1. Prove that if x is an odd integer, then $8 \mid (x^2 - 1)$.

Solution. Let x be an odd integer. Then there is an integer k such that $x = 2k + 1$. Thus, $x^2 - 1 = (2k + 1)^2 - 1 = 4k(k + 1)$. If k is even, then $k = 2j$ for some integer j and $x^2 - 1 = 8j(2j + 1)$, else if k is odd, then $k = 2j + 1$ for some integer j and $x^2 - 1 = 8(j + 1)(2j + 1)$. In either of the possible cases, $8 \mid (x^2 - 1)$, as desired.

1. am (i) Prove that if x and y are natural numbers and $x^2 - y^2 = 1$, then $x = 1$ and $y = 0$.

Solution. Let x and y be natural numbers and $x^2 - y^2 = 1$. So $(x - y)(x + y) = 1$ which implies that both factors are 1 (in which case $x = 1$ and $y = 0$) or both factors are -1 (in which case $x = -1$ and $y = 0$). Since $x \in \mathbb{N}$ we conclude $x = 1$ and $y = 0$.

(ii) Find, with brief justification, all pairs of natural numbers x and y satisfying $x^2 - y^2 = 15$.

Solution. Let x and y be natural numbers and $x^2 - y^2 = 15$. So $(x - y)(x + y) = 15$ which implies that the factors are ± 1 and ± 15, or ± 3 and ± 5. Checking cases yields $x = 4$ and $y = 1$ or $x = 8$ and $y = 7$.

1. pm Prove that if x and y are odd integers, then $4 \mid (x^2 - y^2)$.

Solution. Let x and y be odd integers. Then there is an integer k such that $x = 2k + 1$ and an integer l such that $y = 2l + 1$. Thus, $x^2 - y^2 = 4(k - l)(k + l + 1)$ and $4 \mid (x^2 - y^2)$, as desired.

2. Supply proofs or counterexamples (with explanation) for each of the following statements:

(i) $\forall x \in \mathbb{R}^+ \ \exists y \in \mathbb{R} \ \ [(x = y^2) \ \wedge \ (y - |y| \neq 0)]$

Solution. This is true. Let $x \in \mathbb{R}^+$. Consider the real number $y = -\sqrt{x}$. Then $x = (-\sqrt{x})^2$ and $y - |y| = -\sqrt{x} - |-\sqrt{x}| = -2\sqrt{x} < 0$ (in particular $y - |y| \neq 0$).

(ii) $\forall x \in \mathbb{R} \ \exists y \in \mathbb{R} \ \ [x^2 - y^2 > 0]$

Solution. This is false. Consider $x = 0$. Let $y \in \mathbb{R}$. We have $x^2 - y^2 = 0^2 - y^2 \leq 0$. This shows that the negation of the original statement is true.

2. am Supply answers, with justification, for each of the following:

(i) Create a function $f : \{1, 2, 3, 4\} \to \{a, b, c\}$ such that $f[\{1, 2\} \cap \{2, 3\}] \neq f[\{1, 2\}] \cap f[\{2, 3\}]$.

Solution. There are many examples.

(ii) Explain why $(\forall x \in \mathbb{R} \setminus \{0\})\,(\exists! y \in \mathbb{R})\,(x^2y^2 = 1)$ is false.

Solution. The negation is true. Consider $x = 1$. Then both $y = 1$ and $y = -1$ make $x^2y^2 = 1$ true. That is to say, there is not a unique y when $x = 1$.

2. pm Supply proofs or counterexamples (with explanation) for each of the following statements:

(i) $\forall x \in \mathbb{R}^+ \ \exists y \in \mathbb{R} \ \ [(x - 2 = y^2) \ \wedge \ (y - |y| \neq 0)]$

Solution. The negation is true. Consider $x = 1$. Let $y \in \mathbb{R}$. We have $x - 2 = -1 \neq y^2$, so the negation of the predicate (via de Morgan's) is true.

(ii) $\forall x \in \mathbb{R} \ \exists! y \in \mathbb{R} \ \ [x^2 - y^2 \geq 0]$

Solution. The negation is true. Consider $x = 1$. Then for both $y = 1$ and $y = -1$ we have that $x^2 - y^2 \geq 0$.

3. Prove that for all sets A, B, and C

$$(A \cup B) \setminus C \ \subseteq \ [A \setminus (B \cup C)] \ \cup \ [(B \setminus (A \cap C)].$$

Solution. Let $x \in (A \cup B) \setminus C$. Then $x \in (A \cup B)$ and $x \notin C$.

Case 1) $x \in B$. Since $x \notin C$, we know that $x \notin A \cap C$. Thus $x \in B \setminus (A \cap C)$ and hence $x \in [A \setminus (B \cup C)] \ \cup \ [(B \setminus (A \cap C)]$.

Case 2) $x \notin B$. Since $x \notin C$, we know that $x \notin B \cup C$. Moreover, since $x \in A \cup B$ we have $x \in A$. Thus $x \in A \setminus (B \cup C)$ and hence $x \in [A \setminus (B \cup C)] \ \cup \ [(B \setminus (A \cap C)]$.

