21-128 and 15-151 problem sheet 1

Solutions to the following seven exercises and optional bonus problem are to be submitted through gradescope by 11PM on

Wednesday 7th September 2022.

Problem 1

For each statement below, decide whether it is true or false. Prove your claim using only properties of the natural numbers.

(a) If $n \in \mathbb{N}$ and $n^2 + (n+1)^2 = (n+2)^2$, then $n = 3$.

(b) For all $n \in \mathbb{N}$, it is false that $(n-1)^3 + n^3 = (n+1)^3$.

Solution.

(a) This statement is true. To see this, note that given $n \in \mathbb{N}$ we have

$$\begin{aligned}
& n^2 + (n+1)^2 = (n+2)^2 \\
\Leftrightarrow\ & n^2 + (n^2 + 2n + 1) = n^2 + 4n + 4 && \text{(expanding)} \\
\Leftrightarrow\ & 2n^2 + 2n + 1 = n^2 + 4n + 4 && \text{(simplify LHS)} \\
\Leftrightarrow\ & n^2 - 2n - 3 = 0 && \text{(subtract RHS from both sides)} \\
\Leftrightarrow\ & (n-3)(n+1) = 0 && \text{(factoring)} \\
\Leftrightarrow\ & n = 3 \text{ or } n = -1
\end{aligned}$$

Since $n \in \mathbb{N}$, it follows that $n = 3$.

(b) This statement is true. Notice that

$$\begin{aligned}
& (n-1)^3 + n^3 = (n+1)^3 \\
\Leftrightarrow\ & (n^3 - 3n^2 + 3n - 1) + n^3 = n^3 + 3n^2 + 3n + 1 && \text{(expanding)} \\
\Leftrightarrow\ & 2n^3 - 3n^2 + 3n - 1 = n^3 + 3n^2 + 3n + 1 && \text{(simplify LHS)} \\
\Leftrightarrow\ & 2n^3 - 3n^2 + 3n - 1 = n^3 + 3n^2 + 3n + 1 && \text{(simplify LHS)} \\
\Leftrightarrow\ & n^3 - 6n^2 - 2 = 0 && \text{(subtract RHS from both sides)} \\
\Leftrightarrow\ & n^3 - 6n^2 = 2 && \text{(add 2 to both sides)} \\
\Leftrightarrow\ & n^2(n-6) = 2 && \text{(factor out } n^2\text{)}
\end{aligned}$$

From here, there are multiple approaches:

Approach 1: Division Rules

Hence if there were to exist $n \in \mathbb{N}$ such that $(n-1)^3 + n^3 = (n+1)^2$, then n^2 would have to divide 2, so that $n = 1$. But this implies that $0^3 + 1^3 = 2^3$, which is false. Hence no such n exists, and statement (b) is true.

Approach 2: Casing on n

Note $n \in \mathbb{N}$. We case on n:

- $n = 0$. Clearly the result doesn't hold in this case, as $n^2(n-6) = 0 \neq 2$
- $1 \leq n \leq 6$. $n \leq 6 \implies n - 6 \leq 0 \implies n^2(n-6) \leq 0$ as $n^2 \geq 0\ \forall n \in \mathbb{N}$. So this result cannot be 2 either.
- $n > 6$. Note that $n > 6 \implies (n - 6 > 0) \wedge (n^2 > 6)$ because n is positive. Since $n \in \mathbb{N}$ we have $n - 6 \geq 1$. $(n - 6 \geq 1) \wedge (n^2 \geq 6) \implies n^2(n-6) \geq 6 \implies n^2(n-6) \neq 2$

We have considered all cases, as all $n \in \mathbb{N}$ must be 0, between 1 and 6, or greater than 6, so the result holds.

Problem 2

(a) Show that $(p \Rightarrow q) \vee (p \Rightarrow r)$ and $p \Rightarrow (q \vee r)$ are logically equivalent.

