21-128/15-151 final practice questions

Clive Newstead

Question 1 — logic, sets and induction

(a) Let A, B, X be sets and suppose that $A \subseteq X$ and $B \subseteq X$. Prove that $A \subseteq B$ if and only if $X \setminus B \subseteq X \setminus A$.

Solution. We prove the two directions separately.

- ($\Rightarrow$) Suppose $A \subseteq B$. Let $x \in X \setminus B$; we wish to prove that $x \in X \setminus A$. First note that $x \in X$, since $x \in X \setminus B$. Moreover, if $x \in A$, then $x \in B$ since $A \subseteq B$; but this contradicts $x \in X \setminus B$. Hence $x \notin A$. Since $x \in X$ and $x \notin A$, we have $x \in X \setminus A$, as required.
- ($\Leftarrow$) Suppose $X \setminus B \subseteq X \setminus A$. Let $x \in A$; we wish to prove $x \in B$. Suppose $x \notin B$—we derive a contradiction. Indeed, $x \in X$ since $A \subseteq X$; so if $x \notin B$ then $x \in X \setminus B$. Since $X \setminus B \subseteq X \setminus A$, it follows that $x \notin A$, contradicting the assumption that $x \in A$. So $x \in B$, as required.

(b) Let (f_n) be the Fibonacci sequence, defined recursively by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. Prove, for all $n \in \mathbb{N}$, that f_n is divisible by 4 if and only if n is divisible by 6.

Solution. For each $k \in \mathbb{N}$, let $p(k)$ be the following statement:

The remainders modulo 4 of the numbers $f_{6k}, f_{6k+1}, \cdots, f_{6k+5}$ are $0, 1, 1, 2, 3, 1$, respectively.

We prove $p(k)$ is true for all $k \in \mathbb{N}$ by induction on k.

- (**BC**). The numbers $f_0, f_1, \ldots, f_5$ are $0, 1, 1, 2, 3, 5$, which leave remainders of $0, 1, 1, 2, 3, 1$, respectively, when divided by 4. Hence $p(0)$ is true.
- (**IS**). Fix $k \geq 0$ and suppose $p(k)$ is true. We need to prove that the numbers $f_{6k+6}, \ldots, f_{6k+11}$ leave remainders of $0, 1, 1, 2, 3, 5$ when divided by 4. Well

$$f_{6k+6} \equiv f_{6k+5} + f_{6k+1} \equiv 1 + 3 \equiv 4 \equiv 0 \bmod 4$$

$$f_{6k+7} \equiv f_{6k+6} + f_{6k+5} \equiv 0 + 1 \equiv 1 \bmod 4$$

$$f_{6k+8} \equiv f_{6k+7} + f_{6k+6} \equiv 1 + 0 \equiv 1 \bmod 4$$

$$f_{6k+9} \equiv f_{6k+8} + f_{6k+7} \equiv 1 + 1 \equiv 2 \bmod 4$$

$$f_{6k+10} \equiv f_{6k+9} + f_{6k+8} \equiv 2 + 1 \equiv 3 \bmod 4$$

$$f_{6k+11} \equiv f_{6k+10} + f_{6k+9} \equiv 3 + 2 \equiv 5 \equiv 1 \bmod 4$$

as required.

In particular, $f_{6k} \equiv 0 \bmod 4$ for all $k \in \mathbb{N}$ and $f_{6k+i} \equiv 0 \bmod 4$ for all $i \in [5]$, so that for all $n \in \mathbb{N}$ we have $4 \mid f_n$ if and only if $6 \mid n$.

Question 2 — number theory

(a) Find all integers x satisfying the congruence $385x \equiv 21 \bmod 588$.

Solution. Note first that $385x \equiv 21 \bmod 588$ if and only if $385x + 588y = 21$ for some $y \in \mathbb{Z}$. Running the extended Euclidean algorithm on the pair $(385, 588)$ yields

$$\begin{aligned}
588 &= 1 \cdot 385 + 203 & &\Rightarrow & 7 &= 19 \cdot 588 - 29 \cdot 385 \\
385 &= 1 \cdot 203 + 182 & &\Rightarrow & 7 &= 19 \cdot 203 - 10 \cdot 385 \\
203 &= 1 \cdot 182 + 21 & &\Rightarrow & 7 &= 9 \cdot 203 - 10 \cdot 182 \\
182 &= 8 \cdot 21 + 14 & &\Rightarrow & 7 &= 9 \cdot 21 - 1 \cdot 182 \\
21 &= 1 \cdot 14 + 7 & &\Rightarrow & 7 &= 21 - 1 \cdot 14 \\
14 &= 2 \cdot 7 + 0
\end{aligned}$$

