers satisfying $p = x^2 - y^2$.

To show uniqueness, let x and y be positive integers such that $p = x^2 - y^2 = (x - y)(x + y)$. Since p and $x + y$ are positive, so too is $x - y$ and hence $(x - y)(x + y)$ is a positive integer factorization of p. Since p is prime, we conclude that $x - y = 1$ and $x + y = p$ (we used that fact that $x - y < x + y$ here). Adding the two equations and dividing by two yields $x = \frac{p+1}{2}$. Subtracting the first equation from the second and dividing by two yields $y = \frac{p-1}{2}$, as desired.

4. Define $f : \mathbb{R} \to \mathbb{R}^+$ via

$$f(x) = \begin{cases} 3 - 2x & \text{if } x \leq 1 \\ 1/x & \text{if } x > 1 \end{cases}$$

Show that f is a bijection. You needn't show well-definedness of any functions that you use.

Solution. Define $g : \mathbb{R}^+ \to \mathbb{R}$ via

$$g(x) = \begin{cases} 1/x & \text{if } x \leq 1 \\ \frac{3-x}{2} & \text{if } x > 1 \end{cases}$$

To see that g is a right inverse for f, let $x \in \mathbb{R}^+$ and consider $f(g(x))$. If $x \leq 1$, then $f(g(x)) = f(1/x) = 1/(1/x) = x$, where we used the fact that $1/x \geq 1$ in the selection of the formula for f. If $x > 1$, then $f(g(x)) = f(\frac{3-x}{2}) = 3 - 2(\frac{3-x}{2}) = x$, where we used the fact that $1/x < 1$ in the selection of the formula for f.

To see that g is a left inverse for f, let $x \in \mathbb{R}$ and consider $g(f(x))$. If $x \leq 1$, then $g(f(x)) = g(3 - 2x) = \frac{3-(3-2x)}{2} = x$, where we used the fact that $3 - 2x \geq 1$ in the selection of the formula for g. If $x > 1$, then $g(f(x)) = g(1/x) = 1/(1/x) = x$, where we used the fact that $1/x < 1$ in the selection of the formula for g.

Since f has a two-sided inverse, f is a bijection.

5. Let X be a set with a function $f : X \to X$.

(i) Show that

$$(\forall B \subseteq X)(B \subseteq f[X]) \implies f \text{ is surjective.}$$

Solution. Assume the hypothesis. Let $y \in X$. Then $\{y\} \subseteq X$ and so $\{y\} \subseteq f[X]$. This means that $y \in f[X]$ and hence that there is $x \in X$ such that $f(x) = y$, as desired.

(ii) Show that

$$f \text{ is surjective} \implies (\forall B \subseteq X)(B \subseteq f[X]).$$

Solution. Assume the hypothesis. Let $B \subseteq X$. Let $y \in B$. Then $y \in X$ and since f is a surjection, there is $x \in X$ such that $f(x) = y$. This means that $y \in F[X]$ so $B \subseteq f[X]$, as desired.

Bonus. Determine, with proof, whether the following statement is true:

There exist functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y

$$f(x)g(y) - g(x)f(y) = x.$$

Solution. Assume for sake of contradiction that such functions, f and g, exist. Then taking $x = y = 1$, we have $f(1)g(1) - g(1)f(1) = 0 = 1$, a contradiction. Thus, no such functions exist.