1. (i) (10 points) Let a and b be unspecified integers. Find, in terms of a and b, all integers x which satisfy $x \equiv a \mod 2$ and $x \equiv b \mod 3$.

Solution. This is problem 5 from pset 7 with $n = 2$ and $m = 3$. The solution set consists of all x congruent to $a + 4(b - a)$ mod 6.

(ii) (10 points) Let a, b, and c be unspecified integers. Find, in terms of a, b, and c, all integers x which satisfy $x \equiv a \mod 2$, $x \equiv b \mod 3$, and $x \equiv c \mod 7$. You may use your result from part (i).

Solution. This is problem 5 from pset 7 with $n = 6$ and $m = 7$. The solution set consists of all x congruent to $4b - 3a + 36(c - (4b - 3a))$ mod 42. This can be simplified to $105a - 140b + 36c$ mod 42. Note that other integer linear combinations of a, b, and c work.

2. Determine whether each of the following sets is finite. Assume that $\mathbb{N}$ and $\mathbb{Z}$ exist, and are infinite. You needn't prove well-definedness or any properties of functions that you display, but you must clearly state any properties of such functions and how they are used.

(i) (10 points) $\{(x, y) \in \mathbb{N} \times \mathbb{N} : x + y = 100\}$

Solution. Let $f : \{(x, y) \in \mathbb{N} \times \mathbb{N} : x + y = 100\} \to [101]$ via $f(a, b) = a + 1$. Since f is an injection and $[101]$ is finite, the domain is finite.

(ii) (10 points) $\{(x, y) \in \mathbb{N} \times \mathbb{Z} : x + y = 1\}$

Solution. Let $g : \{(x, y) \in \mathbb{N} \times \mathbb{Z} : x + y = 1\} \to \mathbb{N}$ via $g(a, b) = a$. Since g is a bijection and $\mathbb{N}$ is infinite, the domain is infinite. Note: One could go further and assume that the set in question is finite, and that $h : [n] \to \{(x, y) \in \mathbb{N} \times \mathbb{Z} : x + y = 1\}$ is a bijection for some natural number n. This yields the contradiction that $\mathbb{N}$ is finite by considering h composed with g.

3. (20 points) Determine the remainder when $(20!)^{2201}$ is divided by 23.

Solution.

$$
\begin{aligned}
& (20!)^{2201} & \\
\equiv_{23} & (20!)(20!)^{2200} & \\
\equiv_{23} & (20!)((20!)^{100})^{22} & \\
\equiv_{23} & 20! & \text{FLT, } 20! \perp 23 \\
\equiv_{23} & (22 * 21)^{-1} * (22 * 21) * 20! & \\
\equiv_{23} & (-1 * -2)^{-1} * 22! & \\
\equiv_{23} & 2^{-1} * 22! & \\
\equiv_{23} & 12 * (-1) & \text{Wilson's Theorem} \\
\equiv_{23} & 11 &
\end{aligned}
$$

4. (i) Count the number of 4 card hands with **exactly** 2 suits.

Solution. First, we choose 2 suits $\binom{4}{2}$, then we partition based on AABB or AAAB.

- In the first case, we choose 2 cards from the higher alphabetical suit then the 2 cards for the second suit. by MP this yields $\binom{13}{2}\binom{13}{2}$
- In the second case, we choose which suit has 3 cards ($\binom{2}{1}$), then pick cards for each suit: by MP, this yields $2\binom{13}{3}\binom{13}{1}$

by AP This yields $\binom{4}{2}\left(\binom{13}{2}\binom{13}{2} + 2\binom{13}{3}\binom{13}{1}\right)$

(ii) Count the number of 2 card hands with **more** red cards than face cards.

Solution. We partition on the number of red cards. If we have 1 red card, we cannot have a face card. If we have 2 red cards, we must have either no face cards or only 1 face card (which must be red).

- In the first case, we pick a red non face card, then a black non face card. By MP:
$$\binom{20}{1}\binom{20}{1}$$
- In the next case, we pick 2 red cards that are not face cards:
$$\binom{20}{2}$$
- Finally, in the last case we pick 1 red face card then 1 red non face card. By MP:
$$\binom{6}{1}\binom{20}{1}$$

In total by AP, we have $\binom{20}{1}\binom{20}{1} + [\binom{20}{2} + \binom{6}{1}\binom{20}{1}]$

5. Count the number of **injections** $f : [20] \to [40]$ s.t. $\forall x \in [20]$, $x \equiv f(x) \bmod 5$.

Solution. Every value in the image of f will either be $0, 1, 2, 3, 4 \bmod 5$. So suppose we want to count the number of ways we can map all $x \in [20]$ that satisfy $x \equiv 0 \bmod 5$. In this case, we have $5, 10, 15, 20$, which then each have to map to a different value in $5, 10, 15, 20, 25, 30, 35, 40$. We define a process as follows:

- Choose where the lowest element maps to (8 choices)
- Choose where the second lowest element maps to (7 choices)
- Choose where the third lowest element maps to (6 choices)
- Choose where the last element maps to (5 choices)

Thus, by MP we have $8 * 7 * 6 * 5$ choices. We consider the same process for numbers mod 1, 2, 3, and 4. Then by MP we have a final solution of $(8 * 7 * 6 * 5)^5$.

Another way to think of this problem is that each set of 4 domain elements equivalent mod 5 form their own injection to a set of 8 elements in the codomain.

Bonus. compute $7^{11!} \bmod 23$ (simplify to a value between 0 and 22)
Solution:

$$
\begin{aligned}
& 7^{11!} \\
\equiv_{23} & \left(7^{3*4*5*6*7*8*9*10}\right)^{2*11} \\
\equiv_{23} & \left(7^{3*4*5*6*7*8*9*10}\right)^{22} \\
\equiv_{23} & 1 && \text{FLT, } 7^k \perp 23 \forall k \in \mathbb{N}
\end{aligned}
$$