Sinec $385 \cdot (-29) + 588 \cdot 19 = 7$, it follows by multiplying by 3 that

$$385 \cdot (-87) + 588 \cdot (57) = 21$$

Now $\frac{385}{7} = 55$ and $\frac{588}{7} = 84$, so the integer solutions (x, y) to the equation $385x + 588y = 21$ are precisely those of the form

$$x = -87 + 84k \text{ and } y = 57 - 55k \text{ for some } k \in \mathbb{Z}$$

So $x \in \mathbb{Z}$ is a solution to the congruence $385x \equiv 21 \bmod 588$ if and only if $x \equiv -87 \bmod 84$.

(b) Let a, b, c, d be positive integers and suppose that a and c are coprime, that b and d are coprime, and that $ab = cd$. Prove that $a = d$ and $b = c$.

Solution. Since a and c are coprime and $a \mid cd$, we have $a \mid d$. Since d and b are coprime and $d \mid ab$, we have $d \mid a$. Hence $a = \pm d$. Since $a, d > 0$, it follows that $a = d$. Likewise, since b and d are coprime and $b \mid cd$, we have $b \mid c$. Since c and a are coprime and $c \mid ab$, we have $c \mid b$. Hence $b = \pm c$. Since $b, c > 0$, it follows that $b = c$.

(c) Find the remainder of $76! \cdot 2^{76}$ when divided by 79.

Solution. By Wilson's theorem we have

$$-1 \equiv 78! \equiv 78 \cdot 77 \cdot 76! \equiv (-1)(-2) \cdot 76! \equiv 2 \cdot 76! \bmod 79$$

Multiplying both sides by 40 yields

$$-40 \equiv 80 \cdot 76! \equiv 76! \bmod 79$$

so $76! \equiv -40 \equiv 39 \mod 79$.

By Euler's theorem, we have

$$1 \equiv 2^{78} \equiv 4 \cdot 2^{76} \mod 79$$

Multiplying both sides by 20 yields

$$20 \equiv 80 \cdot 2^{76} \equiv 2^{76} \mod 79$$

so that $2^{76} \equiv 20 \mod 79$. Hence

$$76! \cdot 2^{76} \equiv 39 \cdot 20 \equiv 69 \mod 79$$

Question 3 — functions and countability

Let A be a set of real numbers with the property that $a - b \in \mathbb{Q}$ for all $a, b \in A$. Prove that A is countable.

Solution. If A is empty then this is immediate, so suppose A is inhabited, and fix $a_0 \in A$. For all $a \in A$, we have $a - a_0 \in \mathbb{Q}$. Define a function $f : A \to \mathbb{Q}$ by

$$f(a) = a - a_0 \text{ for all } a \in A$$

Then f is well-defined, since we have seen that $a - a_0 \in \mathbb{Q}$ for all $a \in A$. Moreover, if $a, b \in A$ then

$$f(a) = f(b) \quad \Rightarrow \quad a - a_0 = b - a_0 \quad \Rightarrow \quad a = b$$

so f is injective. Since $\mathbb{Q}$ is countable, it follows from injectivity of f that A is countable.

Question 4 — counting principles

(a) Find the number of 5-card poker hands from a regular 52-card deck which contain exactly two suits and no repeated ranks.

Solution. The suits of the hand must take the form ABBBB or AABBB—that is, there must be one card of one suits and four of another, or two of one suits and three of another. These cases partition our set. Now

- A hand of the form ABBBB can be specified by the following four-step process: first select the single suit (4 choices), then choose a rank for that card (13 choices), then choose the quadrupled suit (3 choices), then choose ranks for those cards ($\binom{12}{4}$ choices, since ranks may not be repeated and one rank has already been chosen). So there are $4 \cdot 13 \cdot 3 \cdot \binom{12}{4}$ such hands of the form ABBBB.

- A hand of the form AABBB can be specified by the following four-step process: first select the doubled suit (4 choices), then choose ranks for those card ($\binom{13}{2}$ choices), then choose the tripled suit (3 choices), then choose ranks for those cards ($\binom{11}{3}$ choices, since ranks may not be repeated and one rank has already been chosen). So there are $4 \cdot \binom{13}{2} \cdot 3 \cdot \binom{11}{3}$ such hands of the form ABBBB.

