General review:

A warning about logical soundness of proofs of inequalities.

Remember how way back in unit 1, we saw that the truth table of $\implies$ gives that false implies true? Well, if we have an inequality of the form $LHS \leq RHS$, it is not logically sound to say that $LHS \leq RHS \implies$ blah $\implies$ blah $\implies 0 < 1$ (or some other true statement), because we know that false implies true.
It is reasonable to do something like that for scratch work, but instead, we want to start at **true** statements and use inequalities to arrive at the conclusion (note: if all steps can be turned into $\Leftrightarrow$ above, then we're chilling, but that's not always gonna be the case).

So, instead, your proof should look something like $LHS \leq$ blah $\leq$ blah $\leq RHS$, where each step blah is justified by either algebraic manipulations, or some inequality that we know.

With that in mind, lets get to the actual inequalities tools.

Means:

AM: $\frac{\sum_{i=1}^{n} a_i}{n}$ - the average of n numbers, where $n \in \mathbb{N}$ and $a_i \in \mathbb{R}$ for each i.
GM: $\sqrt[n]{\prod_{i=1}^{n} a_i}$ - the nth root of the product of n numbers, $n \in \mathbb{N}$, $a_i \in \mathbb{R}^+$ (must be positive, otherwise it doesn't make sense to talk about the root of them).
HM: $\frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}}$ - the reciprocal of the average of the reciprocals of the numbers, $a_i > 0$ for each i.
QM: $\sqrt{\frac{\sum_{i=1}^{n} a_i^2}{n}}$ - the square root of the average of the squares of the numbers.

Inequalities with means:

HGAQ: HM $\leq$ GM $\leq$ AM $\leq$ QM.
Equality holds iff all a_i are equal. So, to show that a value can actually be attained from using one of these, find how to make each term equal.
I like to remember this by seeing the exponent that each term a_i is raised to (ie., in AM they're raised to the power of 1, QM they're raised to the power of 2, etc.,), and the higher the exponent, the higher the mean. And, the $-\infty$ mean is the min of the n numbers and the ∞ mean is the max of the n numbers, and the min is always $\leq$ the max (with equality only when all numbers equal). This isn't a proof, but is how I like to remember this is true.

Vectors:

Let v, w be vectors in $\mathbb{R}^N, N \in \mathbb{N}$. These look like tuples of n real numbers $v = (v_1, v_2, \cdots, v_n), w = (w_1, w_2, \cdots, w_n)$.

Norm: The norm of a vector is basically its size. It's denoted by $\|v\|$, and we have that $\|v\| = \sqrt{\sum_{i=1}^{n} a_i^2}$.

Dot product: We define the dot product $v \cdot w$ to be the sum of the products of the ith component of each vector. So, it looks like $\sum_{i=1}^{n} v_i w_i = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$. This roughly gives a measure of "how aligned" 2 vectors are.

Vector inequalities

Triangle inequality:

$$\|v+w\| \leq \|v\| + \|w\|$$

. Intuitively, tells us that "the shortest path between two points is a line", or "the sum of the shorter two sides on a triangle can't be longer than the longest side). In practice, this looks like $\sqrt{(v_1+w_1)^2+(v_2+w_2)^2+\cdots+(v_n+w_n)^2} \leq \sqrt{v_1^2+v_2^2+\cdots+v_n^2}+\sqrt{w_1^2+w_2^2+\cdots+w_n^2}$.
Equality condition: we have that equality in this case holds when v is a scalar multiple of w ie., $v = cw,\ c \in \mathbb{R}$. To show this, it suffices to set $v_1 = cw_1, v_2 = cw_2, \cdots v_n = cw_n$.

