21-128 and 15-151 problem sheet 8

Solutions to the following seven exercises and optional bonus problem are to be submitted through gradescope.

Note: You don't need to simplify your answers. Sums and products of integers, binomial coefficients, exponents, factorials, etc. are acceptable.

Problem 1

How many ways are there to pick two cards from a standard 52-card deck, without replacement, such that the first card is spades and the second card is not a face card?

Solution. We count the number of ways of picking such a pair of cards by means of the following procedure.

Either the first card is a face card or it is not, so we clearly have a mutually exclusive and exhaustive partition. By the rule of sum, the number of hands we seek is the sum of the number of hands in each of these cases. We know there are $3 \cdot 4 = 12$ face cards by Multiplication Principle (first pick J, Q, or K, then pick a suit).

- If the first card is a face card, then there are 3 face cards with spades as choices. There are then $52 - 12 = 40$ non-face cards possible for the second card. Since we have a k-step process, there are $3 \cdot 40$ ways to pick such pairs of cards by the Multiplication Principle.
- If the first card is not a face card, it must also be spades, so there are $13 - 3 = 10$ choices. There are $40 - 1 = 39$ non-face cards remaining, so there are 39 choices of second card. Since we have a k-step process, there are $10 \cdot 39$ ways to pick such pairs of cards by the Multiplication Principle.

By the Addition Principle, there are $3 \cdot 40 + 10 \cdot 39$ ways of picking two cards from a standard 52-card deck, such that the first card is spades and the second card is a non-face card.

Problem 2

Count the number of 6 card hands where each suit in the hand occurs at least twice AND each rank in the hand is unique.

Solution. Consider the suit that occurs most often in a 6 card hand where each suit occurs at least twice. The frequently occurring suit cannot occur 5 times because then there would be only

1 occurrence of the other suit. Thus we can partition the 6 card hands we want to count into whether the most frequently occurring suit occurs twice, three times, four times, or six times. We already justified why the cases are exhaustive, and it is mutually exclusive because the frequency of the most frequently occurring suit is unique. Let us consider these 4 cases:

- **The most frequently occurring suit occurs 2 times.** Then the other suits must occur twice as well (since all suits must occur at least twice). Thus the hand is of the form AABBCC. First we pick the 3 suits, there are $\binom{4}{3}$ ways to do this. Then we pick the ranks for the suits (we will do this in the order of clubs before diamonds before hearts before spades). There are $\binom{13}{2}$ ways to select 2 ranks for set A, but now there are 11 ranks left over, since the ranks all have to be unique. Thus there are $\binom{11}{2}$ ways to pick the rank for set B and $\binom{9}{2}$ ways to pick the rank for set C. Since we have a k-step process, there are $\binom{4}{3}\binom{13}{2}\binom{11}{2}\binom{9}{2}$ hands in this case.
- **The most frequently occurring suit occurs 3 times.** If one of the other suits occurs exactly twice, there would be a suit that occurs once, so there cannot be any valid hands in that case. So the hand must be of the form AAABBB. First we pick 2 suits, there are $\binom{4}{2}$ ways to do this, and by the same process from the first case, there are $\binom{13}{3}$ ways to pick ranks for suit A and $\binom{10}{3}$ ways to pick ranks for suit B. Since we have a k-step process, there are $\binom{4}{2}\binom{13}{3}\binom{10}{3}$ hands in this case.
- **The most frequently occurring suit occurs 4 times.** We require every set to occur at least twice, so the hand must be of the form AAAABB. First we pick the 2 suits, there are 4 ways to pick the suit for A and 3 ways to pick the suit for B. There are $\binom{13}{4}$ ways to pick ranks for suit A and $\binom{9}{2}$ ways to pick ranks for set B. Since we have a k-step process, there are $4 \cdot 3 \cdot \binom{13}{4}\binom{9}{2}$ hands in this case.
- **The most frequently occurring suit occurs 6 times.** First, we pick the suit, there are 4 ways to do this. Then we pick the ranks, there are $\binom{13}{6}$ ways to do this. Since we have a k-step process, there are $4 \cdot \binom{13}{6}$ hands in this case.

