Solution 1. The successive divisions are $252 = 198 \cdot 1 + 54$, $198 = 54 \cdot 3 + 36$, $54 = 36 \cdot 1 + 18$, and $36 = 18 \cdot 2 + 0$, showing that the greatest common divisor is 18. Back substitution of non-zero remainders yields $18 = 252 \cdot 4 + 198 \cdot (-5)$.

Every solution has the form $x = 4 + 11k$ and $y = -5 - 14k$, for some $k \in \mathbb{Z}$.

Solution 2. (i) We prove the contrapositive and assume that 3 doesn't divide both x and y. Without loss of generality, 3 doesn't divide x. This yields 6 possibilities: $x = 3k + 1$ and $y = 3j$, $x = 3k + 1$ and $y = 3j + 1$, $x = 3k + 1$ and $y = 3j + 2$, $x = 3k + 2$ and $y = 3j$, $x = 3k + 2$ and $y = 3j + 1$, $x = 3k + 2$ and $y = 3j + 2$, for some integers k and j. For each of these possibilities, respectively, we see that $x^2 + y^2$ leaves remainders of 1, 2, 2, 1, 2, and 2. Note: students should display the computations of these remainders, explicitly, for at least one case. For example, $(3k + 1)^2 + (3j)^2 = 3(3k^2 + 2k + 3j^2) + 1$.

(ii) Assume that p^2 divides $(x - 1)(x + 1)$. p can not divide both $x - 1$ and $x + 1$, since it would then divide their difference, ie it would divide 2. Since p is an odd prime, this is impossible. It follows that p^2 is coprime with $x - 1$ or $x + 1$. Using the proposition which states $a \mid bc$ and a, b coprime implies that a divides c, we deduce that p^2 divides $x - 1$ or p^2 divides $x + 1$, as desired.

Solution 3. (i) By FLT, $3^{104} \equiv (3^{52})^2 \equiv 1^2 \equiv 1 \mod 53$. Thus, $18 \cdot 3^{104} \equiv 3^{103} \equiv 18 \mod 53$. Hence the remainder is 18.

(ii) $\gcd((p-1)! + 1, p!) = \gcd((p-1)! + 1, p! - p((p-1)! + 1)) = \gcd((p-1)! + 1, -p)$. So the gcd is at most p, but since by Wilson's theorem p divides $(p-1)! + 1$, we see that the gcd is p.

(iii) 72 divides the number iff both 8 and 9 divide the number iff $(8 \mid 300 + 10a + b$ and $9 \mid 2 + a + b)$. The second condition means that $a + b = 7$ or $a + b = 16$, thus $8 \mid 300 + 9a + 7$ or $8 \mid 300 + 9a + 16$, ie $8 \mid a + 3$ or $8 \mid a + 4$. Thus, $(a = 5$ and $a + b = 7)$ or $(a = 4$ and $a + b = 16)$. The second condition is impossible, and the first condition yields $a = 5$ and $b = 2$.

Solution 4. Let $p(n) :=$ "$\sum_{i=1}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$".

- (**BC**) $p(1)$ is true, since

$$\sum_{i=1}^{1} \frac{i}{2^i} = \frac{1}{2} = 2 - \frac{1+2}{2^1}.$$

- (**IS**) Fix $n \geq 1$ and suppose that $p(n)$ is true. We want to show

$$\sum_{i=1}^{n+1} \frac{i}{2^i} = 2 - \frac{n+3}{2^{n+1}}.$$

This is true, since:

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{i}{2^i} &= \sum_{i=1}^{n} \frac{i}{2^i} + \frac{n+1}{2^{n+1}} && \text{by definition of summation} \\
&= 2 - \frac{n+2}{2^n} + \frac{n+1}{2^{n+1}} && \text{by } p(n) \text{ (i.e. the IH)} \\
&= 2 - \frac{2(n+2) - (n+1)}{2^{n+1}} && \text{combining fractions} \\
&= 2 - \frac{n+3}{2^{n+1}}.
\end{aligned}$$

This completes the induction step.

Solution 5. We show by induction that $\forall n \in \mathbb{N}, 0 \sim n$.

Let $P(n) :=$ "$0 \sim n$".

<u>Base Case</u>: Consider $n = 0$:

We know by the reflexivity of $\sim$ that $0 \sim 0$. Thus, $P(0)$ holds.

<u>Induction Step</u>:

Suppose $\forall 0 \leq m \leq n$, $P(m)$ holds. We want to show $P(n+1)$ holds.

We know from the condition above that $\exists k \in \mathbb{N}$ with $2k \sim n+1$ and $2k < n+1$.

We know by our induction hypothesis that $P(2k)$ holds (since $2k < n+1$).

Thus $0 \sim 2k$. Also, $2k \sim n+1$. By transitivity, we have that $0 \sim n+1$.

Hence $P(n+1)$ holds, and so $0 \sim n+1$.

By SPMI, we then have that $\forall n \in \mathbb{N}, 0 \sim n$.

Bonus Solution. Let p be a prime and assume that $p = ab$ for some integers a, b. Since p is prime, p must divide a or b. Without loss of generality, $p \mid a$. Hence there is an integer k such that $a = pk$. Substitution yields $p = pkb$. Since $b \neq 0$, this means that $1 = kb$ and we conclude that b is a unit, as desired.