Solution.

$$\begin{aligned} & (p \Rightarrow q) \vee (p \Rightarrow r) \\ \equiv\ & (q \vee \neg p) \vee (r \vee \neg p) \\ \equiv\ & (q \vee r) \vee \neg p \\ \equiv\ & p \Rightarrow (q \vee r) \end{aligned}$$

(b) Show that $(\forall x \in S\ P(x)) \vee (\forall x \in S\ Q(x))$ and $\forall x \in S\ (P(x) \vee Q(x))$ are not logically equivalent, where $P(x)$ and $Q(x)$ are logical formulae and S is a set.

Solution. Let $S = \{0, 1\}$. Let $P(x) :=$ '$x = 0$' and let $Q(x) :=$ '$x = 1$'. The first statement is false because not all elements in S are 0 and not all elements in S are 1. However, the second statement is true since all elements of S are 0 or 1. Therefore, the statements are not logically equivalent.

Problem 3

Let $p(x, y)$ be the predicate '$x + y$ is even', where x and y range over the integers.

(a) Prove that $\forall x\, \exists y\ p(x, y)$ is true.

(b) Prove that $\exists\, y\, \forall\, x\ p(x, y)$ is false.

Solution.

(a) Let $x \in \mathbb{Z}$. Since all integers are even or odd by the Division Theorem, the following cases are exhaustive:
Case 1: x is even. Thus, $x = 2k$ for some $k \in \mathbb{Z}$. Consider $y = 0 \in \mathbb{Z}$.
Then $x + y = 2k + 0 = 2k = x$, which is even.
Case 2: x is odd. Thus, $x = 2k + 1$ for some $k \in \mathbb{Z}$. Consider $y = -1 \in \mathbb{Z}$. Then $x + y = 2k + 1 - 1 = 2k$, which is even since 2 divides $2k$ and $k \in \mathbb{Z}$. Since the initial choice of x was arbitrary, we have verified that $\forall\, x\, \exists\, y\ p(x, y)$ is true.

(b) We prove the negation, so we aim to show $\forall\, y\, \exists\, x\ \neg p(x, y)$, where $\neg p(x, y) :=$ 'n is odd'. Let y be an integer. Since all integers are even or odd by the Division Theorem, the following cases are exhaustive:
Case 1: y is even. Thus $y = 2k$ for some $k \in \mathbb{Z}$. Consider $x = 1 \in \mathbb{Z}$. Then, $x + y = 2k + 1$ is odd since k is an integer and $2k + 1$ is not divisible by 2 by the Division Theorem.
Case 2: y is odd. Thus $y = 2k + 1$ for some $k \in \mathbb{Z}$. Consider $x = 0 \in \mathbb{Z}$. Then, $x + y = 2k + 1$ is odd since k is an integer and $2k + 1$ is not divisible by 2 by the Division Theorem.
In both cases, we have found an $x \in \mathbb{Z}$ such that $p(x, y)$ is false. Therefore, the original statement is false.

Problem 4

You have m indistinguishable marbles, $m \in \mathbb{Z}^+$, and 5 indistinguishable bags.

(a) What is the smallest number of marbles such that you are guaranteed to have 5 marbles in the same bag or 5 different bags with at least one marble? Give proof.

(b) Suppose there are m marbles and k bags, where m and k are positive integers. Prove that there is a unique way to distribute the marbles such that the number of marbles in each bag differs by at most one.

Solution.

(a) The smallest number of marbles is 17. First, we show that for fewer than 17 marbles, it is possible to come up with a way to divide the marbles into bags such that the conditions are not met. This can be achieved by making sure we use no more than 4 bags and put no more than 4 marbles in each bag. If we have m marbles where $m \in \mathbb{Z}^+$, $m < 17$, by Division Theorem we have that $m = qk + r$ for some $q, r \in \mathbb{Z}, 0 \leq r < k$. If we have $k = 4$, each bag will have at most 4 marbles. When there are 16 marbles, we can fill 4 bags with 4 marbles, and when there are fewer than 16 marbles, we know that q must be less than or equal to 3 (or m would be at least 16), so we use at most 4 bags.

