Maxwell's Ginormous Countability Review

Maxwell Jones

General Tips

To prove set S Countable
(main idea: show S is "smaller" than a countable set):

- Injection from set S to a countable set
- Surjection from a countable set to set S
- Countable sets that could be helpful:
  - $\mathbb{N}, \mathbb{Z}$
  - Finite cartesian product of countable sets
    - Example: $(a_1, a_2, ..., a_n)$ for fixed $n \in \mathbb{N}$, with all elements naturals/integers
  - Countable union of any countable sets
    - for instance $\bigcup_{i=0}^{\infty} \mathbb{N}^i$, with $\mathbb{N}^i$ being $\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$ i times. This set corresponds to the set of all tuples of naturals with finite size(not just a fixed size).
      (probably most useful/versatile countable set)

To prove set S Uncountable:
(main idea: show S "bigger" than an uncountable set)

- Injection from uncountable set to set S
- Surjection from S to an uncountable set
- **NEW IDEA:** Show S has a subset that is the powerset of an infinite set
  - for examples, look at alternate solutions to questions 1 and 3 of the sets questions.
- Uncountable sets that could be helpful:
  - $\mathbb{R}$
  - $\mathcal{P}(\mathbb{N})$
  - $\{0, 1\}^{\infty}$

Intuition for determining if a set is countable or uncountable/proving it
(PLEASE READ)

In order to determine whether a set is countable or uncountable, it is usually a good idea to see if you can represent each element in the set in a finite way. If you can, the set is probably countable, if you cannot,then it is probably uncountable.

If you have devised a way to represent each item in a finite way, probably inject to tuples of integers where each tuple stores a finite amount of information about your set. Remember - these tuples that you are injecting to don't have to be the same size, as long as they all have finite size. This is because you can inject your set to $\bigcup_{i=0}^{\infty} \mathbb{N}^i$, which contains tuples of naturals for any finite natural size.

They key thing for showing a set is uncountable is finding an infinite amount of DECISIONS for EVERY item in your set. Note that this is different from having an infinite amount of items(which is true for both countable and uncountable sets).

In this case, I usually suggest injecting from the set of infinite binary strings, where we map $b_0b_1b_2...$ to an element where each bit represents one decision about that element. Since we have an infinite amount of decisions per element, each binary string maps to a unique element.

Don't forget, injections are NOT bijections!!! If you are trying to prove something uncountable and injecting from the binary strings to some set, its OK if you dont map to every item in your set as long as each string maps to a unique item. Alternatively, if you are proving countability and injecting from your set to some countable set, its OK if you don't map to every item in the countable set as long as you have an injection.

PLEASE READ BEFORE DOING PROBLEMS

I thought that more problems would be better than less, so I have come up with a lot of different problems for you guys to do. I provide full solutions for some problems that I think are especially instructive, but for the rest I just have general outlines of how to solve them since writing up full solutions to all of them is not a good use of time.

Note: Unless I say something holds for an arbitrary k in the naturals, assume k is FIXED if I just write $k \in \mathbb{N}$.

For each problem, determine whether the set in question is countable or not, and prove your results.

* : easier than exam difficulty/exam difficulty
** : exam difficulty/slightly above exam difficulty
***: above exam difficulty
****: way above exam difficulty
*****: Just for fun, not for studying
Good luck studying!!

PROBLEMS

Binary Strings Countability

1. * Set of Binary strings with a finite amount of 1s per string
2. * Set of Binary strings with an infinite amount of 1s per string
3. * Set of binary string such that after some fixed point k, values alternate between 1 and 0. Example, let $k = 5$: 1011110101010101010...
4. ** Set of binary strings such that for an arbitrary string s in this set, $\forall k \in \mathbb{N}$, the number of 1's between position 10^k(inclusive) and 10^{k+1}(exclusive) of s is 1 (let the 0 position of every string in the set always be a 1)
5. ** Set of all Binary strings such that at some fixed point k and on, values alternate in the form $1^n 0^n$ for some n in the naturals. Example: let $k = 5, n = 4$. A string could be 100101111000011110000111110000...
6. *** Set of binary strings such that for an arbitrary string in our set$b_0 b_1 b_2 b_3 ...$, $b_{3x} b_{3x+1} b_{3x+2} = 110$ or $b_{3x} b_{3x+1} b_{3x+2} = 101$ or $b_{3x} b_{3x+1} b_{3x+2} = 111$ for all x in the naturals. Example: 110101111101110110110111...
7. *** Set of binary strings such that for all points n, the number of 1's before n is greater than or equal to the number of 0's before n (Get help from a previous question possibly)

