21-128/15-151 final practice questions

Clive Newstead

Question 1 — logic, sets and induction

(a) Let A, B, X be sets and suppose that $A \subseteq X$ and $B \subseteq X$. Prove that $A \subseteq B$ if and only if $X \setminus B \subseteq X \setminus A$.

(b) Let (f_n) be the Fibonacci sequence, defined recursively by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. Prove, for all $n \in \mathbb{N}$, that f_n is divisible by 4 if and only if n is divisible by 6.

Question 2 — number theory

(a) Find all integers x satisfying the congruence $385x \equiv 21 \bmod 588$.

(b) Let a, b, c, d be positive integers and suppose that a and c are coprime, that b and d are coprime, and that $ab = cd$. Prove that $a = d$ and $b = c$.

(c) Find the remainder of $76! \cdot 2^{76}$ when divided by 79.

Question 3 — functions and countability

Let A be a set of real numbers with the property that $a - b \in \mathbb{Q}$ for all $a, b \in A$. Prove that A is countable.

Question 4 — counting principles

(a) Find the number of 5-card poker hands from a regular 52-card deck which contain exactly two suits and no repeated ranks.

(b) By defining a finite set and computing its size in two ways, prove that

$$\sum_{i=1}^{n} \sum_{j=1}^{i} j = \binom{n+2}{3}$$

Question 5 — inequalities

(a) Let $x, y > 0$. Define the *harmonic mean* and the *geometric mean* of x and y.

(b) Let $x, y > 0$. Prove that the harmonic mean of x and y is less than or equal to the geometric mean of x and y, and state (without proof) when equality holds.

(c) Let $a, b, c, d \in \mathbb{R}$ with $a > c \geq 0$ and $b > d \geq 0$. Prove that

$$\left(\frac{(a+b+c+d)(a+b-c-d)}{a+b}\right)^2 \leq (a+b)^2 - (c+d)^2$$

and determine when equality holds.

Question 6 — relations

(a) Let X be a set and let R be a reflexive, transitive relation on X. Define a new relation $\sim$ on X by letting

$$x \sim y \quad \Leftrightarrow \quad x \mathrel{R} y \text{ and } y \mathrel{R} x$$

for all $x, y \in X$. Prove that $\sim$ is an equivalence relation on X.

(b) Describe the equivalence classes of $\sim$ when $X = \mathbb{Z}$ and R is the divisibility relation on $\mathbb{Z}$; that is, R is defined for $x, y \in \mathbb{Z}$ by letting $x \mathrel{R} y$ if and only if x divides y.

Question 7 — probability

You roll a fair six-sided die twice. Each time you roll, you gain an amount of money in dollars equal to the number shown on the second roll minus the number shown on the first roll. (For example, if the die shows 3 on the first roll and 5 on the second roll, then you win \$2; but if the first die shows 6 and the second die shows 1, then you lose \$5.) Find the probability that the first roll showed 5 given that you lost money overall.

Question 8 — probability

There are a hundred quarters in a bowl, of which twenty commemorate the British Surrender in 1777. Clive wishes to remove all these quarters from circulation, and begins drawing quarters from the bowl. With each draw, if the coin drawn commemorates the British Surrender, he puts it in his pocket; otherwise, he returns it to the bowl. He stops when all twenty British Surrender quarters are in his pocket. Find the expected number of times that Clive draws a quarter from the box.