FINITE SET REVIEW

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General Tips

- to show that a set A has size n for some natural n, you should find a bijection from A to $[n]$
- to show that sets A and B have the same size (denoted $|A| = |B|$), you should find a bijection between A and B
  - note: $[n]$ is defined to have size n ($|[n]| = n$)
- to show that $|A| \leq |B|$, we have two main approaches:
  - finding an **injection** from A to B
  - finding a **surjection** from B to A
- to show that set A is finite given a set B is finite, it suffices to show $|A| \leq |B|$
- given sets A and B is finite, it is proven in the textbook that $A \times B$ is finite and that it has cardinality $|A| * |B|$(proposition 7.1.22)

Practice Questions

1. Show that the number of injections from some sets A to B finite $\implies$ the number of bijections from sets A to B finite
2. let n be a fixed natural. Show that
   $|\{(x, y) \in [n] \times [n] \mid x < y\}| = |\{(x, y) \in [n] \times [n] \mid x > y\}|$
3. Let set $S \subset \mathbb{Z}$, and let set $Y \subset \mathbb{Z}$ have the property that $(\exists z \in \mathbb{Z})(\forall y \in Y)(y - z \in S)$
   Show S Finite $\implies$ Y Finite with size less than or equal to S
4. Given non-empty $A, B \subset \mathbb{N}$, let $A - B$ be the set $\{a - b | a \in A, b \in B, a - b \in \mathbb{N}^{+}\}$.
   Show A finite $\Leftrightarrow$ $A - B$ finite

Solutions

Problem 1

Proof 1

Note that all bijections are necessarily injective, so the set of bijections is a subset of the set of injections. Since we have that the set of bijections is a subset of a finite set, it too must be finite by definition.

Proof 2

We find an injection from the set of bijections from A to B to the set of injections from A to B via $f(x) = x$.

Note here that x is a function. This function is defined uniquely in the general case, so it is both total and unique.

To show existence, we need to argue that $x \in \text{domain} \implies f(x) \in \text{codomain}$. since $f(x) = x$, we are showing that if x is bijective, then x is injective. This is clearly true since all bijections are injections.

This function is clearly injective since $f(x) = f(y) \implies x = y$

Problem 2

To show the sets are the same size, we find a bijection between them. Let $A = \{(x, y) \in [n] \times [n] \mid x < y\}$ and let $B = \{(x, y) \in [n] \times [n] \mid x > y\}$. Define $f : A \to B$ via $f((x, y)) = (y, x)$. f is clearly total and has a unique output for each input. f satisfies existence because if $x < y$, then $y > x$. Also, see that f is a bijection because we have a two sided inverse $f' : B \to A$ via $f'((x, y)) = (y, x)$. f' satisfies existence because if $x > y$, then $y < x$. See that $\forall (x, y) \in [n] \times [n], f(f'(x, y)) = f((y, x)) = (x, y)$ and $f'(f(x, y)) = f'((y, x)) = (x, y)$ so that f' is a two sided inverse as desired. Thus, f is a bijection.

Problem 3

We can find an injection from Y to S via $f(y) = y - z$, with z being a fixed element such that $y - z \in S \forall y$.

Note that if $y \in Y$, then $y - z \in S$ by definition of S, so existence holds. Since we have defined all $f(x)$ uniquely and exactly once, totality and uniqueness hold.

Next, note that $y - z = y' - z \implies y = y'$, so the function is injective. Since we have an injection to a countable set(namely S), we have that Y is finite with size less than or equal to Z

Problem 4

A finite $\implies$ A-B finite

we will construct an injection from $A - B$ to a finite set.

Since A is finite, we have that A has a maximum element. We claim that we can inject $A - B \to [max(A)]$ via $f(x) = x$

Clearly this function is an injection($f(x) = f(y) \implies x = y$, so we just need to show that it is well defined.

It is unique and total since we defined one and only one value for every x, now we just need to show that it exists.

$$\begin{aligned} & a \leq \text{max(A)} \forall a \in A \\ & \implies a - b \leq max(A) \forall a \in A, \forall b \in B \\ x \in A - B & \implies x = a - b, a \in A, b \in B \wedge x \in \mathbb{N}^{+} \\ & \implies x \leq max(A) \wedge x \in \mathbb{N} \\ & \implies x \in [max(A)] \end{aligned}$$

We have a valid injection to a finite set, so $A - B$ is finite

A-B finite $\implies$ A finite

We will construct an injection from A to a finite set.

since $A - B$ is finite, we have that $A - B$ has a maximum element $max(A - B)$
Since B is a subset of the naturals, we have by well ordering principle of the naturals that B has a minimum element $\min(B)$(even if B is inifinte!)
We claim we have an injection from A to $[max(A - B) + min(B)]$ via f(x) = x Again, we see clearly that this is an injection, as f(x) = f(y) $\implies$ x = y, and also that it is total and unique. Now we just need to show that it exists.

The first thing we note is that for any a, if $a - min(B)$ is less than 0, then it must be the case that no matter what $b \in B$ we choose, $a - b$ is less than 0, as $mn(B) \leq b \forall b \in B$. We formalize this as follows:

$$
\begin{aligned}
&\forall a \in A((a - min(B) \in A - B) \vee (\forall b \in B, a - b < 0)) \\
&\text{(the second case of the or is when a is not represented in A-B)} \\
&a - min(B) \in A - B \\
\implies &a - min(B) \leq max(A - B) \\
\implies &a \leq max(A - B) + min(B) \\
&a \leq max(A - B) + min(B) \forall a \wedge a \in \mathbb{N}^{+} \\
\implies &a \in [max(A - B) + min(B)] \forall a \in A
\end{aligned}
$$

We have now shown existence, so we have that the function is well defined and injective. Since we have an injection to a finite set, we have that our set A is finite.