Functions Countability

1. * the set of functions from $[k]$ to $\mathbb{N}$, $k \in \mathbb{N}$
2. * the set of functions from $[k]$ to $\mathbb{R}, k \in \mathbb{N}$
3. ** the set of functions $f : \mathbb{R} \to \mathbb{R}$ s.t. $f(x) \neq x$ a finite amount of times
4. ** the set of functions $f : \mathbb{R} \to \mathbb{R}$ s.t. the image of f is a subset of the naturals
5. ** the set of functions $f : \mathbb{N} \to \mathbb{N}$ s.t. $f(x) \neq x$ a finite amount of times
6. ** the set of functions $f : \mathbb{Z} \to [k], k \in \mathbb{N} \setminus \{0, 1\}$
7. ** the set of functions $f : \mathbb{N} \to \mathbb{Z}$ s.t. the image of f is a subset of the evens
8. ** the set of functions $f : \mathbb{N} \to \mathbb{N}$ s.t. the image of f is finite
9. *** the set of functions $f : \mathbb{Z} \to \mathbb{Z}$ s.t. $\forall x \in \mathbb{Z}, |f(x) - x| < k, k \in \mathbb{N}, k \geq 2$
10. *** the set of functions $f : [k] \to [0, 10]$(interval of the reals) s.t. for all $0 < i \leq k$, $\exists n \in \mathbb{N}$ s.t. $f(i)^n \in \mathbb{N}$
11. **** the set of functions $f : \mathbb{Z} \to \{1, 2, 4, 9, 16, 25, ...\}$ s.t. $\forall y \in \{1, 2, 4, 9, 16, ...\}, \exists x \in \mathbb{Z}$ with $f(x) = y$ and $x^2 = y$

Sets Countability

1. * Subset of the powerset of the naturals S such that all elements of S contain only evens

2. * Subset of the powerset of the naturals S such that all elements of S must be finite in size

3. ** Subset of the powerset of the naturals S such that for an abitrary element $A \in S$, $\forall x, y \in A, x|y \vee y|x$

4. *** Cardinality of an arbitrary partition of the naturals(disjoint and exhaustive, nothing empty)

5. *** Cardinality of all possible partitions of the naturals

6. **** Given fixed $k \in \mathbb{R}$, Subset of the Reals S such that $\forall x, y \in S\mathbb{N}, |x - y| > k$

For these, I'll just tell you they're both countable. You just have to show it.
These are for if you're bored and love math, not really for studying for the final.

1. ***** set of HALF periodic functions: functions $f : \mathbb{N} \to \mathbb{N}$ such that for some k, $f(x) = f(x + k)$ or $f(x) = f(x + 2k)$ for all $x \in \mathbb{N}$

2. *****some $S \subseteq \mathbb{R}$ such that for any arbitrary subset $S' \subseteq S$, S' has a minimum element

Solutions

Strings Countability

1. YES. inject to the set of all finite tuples via a string s maps to tuple $(a_1, a_2, ..., a_n)$ if a_1 is the first location of a 1, a_2 is the second location, and so on.

2. NO. inject from the binary strings to this set via $b_0b_1b_2b_3... \rightarrow b_01b_11b_21...$

3. YES. Map each string with k elements before the pattern starts to a k tuple containing those elements

4. NO. inject from the set of all binary strings to this set in the following way: map from string s to s' such that s_n is a 1 implies that the 10^nth spot of $s'$is a one, and $s_n = 0$ implies that the $10^n + 1$st spot is a one.

5. YES. inject from this set to tuples in the following way. Consider an abitrary string with k elements occuring before the pattern begins. map to a $k + 1$ tuple where the first k elements are the elements 1 through k of the binary string, and the $k + 1$st element is the n for which the binary string takes the pattern 1^n0^n

6. NO. inject from the set of all binary strings via b maps to b' if for all n, $b_n = 1$ if $b'_{3n}b'_{3n++1}b'_{3n+2} = 100$ and $b_n = 0$ if those three are 101

7. NO. Note that the previous set had this property, and it was uncountable, so we can inject from the previous set to this set via the identity.

