151/128 Counting Review [Solutions]

Ryan Ravitz

Tips and Tricks

Things You Should Know

- Know the difference between **permutations**, **arrangements**, and **selections**.
- **Multiplication Principle**
- **Addition Principle**
  - Know when you need to partition the set you are counting.
  - Be able to justify why your partition is mutually exclusive and exhaustive.
- Counting Techniques:
  - Stars and Bars
  - Counting Lattice Paths
  - Complementary Counting
- Feeling stuck? Know how to debug when you overcount or undercount (I like using the tuple technique).

How To Use This Packet

You may notice that all of these problems have multiple parts (scary!). When I took Concepts (a full year ago), I would get stuck on review problems and just look at the solution, which takes away a good opportunity to practice. If you start a problem and get stuck, don't be afraid to read through the solution for that part (these problems are hard!!!). Make sure you understand it (and ask for help if you don't!). Then, you can still attempt the other part(s) of the question and get some good practice.

Feel free to email me at rravitz@andrew.cmu.edu if you have any questions!

Problem 1: Addition

Consider the following equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 39$$

How many solutions $(x_1, x_2, x_3, x_4, x_5)$ are there if:

(a) $x_1, \ldots, x_5 \in \mathbb{N}$?

We can use stars and bars. We have 39 stars, and we want to split them between 5 variables, so there are $\binom{39+5-1}{5-1} = \binom{43}{4} = 123{,}410$ solutions.

(b) $x_1, \ldots, x_5 \in \mathbb{N},\ x_2 = 3$?

Note that this is like finding the number of solutions to $x_1 + x_3 + x_4 + x_5 = 36$, having 4 variables and a total of 39 - 3 = 36. By stars and bars, there are $\binom{36+4-1}{4-1} = \binom{39}{3} = 9{,}139$ solutions.

(c) $x_1, \ldots, x_5 \in \mathbb{Z}^+$?

We know each variable must be at least 1 if it is in $\mathbb{Z}^+$, so we remove 5 from 39 and give 1 to each term in the sum. So we've reduced this problem to one similar to (a): $(x_1 + 1) + (x_2 + 1) + (x_3 + 1) + (x_4 + 1) + (x_5 + 1) = 39 \implies x_1 + x_2 + x_3 + x_4 + x_5 = 34$, where $x_1, \ldots, x_5 \in \mathbb{N}$. By stars and bars, there are $\binom{34+5-1}{5-1} = \binom{38}{4} = 73{,}815$ solutions.

(d) $x_1, \ldots, x_5 \in \mathbb{Z},\ x_1, \ldots, x_5 \geq -5$?

Let's apply the technique from (c) and try to reduce this to a problem where $x_1, \ldots, x_5 \in \mathbb{N}$. If we add 5 to every term in the equation, the minimum value becomes 0, and thus $x_1, \ldots, x_5$ would be in $\mathbb{N}$. So we solve the following instead: $(x_1 + 5) + (x_2 + 5) + (x_3 + 5) + (x_4 + 5) + (x_5 + 5) = 39 + 5 \cdot 5 \implies x_1' + x_2' + x_3' + x_4' + x_5' = 64,\ x_i' = x_i + 5$. Now we have $x_1, \ldots, x_5 \in \mathbb{N}$, so we use stars and bars to get $\binom{64+5-1}{5-1} = \binom{68}{4} = 814{,}385$ solutions.

(e) $x_1, \ldots, x_5 \in \mathbb{N},\ x_4 = 15x_2$

Note that this is the same as solving $x_1 + 16x_2 + x_3 + x_5 = 39$. If $x_2 \in \mathbb{N}$, the only possible values for x_2 are 0, 1, and 2. We can use stars and bars for each case and invoke the Addition Principle to get our answer:

- $x_2 = 0.$ $\quad x_1 + 16(0) + x_3 + x_5 = 39 \implies x_1 + x_3 + x_5 = 39 \longleftarrow \binom{39+3-1}{3-1} = \binom{41}{2}$
- $x_2 = 1.$ $\quad x_1 + 16(1) + x_3 + x_5 = 39 \implies x_1 + x_3 + x_5 = 23 \longleftarrow \binom{23+3-1}{3-1} = \binom{25}{2}$
- $x_2 = 2.$ $\quad x_1 + 16(2) + x_3 + x_5 = 39 \implies x_1 + x_3 + x_5 = 7 \longleftarrow \binom{7+3-1}{3-1} = \binom{9}{2}$

By the Addition Principle, there are $\binom{41}{2} + \binom{25}{2} + \binom{9}{2} = 820 + 300 + 36 = 1,156$ solutions.