Next, we show that we are guaranteed to meet the condition with 17 marbles. AFSOC we use no more than 4 bags and put no more than 4 marbles in each bag. Then the maximum number of marbles that there could be is $4 * 4 < 17$, contradiction.

(b) **Existence.** By Division Theorem, we have that $m = qk + r$ for some $q, r \in \mathbb{Z},\ 0 \leq r < k$. Then $m = qk + r = qk + (qr - qr) + r = q(k - r) + (q + 1)r$. So we can have $k - r$ bags of q marbles and r bags of $q + 1$ marbles, and clearly q and $q + 1$ differ by 1.

Uniqueness. If there is a bag of size $\geq q+2$, then the other bags must have size at least $q+1$, so we end up with $m \geq (q+2)+(q+1)(k-1) = q+2+qk+k-q-1 = (q+1)k+1 > qk+r = m$, contradiction.

Similarly, if there is a bag of size $q - 1$ or less, the other bags have size at most q, so we end up with $m \leq (q - 1) + q(k - 1) = q - 1 + qk - q = qk - 1 = q(k - 1) + (k - 1) < qk + r = m$, contradiction.

Now our possible bag sizes are q and $q + 1$, so $m = q(k - x) + (q + 1)x$ since we have k bags. Note that $x < k$ because $(q + 1)k > qk + r = m$. So $q(k - x) + (q + 1)x = qk + r \implies qk - qx + qx + x = qk + r \implies qk + x = qk + r \implies x = r$. So we proved that having $k - r$ bags of q marbles and r bags of $q + 1$ marbles is the unique way to satisfy the conditions.

Problem 5

A Pythagorean triple is a triple of positive integers (a, b, c) such that $a^2 + b^2 = c^2$. Let (x, y, z) be a Pythagorean triple, and let $P = x + y + z$ and $A = \frac{1}{2}xy$ be the perimeter and area, respectively, of the right-angled triangle whose side lengths are x, y and z.

(a) Find the possible values of (x, y, z) when $P = A$.

(b) Find the possible values of (x, y, z) when $P = 2A$.

Solution. Recall from the definition of Pythagorean triples that x, y, z are all positive integers. Without loss of generality, assume $x \geq y$.

(a) Since $P = A$, we have $x + y + z = \frac{1}{2}xy$, and $z = \frac{1}{2}xy - x - y$. Since (x, y, z) is a Pythagorean triple, we have

$$
\begin{aligned}
x^2+y^2 &= (\frac{1}{2}xy-x-y)^2 \\
&= \frac{1}{4}x^2y^2+x^2+y^2+2xy-x^2y-xy^2 \\
0 &= \frac{1}{4}x^2y^2+2xy-x^2y-xy^2 \\
0 &= \frac{1}{4}xy+2-x-y && (x>0, y>0) \\
0 &= xy+8-4x-4y && \text{(Multiplying throughout by 4)} \\
8 &= (x-4)(y-4)
\end{aligned}
$$

Since the integers are closed under addition, both $x-4$ and $y-4$ must be integers. Therefore, we only need to concern ourselves with integral factors of 8 when solving the equation above. After doing so, we arrive at the only possible integral solutions: $(x, y, z) = (12, 5, 13)$ or $(8, 6, 10)$. We still need to verify that these are valid solutions. Substitution of these values into the equations shows that these triples are, in fact, solutions to the problem.