Functions Countability

1. YES, inject from this set to the set of k tuples with natural elements
2. NO, surject from this set to the reals via $h(f) = f(1)$
3. NO. inject from the reals to this set via $f(x) =$ a function where all reals map to themselves, except x maps to $x+1$
4. NO, surject from this set to the powerset of the naturals. Let h : Our set $\rightarrow \mathcal{P}(\mathbb{N})$ via $h(f) = image(f)$
5. YES, inject from this set to the set of all finite k tuples of naturals by simply listing the elements that do not satisfy $f(x) = x$ in order with both x and $f(x)$. For instance, if x, y, and z did not map to their own item, then the tuple would be $((x, f(x)), (y, f(y)), (z, f(z)))$. Note we need to keep track of both what values do not map to themselves(x, y, z) and what these erroneous values map to $f(x), f(y), f(z)$
6. NO, inject from the set of binary strings via $f(b_0b_1b_2b_3...) =$a function where $f(n) = 1$if $b_n = 0$ and 2 if $b_n = 1$
7. NO, inject from the binary strings to this set via a string s maps to function f if for all n in the naturals, the nth element in s is a 1 if $2n$ is in the image, and a 0 otherwise
8. NO, inject from the binary strings to the set of functions via a string s maps to a function f if for all n, the nth position of the string is a 1 if $f(n) = 1$ and a 0 if $f(n) = 0$
9. NO, note that for all k greater than 2, every x can map to x-1, x+1, or something else. inject from the binary strings to this set via a string s maps to function f if for all n in the naturals, the nth element in s is a 1 if $f(n) = n - 1$, and a 0 if $f(n) = n + 1$
10. YES, inject to the set of tuples of the form $((n_1, f(1)^{n_1}), (n_2, f(2)^{n_2}), ..., (n_k, f(k)^{n_k}))$ where n_i is a natural such that $f(k)^{n_i}$ is also natural. This is really just a tuple of size $2k$, and the set of $2k$ tuples is countable
11. NO. inject from the set of binary strings to the set of these via a string s maps to a function f if for all non-negative x, $f(x) = x^2$, and for all negative x, if the $n$the position of the string is 0, then $f(-n)$ is 0, and if the nth position is 1, then $f(-n)$ is 1.

Sets Countability

1. NO. inject from the powerset of the naturals to this set via $f(S) = \{2x, x \in S\}$

    - **Alternate solution:** Note that the set $\{2x, x \in S\}$ is countably infinite via injection $f(x) = 2x$ from the naturals to this set. Since any subset of this set will be viable, $\mathcal{P}(\{2x, x \in S\}) \subseteq |S| \implies |\mathcal{P}(\{2x, x \in S\})| \leq |S| \implies |\mathcal{P}(\mathbb{N})| \leq |S| \implies S$ uncountable

2. YES. inject from this set to the set of finite tuples of naturals via $f(S) = (min(s), \text{second lowest} S, ...)$

3. NO. inject from the powerset of the naturals to this set via $f(S) = \{2^x, x \in S\}$. note that for any two items in one of these sets, one must divide the other. More details in full explanation.

    - **Alternate solution:** Note that the set $\{2^x, x \in S\}$ is countably infinite via injection $f(x) = 2^x$ from the naturals to this set. Since any subset of this set will be viable, $\mathcal{P}(\{2^x, x \in S\}) \subseteq |S| \implies |\mathcal{P}(\{2x, x \in S\})| \leq |S| \implies |\mathcal{P}(\mathbb{N})| \leq |S| \implies S$ uncountable

4. YES. let $f(S) = min(S)$ where min is the minimum element. This will be an injection to the naturals

5. NO. We can inject from the set of infinite binary strings to this set via $b_0b_1b_2...$ maps to the partition where elements that are 1 in the binary string are a minimum element in the partition, with sequential items in the same set. For example: $1001100101011... \rightarrow \{0, 1, 2\}, \{3\}, \{4, 5, 6\}, \{7, 8\}, \{9, 10\}, \{11\}, ...$

6. YES. let $f(x) = \lfloor \frac{x}{k} \rfloor$. Basically, were mapping all x between 0 and k to the same number, all x between k and 2k to the same number, etc... But wait, there can be at most one of these per interval of size k. So it must be injective.

Full Solution Functions #6

We will find an injection from an uncountable set to our set, thus showing that our set is uncountable. We choose the inifnite binary strings as this set.
Our mapping is as follows:

$$h(b) = f(\mathbb{Z} \to [k])(x) = \begin{cases} 1 & b_x = 0 \\ 2 & b_x = 1 \end{cases} \text{ where } b_x \text{ is the xth term in the string b}$$

Note that this function is well defined because it is trivially total and unique, and to prove the function exists note that the function f that gets outputted takes in an integer, and necessarily gives an element in $[k]$ since k must be at least 2 and our function returns 1 or 2 always.