(f) *Application.* The House of Representatives has 435 seats divided between the 50 states. How many ways can we partition the seats between the states with at least 1 representative per state?

This problem is very similar to (c), because the value of each of our variables (number of representatives per state) must be at least 1. So we solve using the same technique: finding the number of solutions to $x_1 + \ldots + x_{50} = 435$, where $x_1, \ldots, x_{50} \in \mathbb{Z}^+$ is the same as finding the number of solutions to $(x_1+1) + \ldots + (x_{50}+1) = 435 \implies x_1 + \ldots + x_{50} = 385$, where $x_1, \ldots, x_{50} \in \mathbb{N}$. By stars and bars, there are $\binom{385+50-1}{50-1} = \binom{434}{49}$ ($> 10^{65}$, there are very many ways to do this!).

Problem 2: Blocks

You have 10 blocks: 1 red, 2 yellow, 3 green, and 4 blue.

(a) How many unique permutations of the blocks can you make?

- Technique 1. We can choose the locations of each color relative to one another using a k-step process:

  1. Choose the locations of the blue blocks. $\longleftarrow \binom{10}{4}$
  2. Since there are 10 - 4 = 6 blocks remaining, choose the locations of the green blocks relative to the spots which are open. $\longleftarrow \binom{10-4}{3} = \binom{6}{3}$
  3. Choose the locations of the yellow blocks. $\longleftarrow \binom{6-3}{2} = \binom{3}{2}$

  Since there is only 1 spot remaining and 1 red block, the location of the red block is predetermined by our other choices. By the Multiplication Principle, there are $\binom{10}{4}\binom{6}{3}\binom{3}{2} = 12,600$ unique permutations.

- Technique 2. We can start with *all* permutations of the blocks and divide to account for repeats to get the number of *unique* permutations. We use a k-step process here too:

  1. Find the total number of permutations. $\longleftarrow$ 10!
  2. Account for the repeated blue blocks through division. $\longleftarrow \frac{1}{4!}$
  3. Account for the repeated green blocks through division. $\longleftarrow \frac{1}{3!}$
  4. Account for the repeated yellow blocks through division. $\longleftarrow \frac{1}{2!}$

  By the Multiplication Principle, there are $\frac{10!}{4!3!2!} = 12,600$ unique permutations.

(b) Count the number of permutations of the blocks such that the last red block comes before the last yellow block which comes before the last green block which comes before the last blue block.

Ex: **Y**, **B**, **B**, **G**, **B**, **R**, **G**, **Y**, **G**, **B**

We can work backward starting with placing the blue blocks. Note that the last block **must** be blue because if anything comes after it, then the last blue block will come before the last block of a different color. After placing 1 blue block at the end of the permutation, there are 9 blocks left to be placed total and 3 blue blocks remaining, so there are $\binom{9}{3}$ ways to allocate the blue blocks. Next we move on to green. Note that there are 6 total blocks remaining and the last of these 6 **must** be green. By the same logic, there are 5 total blocks remaining and 2 green blocks, so there are $\binom{5}{2}$ ways to place them. Continuing on, there will be $\binom{2}{1}$ ways to place the yellow blocks and only 1 way to place the red block (is predetermined by placing the yellow, green, and blue blocks). Since this is a k-step process, we can use the Multiplication Principle to get $\binom{9}{3}\binom{5}{2}\binom{2}{1} = 1,680$ permutations.

Problem 3: Chess I

The game of chess is played on an 8x8 grid (for a total of 64 grid squares). One player has black pieces and the other has white pieces. Each player has 1 king, 1 queen, 2 rooks, 2 bishops, 2 knights, and 8 pawns.

(a) How many arrangements of chess pieces are possible if both players place their pawns and king on the board?

