15-151 Additional Practice Final

Problem 1:

Prove that every integer greater than 55 can be written as the sum of three composite numbers.

Problem 2:

Prove that for any positive integer n,

$$\frac{n(n+1)(2n+1)}{6}$$

is an integer.

Problem 3:

Let

$$X = \sum_{k=1231}^{1985} F_k$$

where F_k is the k^{th} Fibonacci number.

It can be shown that X can be written in the form $F_i - F_j$ for some non-negative integers i and j.

Find any such pair (i, j).

Problem 4:

Show that

$$\forall a, b \in \mathbb{Z}^+, a \leq b \implies \exists k \in \mathbb{Z}^+ \text{ such that } a \bmod k = k \bmod b,$$

but that it is not true that

$$\forall a, b \in \mathbb{Z}^+, \exists k \in \mathbb{Z}^+ \text{ such that } a \bmod k = k \bmod b \implies a \leq b.$$

Problem 5:

You have a deck of 20 numbered cards, with card 1 on top, card 2 being 2^{nd} from the top, and so on, up to card 20 being 20^{th} from the top (at the bottom).

You shuffle them, but in a predictable manner: the card currently k^{th} from the top in the deck becomes p_k^{th} from the top after one shuffle, where $p_1, p_2, \ldots, p_{20}$ is a permutation of the positive integers not exceeding 20.

It can be shown that, regardless of p, there always exists an $m \in \mathbb{Z}^+$ such that doing m shuffles gives back the original arrangement.

Across all 20! permutations $p_1, p_2, \ldots, p_{20}$, what is the maximum number of shuffles required for a deck of 20 cards to give back the original arrangement?

Problem 6:

Call a binary relation R on a set S an *almost equivalence relation* if it satisfies *exactly two* of the properties:

1. Reflexive: $\forall x \in S, x \; R \; x$,
2. Symmetric: $\forall x, y \in S, x \; R \; y \implies y \; R \; x$,
3. Transitive: $\forall x, y, z \in S, (x \; R \; y) \wedge (y \; R \; z) \implies x \; R \; z$.

Define the *opposite* of a binary relation R on a set S as the binary relation R' on S such that

$$\forall x, y \in S, x \; R \; b \iff \neg(a \; R' \; b)$$

Show that, for all sets S, there is no binary relation R on S such that both R and R' are almost equivalence relations.

Problem 7:

Consider the set of all non-empty finite-length strings consisting of lowercase alphabet characters S, and define the binary relation R on S as

$$\forall a, b \in S, a \; R \; b \iff a \text{ is a subsequence of } b$$

A string s is called a *subsequence* of another string t if and only if we can remove zero or more letters from t to get s. For example, "`make`" is a subsequence of "`mackey`", since we can remove two letters in "`mackey`" to get "`make`", but "`cs`" is not a subsequence of "`cmu`".

(a) How many different $x \in S$ satisfy $x \; R$ "`alice`"?

(b) How many different $x \in S$ satisfy $x \; R$ "`aaron`"?

(c) How many different $x \in S$ satisfy $x \; R$ "`terence`"?