Since we have a valid partition, we can apply the Addition Principle to get the total number of such hands:

$$\binom{4}{3}\binom{13}{2}\binom{11}{2}\binom{9}{2} + \binom{4}{2}\binom{13}{3}\binom{10}{3} + 4 \cdot 3 \cdot \binom{13}{4}\binom{9}{2} + 4 \cdot \binom{13}{6}.$$

Problem 3

Find the number of functions $f : [6] \to [6]$ such that f contains exactly three elements in its image.

Solution. Such functions are the result of a 2-step process. First choose a 3-element subset S of $[6]$ and then choose a surjection from $[6]$ to S. There are $\binom{6}{3}$ ways to perform the first step.

Once the first step has been completed, the number of surjections from $[6]$ to S is the number of functions from $[6]$ to S, minus the number of functions from $[6]$ to S having exactly one element in their image, minus the number of functions from $[6]$ to S having exactly two elements in their image. This is because we can partition the functions from the first step into functions into those with an image of size 1, 2, or 3. We consider all possible sizes of images and the size of the image for a particular function is unique, so this partition is valid.

To count all functions $[6]$ to S, we use a 6-step process of choosing $f(1), f(2), \ldots, f(6)$, in order, from among the three elements of S. By the Multiplication Principle, there are 3^6 ways to do this.

If the size of the image is 1, we just pick the element of S to which everything gets mapped, so there are 3 ways to do this.

If the size of the image is 2, we first pick two elements of S for the image, there are $\binom{3}{2}$ ways to do this. Then we need to pick all functions that map to both selected elements by considering all functions and then subtracting those having only one element in their image. There are 2^6 ways to pick one of 2 elements for the 6 elements of the domain and there are only 2 ways to map everything to one of the 2 elements of the codomain, so there are $2^6 - 2$ ways to do this. Since we have a k-step process, the total number of ways to do this is $\binom{3}{2}(2^6 - 2)$ by the Multiplication Principle.

Since we have a valid partition, we know that $\sum_{i=1}^{3}$ [number of functions with an image of size i] = number of functions from $[6]$ to S by the Addition Principle, so we can subtract the first 2 terms of the sum from both sides to get that there are $3^6 - 3 - \binom{3}{2}(2^6 - 2)$ functions from $[6]$ to S. Since the number of functions going to S does not depend on the elements of S, we can apply the Multiplication Principle to say there are $\binom{6}{3}(3^6 - 3 - \binom{3}{2}(2^6 - 2))$ such functions.

Problem 4

Theo wants to show his appreciation to the TAs for their hard work this semester, so he goes to the 151/128 store to buy them gifts! Note that there are 28 TAs on staff this year.

He buys 100 hedgehog plushies and 100 bottles of Diet Coke. He wants to give away all of the gifts and make sure each TA gets a least one gift. But, no TA should receive both plushies and Diet Coke. How many ways can Theo do this?

Solution. We can partition on the number of TAs receiving plushies (between 1 and 27, since at least one TA has to receive Diet Coke). This is an exhaustive partition because we consider all possible numbers of TAs we can give plushies to, and for each distribution of gifts there is a unique number of TAs who get plushies, so the partition is mutually exclusive. If Theo decides to give the plushies to t TAs, we first have to pick t of the 28 TAs to get plushies, there are $\binom{28}{t}$ ways to do this. Then since we have to give each of these TAs 1 plushy, there are $100-t$ plushies to give and t TAs to give them to. By stars and bars, there are $\binom{(100-t)+(t-1)}{t-1}$ ways to do this. The procedure is the same for distributing the Diet Coke to the remaining $28-t$ TAs, so there are $\binom{(100-(28-t))+((28-t)-1)}{(28-t)-1}$ ways to do this. Since we have a k-step process, we can apply the Multiplication Principle, and we can sum over t by the Addition Principle since we have a valid partition. Thus, the number of ways Theo can distribute gifts is:

$$\sum_{t=1}^{27}\binom{28}{i}\binom{99}{t-1}\binom{99}{27-t}.$$

Problem 5

Consider a set of 1×1 squares and 1×2 dominoes. The 1×1 squares are colored either red or blue; the 1×2 dominoes are colored green, yellow, or black. Let A_n denote the number of ways to tile a $1\times n$ board of unit squares with these squares and dominoes such that no two pieces overlap.