There are 64 grid squares on a chess board, so imagine they are labeled from 1 to 64. Each player will be placing 9 pieces (1 king and 8 pawns) for a total of 18 pieces. We can use a k-step process to count the number of arrangements:

1. Select the squares which will have chess pieces on them. $\longleftarrow \binom{64}{18}$
2. From the previously selected squares, select the squares that will have white chess pieces on them (note that the rest will be black). $\longleftarrow \binom{18}{9}$
3. Select a king from the 9 white pieces. Do the same for the black pieces. $\longleftarrow 9^2$

By the Multiplication Principle, there are $\binom{64}{18}\binom{18}{9} \cdot 9^2$ ($> 10^{22}$) such arrangements.

(b) What if both players put all of their pieces on the board?

This time, each player will be placing 16 pieces for a total of 32. Similar to (a), we can use a k-step process to count the number of arrangements:

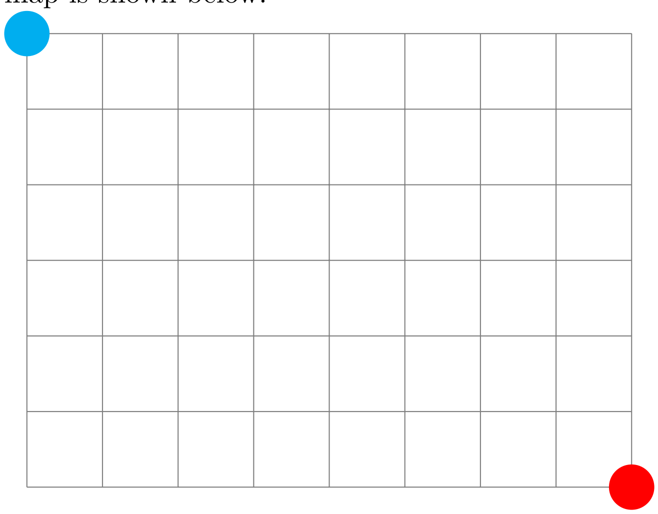
1. Select the squares which will have chess pieces on them. $\longleftarrow \binom{64}{32}$
2. From the previously selected squares, select the squares that will have white chess pieces on them (note that the rest will be black). $\longleftarrow \binom{32}{16}$
3. Select a king from the 16 white pieces. Do the same for the black pieces. $\longleftarrow 16^2$
4. Select a queen from the 16 - 1 = 15 remaining white pieces. Do the same for the black pieces. $\longleftarrow 15^2$
5. Select the rooks. $\longleftarrow \binom{14}{2}^2$
6. Select the bishops. $\longleftarrow \binom{12}{2}^2$
7. Select the knights (the remaining pieces will be pawns). $\longleftarrow \binom{10}{2}^2$

By the Multiplication Principle, there are $\binom{64}{32}\binom{32}{16} \cdot 16^2 \cdot 15^2 \cdot \binom{14}{2}^2\binom{12}{2}^2\binom{10}{2}^2$ ($> 10^{42}$) such arrangements.

Problem 4: Among Us

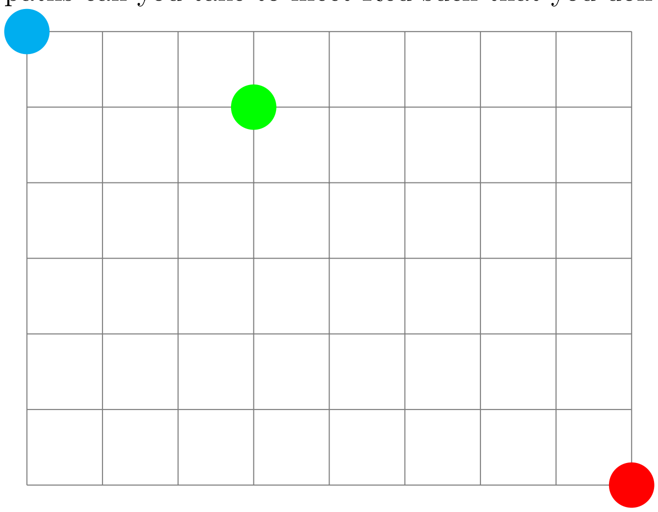
You, Cyan, are in Cafeteria, and you want to meet Red in O2. Your controls are broken and you can only move right one unit or down one unit per move.

