21-128 and 15-151 problem sheet 2

Solutions to the following seven exercises and optional bonus problem are to be submitted through gradescope by 11PM on

Wednesday 14th September 2022.

Problem 1

Determine which of the following assertions are true, where A, B and C are non-empty subsets of $\mathbb{Z}$.

(a) $(A \cap B \cap C) \subset (A \cup B)$

(b) $(A \setminus B) \cap (B \setminus A) = \varnothing$

(c) $(A \cap B \neq \varnothing) \implies ((A \setminus B) \subset A)$

Problem 2

Prove that

$$\{x \in \mathbb{Z} : 5 \mid x\} = \{x \in \mathbb{Z} : 5 \mid (10 - 4x)\}.$$

Problem 3

Let $A, B \subseteq \mathbb{N}$ be finite and let $C = \{\, a \in A \mid \exists b \in B \ (a + b \in A) \,\}$.
For what sets A, B does $A = C$?

Problem 4

Let $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$. Prove

$$(A \triangle B) \setminus (B \triangle C) \subseteq ((B \cap C) \setminus A) \cup (A \setminus (B \cup C)).$$

Problem 5

Find a non-empty set A such that

1. $\forall S \in A \; (S \subseteq \mathbb{N})$
2. $\forall S \in A \; \exists S' \in A \; \big((S \neq S') \wedge (S \cup S') = S\big)$

Problem 6

Let $\mathbb{N}^+$ denote the set of positive integers and consider the function $f : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{R}$ defined by

$$f(a,b) = \frac{(a+1)(a+2b)}{2}$$

(a) Show that the image of f is a subset of $\mathbb{N}^+$.

(b) Determine exactly which positive integers are elements of the image of f. (**Hint:** Formulate a hypothesis by trying values.)

Problem 7

For $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, show that (a) and (b) below have different meanings.

(a) $\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in \mathbb{R} \; \big(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\big)$.

(b) $\exists \delta > 0 \; \forall \varepsilon > 0 \; \forall x \in \mathbb{R} \; \big(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\big)$.

(**Hint:** Find a function $f : \mathbb{R} \to \mathbb{R}$ and an element $a \in \mathbb{R}$ for which (a) and (b) have different truth values.)

Bonus Problem (2 points)

A function $g : \mathbb{R} \to \mathbb{R}$ is *even* if $g(-x) = g(x)$ for all $x \in \mathbb{R}$, or *odd* if $h(-x) = -h(x)$ for all $x \in \mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$.

(a) Prove that there exists a unique pair of functions $g, h : \mathbb{R} \to \mathbb{R}$ such that g is even, h is odd, and $f = g + h$. (**Hint:** Express both $f(x)$ and $f(-x)$ in terms of $g(x)$ and $h(x)$, and solve the resulting system of equations.)

(b) When f is a polynomial function, express g and h as in (a) in terms of the coefficients of f.