1. Show that $A_n = 2A_{n-1} + 3A_{n-2}$ for all $n \geq 2$.
2. Deduce that
$$A_n = \frac{3^{n+1} + (-1)^n}{4} \quad \text{for all } n \geq 1.$$

Solution.

1. Fix $n \geq 2$ and let T_n denote the set of all tilings of a $1\times n$ board with the given set of 1×1 and 1×2 tiles. Define the auxiliary sets
$$S_n = \{t \in T_n : t \text{ ends with a square}\} \quad \text{and} \quad D_n = \{t \in T_n : t \text{ ends with a domino}\}.$$
Note that $T_n = S_n \sqcup D_n$, since every tiling must end with one of those two pieces and no single tiling can end with both. As a result,
$$|T_n| = |S_n \sqcup D_n| = |S_n| + |D_n|.$$
It suffices to compute $|S_n|$ and $|D_n|$. To compute $|S_n|$, note that every tiling $s \in S_n$ consists of an arbitrary tiling of the first $n-1$ squares of the board followed by a 1×1 square. There are A_{n-1} possibilities for the former case and 2 possibilities for the latter. By the

multiplication principle, this yields $|S_n| = 2A_{n-1}$. Similar reasoning yields $|D_n| = 3A_{n-2}$, where in the case that $n = 2$ we utilize $A_0 = 1$. As a result, we obtain

$$A_n = 2A_{n-1} + 3A_{n-2},$$

as desired.

2. Let $P(n) =$ "$A_n = \dfrac{3^{n+1} + (-1)^n}{4}$". We prove that $P(n)$ is true for all positive integers n, by induction.

   - **Base case.** First suppose that $n = 1$. The only way that one can tile a 1×1 board is with one of the two 1×1 squares; this gives $A_1 = 2$. We also have

     $$\frac{3^{1+1} + (-1)^1}{4} = \frac{9 - 1}{4} = 2,$$

     so $P(1)$ is true.

     Now suppose that $n = 2$. There are two ways to tile a 1×2 board: either with one domino or with two squares. The former case has 3 possibilities; the latter has $2 \cdot 2 = 4$. This implies $A_2 = 3 + 4 = 7$. Once again, this checks, as

     $$\frac{3^{2+1} + (-1)^2}{4} = \frac{28}{4} = 7.$$

     Thus $P(2)$ is also true.

   - **Induction step.** Let n be a positive integer, and assume that $P(k)$ holds for all $1 \leq k \leq n$. We wish to show that $P(n+1)$ is true. If $n = 1$ then we refer to the base case where the truth of $P(2)$ was established. If $n > 1$, we may use the recursion relation as follows:

     $$\begin{aligned} A_{n+1} &= 2A_n + 3A_{n-1} \\ &\overset{IH}{=} 2\left(\frac{3^{n+1} + (-1)^n}{4}\right) + 3\left(\frac{3^n + (-1)^{n-1}}{4}\right) \\ &= \frac{2 \cdot 3^{n+1} + 2(-1)^n + 3^{n+1} - 3(-1)^n}{4} \\ &= \frac{3^{n+2} + (-1)^{n+1}}{4}. \end{aligned}$$

     Hence $P(n+1)$ is true, and the induction step is complete.

Problem 6

How many nine-digit integers consisting of each of the digits 1-9 exactly once are divisible by 36?

Solution. Let N be such an integer. Note that since each of the digits 1 through 9 is used once, we have

$$N \equiv 1 + 2 + \cdots + 9 \equiv 45 \equiv 0 \pmod 9.$$

So N is divisible by 9 and $N = 9j$ for for some integer j. Since 4 and 9 are coprime, we see that $4 \mid N \Leftrightarrow 4 \mid j \Leftrightarrow 36 \mid N$. Thus, it remains only to compute the number of such integers N which are divisible by 4.