Next we must show that this function is an injection. Consider we have two functions $f = h(b)$ and $f' = h(b')$ We must show that $f = f' \implies b = b'$. To do so, we consider an arbitrary index x. If we can show for an arbitrary index x, $b_x = b'_x$, then we have that for all indexes x, $b_x = b'_x$, indicating that $b = b'$.

note that $f = f' \implies f(x) = f'(x)$. However, if $f(x) = f'(x) = 1$, then $b_x = b'_x = 0$. If $f(x) = f'(x) = 2$, then $b_x = b'(x) = 1$. If $f(x) = f'(x) \neq 1, 2$, then we can conclude that these functions are not in the image of h, and thus are not a case we worry about(since h returns functions that only give 1 or 2).

We have now shown that any arbitrary index of b and b' must be the same, indicating that all indexes are the same. Since we now have that $f = f' \implies b = b'$, or $h(b) = h(b') \implies b = b'$,we have shown h injective. Since we have an injection from an uncountable set, our set must be uncountable as well.

Full Solution Strings #5

First we note that the set of natural k tuples for a fixed k is countable, and so the union of all k tuples from $k = 0$ to infinity for $k \in \mathbb{N}$ is the countable union of countable sets, and so it itself is countable. We now have that the total set of all k tuples for all naturals k is countable, so if we find an injection to this set, we will have proven our original set is countable.

We will construct an injection as follows:

$$h(b) = (b_1, b_2, ..., b_k, n_b)$$

where the first k digits are unrestricted and the pattern 1^n0^n holds for our string b with $n = n_b$. Basically, my finite tuples store information about the first k elements and also the n that is used for the rest of the string

To show well definedness, note that the function is total and unique trivially. Also note our function satisfies existence because we are mapping to a tuple with $k + 1$ elements, and we know that the k we have must be a natural as the problem statement says, implying that we have a $k + 1$ tuple which is in our set of all finite tuples. We also have that every item b_i is either 0 or 1 so it is a natural, and n_b is also a natural and exists by the problem statement, so the tuple is a finite size and contains only naturals.

Now to show that this is an injection. Consider $h(b) = h(b')$. We have that $(b_1, b_2, ..., b_k, n_b) = (b'_1, b'_2, ..., b'_k, n'_b)$. In order for tuples to be equal, we must have that all elements of the tuple are equal, and so we immediately have that the first k elements of the two strings are equal. Next we see that $n_b = n'_b$, so we have after the first k elements of b and b', they both follow the pattern $0^{n_b}1^{n_b}$ forever. This constrains the rest of their digits forever and thus $b = b'$. Since we have that $h(b) = h(b') \implies b = b'$, we have an injection. Since we have an injection to a countable set, our set must also be countable.

Full Solution Sets #3

We will find an injection from the powerset of the naturals to this set. Since the powerset of the naturals is uncountable, finding an injection to an uncountable set will show that our set is also uncountable.
We will define our function as follows:

$$h(S) = \{2^x, x \in S\}$$

We must show that this function is total. In order to do that, first note that the function trivially is total and unique. Next to show it exists. It is clearly a subset of the naturals for all sets S, since $x \in \mathbb{N} \implies 2^x \in \mathbb{N}$. Next to show that for any two arbitrary items, one divides the other. Note that if we have two of the same item, i.e. $2^x, 2^x$, note that $2^x * 1 = 2^x \implies 2^x|2^x$. Now consider two different items $2^x, 2^y$. Since the naturals are well ordered, WLOG assume $x > y$ which then implies that $x - y > 0$. Then we have that $2^y * 2^{(x-y)} = 2^x \implies 2^y|x$. This holds because $x - y > 0 \implies 2^{x-y} \in \mathbb{N}$

Next to show that this function is an injection. it suffices to show that $h(S) = h(S') \implies S = S'$. To do so, we will show that for any arbitrary element x in $h(S) and h(S')$, S and S' must share some element y with x only being produced by y. If this holds for any arbitrary element, then it will hold for all elements, and thus $S = S'$

Consider x in both $f(S)$ and $f(S')$. I claim that $\log_2 x$ must be in both S and S' and x can only be produced by $\log_2 x$. This holds since $2^{\log_2(x)} = x$ and the function 2^x is injective. We can show this since $2^x = 2^y \implies x = y$(this would be allowed to write on a test).

Since we have that for all items x in $f(S), f(S')$, they require both S and S' to share an element y, and that element x can only be produced by that y, we must have that all elements between S and S' are shared. Since $f(S) = f(S') \implies S = S'$, we have an injection.

The Final Two

The last two proofs would take too long to write out, so I didn't. If you think you know the answers, nice. Maybe email me after you're done studying for/taking finals if you want to talk about it.