151/128 Counting Review

Ryan Ravitz

Tips and Tricks

Things You Should Know

- Know the difference between **permutations**, **arrangements**, and **selections**.
- **Multiplication Principle**
- **Addition Principle**
  - Know when you need to partition the set you are counting.
  - Be able to justify why your partition is mutually exclusive and exhaustive.
- Counting Techniques:
  - Stars and Bars
  - Counting Lattice Paths
  - Complementary Counting
- Feeling stuck? Know how to debug when you overcount or undercount (I like using the tuple technique).

How To Use This Packet

You may notice that all of these problems have multiple parts (scary!). When I took Concepts (a full year ago), I would get stuck on review problems and just look at the solution, which takes away a good opportunity to practice. If you start a problem and get stuck, don't be afraid to read through the solution for that part (these problems are hard!!!). Make sure you understand it (and ask for help if you don't!). Then, you can still attempt the other part(s) of the question and get some good practice.

Feel free to email me at rravitz@andrew.cmu.edu if you have any questions!

Problem 1: Addition

Consider the following equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 39$$

How many solutions $(x_1, x_2, x_3, x_4, x_5)$ are there if:

(a) $x_1, \ldots, x_5 \in \mathbb{N}$?

(b) $x_1, \ldots, x_5 \in \mathbb{N},\ x_2 = 3$?

(c) $x_1, \ldots, x_5 \in \mathbb{Z}^+$?

(d) $x_1, \ldots, x_5 \in \mathbb{Z},\ x_1, \ldots, x_5 \geq -5$?

(e) $x_1, \ldots, x_5 \in \mathbb{N},\ x_4 = 15x_2$

(f) *Application.* The House of Representatives has 435 seats divided between the 50 states. How many ways can we partition the seats between the states with at least 1 representative per state?

Problem 2: Blocks

You have 10 blocks: 1 red, 2 yellow, 3 green, and 4 blue.

(a) How many unique permutations of the blocks can you make?

(b) Count the number of permutations of the blocks such that the last red block comes before the last yellow block which comes before the last green block which comes before the last blue block.

(Ex: $\boxed{\textbf{Y}}$, $\boxed{\textbf{B}}$, $\boxed{\textbf{B}}$, $\boxed{\textbf{G}}$, $\boxed{\textbf{B}}$, $\boxed{\textbf{R}}$, $\boxed{\textbf{G}}$, $\boxed{\textbf{Y}}$, $\boxed{\textbf{G}}$, $\boxed{\textbf{B}}$)

Problem 3: Chess I

The game of chess is played on an 8x8 grid (for a total of 64 grid squares). One player has black pieces and the other has white pieces. Each player has 1 king, 1 queen, 2 rooks, 2 bishops, 2 knights, and 8 pawns.

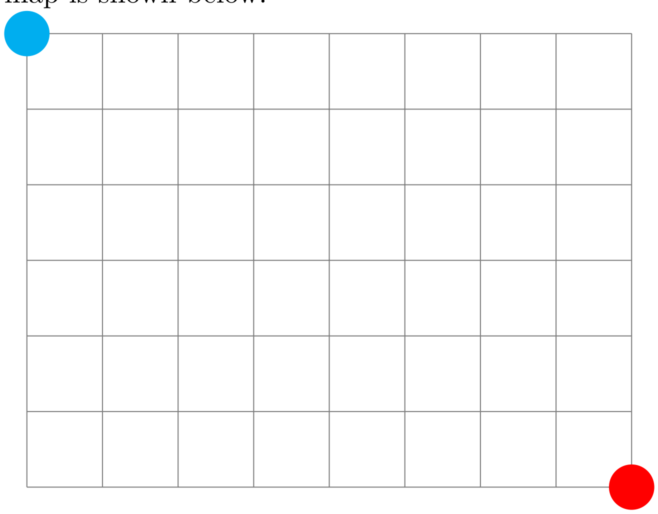
(a) How many arrangements of chess pieces are possible if both players place their pawns and king on the board?

(b) What if both players put all of their pieces on the board?

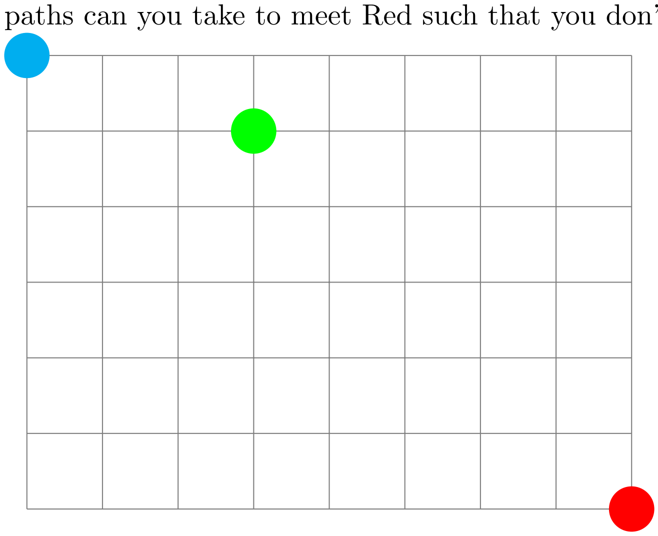
Problem 4: Among Us

You, Cyan, are in Cafeteria, and you want to meet Red in O2. Your controls are broken and you can only move right one unit or down one unit per move.

