21-128 and 15-151 problem sheet 6

Solutions to the following two exercises and optional bonus problem are to be submitted through gradescope.

Problem 1

Rose, Emily, Susan, Micah, and Harrison are sailors on a tropical island. They spend the day gathering a pile of coconuts. Exhausted, they postpone dividing it until the next morning. Suspicious, each decides to take their share during the night. Rose divides the pile into five equal portions plus one extra coconut, which she gives to a monkey. She takes one pile and leaves the rest in a single pile. Emily later does the same; again the monkey receives one leftover coconut. Susan, Micah, and Harrison all do this; each time, a remainder of one goes to the monkey. In the morning, they split the remaining coconuts into five equal piles, and each sailor gets their "share". (Each knows some were taken, but none complains, since each is guilty!) What is the smallest number of coconuts in the original pile?

Hint: The recursive calculations of the number of coconuts remaining are substantially simplified by assuming that there are $n-4$ coconuts initially.

Solution. Let $n-4$ be the total number of coconuts. The number of coconuts remaining after the first sailor is $(4/5)n-4$, the number of coconuts remaining after the second sailor is $(4/5)^2n-4$, the number of coconuts remaining after the third sailor is $(4/5)^3n-4$, the number of coconuts remaining after the fourth sailor is $(4/5)^4n-4$, and the number of coconuts remaining after the fifth sailor is $(4/5)^5n-4$.

Since 4 and 5 are relatively prime, n must be divisible by $5^5 = 3125$. Hence $n \geq 5^5$ and $n-4 \geq 3121$. An initial number of 3121 coconuts meets the conditions of the problem, since after the sailors take their shares, there will be 2496, 1996, 1596, 1276 and 1020 coconuts remaining, and 1020 is evenly divisible by 5. Thus, the minimum initial number of coconuts meeting the conditions of the problem is 3121.

Problem 2

Let p be a prime number greater than 2. Prove that $\frac{2}{p}$ can be expressed in exactly one way in the form $\frac{1}{m}+\frac{1}{n}$ where m and n are positive integers with $n > m$.

Hint: For proving uniqueness, try to get the equation in a form where you can case on whether $p \mid n$ or $p \mid m$.

Solution. Let p be a prime number greater than 2 and assume that $\frac{2}{p} = \frac{1}{m} + \frac{1}{n}$ where m and n are positive integers with $n > m$. Multiplying by pmn yields the equation $2mn = p(n+m)$. Thus p divides $2mn$ and since p is prime we know that p divides 2, m or n. Since $p > 2$, p does not divide 2, and thus p divides m or n.

Case 1) p divides n. We write $n = px$ for some positive integer x and substitute into $2mn = p(n+m)$ to obtain $2mx = px + m$, which can be written as $m(2x-1) = px$. Note that $\gcd(x, 2x-1) = \gcd(x, -1) = 1$. There are two common paths to completion from here:

Path i) $2x - 1$ must divide p, so $x = 1$ or $x = \frac{p+1}{2}$. $x = 1$ implies that n and m are both p, in conflict with $n > m$, so we must have $x = \frac{p+1}{2}$. Upon substituting into $n = px$ this yields $n = \frac{p(p+1)}{2}$. We also have $m = xy = x = \frac{p+1}{2}$. Note that n and m, defined as above, satisfy the hypothesis.

Path ii) x must divide m. We write $m = xy$ for some positive integer y and substitute into $m(2x-1) = px$ to obtain $y(2x-1) = p$. So $p = y$ or $p = 2x - 1$. If $p = y$, then $x = 1$, so from $m(2x-1) = px$ we see that $m = p$ and from $n = px$ we see that $n = p$. This means that $m = n$, contradicting $n > m$. So it must be the case that $y = 1$ and $p = 2x - 1$, ie $x = \frac{p+1}{2}$, which upon substituting into $n = px$ yields $n = \frac{p(p+1)}{2}$. We also have $m = xy = x = \frac{p+1}{2}$. Note that n and m, defined as above, satisfy the hypothesis.

Case 2) p divides m. The same argument as in Case 1 yields $m = n$ or ($m = \frac{p(p+1)}{2}$ and $n = \frac{p+1}{2}$), neither of which satisfy $n > m$. So, no additional solutions are found.

Thus, the unique pair of positive integers satisfying the hypothesis are $n = \frac{p(p+1)}{2}$ and $m = \frac{p+1}{2}$.

Bonus Problem - 2 points

A census taker interviews a woman in a house. “Who lives here?” he asks. “My husband and I and my three daughters,” she replies. “What are the ages of your daughters?” “The product of their ages is 36 and the sum of their ages is the house number.” The census taker looks at the house number, thinks, and says, “You haven’t given me enough information to determine the ages.” “Oh, you’re right,” she replies, “Let me also say that my eldest daughter is asleep upstairs.” “Ah! Thank you very much!” What are the ages of the daughters? (The problem requires “reasonable” mathematical interpretations of its words.)

Knowing that the product of the daughters’ ages is 36 (and assuming ages are represented as natural numbers, not with fractions) then we can write the following list of triplets of numbers and know that these are the only possibilities for their ages:

Ages	Sum of Ages
1, 1, 36	38

1, 2, 18	21
1, 3, 12	16
1, 4, 9	14
1, 6, 6	13
2, 2, 9	13
2, 3, 6	11
3, 3, 4	10

The census-taker says that knowing the product is 36 and the sum is the house number (whatever it is) is not enough information, so there must still be some ambiguity. The only two possibilities in the list above that share a sum are $1 + 6 + 6 = 13$ and $2 + 2 + 9 = 13$, so the house number must be 13 (otherwise, the census-taker would be able to answer the riddle already). The last clue makes reference to the eldest daughter, which means that one daughter must be older than the others. Therefore, 1, 6, 6 is not a possibility (there is no eldest daughter), so it must be that the daughters' ages are 2, 2, 9.