21-128 and 15-151 Exam 3 Review Problems

1. Consider the set R of all 6-digit numbers where each digit is non-zero.
a) How many numbers are there in the set R?
b) How many numbers in R have distinct digits?
c) How many numbers in R have 1 as their first digit?
d) How many numbers in R have distinct digits as well as 2 as their first digit and 4 as their last digit?

Solution.
a) 9^6 since they are the result of a 6-step process which has 9 options at each step.
b) $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$ since it's an arrangement of 6 from 9.
c) 9^5 since they are the result of a 5-step process which has 9 options at each step.
d) $7 \cdot 6 \cdot 5 \cdot 4$ since it's an arrangement of 4 from 7.

2. Determine the number of ways of arranging the letters of Mississippi and leave your answer in terms of factorials.

Solution. $11!/(4!4!2!)$. Distinguish the letters by labeling them $\{M, i_1, i_2, i_3, i_4, s_1, s_2, s_3, s_4, p_1, p_2\}$. There are 11! permutations of this set, and they are the result of the 2-step process of first choosing a rearrangement of the letters of Mississippi, followed by applying subscripts to the letters. Since there are 4!4!2! ways to apply the subscripts, we conclude that
$11! =$ (The number of arrangments of the letters of Mississippi) $\times$ (4!4!2!), and the result follows by division.

3. In how many ways can one choose 8 people from 18 people and seat them
a) in a row from left to right?
b) in a circle?
c) in a square with 2 on each side?
d) in two rows of 4 facing each other?

Solution.
a) $18!/10!$ since it's an arrangement of 8 from 18.
b) $\binom{18}{8}8!/8$ since we can choose the 8 people to seat, then assign them to the seats labeled $1, 2, \ldots, 8$ and divide by 8 since the seatings are partitioned into equivalence classes of size 8 by rotation.
c) $\binom{18}{8}8!/4$ since we can choose the 8 people to seat, then assign them to the seats labeled $1, 2, \ldots, 8$ and divide by 4 since the seatings are partitioned into equivalence classes of size 4 by rotation.
d) $\binom{18}{8}8!/2$ since we can choose the 8 people to seat, then assign them to the seats labeled

$1, 2, \ldots, 8$ and divide by 2 since the seatings are partitioned into equivalence classes of size 2 by rotation.

4. Consider the experiment of flipping a fair coin 9 times.
a) What is the probability of exactly i heads for $i = 0, 1, 2$?
b) What is the probability of obtaining 8 or more heads?

Solution.
a) $1/2^9$, $9/2^9$, $\binom{9}{2}/2^9$
b) $(\binom{9}{8} + 1)/2^9$

5. Let $S = \{T \subseteq [n+1] : |T| = k+1\}$ and let $S_i = \{T \subseteq [n+1] : |T| = k+1\,$, and i is the least element of T$\}$. Show that $\{S_1, S_2, \ldots, S_{n-k+1}\}$ is a partition of S.

Solution. Consider a subset T of $[n+1]$ such that $|T| = k+1$. If x is the least element of T, then $T \in S_x$, so the union of the S_i is S. Moreover, if $y \neq x$, then $T \notin S_y$ since the least element in T is not y. Hence the S_i are pairwise disjoint and thus partition S.

6. How many ways can 6 people be partitioned into two groups of 3?

Solution. The process of choosing 3 people from 6 creates each partition exactly twice, since each of the two groups can be selected. Thus, the answer is $\binom{6}{3}/2 = 10$.

7. Consider the 36 equally likely outcomes when a fair pair of dice is rolled.
a) What is the probability of doubles?
b) What is the probability that the sum is prime?
c) What is the probability that the sum is even or greater than 8?
d) What is the probability that the product is greater than 15?

Solution.
a) 6/36
b) 15/36
c) 24/36
d) 11/36

8. Consider the 16 equally likely outcomes when a fair coin is flipped 4 times.
a) What is the probability of at least one head?
b) What is the probability of exactly 2 heads?
c) What is the probability that no two heads occur consecutively?

d) What is the probability that the first head occurs on the third flip?

Solution.
a) 15/16
b) 6/16
c) 8/16
d) 2/16

9. We wish to choose 9 cards from a usual deck of 52 playing cards.
a) In how many ways can be achieve this?
b) In how many ways can we achieve this if we are required to choose all cards from the same suit?
c) In how many ways can we achieve this if we are required to choose exactly 3 aces and 3 kings?
d) In how many ways can we achieve this if we are required to choose cards of different values (assuming that the 13 cards in each suit are of different values)?

Solution.
a) $\binom{52}{9}$
b) $4\binom{13}{9}$ choose the suit, then choose the cards from the suit.
c) $\binom{4}{3}\binom{4}{3}\binom{44}{3}$ choose the aces, then choose the kings, then the remaining 3 cards.
d) $\binom{13}{9}4^9$ choose the values, then choose the suits for the values, going from smallest to largest value.