(a) How many unique paths can you take to meet Red? The (admittedly not very accurate) map is shown below:



The lattice is 6 x 8. You have to move 6 + 8 = 14 units to get to red, and you need exactly 8 of them to be moves to the right (leaving 6 remaining down moves). Therefore, there are $\binom{6+8}{8} = \binom{14}{8} = 3003$ unique paths.

(b) Oh no! You see Green is on the way to O2, and he's acting kinda sus. How many unique paths can you take to meet Red such that you don't end up on the same corner as Green?

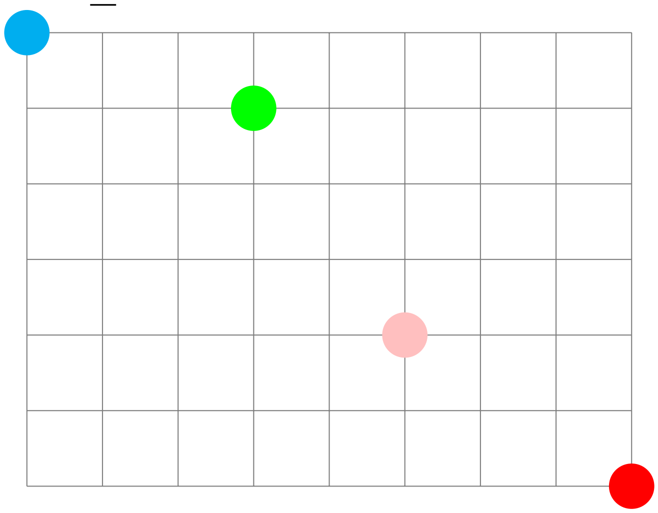


There are 3003 paths to Red, so we can use complementary counting to subtract out the paths that go through Green. But how many paths go through Green? We can view this as the number of unique paths from Cyan to Green and the number of unique paths from Green to Red. This is a k-step process:

1. Count the number of unique paths from Cyan to Green. This is a 1 x 3 lattice, so there are $\binom{1+3}{1} = 4$ ways to do this.
2. Count the number of unique paths from Green to Red. This is a 5 x 5 lattice, so there are $\binom{5+5}{5} = \binom{10}{5}$ ways to do this.

By the Multiplication Principle, there are $4 \cdot \binom{10}{5}$ unique paths to Red which go through Green. Using complementary counting, there are $3003 - 4 \cdot \binom{10}{5} = 3003 - 1008 = 1995$ unique paths to Red which do not go through Green.

(c) What?!? You see Pink is also on the way to O2, and she's definitely acting sus. How many unique paths can you take to meet Red such that you don't end up on the same corner as Green <u>or</u> Pink?



We can count the number of paths by computing the following: all paths to Red - paths going through Green - paths going through Pink + paths going through Green <u>and</u> Pink. This is because when we subtract out the "Green paths" and "Pink paths", we subtract the paths that go through both twice, so we have to add them back in to make sure they are only subtracted out once (see Inclusion/Exclusion Principle). We already know there are 3003 total paths and 1008 going through Green. Using the same technique as shown in (b), "Pink paths" can be counted by considering paths through a 4 x 5 lattice (Cyan to Pink) and a 2 x 3 lattice (Pink to Red). There are $\binom{4+5}{4}\binom{2+3}{2} = \binom{9}{4}\binom{5}{2}$ "Pink Paths". Finally, to count the paths that go through Green <u>and</u> Pink, we consider the number of unique paths that first go through a 1 x 3 lattice (Cyan to Green), then a 3 x 2 lattice (Green to Pink), and then a 2 x 3 lattice (Pink to Red, will also be the same count as Green to Pink). Therefore, there are $4 \cdot \binom{5}{2}^2$ paths that go through Green <u>and</u> Pink. Using the formula described at the beginning, there are $3003 - 1008 - \binom{9}{4}\binom{5}{2} + 4 \cdot \binom{5}{2}^2 = 3003 - 1008 - 1260 + 400 = 1135$ paths that do not go through Green <u>or</u> Pink.

