21-128 and 15-151 problem sheet 3

Solutions to the following seven exercises and optional bonus problem are to be submitted through gradescope by 11PM on

Wednesday 21st September 2022.

Problem 1

Consider a function $f : A \to A$. Prove that if $f \circ f$ is injective, then f is injective.

Problem 2

Consider functions $f : A \to B$ and $g : B \to A$. Prove that

(a) If $f \circ g$ is the identity function on B, then f is surjective.

(b) If $g \circ f$ is the identity function on A, then f is injective.

To remind you: given a set X, the identity function on X is the function $\mathrm{id}_X : X \to X$ defined by $\mathrm{id}_X(x) = x$ for all $x \in X$.

Problem 3

Let $a, b, c, d \in \mathbb{R}$ be constants with $a \neq 0$ and $c \neq 0$. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ with $f(x) = ax+b$ and $g(x) = cx + d$ for all $x \in \mathbb{R}$. Prove that f and g are injective and surjective, but that the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h = g \circ f - f \circ g$ is neither injective nor surjective.

Problem 4

Verify that the function $f : (0, 1) \to \mathbb{R}$ defined by

$$f(x) = \frac{2x - 1}{2x(1 - x)} \quad \text{for all } x \in (0, 1)$$

is a bijection.

Problem 5

(a) Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ where f and g are surjective and let $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ s.t. $\forall\,(x, y) \in \mathbb{R} \times \mathbb{R},\ h(x, y) = f(x, y) + g(x, y)$. Determine, with proof, whether or not h is necessarily surjective.

(b) For all positive integers n, define S_n to be the set of bijections on $[n]$. Let

$$A = \{\, f \in S_n \mid \forall\, x \in [n]\, (f(x) \neq x) \,\}$$
$$B = \{\, f \in S_n \mid \exists\, g \in S_n\, \forall\, x \in [n]\, (f(g(x)) \neq g(x)) \,\}$$

Show that $A = B$.

Problem 6

(a) Find a function $f : \mathbb{N} \to \mathbb{N}$ that is surjective and not injective, and show that it has at least two right-inverses.

(b) Suppose that $f : A \to B$ is surjective but not injective. Prove that f has at least two right-inverses.

Problem 7

Let $g : \mathbb{R} \to \mathbb{R}$ and define $f : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ via $f(S) = \{g(x) \mid x \in S\}$.

a) Determine necessary and sufficient conditions on g to have

$$\forall\, A, B \subseteq \mathbb{R}\ \ f(A \cap B) = f(A) \cap f(B)$$

b) Show that if g is surjective, then f is surjective.

Bonus Problem - 2 points

Let $f : A \to B$ be a function.

(a) Prove that there exists a set X and functions $p : A \to X$ and $i : X \to B$, with p surjective and i injective, such that $f = i \circ p$.

(b) Prove that there exists a set Y and functions $j : A \to Y$ and $q : Y \to B$, with j injective and q surjective, such that $f = q \circ j$.