Cauchy-Schwarz inequality: by magic, we have that $v \cdot w = \|v\|\|w\|\cos(\theta)$, where θ is a number denoting "the angle" between two vectors (it isn't exactly relevant what θ represents). We know $|\cos(\theta)| \leq 1$, so Cauchy-Schwarz uses this to get that

$$|v \cdot w| \leq \|v\|\|w\|$$

.
In practice, this looks like $|(v_1w_1+v_2w_2+\cdots v_nw_n)| \leq \sqrt{v_1^2+v_2^2+\cdots+v_n^2}\sqrt{w_1^2+w_2^2+\cdots+w_n^2}$.
You may also see the square of this inequality being used, ie., $(v \cdot w)^2 \leq (\sum_{i=1}^n v_i^2)(\sum_{i=1}^n v_i^2)$.
To use this in practice, your goal is to find two vectors v and w whose dot product can look like one side of the equation and whose product of magnitudes looks like the other side.
Equality condition: we have the same equality condition as before.

Problems

(1) Let $x, y \in \mathbb{R}^+$. Find the minimum possible value of $3x^2 + \frac{2}{xy} + y^2$.

(2) Let $x, y, z \in \mathbb{R}$ such that $x^2 + y^2 + z^2 \leq 1$. Prove that $x^n + y^n + z^n \leq \sqrt{x^{2n-2} + y^{2n-2} + z^{2n-2}}$.

Solutions

(1) We want to find the minimum value of $3x^2 + \frac{2}{xy} + y^2$. This looks like something where we can multiply all the terms together and get all the xs and ys to cancel, but it doesn't quite work out that way because we'd end up with $\frac{3x^2*2*y^2}{xy} = 3xy$. But, if we split the $\frac{2}{xy}$ term into $\frac{1}{xy} + \frac{1}{xy}$, when we multiply all 4 terms of $3x^2 + \frac{1}{xy} + \frac{1}{xy} + y^2$, we get $\frac{3x^2*1*1*y^2}{xy*xy} = 3$.
So, we can try using AM-GM to get that $3x^2+\frac{1}{xy}+\frac{1}{xy}+y^2 \geq 4\sqrt[4]{\frac{3x^2*1*1*y^2}{xy*xy}} = 4\sqrt[4]{3}$.
I claim that $4\sqrt[4]{3}$ is actually the minimum value of this function. But, we need to show that $4\sqrt[4]{3}$ is actually attainable. Well, we know that $AM = GM \Leftrightarrow$ all terms are equal. So, we can turn the **inequality** above into an **equaility** by seting $3x^2 = \frac{1}{xy} = \frac{1}{xy} = y^2$, since then that would give that $3x^2 + \frac{2}{xy} + y^2 = 4\sqrt[4]{\frac{3x^2*1*1*y^2}{xy*xy}} = 4\sqrt[4]{3}$.
To do this, we just solve $3x^2 = \frac{1}{xy} = y^2$ (**note: we use** $\frac{1}{xy}$ here instead of $\frac{2}{xy}$, since we used AM-GM on terms involving $\frac{1}{xy}$, not the latter).
Well, from the first equality we get $y = \frac{1}{3x^3}$ and plugging that into $3x^2 = y^2$ we get $x = \sqrt[8]{\frac{1}{27}}$ (ugly number, sorry), and $y = \sqrt{3 * \sqrt[4]{\frac{1}{27}}}$.
These values of x and y demonstrate that $4\sqrt[4]{3}$ is actually attainable, and thus it is the minimum.

(2) Since we see that we have the sums of squares of things in square roots (because each term x^{2n-2} is really just $(x^{n-1})^2$), we try to use Cauchy-Schwarz. We use Cauchy-Schwarz on the vectors $v = \langle x, y, z\rangle$, $w = \langle x^{n-1}, y^{n-1}, z^{n-1}\rangle$.
Then, Cauchy Schwarz tells us that $x^n+y^n+z^n \leq \sqrt{x^2+y^2+z^2}\sqrt{x^{2n-2}+y^{2n-2}+z^{2n-2}} \leq \sqrt{x^{2n-2}+y^{2n-2}+z^{2n-2}}$ by the assumption that $x^2+y^2+z^2 \leq 1$.