Problem 5: Counting Cards I

(a) How many 5 card hands can you form where each card has a unique rank?

Ex: {6♥, 5♣, J♣, 2♦, Q♠}

We can use the following k-step process:

1. Select a rank for the first card through the fifth card. $\longleftarrow \binom{13}{5}$
2. Select a suit for the first card through the fifth card (in increasing rank order, say from A to K). $\longleftarrow 4^5$

By the Multiplication Principle there are $\binom{13}{5} \cdot 4^5 = 1,317,888$ such hands.

(b) How many permutations of 5 card hands can you form where each card has a unique rank?

Ex: (6♥, 5♣, J♣, 2♦, Q♠)

We just need to pick an order for each of the $1,317,888$ hands from (a). We can add a step to our k-step process where we select an order for the hand. There are $5!$ ways to do this. By the Multiplication Principle, there are $1,317,888 \cdot 5! = 158,146,560$ such permutations.

(c) How many permutations of 5 card hands are there such that the ranks of the cards are strictly increasing (let A = 1, J = 11, Q = 12, K = 13)?

Ex: (2♦, 5♣, 6♥, J♣, Q♠)

This is just going to be the same number as (a). For every 5 card hand where each card has a unique rank, there is only one way to order the cards such that the rank is strictly increasing, so there are 1,317,888 permutations.

For extra practice, since the sets described in (a) and (c) have the same cardinality, we can define a bijection $f : \_\_\_ \to \_\_\_$ that maps the set from (a) to the set from (c) via $f(S) = \text{sort}(S)$. What are the domain and codomain?

(d) How many permutations of 5 card hands are there such that the ranks of the cards are strictly increasing (let A = 1, J = 11, Q = 12, K = 13) and the ranks of the cards are all consecutive?

Ex: (2♦, 3♣, 4♥, 5♣, 6♠)

This process is very similar to the one from (a). However, once we pick the rank for the first card, the ranks for the following cards are predetermined, since they are consecutive. Furthermore, we can only pick cards from A to 9 as the first card (why?). We can use the following k-step process:

1. Select the rank for the first card. $\longleftarrow$ 9
2. Select a suit for the first card through the fifth card. $\longleftarrow$ 4^5

By the Multiplication Principle, there are $9 \cdot 4^5 = 9,216$ such permutations.

Problem 6: Counting Cards II

(a) How many 5 card hands can you form with exactly 2 Jacks?

Ex: {6♥, 5♣, J♣, J♦, Q♠}

This is a k-step process:

1. Select the suits for the Jacks. $\longleftarrow$ $\binom{4}{2}$
2. Select the remaining 3 cards. $\longleftarrow$ $\binom{52-4}{3} = \binom{48}{3}$

By the Multiplication Principle, there are $\binom{4}{2}\binom{48}{3} = 103,776$ such hands.

(b) How many permutations of 5 card hands can you form with exactly 2 Jacks such that the permutation starts and ends with a Jack?

Ex: (J♣, 6♥, 5♣, Q♠, J♦)

This is the same as (a) except that the order of the middle 3 cards matters and the order of the Jacks matters. We can add two steps to our k-step process where we select an order for the non-Jacks and an order for the Jacks. There are $3! \cdot 2!$ ways to do this. By the Multiplication Principle, there are $103,776 \cdot 3! \cdot 2! = 1,245,312$ such permutations.

(c) How many 5 card hands can you form with at least 2 Jacks?

Ex: {J♥, 5♣, J♣, J♦, Q♠}

We can partition this set into the 5 card hands with exactly 2 Jacks, 3 Jacks, and 4 Jacks. We already know there are 103,776 hands with 2 Jacks from (a). Now we have to compute the number of hands with 3 Jacks and 4 Jacks. The process is the same; select the suits for the Jacks and then select the remaining cards. By the Multiplication Principle, there are $\binom{4}{3}\binom{48}{2} = 4,512$ hands with exactly 3 Jacks and 48 hands with exactly 4 Jacks. By the Addition Principle, there are $103,776 + 4,512 + 48 = 108,336$ such hands.

(d) How many permutations of 5 card hands can you form with at least 2 Jacks such that the permutation starts and ends with a Jack?

