**Problem 1**

For positive real numbers $x, y, z$, prove the following inequality:

$$\frac{x+y}{x^2+y^2} + \frac{y+z}{y^2+z^2} + \frac{z+x}{z^2+x^2} \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

**Problem 2**

For positive real numbers $a, b, c$, assume that $a^2+b^2+c^2 = 3$. Prove the following inequality:

$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \leq \frac{9}{(a+b+c)^2}$$

*This question is pretty difficult; a hint is to try applying Cauchy-Schwarz in the denominator of the LHS.*

**Problem 1**

*Solution:* The first issue is that we have squares in our LHS. We can get rid of these by using the QM-AM inequality.

$$\sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2} \quad\Longrightarrow\quad x^2+y^2 \geq \frac{(x+y)^2}{2} \quad\Longrightarrow\quad \frac{1}{x^2+y^2} \leq \frac{2}{(x+y)^2}$$

Applying this logic to the denominators of all our fractions on the LHS of our original inequality, we have:

$$\begin{aligned}\frac{x+y}{x^2+y^2}+\frac{y+z}{y^2+z^2}+\frac{z+x}{z^2+x^2} &\leq \frac{2(x+y)}{(x+y)^2}+\frac{2(y+z)}{(y+x)^2}+\frac{2(z+x)}{(z+x)^2} \\ &= \frac{2}{x+y}+\frac{2}{y+z}+\frac{2}{z+x}\end{aligned} \tag{QM-AM}$$

We can see that the inequality now looks like the sum of the reciprocal of three arithmetic means. We basically have two choices here: AM-GM or AM-HM. Since there are no square roots in our final RHS, intuitively we should try AM-HM first.

$$\frac{x+y}{2} \geq \frac{2}{\frac{1}{x}+\frac{1}{y}} \quad\Longrightarrow\quad \frac{2}{x+y} \leq \frac{\frac{1}{x}+\frac{1}{y}}{2}$$

Again, we use the same logic on all three of our fractions.

$$\begin{aligned}\frac{2}{x+y}+\frac{2}{y+z}+\frac{2}{z+x} &\leq \frac{\frac{1}{x}+\frac{1}{y}}{2}+\frac{\frac{1}{y}+\frac{1}{z}}{2}+\frac{\frac{1}{z}+\frac{1}{x}}{2} \\ &= \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\end{aligned} \tag{AM-HM}$$

And we are done!

**Problem 2**

*Solution:* This problem is pretty difficult since the use of Cauchy Schwarz is a bit of a clever trick. We can start by rewriting our fractions in more suggestive way.

$$\begin{aligned}
&\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2} \\
&\qquad = \frac{1}{(a^2+1^2+1^2)}+\frac{1}{(1^2+b^2+1^2)}+\frac{1}{(1^2+1^2+c^2)} \\
&\qquad = \frac{1^2+b^2+c^2}{(a^2+1^2+1^2)(1^2+b^2+c^2)}+\frac{a^2+1^2+c^2}{(1^2+b^2+1^2)(a^2+1^2+c^2)}+\frac{a^2+b^2+1}{(1^2+1^2+c^2)(a^2+b^2+1)}
\end{aligned}$$

If we examine the first fraction's denominator, we can see that this is equal to $(\|u\|\|v\|)^2$ where $u = \langle a, 1, 1\rangle$ and $v = \langle 1, b, c\rangle$. Then by Cauchy-Schwarz, we know that $(\|u\|\|v\|)^2 \geq |u \cdot v|^2$, so

$$(a^2+1^2+1^2)(1^2+b^2+c^2) \geq (a+b+c)^2$$

Similar logic applies for the other fractions. Substituting, we get:

$$\begin{aligned}
\frac{1^2+b^2+c^2}{(a^2+1^2+1^2)(1^2+b^2+c^2)}&+\frac{a^2+1^2+c^2}{(1^2+b^2+1^2)(a^2+1^2+c^2)}+\frac{a^2+b^2+1}{(1^2+1^2+c^2)(a^2+b^2+1)} \\
&\leq \frac{1^2+b^2+c^2}{(a+b+c)^2}+\frac{a^2+1^2+c^2}{(a+b+c)^2}+\frac{a^2+b^2+1^2}{(a+b+c)^2} \\
&= \frac{2(a^2+b^2+c^2)+3}{(a+b+c)^2} \\
&= \frac{9}{(a+b+c)^2}
\end{aligned}$$

Definitely a tough question, but we are done!