Solution 1. Note that the inequality holds if $x = y$. If $x \neq y$ we can divide both sides by $(x-y)^2$ to obtain equivalent inequalities. This yields $(x+y)^2 \geq 4xy$ which is equivalent to $(x-y)^2 \geq 0$ for the 9am exam inequality. It yields $(x+y)^2 \geq (x-y)^2$ which is equivalent to $xy \geq 0$ for the inequality on the other exams.

Solution 2.

(i) To show that A finite implies B finite, we define an injection f from B to A via $f(x) = \sqrt{x}$, noting that everything in B is positive.

(ii) B is finite, so we know that there exists a bijection f from B to $[n]$. We define a bijection g from A to $[2n]$ as follows:

$$g(x) = \begin{cases} f(x^2) & x < 0 \\ f(x^2) + n & x > 0 \end{cases}$$

From here we note that g is well-defined, injective and surjective, remembering that by definition of A, if we have a positive element like $i \in A$, then we must also have $-i \in A$.

Solution 2. This is very similar to the $A + B$ question from the homework. For the forward direction we can either define a surjection from 2 tuples to products or note that the RHS is a subset of $[max(X)max(Y)]$.

For the reverse direction, suppose that we want to show X to be finite. We pick an arbitrary element t from Y, and define an injection f from X to the RHS via $f(x) = tx$ for every $x \in X$. Showing Y to be finite is similar.

Solution 3. We partition as follows:

- $\{$heart, black$\}$: $\binom{13}{1}\binom{26}{1}$
- $\{$heart, diamond : $\binom{13}{1}\binom{13}{1}\}$
- $\{$diamond, diamond : $\binom{13}{2}\}$
- $\{$diamond, black : $\binom{13}{1}\binom{26}{1}\}$

The solution is the sum of these:

$\binom{13}{1}\binom{26}{1} + \binom{13}{1}\binom{13}{1} + \binom{13}{2} + \binom{13}{1}\binom{26}{1}$.

Solution 3. We use complementary counting here, and let the solution be the total number of hands minus the number of hands that have no spades and no hearts:

$\binom{52}{5} - |\text{hands with no spades or no hearts}|$

We now use $|A \cup B| = |A| + |B| - |A \cap B|$ to arrive at:
$|\text{hands with no spades or no hearts}| = |\text{no spades}| + |\text{no hearts}| - |\text{no spades and no hearts}| = \binom{39}{5} + \binom{39}{5} - \binom{26}{5}$

So our solution is:

$\binom{52}{5} - [\binom{39}{5} + \binom{39}{5} - \binom{26}{5}]$

Solution 4. Such functions g are the result of the following 2 step process:

- pick a value $x \in [n]$ that will be the unique x such that $f(x) = g(x)$: There are n ways to do this.
- pick values for the $n-1$ other elements in the domain of g, noting that these must come from $[n] \setminus \{g(x)\}$: There are $(n-1)^{n-1}$ ways to do this (this second step is an n-1 step process with n-1 possible choices at each step).

Thus, the number of such functions is $n * (n-1)^{n-1}$.

Solution 5. From a set of n people, choose a supervisor and a person or two (not the supervisor) to complete distinct tasks A and B.

This can be done with the three step process of first picking the supervisor, then picking the person for task A and then picking the person for task B. There are $n(n-1)(n-1)$ ways to do this.

Alternatively, such work groups can be partitioned into two kinds based on whether one person completes tasks A and B, or whether distinct people complete tasks A and B. The number of work groups of the first kind is $\binom{n}{2} * 2$ via the 2 step process of first picking the group of 2 people and then deciding which will be the supervisor and which will do the tasks. The number of work groups of the second kind is $\binom{n}{3} * 3 * 2 * 1$ via the 4 step process of picking the group of 3 people, picking the supervisor, picking the person to do task A and then picking the person to do task B.

Invoking the addition principle completes the problem.

Solution 5. Let S be the set of teams of size at least 2 from a group of n, with 2 co-presidents. Let S_i be the set of teams with i members. Note that $\{S_2, \ldots, S_n\}$ is a partition of S.

$|S| = \binom{n}{2}2^{n-2}$ since teams can be chosen with the 2 step process of first choosing the co-presidents and then choosing a subset of the remaining $n-2$ people to complete the team.

$S_i = \binom{n}{i}\binom{i}{2}$ since teams with i members can be formed with the 2 step process of first picking the i team members from the n people, and then choosing 2 people from among the i to be co-presidents.

Invoking the addition principle completes the problem.

Bonus. Seven choose three, ie 35.