Ex: (J♥, 5♣, J♣, Q♠, J♦)

You can solve this problem in a way similar to (c) where we use the Addition Principle, but it's actually not necessary here. In the cases with exactly 3 or 4 Jacks, we can consider the Jacks not placed at the beginning or end of the permutation to be a part of the remaining cards in the deck. So we have the following k-step process:

1. Select the suit for the first Jack. $\longleftarrow$ 4
2. Select the suit for the second Jack. $\longleftarrow$ 3
3. Select the second, third, and fourth cards. $\longleftarrow$ $50 \cdot 49 \cdot 48$

By the Multiplication Principle, there are $4 \cdot 3 \cdot 50 \cdot 49 \cdot 48 = 1,411,200$ such permutations.

Problem 7: I Before E

For this problem, consider strings of 4 uppercase Latin letters.

(a) How many strings can you form with exactly 1 “I” and exactly 1 “E”?

(Ex: “DELI”)

This is a k-step process:

1. Pick the location of the “I” $\longleftarrow$ 4
2. Pick the location of the “E” $\longleftarrow$ 3
3. Pick the letter (that is not “I” or “E”!) for the first remaining space then the second remaining space $\longleftarrow$ 24^2

By the Multiplication Principle, there are $4 \cdot 3 \cdot 24^2 = 6,912$ such strings.

(b) How many strings can you form with exactly 1 “I” and exactly 1 “E” where “I” comes before “E”?

(Ex: “BIKE”)

First of all, we know that regardless of the positions of the “I” and “E” we are going to have to pick letters for the other 2 blanks as we did in (a), and we know there are 24^2 ways to do this. Then, we can partition based on the location of the “I” in the string, and then use the Addition Principle. When the “I” is first, we can pick the second, third, or fourth slot to be the “E”, so we have 3 choices there. Similarly, when “I” is second we have 2 choices, when “I” is third we have 1 choice, and “I” cannot be last (why?). By the Addition Principle, there are $3 + 2 + 1 = 6$ ways to arrange the “I” and the “E”. Since we have a k-step process, we can use the Multiplication Principle, so there are $6 \cdot 24^2 = 3,456$ such strings.

Note that this is exactly half of the strings from (a)! In the other half of the strings with exactly one “I” and one “E”, the “E” comes before the “I”.

(c) **Challenge.** As the old saying goes, *"I before E except after C"*. Count the number of strings with exactly 1 "I" and 1 "E" where "I" comes before "E" except if the "E" and "I" directly follow a "C".

(Ex: "CEIL")

This looks tricky, but we will actually get the same count as the one from (b)! Consider the string "CIEL" which violates our new rule that "I" comes before "E" except after "C". We can switch the "E" and the "I" in the string to make it comply with the new rules (here we get "CEIL"). Therefore, there are 3,456 such strings.

For extra practice, try to define a bijection between the sets described in (b) and (c). Hint: your function should take in a string *abcd*, and you may want it to be piecewise to account for the new rule.

Problem 8: Balls and Bins

You have 20 balls and 4 bins (the bins are labeled from 1 to 4, so they are distinct).

(a) If the balls are indistiguishable, how many ways are there to put exactly one ball in each bin?

The balls are indistinguishable, so there is only 1 way to do this.

(b) If the balls are all distinct (say, labeled from A to T), how many ways are there to put exactly one ball in each bin?

This is a k-step process, pick the first ball, then the second ball, then the third ball, then the fourth ball. By the Multiplication Principle, there are $20 \cdot 19 \cdot 18 \cdot 17 = 116,280$ ways to do this.

(c) If the balls are indistiguishable, how many ways are there to distribute all of the balls among the bins (no restriction on number of balls per bin)?

We can use stars and bars here. Therefore, there are $\binom{20+4-1}{4-1} = \binom{23}{3} = 1,771$ ways to do this.

(d) Upon closer inspection, 6 of the balls are green and the rest are blue. If the green balls are indistinguishable from each other and the blue balls are indistinguishable from each other, how many ways are there to distribute all of the balls among the bins (no restriction on number of balls per bin)?