(b) Proceeding as in the first part, we have $z = xy - x - y$.

$$
\begin{aligned}
x^2+y^2 &= (xy-x-y)^2 \\
&= x^2y^2+x^2+y^2-2x^2y-2xy^2+2xy \\
0 &= x^2y^2-2x^2y-2xy^2+2xy \\
0 &= xy-2x-2y+2 && (x>0, y>0) \\
2 &= (x-2)(y-2)
\end{aligned}
$$

Since the integers are closed under addition, both $x-2$ and $y-2$ are integers. Therefore, we only need to concern ourselves with integral factors of 2 when solving the equation above. After doing so, we arrive at the only possible integral solution: $(x, y, z) = (4, 3, 5)$. We still need to verify that this is a valid solution. Substitution of these values into the equations shows that this triple is, in fact, a solution to the problem.

Problem 6

(a) Show that the following statement is false:

For all $a, x \in \mathbb{R}$ there is a unique $y \in \mathbb{R}$ such that $x^4y + ay + x = 0$

(b) Find the set of real numbers a such that the following statement is true:

For all $x \in \mathbb{R}$ there is a unique $y \in \mathbb{R}$ such that $x^4y + ay + x = 0$

Solution.

(a) Let $a = x = 0$. Then $x^4y + ay + x = 0^4 \cdot y + 0 \cdot y + 0 = 0$ for *all* values of $y \in \mathbb{R}$, so in particular there is not a unique $y \in \mathbb{R}$ such that $x^4y + ay + x = 0$. Hence (a) is false.

(b) First, we give the claim:

The statement is true for all $a > 0$ and false for all $a \leq 0$.

- Suppose $a > 0$, and let $x \in \mathbb{R}$. Then $x^4 \geq 0$, so $a + x^4 \geq a > 0$.
  - **Existence.** Let $y \in \mathbb{Z},\ y = \frac{-x}{a+x^4}$. Then

$$\begin{aligned} & y = \frac{-x}{a + x^4} \\ \Rightarrow\ & (a + x^4)y = -x && \text{(multiply both sides by } a + x^4\text{)} \\ \Rightarrow\ & x^4y + ay = -x && \text{(simplify)} \\ \Rightarrow\ & x^4y + ay + x = 0 && \text{(subtract RHS from both sides)} \end{aligned}$$

    as required.
  - **Uniqueness.** Suppose $u, v \in \mathbb{R}$ both satisfy the equation. Then

$$x^4u + au + x = x^4v + av + x = 0 \quad \Rightarrow \quad (x^4 + a)u = (x^4 + a)v$$

    by factoring. Since $x^4 + a > 0$ we can cancel it to reveal that $u = v$.

  So the statement is true when $a > 0$.
- Suppose $a \leq 0$. In order to disprove the statement: "for all $x \in \mathbb{R}$ we have a unique y with $x^4y + ay + x = 0$", we show that the negation holds by showing there exists such an x that does not have a unique y that satisfies the equation.
  - **Case 1:** $a = 0$**.** Let $x = 0$. Then, all values of $y \in \mathbb{R}$ satisfy $x^4y + ay + x = 0$.
  - **Case 2:** $a < 0$**.** Let $x = \sqrt[4]{-a}$ (keep in mind a is negative so $-a$ is positive and we can take a root). Then

$$x^4y + ay = (x^4 + a)y = (-a + a)y = 0$$

    by factoring and substituting in $\sqrt[4]{-a}$ for x. This leaves $x = 0$, which is a contradiction. Then, no values of $y \in \mathbb{R}$ satisfy $x^4y + ay + x = 0$.

  In both cases, we have shown that there exists an x that has no *unique* value of y satisfying $x^4y + ay + x = 0$.
  So the statement is false when $a \leq 0$.

Problem 7

Which of the following numbers are irrational for every choice of numbers r, a and b, such that r is rational and a and b are irrational?

$$a+r \qquad a+b \qquad ar \qquad ab \qquad a^r \qquad r^a \qquad a^b$$

Prove your claims, either by proving that the number must always be irrational or by providing a counterexample. If you claim that a number is irrational, then you should prove it.