As such, the size of the set is

$$4 \cdot 13 \cdot 3 \cdot \binom{12}{4} + 4 \cdot \binom{13}{2} \cdot 3 \cdot \binom{11}{3}$$

(b) By defining a finite set and computing its size in two ways, prove that

$$\sum_{i=1}^{n} \sum_{j=1}^{i} j = \binom{n+2}{3}$$

Solution. Let X be the set of subsets of $[n+2]$ with three elements. We count X in two ways.

- Procedure 1 (RHS). By definition of binomial coefficients, $|X| = \binom{n+2}{3}$.
- Procedure 2 (LHS). For each $i \in [n]$, let X_i be the set of 3-element subsets of $[n+2]$ whose greatest element is $i+2$. These sets partition X since the greatest element of a 3-element subset must be at least $3 = 1+2$ and cannot exceed $n+2$. By the addition principle, we have

  $$|X| = \sum_{i=1}^{n} |X_i|$$

  Fix $i \in [n]$. For each $j \in [i]$, let $X_{i,j}$ be the set of 3-element subsets of $[n+2]$ whose greatest element is $i+2$ and whose second-greatest element is $j+1$. The sets $X_{i,j}$ partition X_i, since the second-least element must be at least $2 = 1+1$ and cannot exceed $i+1$ (since $i+2$ is the greatest element). Hence

  $$|X_i| = \sum_{j=1}^{i} |X_{i,j}|$$

  Fix $i \in [n]$ and $j \in [i]$. A 3-element subset of $[n+2]$ whose greatest element is $i+2$ and whose second-greatest element is $j+1$ is determined by specifying its last element, which must be an element of $[j]$. Hence $|X_{i,j}| = j$.
  Putting this all together yields

  $$|X| = \sum_{i=1}^{n} |X_i| = \sum_{i=1}^{n} \sum_{j=1}^{i} |X_{i,j}| = \sum_{i=1}^{n} \sum_{j=1}^{i} j$$

By counting in two ways, the equation is proved.

Question 5 — inequalities

(a) Let $x, y > 0$. Define the *harmonic mean* and the *geometric mean* of x and y.

Solution. Their harmonic mean is $\frac{2xy}{x+y}$ and their geometric mean is $\sqrt{xy}$.

(b) Let $x, y > 0$. Prove that the harmonic mean of x and y is less than or equal to the geometric mean of x and y, and state (without proof) when equality holds.

Solution. Note first that

$$0 \leq (\sqrt{x} - \sqrt{y})^2 = x - 2\sqrt{xy} + y$$

Rearranging yields

$$2\sqrt{xy} \leq x + y$$

Multiplying both sides by $\frac{\sqrt{xy}}{x+y}$ yields

$$\frac{2xy}{x+y} \leq \sqrt{xy}$$

as required.

Equality holds if and only if $x = y$.

(c) Let $a, b, c, d \in \mathbb{R}$ with $a > c \geq 0$ and $b > d \geq 0$. Prove that

$$\left(\frac{(a+b+c+d)(a+b-c-d)}{a+b} \right)^2 \leq (a+b)^2 - (c+d)^2$$

and determine when equality holds.

Solution. Let $x = a+b+c+d$ and $y = a+b-c-d$. Then $x + y = 2(a+b)$. As such we have

$$\begin{aligned}
\frac{(a+b+c+d)(a+b-c-d)}{a+b} &= \frac{2xy}{x+y} \\
&\leq \sqrt{xy} \\
&= \sqrt{((a+b)+(c+d)}\sqrt{(a+b)-(c+d)} \\
&= \sqrt{(a+b)^2 - (c+d)^2}
\end{aligned}$$

Squaring both sides yields the desired result.

Question 6 — relations

(a) Let X be a set and let R be a reflexive, transitive relation on X. Define a new relation $\sim$ on X by letting

$$x \sim y \quad \Leftrightarrow \quad x \mathrel{R} y \text{ and } y \mathrel{R} x$$

for all $x, y \in X$. Prove that $\sim$ is an equivalence relation on X.

Solution. We prove that $\sim$ is reflexive, symmetric and transitive.

- **Reflexivity.** Let $x \in X$. Then $x \mathrel{R} x$ and $x \mathrel{R} x$ by reflexivity of R, so that $x \sim x$.
- **Symmetry.** Let $x, y \in X$ and suppose $x \sim y$. Then $x \mathrel{R} y$ and $y \mathrel{R} x$. But then $y \mathrel{R} x$ and $x \mathrel{R} y$, so that $y \sim x$.
- **Transitivity.** Let $x, y, z \in X$ and suppose $x \sim y$ and $y \sim z$. Then $x \mathrel{R} y$, $y \mathrel{R} x$, $y \mathrel{R} z$ and $z \mathrel{R} y$. by the first and third of these, we have $x \mathrel{R} z$ by transitivity of R; and by the fourth and second, we have $z \mathrel{R} x$ by transitivity of R. hence $x \sim z$.