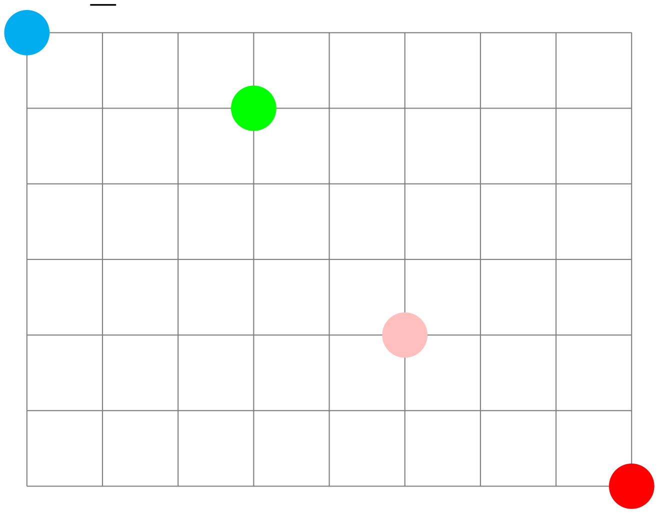
(a) How many unique paths can you take to meet Red? The (admittedly not very accurate) map is shown below:



(b) Oh no! You see Green is on the way to O2, and he's acting kinda sus. How many unique paths can you take to meet Red such that you don't end up on the same corner as Green?



(c) What?!? You see Pink is also on the way to O2, and she's definitely acting sus. How many unique paths can you take to meet Red such that you don't end up on the same corner as Green <u>or</u> Pink?



Problem 5: Counting Cards I

(a) How many 5 card hands can you form where each card has a unique rank?

Ex: {6♥, 5♣, J♣, 2♦, Q♠}

(b) How many permutations of 5 card hands can you form where each card has a unique rank?

Ex: (6♥, 5♣, J♣, 2♦, Q♠)

(c) How many permutations of 5 card hands are there such that the ranks of the cards are strictly increasing (let A = 1, J = 11, Q = 12, K = 13)?

Ex: (2♦, 5♣, 6♥, J♣, Q♠)

(d) How many permutations of 5 card hands are there such that the ranks of the cards are strictly increasing (let A = 1, J = 11, Q = 12, K = 13) and the ranks of the cards are all consecutive?

Ex: (2♦, 3♣, 4♥, 5♣, 6♠)

Problem 6: Counting Cards II

(a) How many 5 card hands can you form with exactly 2 Jacks?

Ex: {6♥, 5♣, J♣, J♦, Q♠}

(b) How many permutations of 5 card hands can you form with exactly 2 Jacks such that the permutation starts and ends with a Jack?

Ex: (J♣, 6♥, 5♣, Q♠, J♦)

(c) How many 5 card hands can you form with at least 2 Jacks?

Ex: {J♥, 5♣, J♣, J♦, Q♠}

(d) How many permutations of 5 card hands can you form with at least 2 Jacks such that the permutation starts and ends with a Jack?

Ex: (J♥, 5♣, J♣, Q♠, J♦)

Problem 7: I Before E

For this problem, consider strings of 4 uppercase Latin letters.

(a) How many strings can you form with exactly 1 "I" and exactly 1 "E"?

(Ex: "DELI")

(b) How many strings can you form with exactly 1 "I" and exactly 1 "E" where "I" comes before "E"?

(Ex: "BIKE")

(c) **Challenge.** As the old saying goes, *"I before E except after C"*. Count the number of strings with exactly 1 "I" and 1 "E" where "I" comes before "E" except if the "E" and "I" directly follow a "C".

(Ex: "CEIL")

Problem 8: Balls and Bins

You have 20 balls and 4 bins (the bins are labeled from 1 to 10, so they are distinct).

(a) If the balls are indistiguishable, how many ways are there to put exactly one ball in each bin?

(b) If the balls are all distinct (say, labeled from A to T), how many ways are there to put exactly one ball in each bin?

(c) If the balls are indistiguishable, how many ways are there to distribute all of the balls among the bins (no restriction on number of balls per bin)?

(d) Upon closer inspection, 6 of the balls are green and the rest are blue. If the green balls are indistinguishable from each other and the blue balls are indistinguishable from each other, how many ways are there to distribute all of the balls among the bins (no restriction on number of balls per bin)?

Problem 9: Concepts Strings

For this problem, we are going to consider strings of length 10. Characters in these strings can be digits (0-9) or uppercase Latin letters.

(a) Without any restrictions, how many such strings exist?

(Ex: "A1B2C3D4E5")

(b) How many of these strings contain "MACKEY" as a substring?

(Ex: "MACKEY4005")

(c) Let's define a *151-string* (sorry 128, the sum of your digits exceeds 10) as a string which contains a substring of the form ABBBBBA. How many of these strings are *151-strings*?

(Ex: "AB15555511")

(d) **Challenge.** If we change the length of the strings to 12, how many *151-strings* can we make?

(Ex: "RJR1555551JM")

Problem 10: Chess II

The game of chess is played on an 8x8 grid. You have 8 rooks of the same color.

(a) How many unique ways are there to place all of the rooks on a chess board?

(b) How many unique ways are there to place all of the rooks such that no 2 rooks share a row or column?

(c) Now one of the rooks is red and the other 7 are blue. How many unique ways are there to place all of the rooks such that the red rook does not share a row or column with any of the other rooks?