21-128 and 15-151 problem sheet 7

Solutions to the following seven exercises and optional bonus problem are to be submitted through gradescope by 11PM on

Wednesday 9th November 2022.

Problem 1

For $x, y, z \in \mathbb{Z}$, suppose that 5 divides $x^2 + y^2 + z^2$. Prove that 5 divides at least one of x, y or z.

Hint: What remainders can be left when the square of an integer is divided by 5?

Problem 2

The base 10 representation of an integer is *palindromic* if the digits read the same when written forward or backward. Prove that every palindromic integer with an even number of digits is divisible by 11.

Problem 3

Show your work in the following computations.

(a) Determine the last two digits of 14^{2022}.

(b) Compute $\frac{53!}{27}$ mod 27.

(c) Find all integers x such that $x^2 + 3x \equiv 3^{31} \mod 29$.

Problem 4

Show that the equation $x^2 + 1 \equiv 0 \pmod{p}$ has a solution when p prime and $p \equiv 1 \pmod{4}$.

Hint: Wilson's Theorem.

Problem 5

Let m and n be positive, relatively prime integers, and r and s be integers such that $mr \equiv 1 \bmod n$ and $ns \equiv 1 \bmod m$. For integers a, b, find an integer value of x in terms of a, b, m, n, r, s satisfying $x \equiv a \bmod n$ and $x \equiv b \bmod m$.

Problem 6

Let $A \subseteq \mathbb{N}^+$ and $B \subseteq \mathbb{N}^+$ be nonempty sets of positive integers. Define

$$A + B \stackrel{\text{def}}{=} \{a + b : a \in A, b \in B\}.$$

Show that $A + B$ is finite if and only if both A and B are finite.

Problem 7

For arbitrary $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$, show that if the image of g is finite, then the image of $f \circ g$ is finite with size less than or equal to size of the image of g.

Bonus Problem - 2 points

Find all positive integers a for which there exist non-negative integers $x_0, x_1, \ldots x_{2020}$ satisfying the equation

$$a^{x_0} = a^{x_1} + a^{x_2} + \cdots + a^{x_{2020}}.$$