Recall that an integer N is divisible by 4 if and only if the number formed by the last two digits of N is divisible by 4 (since the N minus its last 2 digits is a multiple of 100 and is congruent to zero mod 4). Such an ending cannot have any zeroes or use the same digit twice. There are 25 possible endings (of the form $4n$ for $0 \leq n \leq 24$). Seven of them include a zero (00, 04, 08, 20, 40, 60, 80), and two others have the same digit repeated twice (44, 88). This gives a count of $25 - 7 - 2 = 16$ 2-digit endings that are divisible by 4.

Note that there are 7! ways to arrange each of the remaining seven digits, giving an answer of $\boxed{16 \cdot 7! = 80640}$, by the multiplication principle.

Problem 7

By counting in two ways, prove that $\sum_{k=1}^{n} 2^{k-1} = 2^n - 1$ for all $n \in \mathbb{N}$.

Solution. Let $X = \mathcal{P}([n]) - \{\varnothing\}$ be the set of nonempty subsets of $[n]$, and for each $i \in [n]$ let Y_i be the set of subsets of $[n]$ whose greatest element is i.

Note that $Y_i \cap Y_j = \varnothing$ when $i \neq j$, since a subset of $[n]$ can only have one greatest element. Moreover $\bigcup_{i=1}^{n} Y_i = X$. We prove this by double-containment:

- ($\subseteq$). Let $S \in \bigcup_{i=1}^{n} Y_i$. Then $S \in Y_i$ for some $i \in [n]$. But then $i \in S$, so $S \neq \varnothing$, and hence $S \in X$.
- ($\subseteq$). Let $S \in X$. Since $S \neq \varnothing$, S has a greatest element, say i. But then $S \in Y_i$, so $S \in \bigcup_{i=1}^{n} Y_i$.

Hence $\{Y_i : 1 \leq i \leq n\}$ is a partition of X.

By the rule of sum,

$$\sum_{i=1}^{n} |Y_i| = |X|$$

Now $|X| = 2^n - 1$, since $[n]$ has 2^n subsets of which exactly one is empty.

For each $i \in [n]$, $|Y_i| = 2^{i-1}$. To see this, note that the function $f : \mathcal{P}([i-1]) \to Y_i$ defined by $f(T) = T \cup \{i\}$ is a bijection. Certainly f is well-defined: if $S \in f(T)$ then $i \in S$, and for all $x \in S$ we have either $x = i$ or $x \leq i - 1$; so i is the greatest element of S and $S \in Y_i$. Moreover, f is injective: if $f(T) = f(T')$ then $T \cup \{i\} = T' \cup \{i\}$, so $T = T'$ since $i \notin T$ and $i \notin T'$. Finally, f is surjective: if $S \in Y_i$ then $S = f(S - \{i\})$; note that $S - \{i\} \subseteq [i-1]$ since $i > x$ for all $x \in S$ with $x \neq i$.

It follows that $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$.

Bonus Problem - (2 points)

Tram tickets have six-digit numbers (from 000000 to 999999). A ticket is called *lucky* if the sum of its first three digits is equal to the sum of its last three digits. A ticket is called *medium* if the sum of all its digits is 27. Let A and B denote the numbers of lucky tickets and medium tickets respectively. Prove that $A = B$.

Solution. Let L and M be the sets of lucky and medium tickets, respectively. Define $f : L \to M$ via $f(abcdef) = abc(9-d)(9-e)(9-f)$. This specification is valid since $abcdef$ lucky means that $a+b+c = d+e+f$, so $a+b+c+(9-d)+(9-e)+(9-f) = (a+b+c-d-e-f)+27 = 27$ and $abc(9-d)(9-e)(9-f)$ is medium.

Similarly, the function $g : M \to L$ defined via $g(abcdef) = abc(9-d)(9-e)(9-f)$ is well-defined, and we see that g and f compose to the identity in both orders, and are hence inverses of one another. Thus, g and f are both bijections and $|L| = A = B = |M|$.