Solution. We know from class that $\sqrt{2}$ is irrational. It follows that $-\sqrt{2}$ is irrational, since if $-\sqrt{2} = \frac{m}{n}$ for integers m and n then $\sqrt{2} = \frac{-m}{n}$, which is the ratio of two integers (and this is impossible).

All of the following will only use the irrationality of $\sqrt{2}$ or $-\sqrt{2}$.

- If a is irrational and r is rational, then $a+r$ is irrational. To see this, suppose there exist irrational a and rational r such that $a+r$ is rational. Write $r = \frac{m}{n}$ and $a + r = \frac{s}{t}$, where m, n, s, t are integers and n and t are nonzero. Then

  $$a = (a+r) - r = \frac{s}{t} - \frac{m}{n} = \frac{sn - tm}{tn}$$

  which is the ratio of two integers. Hence a is rational, contradicting the assumption that a is irrational.

- Given irrational numbers a, b, it is possible that $a + b$ is rational. To see this, note that $\sqrt{2} + (-\sqrt{2}) = 0$, which is rational.

- Given an irrational number a and a rational number r, it is possible that ar is rational. To see this, note that $\sqrt{2} \cdot 0 = 0$, which is rational.

- Given irrational numbers a, b, it is possible that ab is rational. To see this, note that $\sqrt{2} \cdot \sqrt{2} = 2$, which is rational.

- Given an irrational number a and a rational number r, it is possible that a^r is rational. To see this, note that $(\sqrt{2})^2 = 2$, which is rational.

- Given an irrational number a and a rational number r, it is possible that r^a is rational. To see this, note that $1^{\sqrt{2}} = 1$, which is rational.

- Given irrational numbers a, b, it is possible that a^b is rational. To see this, note that either

  - $(\sqrt{2})^{\sqrt{2}}$ is rational, in which case we've proved the claim; or
  - $(\sqrt{2})^{\sqrt{2}}$ is irrational, in which case

    $$2 = (\sqrt{2})^2 = (\sqrt{2})^{\sqrt{2}\cdot\sqrt{2}} = ((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}}$$

    is rational and of the form a^b where a and b are irrational.

Note. There is a powerful theorem called the Gelfond-Schneider theorem that shows that when $a = b = \sqrt{2}$, a^b is actually irrational (even transcendental).

Bonus Problem (2 points)

Three brilliant, flawless logicians - Aimee, Brad, and Cindy were blindfolded and each had a hat with a positive integer (possibly different for each) written on it placed on their heads.

Their blindfolds were then removed; they faced each other in a circle and each could see the hats the others were wearing, but not their own hat.

They were told that two of the numbers added up to the third. In order to be generously rewarded they needed to figure out what number was written on their hats.

Here is the conversation that took place:

Aimee: I don't know what my number is.
Brad: I don't know what my number is.
Cindy: I don't know what my number is.
Aimee: Now I know what my number is. It is 50.

(a) What are the other numbers?

(b) What combination(s) of numbers would allow Cindy to solve the problem in round 1?

Solution.

(a) First, we note that no matter what numbers are on B's [Brad's] and C's [Cindy's] hats, A [Aimee] knows that their number is one of at most two options (specifically $\mid B - C \mid, B + C$), because of the summation rule.

Now, we consider the following scenario. Let x be a positive integer. If Person A sees the numbers x and x on Person B's and Person C's hats, respectively, then Person A will know that the number on their hat is $2x$, because the only other possibility, 0 (to satisfy the summing condition), is ruled out by the fact that every number on a hat is a positive integer. We will be making use of this fact to solve this problem.

Suppose A's number is either m or n. Now, we make the observation that for A to be sure that their number is m, they need to rule out the possibility that their number is n, and the only way of doing that is if, given the perfectness of A, B and C's logic, and the fact that n was A's number, the sequence of events that actually transpired (A,B,C all taking turns to say "I don't know") was impossible. That is, if n was indeed A's number, then someone else would have guessed their number before A's second guess (or if $n = 0$, which is also impossible).