In either of the possible cases, the desired result holds.

3. pm Prove or disprove that for all sets A, B, and C

$$[A \setminus (B \cup C)] \ \cup \ [(B \setminus (A \cap C)] \ \subseteq \ (A \cup B) \setminus C.$$

Solution. The statement is false. Consider $B = C = \mathbb{R}$ and $A = \mathbb{R}^+$. The left hand side is the set of all non-positive real numbers and the right hand side is the empty set.

4. Let $f : \mathbb{A} \to \mathbb{B}$ be a function. Show that for all $S, T \subseteq \mathbb{B}$,

$$f^{-1}[S \cup T] \ = \ f^{-1}[S] \cup f^{-1}[T].$$

Solution. Let $S, T \subseteq \mathbb{B}$ and let $x \in U$. Then $x \in f^{-1}[S \cup T]$ iff $f(x) \in S \cup T$ iff $[f(x) \in S] \vee [f(x) \in T]$ iff $[x \in f^{-1}[S]] \vee [x \in f^{-1}[T]]$ iff $x \in f^{-1}[S] \cup f^{-1}[T]$.

4. am Let $f : \mathbb{A} \to \mathbb{B}$ be a function. Show that for all $T \subseteq \mathbb{A}$,

$$T \subseteq f^{-1}[f[T]].$$

Solution. Let $T \subseteq \mathbb{A}$ and $x \in T$. Then, by definition of image, $f(x) \in f[T]$. By the definition of preimage, $x \in f^{-1}[f[T]]$.

4. pm Let $f : \mathbb{A} \to \mathbb{B}$ be a function. Show that for all $T \subseteq \mathbb{B}$,

$$f[f^{-1}[T]] \subseteq T.$$

Solution. Let $T \subseteq \mathbb{B}$ and $y \in f[f^{-1}[T]]$. Then, by definition of image, there is $x \in f^{-1}[T]$ such that $f(x) = y$. By the definition of preimage, we have $f(x) \in T$. Since $f(x) = y$, we conclude that $y \in T$, as desired.

5. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Z} \\ 1 - x & \text{if } x \notin \mathbb{Z} \end{cases}$$

Determine, with proof, whether or not f is a bijection.

Solution. f is a bijection because f is its own two-sided inverse. We know that f, as given, is well-defined, so we only need to show that $f(f(x)) = x$ for all $x \in \mathbb{R}$. To that end, let $x \in \mathbb{R}$.

Case 1) $x \in \mathbb{Z}$. Then $f(f(x)) = f(x) = x$.

Case 2) $x \notin \mathbb{Z}$. Then $f(f(x)) = f(1-x) = 1 - (1-x) = x$. In the second to last equality we used the fact that $x \notin \mathbb{Z}$ implies $1 - x \notin \mathbb{Z}$.

In either of the possible cases, the desired result holds.

5. am Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ via

$$f(x, y) = \begin{cases} (x, y) & \text{if } x > 0 \\ (y, x) & \text{if } x \leq 0 \end{cases}$$

(i) Determine, with proof, whether f is an injection.

Solution. f is not an injection since $f(2, -1) = (2, -1) = f(-1, 2)$ but $(2, -1) \neq (-1, 2)$.

(ii) Determine, with proof, whether f is a surjection.

Solution. f is not a surjection since $(-1, 1)$ is not in the image of f. It can't be the image of a point with a positive first coordinate (since the image of such a point has positive first coordinate), and it can't be the image of point with non-positive first coordinate (since the images of points with non-positive first coordinates have non-positive second coordinates).

5. pm Verify that the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(x, y, z) = (x + y,\ 2y + z,\ z - x)$$

is a bijection.

Solution. Define a function $g : \mathbb{R}^3 \to \mathbb{R}^3$ via $g(x, y, z) = (2x - y + z,\ -x + y - z,\ 2x - y + 2z)$. g is well-defined since the component formulae return unique real numbers for all inputs. Moreover, $g(f(x, y, z)) = f(g(x, y, z)) = (x, y, z)$ for every $(x, y, z) \in \mathbb{R}^3$. (short computations should be shown)

Bonus. Assume that on the show Love Island, each contestant must always tell the truth or always lie. If I am watching the show and three contestants **A**, **B**, and **C** make the following statements, which ones (if any) should I believe? Briefly justify your answer.

A: "All three of us are liars."

B: "Exactly two of us are liars."

C: "**A** and **B** are both liars."

Solution. A can't be truthful, so there is at least one truthful among B and C. C thus can't be truthful, hence B is the sole truthful contestant.

Bonus. am Display (without proof) all functions $f : [2] \to [2]$ that satisfy $f(f(x)) = f(x) \quad \forall x \in [2]$.

Solution. The identity, together with the two constant functions.

Bonus. pm Display (without proof) all bijections $f : [4] \to [4]$ that satisfy $f(f(x)) = x \quad \forall x \in [4]$.

Solution. There are 10 altogether!