

21-128 and 15-151 Midterm 3
November 23, 2020

1. Show that for all real numbers x and y with $xy \geq 0$,

$$(x^2 - y^2)^2 \geq (x - y)^4.$$

2. Let $X \subseteq \mathbb{N}^+$ and $Y \subseteq \mathbb{N}^+$ be nonempty sets of positive integers.

Show that X is finite and Y is finite $\Leftrightarrow \{a_1 \cdot a_2 \mid a_1 \in X \text{ and } a_2 \in Y\}$ is finite.

3. Determine the number of 5 card hands with at least one spade and at least one heart.

4. Let n be a positive integer and $f : [n] \rightarrow [n]$ be an unknown, fixed function.

Determine the number of functions $g : [n] \rightarrow [n]$ such that $\exists! x \in [n]$ with $f(x) = g(x)$.

5. Show, by counting in two ways, that for all integers $n \geq 2$,

$$n * (n - 1)^2 = 2 * \binom{n}{2} + 6 * \binom{n}{3}.$$

Bonus. How many positive integers less than 128 have exactly three 1s in their binary (base-2) representation?