(b) Describe the equivalence classes of $\sim$ when $X = \mathbb{Z}$ and R is the divisibility relation on $\mathbb{Z}$; that is, R is defined for $x, y \in \mathbb{Z}$ by letting $x \mathrel{R} y$ if and only if x divides y.

Solution. Note that, for all $a, b \in \mathbb{Z}$, we have $a \sim b$ if and only if $a \mid b$ and $b \mid a$, which holds if and only if $a = b$ or $a = -b$. Hence $[a]_\sim = \{a, -a\}$ for all $a \in \mathbb{Z}$.

Question 7 — probability

You roll a fair six-sided die twice. Each time you roll, you gain an amount of money in dollars equal to the number shown on the second roll minus the number shown on the first roll. (For example, if the die shows 3 on the first roll and 5 on the second roll, then you win \$2; but if the first die shows 6 and the second die shows 1, then you lose \$5.) Find the probability that the first roll showed 5 given that you lost money overall.

Solution. Let A be the event that you lose money overall and, for each $i \in [6]$, let B_i be the event that the first die roll showed the number i. The sets B_i partition the sample space, since the first die must show one of the numbers $1, 2, 3, 4, 5, 6$ and these cases are mutually exclusive. We want to find $\mathbb{P}(B_5 \mid A)$. By Bayes's theorem,

$$\mathbb{P}(B_5 \mid A) = \frac{\mathbb{P}(A \mid B_5)\mathbb{P}(B_5)}{\mathbb{P}(A \mid B_1)\mathbb{P}(B_1) + \cdots + \mathbb{P}(A \mid B_6)\mathbb{P}(B_6)}$$

Now $\mathbb{P}(B_i) = \frac{1}{6}$ for each $i \in [6]$, since this is just the probability that the first die shows i. Moreover, $\mathbb{P}(A \mid B_i) = \frac{i-1}{6}$ for each $i \in [6]$, since this is the probability that the second die shows a value less than i. Hence

$$\mathbb{P}(B_5 \mid A) = \frac{\frac{4}{6} \cdot \frac{1}{6}}{\frac{0}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} + \frac{2}{6} \cdot \frac{1}{6} + \frac{3}{6} \cdot \frac{1}{6} + \frac{4}{6} \cdot \frac{1}{6} + \frac{5}{6} \cdot \frac{1}{6}} = \frac{4}{0+1+2+3+4+5} = \frac{4}{15}$$

Question 8 — probability

There are a hundred quarters in a bowl, of which twenty commemorate the British Surrender in 1777. Clive wishes to remove all these quarters from circulation, and begins drawing quarters from the bowl. With each draw, if the coin drawn commemorates the British Surrender, he puts it in his pocket; otherwise, he returns it to the bowl. He stops when all twenty British Surrender quarters are in his pocket. Find the expected number of times that Clive draws a quarter from the box.

Solution. Let X be the total number of times Clive draws a quarter from the box and, for each $i \in [20]$, let X_i be the number of times he draws from the box between collecting the $(i-1)^{\text{th}}$ British Surrender quarter to having the i^{th}. Then

$$X = X_1 + X_2 + \cdots + X_{20}$$

By linearlity of expectation, it follows that

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_{20}]$$

To compute $\mathbb{E}[X_i]$ for each $i \in [20]$, we consider its distribution. If Clive has drawn $i-1$ British Surrender quarters from the bowl, then there are $101-i$ quarters remaining, $21-i$ of which are British Surrender quarters. Thus, for each $k \geq 1$, the event $\{X_i = k\}$ corresponds to the situation in which Clive fails to draw a British Surrender quarter $k-1$ times, and then succeeds to draw a British Surrender quarter on the k^{th} draw. Hence

$$\mathbb{P}\{X_i = k\} = \left(1 - \frac{21-i}{101-i}\right)^{k-1} \cdot \left(\frac{21-i}{101-i}\right)$$

Hence X_i is geometrically distributed, with parameter $p = \frac{21-i}{101-i}$. Its expectation is therefore equal to $\frac{101-i}{21-i}$. In summary, we have

$$\mathbb{E}[X] = \frac{100}{20} + \frac{99}{19} + \cdots + \frac{81}{1} \approx 307.8$$