151 and 128 Induction Review
12-8-2022

1 Induction Review

- Weak Induction:
  - $P(n) :=$ "..."
  - BC: Show $P(n_0)$ where n_0 is the smallest number in our WTS
  - IH: Assume $P(n)$ for some $n \geq n_0$
  - IS: Show $P(n+1)$
- Strong Induction
  - $P(n) :=$ "..."
  - BC: Show $P(n_0) \wedge P(n_1)...P(n_m)$, as many base cases as you need
  - IH: Assume $P(k)$ for all $n_0 \leq k \leq n$ for some $n \geq n_m$
  - IS: Show $P(n+1)$
- Both show P for all integers greater or equal to n_0

2 Induction Problems

1. Show that $n^3 - n$ is divisible by 6 for all $n \in \mathbb{N}$

   Proof.
   $$P(n) := \text{"}n^3 - n \text{ is divisible by 6"}$$
   BC: $n = 0$
   $$0^3 - 0 = 0$$
   $$6 \mid 0$$
   IH: Assume $p(n)$ for some $n \geq 0$
   IS: WTS $P(n+1)$
   $$\begin{aligned} &(n+1)^3 - (n+1) \\ =&n^3 + 3n^2 + 2n \\ =&(n^3 - n) + 3n^2 + 3n \end{aligned}$$
   By IH, $6 \mid n^3 - n$, so need to show $6 \mid 3n^2 + 3n$.
   Case 1: n is even
   $$\begin{aligned} &\exists k \in \mathbb{Z}, n = 2k \\ &3(2k)^2 + 3(2k) \\ =&12k^2 + 6k \\ =&6(2k^2 + k) \end{aligned}$$

Case 2: n is odd

$$\begin{aligned}
&\exists k \in \mathbb{Z}, n = 2k+1 \\
&3(2k+1)^2 + 3(2k+1) \\
&= 3(4k^2 + 4k + 1 + 2k + 1) \\
&= 6(2k^2 + 3k + 1)
\end{aligned}$$

□

2. Let $T_1 = T_2 = T_3 = 1$, and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 4$
   Prove that $T_n < 2^n$ for all $n \in \mathbb{Z}_+$

   Proof.

   $$P(n) := \text{"}T_n < 2^n\text{"}$$

   BC: n = 1, 2, 3
   IH: Assume $P(k)$ for all $3 \leq k \leq n$ for some n
   IS: WTS $P(n+1)$

   $$\begin{aligned}
   T_{n+1} &= T_n + T_{n-1} + T_{n-2} \\
   &< 2^k + 2^{k-1} + 2^{k-2} && \text{(IH)} \\
   &= 2^{k+1}(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}) \\
   &= 2^{k+1}\frac{7}{8} \\
   &< 2^{k+1}
   \end{aligned}$$

   □

3 Rectypes Review

Rectype List(A): all lists over A

- *nil*
- $prep[a, L]$ (aka $a :: L$) for all $a \in A$

More definitions:

- app takes in an element and a list and defines how to add the element to the end of the list.

  $\mathsf{app}(a, \mathsf{nil}) = a :: \mathsf{nil}$
  $\mathsf{app}(a, b :: L) = b :: \mathsf{app}(a, L)$

- join takes in two lists and defines how to concatenate the.

  $\mathsf{join}(\mathsf{nil}, K) = K$
  $\mathsf{join}(a :: L, K) = a :: \mathsf{join}(L, K)$

- rev takes in a list and defines how to reverse it.

  $\mathsf{rev}(\mathsf{nil}) = \mathsf{nil}$
  $\mathsf{rev}(a :: L) = \mathsf{rev}(L) :: a$

Structural Induction:

- $p(L) :=$ "..."
- BC: $p(\mathsf{nil})$
- IH: $p(L')$ true for some L'
- IS: WTS $p(a :: L')$

4 Structural Induction Problems

Prove that $\mathsf{join}(L, \mathsf{rev}(K)) = \mathsf{rev}(\mathsf{join}(K, \mathsf{rev}(L)))$ for all lists L and K

Lemma 1.

$$\mathsf{join}(L, K) :: a = \mathsf{join}(L, K :: a) \text{ for all lists L and K}$$

We induct on L.

$$P(L) := \mathsf{join}(L, K) :: a = \mathsf{join}(L, K :: a)$$

BC: $L = \mathsf{nil}$

$$\begin{aligned} & \mathsf{join}(\mathsf{nil}, K) :: a \\ =& K :: a && (\mathsf{join}_1) \\ =& \mathsf{join}(\mathsf{nil}, K :: a) && (\mathsf{join}_1) \end{aligned}$$

IH: Assume $P(L')$ for some list L'
IS: WTS $P(b :: L')$

$$
\begin{aligned}
LHS &= \mathsf{join}(b :: L, K) :: a \\
&= (b :: \mathsf{join}(L, K)) :: a && (\mathsf{join}_2) \\
&= b :: (\mathsf{join}(L, K) :: a) && (\mathsf{app}_2) \\
RHS &= \mathsf{join}(b :: L, K :: a) \\
&= b :: \mathsf{join}(L, K :: a) && (\mathsf{join}_2) \\
&= b :: (\mathsf{join}(L, K) :: a) && \text{(IH)}
\end{aligned}
$$

Proof. We induct on K.

$$P(K) = \mathsf{join}(L, \mathsf{rev}(K)) = \mathsf{rev}(\mathsf{join}(K, \mathsf{rev}(L)))$$

BC: $K = \mathsf{nil}$

$$
\begin{aligned}
LHS &= \mathsf{join}(L, \mathsf{rev}(\mathsf{nil})) \\
&= \mathsf{join}(L, \mathsf{nil}) && (\mathsf{rev}_1) \\
&= L && \text{(we showed this in HW as a lemma)} \\
RHS &= \mathsf{rev}(\mathsf{join}(\mathsf{nil}, \mathsf{rev}(L))) \\
&= \mathsf{rev}(\mathsf{rev}(L)) \\
&= L && \text{(also proved in HW)} \\
LHS &= RHS
\end{aligned}
$$

IH: Assume $P(K')$ for some K'
IS: WTS $P(a :: K')$

$$
\begin{aligned}
RHS &= \mathsf{rev}(\mathsf{join}(a :: K', \mathsf{rev}(L))) \\
&= \mathsf{rev}(a :: \mathsf{join}(K', \mathsf{rev}(L))) && (\mathsf{join}_2) \\
&= \mathsf{rev}(\mathsf{join}(K', \mathsf{rev}(L))) :: a && (\mathsf{rev}_1) \\
&= \mathsf{join}(L, \mathsf{rev}(K')) :: a && \text{(IH)} \\
LHS &= \mathsf{join}(L, \mathsf{rev}(a :: K')) \\
&= \mathsf{join}(L, \mathsf{rev}(K') :: a) && (\mathsf{rev}_2) \\
&= \mathsf{join}(L, \mathsf{rev}(K')) :: a && \text{(lemma 1)}
\end{aligned}
$$

□