Suppose that the numbers on the hats of A, B, and C are 10, 20, and 30, respectively.

At the very beginning, Person A does not know whether the number on their hat is 10 or 50, so they say "I don't know". Next, Person B, looking at the numbers 10 and 30, also doesn't know whether their number is 20 or 40, so they also say "I don't know". Now, Person C is looking at the numbers 10 and 20. They know that their number is either 10 or 30. However, if their number were indeed 10, then B would have been looking at the numbers 10 and 10 on A's and C's hats, respectively, and from the argument above we know that B would have declared their own number to be 20. Since they (as a perfect logician) failed to do so, C can conclude that B in fact did not see the numbers 10 and 10, which would lead them to the conclusion that their number was 30.

Now, we have concluded that if the numbers on the hats were 10, 20 and 30, respectively, then C would be able to deduce their hat's number.

Now suppose that the numbers on the hats of A, B, and C are 50, 20, and 30, respectively.

Similar to the reasoning above, A says "I don't know", not knowing whether their number is 10 or 50. B echoes because they can't tell if their number is 20 or 80. C says they don't know as well, unable to figure out if their number is 30 or 70. Using the logic we made earlier, A can deduce from C's inability to discern C's number that the number on A's hat is not 10! Therefore, they can conclude that the number on their hat must be 50.

Note that this solution does not prove uniqueness. We will do so in part b).

(b) As per the solution in a), the only way for person C to be sure of their number is if the other possible value for their number results in someone else guessing their number before them, or that number is zero.

Since C is guessing third, we can exhaustively list all the "wrong" cases. In all cases, we let $x \in \mathbb{Z}^+$.

Case 1: C's other number is zero. Then the actual value of (A, B, C) was $(x, x, 0)$. However, since C won, this did not happen, so a possible set of values would instead feature C's other possible value, ie. $(A, B, C) = (x, x, 2x)$.

Case 2: A would have guessed their number first. The only way for A to know their number by just looking at B's and C's hats is if the other two numbers are equal. That is, the actual value of (A, B, C) was $(2x, x, x)$. However, since C won, this did not happen, so a possible set of values would instead feature C's other possible value, ie. $(A, B, C) = (x, x, 2x)$.

Case 3a: B would have guessed their number first, but not because A did not guess their number. This case would occur if B could conclude on their own what their number was. (Identical to case 2). That is, the actual value of (A, B, C) was $(x, 2x, x)$. However, since C won, this did not happen, so a possible set of values would instead feature C's other possible value, ie. $(A, B, C) = (x, x, 2x)$.

Case 3b: B would have guessed their number first because A did not guess their number. In this case, B knows that A's silence meant that B did not have the same number as C. Thus B's number must be the sum of A's and C's number, that is, the actual value of $(A, B, C) = (2x, 3x, x)$. (recall that we are trying to create a situation in which C did NOT win because B won!)

However, since C won, this did not happen, so a possible set of values would instead feature C's other possible value, ie. $(A, B, C) = (2x, 3x, 5x)$.

Thus, if C were to win, the following sets of values are all the possible ways they could do so: $(A, B, C) : (x, x, 2x), (x, 2x, 3x), (2x, x, 3x), (2x, 3x, 5x)$. This ends the proof of part b).

Now, we prove the uniqueness of part a). Note that for A to have won on their second guess, C must not have won with their first guess! (We don't care about B or A winning with their first guesses, because those have already been accounted for in our above analysis). Therefore, to figure out all the possible cases in which A wins on their second guess, it suffices to "flip" A's numbers in each of the 4 cases. Doing so results in the following sets: $(A, B, C) = (3x, x, 2x), (5x, 2x, 3x), (4x, x, 3x), (8x, 3x, 5x)$. In each of these cases, A's number $= 3x, 5x, 4x$ or $8x$. But 50 is divisible by only 5, and not $3, 4$ or 8. Thus the solution provided in part a) is unique.