10. Suppose that you are one of 12 candidates for election to a small committee of 3 people. Suppose further that each candidate is equally likely to be elected.
a) What is the probability that you will be successful?
b) Your best friend is also one of the candidates. What is the probability that both of you are successful?

Solution.
a) $\binom{11}{2}/\binom{12}{3} = 1/4$
b) $\binom{10}{1}/\binom{12}{3} = 1/22$

11. We wish to elect 10 members to a committee from 30 candidates, and you and two friends are among the candidates.
a) What is the probability that you and exactly one of your two friends are elected?
b) What is the probability that you and at least one of your two friends are elected?
c) What is the probability that both your friends are elected but you are not?

Solution.
a) $2\binom{27}{8}/\binom{30}{10} = 30/203$
b) $(2\binom{27}{8} + \binom{27}{7})/\binom{30}{10} = 36/203$
c) $\binom{27}{8}/\binom{30}{10} = 15/203$

12. Explain the chairperson identity from the notes by counting chaired committees in two ways.

Solution. Consider the set of all chaired committees of size k chosen from n people. They can be formed by the following two procedures.

1) First choose the k committee members out of the n people, and then choose the chair from among them. There are $\binom{n}{k}k$ ways to do this.

2) First choose the chair out of the n people, and then choose $k-1$ additional committee members from the remaining $n-1$ people. There are $n\binom{n-1}{k-1}$ ways to do this.

Since both procedures produce the same set, we conclude that $\binom{n}{k}k = n\binom{n-1}{k-1}$.

13. This problem concerns the Chinese Remainder Theorem. Let a, b, and c be integers. Find (in terms of a, b, and c) all integers x which satisfy $x \equiv a \mod 2$, $x \equiv b \mod 3$ and $x \equiv c \mod 7$.

Solution. First note that x must be $c + 7k$ for some integer k. Substituting into the second congruence yields $7k \equiv b - c \mod 3$ and hence $k \equiv b - c \mod 3$. Thus $k = b - c + 3j$ for some integer j. This yields $x = 7b - 6c + 21j$, and substituting this into the first congruence yields $j \equiv a + b \mod 2$, ie $j = a + b + 2l$ for some integer l. Thus, all solutions x have the form $x = 21a + 28b - 6c + 42l$ for some integer l.

14. Let a and b be real numbers. Show that

$$(a^2 + 1)(b^2 + 1) \geq (a + b)^2.$$

Solution. Apply Cauchy-Schwarz in two variables to obtain

$$(a^2 + 1)(b^2 + 1) = (a^2 + 1)(1 + b^2) \geq (\sqrt{a^2 \cdot 1} + \sqrt{b^2 \cdot 1})^2 = (a + b)^2.$$

Alternatively, note that expanding both sides and simplifying yields that the original inequality is equivalent to $a^2b^2 + 1 \geq 2ab$, which is in turn equivalent to $(ab - 1)^2 \geq 0$.

15. The 128 and 151 TAs decide to enclose a region for themselves so they can grade your exams in privacy. To do this, they have a 100 meter rope. They use this to create three sides of a rectangle, with the fourth side being one the walls in the hallway of Baker/Porter (i.e. a wall which is very long). What is the maximum area of the rectangle formed by this rope?

Solution. Note that in order for this to occur, the rope must be divided into three segments of lengths x, x, and $100 - 2x$ for some $x \geq 0$. This means that the area of the rectangle is

$$x(100-2x) = 2x(50-x) \leq 2\left(\frac{x+(50-x)}{2}\right)^2 = \boxed{1350},$$

where the inequality is an application of AM-GM. Equality holds when $x = 50 - x$, or $x = 25$.

16. A line with negative slope passing through the point $(18, 8)$ intersects the x and y axes at $(a, 0)$ and $(0, b)$ respectively.

1. Let the slope of the line be $-m$ (so that $m > 0$). Find a and b in terms of m.
2. (CMIMC 2016) What is the smallest possible value of $a + b$?

Solution.

1. Note that the equation of the line can be written in point-slope form as

   $$y - 8 = -m(x - 18) = m(18 - x).$$

   Now computing a can be done by setting $y = 0$; this yields

   $$-8 = -m(a - 18) \quad \Rightarrow \quad a = 18 + \frac{8}{m}.$$

   Similarly, computing b can be done by setting $x = 0$; this yields

   $$b - 8 = m(18) \quad \Rightarrow \quad b = 8 + 18m.$$

2. Note that since $m > 0$, we have $\frac{8}{m} > 0$ and $18m > 0$. Thus, by AM-GM,

   $$a+b = \left(18 + \frac{8}{m}\right) + (8 + 18m) = 26 + \frac{8}{m} + 18m \geq 26 + 2\sqrt{(18m)\left(\frac{8}{m}\right)} = 26 + 2\sqrt{8 \cdot 18} = \boxed{50}.$$

   Equality holds when $\frac{8}{m} = 18m \Leftrightarrow m = \frac{2}{3}$, so this is actually a minimum and we are done.