We can view this as a k-step process where we first distribute the green balls and then we distribute the blue balls. As we did in (c), we can use stars and bars for each step. There are $\binom{6+4-1}{4-1} = \binom{9}{3}$ ways to distribute the green balls and $\binom{14+4-1}{4-1} = \binom{17}{3}$ ways to distribute the blue balls. By the Multiplication Principle, there are $\binom{9}{3}\binom{17}{3} = 57,120$ ways to do this.

Problem 9: Concepts Strings

For this problem, we are going to consider strings of length 10. Characters in these strings can be digits (0-9) or uppercase Latin letters.

(a) Without any restrictions, how many such strings exist?

(Ex: "A1B2C3D4E5")

Note that there are a total of 36 characters in our alphabet here (10 digits + 26 uppercase Latin letters). This is a k-step process where we pick a character for each of the 10 spaces in order. By the Multiplication Principle, there are 36^{10} ($> 10^{15}$) such strings.

(b) How many of these strings contain "MACKEY" as a substring?

(Ex: "MACKEY4005")

We can view this instead as picking a string of length 5 where one of the characters is set as "MACKEY" (in our example, that would look like "[MACKEY]4005"). Then we can define a k-step process where we first pick where "MACKEY" will go (there are 5 ways to do this), and then we choose the characters for the remaining 4 spots in order (there are 36^4 ways to do this). By the Multiplication Principle, there are $5 \cdot 36^4 = 8,398,080$ such strings.

(c) Let's define a *151-string* (sorry 128, the sum of your digits exceeds 10) as a string which contains a substring of the form ABBBBBA. How many of these strings are *151-strings*?

(Ex: "AB15555511")

Outline: first we can count the number of possible substrings of the form ABBBBBA and then we can treat it as a character in a 4 character string like we did in (b) to finish constructing the *151-string*. This is a k-step process:

1. Pick the character that goes at the beginning and end of the substring. (A) $\longleftarrow$ 36
2. Pick the character that goes in the middle of the substring. (B) $\longleftarrow$ 35
3. Place the substring into our *151-string*. $\longleftarrow$ 4
4. Pick the remaining 3 characters in order. $\longleftarrow$ 36^3

By the Multiplication Principle, there are $36 \cdot 35 \cdot 4 \cdot 36^3 = 235,146,240$ *151-strings*.

(d) **Challenge.** If we change the length of the strings to 12, how many *151-strings* can we make?

(Ex: "RJR1555551JM")

This is where things start to get tricky. Consider the following string "155555111115". This is a valid *151-string*, but it contains 2 substrings of the form ABBBBBA ("**1555551**11115" and "15555**5111115**"). This means that *151-strings* of the form ABBBBBAAAAAB get counted twice, so we need to make sure to account for this. There are $36 \cdot 35$ strings of this variety, because you have to pick A and B. Therefore, using the exact same process as was used in (c) there are $36 \cdot 35 \cdot 6 \cdot 36^5 - 36 \cdot 35 = 457,124,289,300$ strings.

Problem 10: Chess II

The game of chess is played on an 8x8 grid. You have 8 rooks of the same color.

(a) How many unique ways are there to place all of the rooks on a chess board?

We choose 8 of the 64 squares to have rooks on them, so there are $\binom{64}{8} = 4,426,165,368$ ways to do this.

(b) How many unique ways are there to place all of the rooks such that no 2 rooks share a row or column?

This is admittedly a little tricky. Basically, if you consider the coordinate (x, y) of a given rook, no other rook should be in column x or row y. So since we know that there will be one rook in each column, we can just select the row for each rook in order from column 1 to column 8. Therefore, there are $8! = 40{,}320$ ways to do this.

(c) Now one of the rooks is red and the other 7 are blue. How many unique ways are there to place all of the rooks such that the red rook does not share a row or column with any of the other rooks?

Another tricky one. Consider the following k-step process. First, we place the red rook on the board, and there are 64 ways to do this. Then we know we can't place any blue rooks in the same row or column as the red rook. We can effectively "remove" that row and column, and this leaves us with a 7 x 7 grid (if you don't believe me, try drawing it out on paper). Then we just place the remaining 7 rooks on the 7 x 7 grid, and there are $\binom{49}{7}$ ways to do that. By the Multiplication Principle, there are $64 \cdot \binom{49}{7} = 5,497,637,376$ ways to do this.