151 Excellent Exam 1 Review

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Midterm 1

1 Things to Know

1.1 Basics

1.1.1 Basic proof techniques and templates

- **Direct proofs**: Only use the assumption from the problem to derive the conclusion; i.e. if proving $p \implies q$, only assume p and derive q.
- **Contradiction**: Assume the opposite of what you want to prove, derive a contradiction (bad math fact or violation of some other assumption or theorem)
  - Note that if you're trying to prove $p \implies q$, you assume $p \wedge \neg q$ (which is the same thing as $\neg(p \implies q)$) for contradiction
- **Contrapositive**: ONLY for $p \implies q$ proofs; prove $\neg q \implies \neg p$
- **Casework**: break the assumption or what you're trying to prove into cases
  - Must show that your cases are exhaustive
  - Good to try if you're stuck and haven't used some assumption yet
- **If and only if**: Must prove both directions; sometimes this isn't explicitly stated as an if and only if
  - Look for "solve this equation", "for what conditions", "find the set"; refer to homework and Clive for more examples of iff phrasing
- **Proving quantified statements**:
  - $\exists x \in X, p(x)$: Find an x, show it is in X, and show that $p(x)$ is true
  - $\forall x \in X, p(x)$: Consider a general $x \in X$ and show $p(x)$ is true

1.1.2 Uniqueness

- **Formal statement of uniqueness**: there is one and it is the only one
  - $\exists! x \in X, p(x) \equiv (\exists x \in X, p(x)) \wedge (\forall x_1, x_2 \in X, p(x_1) \wedge p(x_2) \implies x_1 = x_2)$
  - This also tells you the proof structure: first show it exists, then show that it is the only one that exists
- **Negation of uniqueness:** (practice by negating the above yourself!)

1.1.3 Negations

- **Proving statement false**: prove the negation is true
- **Counterexample**: Use this when you are trying to prove a *for all* statement false
  - Note that this is the same as proving the negation
  - DOES NOT work to prove a *there exists* statement false!
- **Negations of all the logical connectives**
  - $\neg(p \wedge q) = \neg p \vee \neg q$
  - $\neg(p \vee q) = \neg p \wedge \neg q$
  - $\neg(p \implies q) = p \wedge \neg q$
  - $\neg(p \iff q) = (p \wedge \neg q) \vee (\neg p \wedge q)$
- **Negations of quantified statements**
  - Work from the outside in
  - Negation of $\forall x \in X, p(x) = \exists x \in X, \neg p(x)$
  - Negation of $\exists x \in X, p(x) = \forall x \in X, \neg p(x)$

1.2 Sets

- Formal definitions of the **set operations**:
  - $A \cup B = \{a \in \mathcal{U} | a \in A \vee a \in B\}$
  - $A \cap B = \{a \in \mathcal{U} | a \in A \wedge a \in B\}$
  - $A \setminus B = \{a \in \mathcal{U} | a \in A \wedge a \notin B\} = \{a \in A | a \notin B\}$
  - $A \times B = \{(a, b) | a \in A \wedge b \in B\}$
  - $A \subseteq B \equiv \forall a \in A, a \in B$
  - $\overline{A} = A^c = \{b \in \mathcal{U} | b \notin A\}$
  - $\mathcal{P}(A) = \{B \in \mathcal{U} | B \subseteq A\}$
- **Proving $A \subseteq B$:** Show that any $x \in A$ must also be in B
- **Proving $A = B$:** Show $A \subseteq B$ and $B \subseteq A$; same as $x \in A \iff x \in B$
- **Converting between sets and propositional logic**: use set-builder notation or consider a general x in the set and use definitions
  - **DO NOT** mix sets with propositional logic connectives!!

1.3 Functions

1.3.1 Images, Pre-images, well-definedness:

- **Definition of function**: outlines the conditions for well-definedness; suppose $f : X \to Y$
  - **Totality:** f knows what to do - f is specified for all $x \in X$
  - **Existence:** f does it right - $f(x) \in Y$ for all $x \in X$
  - **Uniqueness:** every x has a unique value of $f(x)$
- **Definition of image**: For some $f : X \to Y, U \subseteq X$, $f[U] = \{y \in Y \mid \exists x \in U, f(x) = y\}$.
  - When proving that $f[U] = S$, you must show *both* directions
  - For the reverse direction, make sure to show that the x you consider is actually in U

- If you are unsure what the image is:
  * Try out values of x and look for a pattern (a la 6b on PSET2)
  * Use what you know about $x \in U$ to come up with bounds for $f(x)$ - here it's good to remember $x^2 \geq 0$ for all $x \in \mathbb{R}$ and other general inequality rules (looking back at homework solutions might help you see these!)
  * Graph the function specification

- **Definition of pre-image**: For some $f : X \to Y, V \subseteq Y,\ f^{-1}[V] = \{x \in X \mid f(x) \in V\}$.
  - Again, show both directions when proving $f^{-1}[V] = S$
  - Do not confuse this notation with the inverse!
  - If you are unsure what the pre-image is, use what you know about $y \in V$ and how $f(x)$ is defined to come up with bounds for x - here rearranging to solve for x in terms of y may help

1.3.2 *jectivity, *inverse:

- **Definition of injectivity:** $f : X \to Y$ is injective iff $\forall x_1, x_2 \in X, f(x_1) = f(x_2) \implies x_1 = x_2$
- **Definition of surjectivity:** $f : X \to Y$ is surjective iff $\forall y \in Y, \exists x \in X, f(x) = y$
  - Equivalent to saying $f[X] = Y$
- **Definition of bijectivity:** $f : X \to Y$ is bijective iff f is injective and f is surjective
- **Relationships between inverses and *jectivity**:
  - f is injective iff f has a left inverse and the domain of f is inhabited
  - f is surjective iff f has a right inverse
  - f is bijective iff f has a two-sided inverse
- **Finding inverses**: find a specification for g (the inverse of f), prove it is well-defined, prove that is an inverse (left, right, or both depending on what you are trying to show)
  - To find the specification, try setting y equal to $f(x)$ and then solve for x in terms of y

2 Problems

We've put together a bunch of problems for you to use for review. I'd recommend sitting down and solving them without looking at the solutions. The midterm itself is 53 minutes long, and consists of five questions, so time yourself accordingly. In general, you should aim to spend no more than 10 minutes on a problem. Remember, you can be terser and provide less justification (but not *no* justification) than you would on a typical homework.

2.1 Sets

1. Prove that $(A\Delta B)\cup C = (A\cup C)\Delta(B\setminus C)$, where $A\Delta B = (A\setminus B)\cup(B\setminus A)$ is defined as the symmetric difference of A and B.
   Note: This is a pretty involved proof

2.2 Functions: Images/Pre-images/Well-Definedness

2. Let $f : X \to Y$. Prove that $f^{-1}[Y \setminus U] = X \setminus f^{-1}[U]$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = \frac{x}{1+|x|}$. Find the image of f.

4. Let $g : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ via $g(x, y) = xy$. Find $g[[\mathbb{Q} \cap (0, 1)]^2]$.

2.3 Functions: *jectivity/Inverses

5. Let $f : (1, \infty) \to (0, 1)$ be defined via $f(x) = \frac{x-1}{x+1}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined via $g(x) = 2x + 3$.
   (a) What is $g \circ f$?
   (b) What are the domain and codomain of $g \circ f$?
   (c) Prove $g \circ f$ is an injection.
   (d)

6. Let $h : [n] \to \mathcal{P}([n])$ be defined via $h(x) = [n] \setminus \{x\}$. Find a left inverse.

2.4 The Rest

7. Show that the binary expansion of every decimal number is unique.

8. If x and y are distinct real numbers, then $(x + 1)^2 = (y + 1)^2$ iff $x + y = -2$.

9. If you have a total of $2n + 1$ items placed in 2 boxes, show that at least one box must have at least $n + 1$ items.

10. If $a, b, c \in \mathbb{Z}$ such that $a^2 + b^2 = c^2$, either a or b is even.

3 Solutions

These are solutions, as we would expect you to write them on the midterm. Note that these are often shorter and less thoroughly explained than your HW solutions.
Any grayed out text is additional information / extra notes that you would not need to write in your problem.

Problem 1 *Prove that* $(A \Delta B) \cup C = (A \cup C) \Delta (B \setminus C)$*, where* $A \Delta B = (A \setminus B) \cup (B \setminus A)$ *is defined as the symmetric difference of* A *and* B*.*

Solution:
($\subseteq$):
Let $x \in (A \Delta B) \cup C$. Then by the definition of set union, $x \in A \Delta B$ or $x \in C$.

Suppose $x \in A \Delta B$. Then, $x \in (A \setminus B) \cup (B \setminus A)$. Consider the followinng two cases:

(i) If $x \in A \setminus B$, then $x \in A$ and $x \notin B$. So, since $x \in A$, $x \in A \cup C$ and since $x \notin B$, $x \notin B \setminus C$. Thus, $x \in (A \cup C) \setminus (B \setminus C)$ by the definition of set minus. Since $(A \cup C)\Delta(B \setminus C) = ((A \cup C) \setminus (B \setminus C)) \cup ((B \setminus C) \setminus (A \cup C))$, $x \in (A \cup C)\Delta(B \setminus C)$ by the definition of set union.

(ii) If $x \in B \setminus A$, then $x \in B$ and $x \notin A$. Suppose $x \in C$. Then, $x \in A \cup C$ by the definition of set union and $x \notin B \setminus C$ by the definition of set minus. The argument is thus the same as above. Suppose $x \notin C$. Then, $x \in B \setminus C$. Since $x \notin C$ and $x \notin A$, $x \notin A \cup C$, so $x \in (B \setminus C) \setminus (A \cup C)$ and so $x \in (A \cup C)\Delta(B \setminus C)$.

Suppose $x \in C$. Then, $x \in A \cup C$ and $x \notin B \setminus C$. So, $x \in (A \cup C) \setminus (B \setminus C)$ and therefore $x \in (A \cup C)\Delta(B \setminus C)$ by the definition of set union and Δ.

So, $x \in (A \cup C)\Delta(B \setminus C)$.

($\supseteq$)
Suppose $x \in (A \cup C)\Delta(B \setminus C)$. Then, $x \in ((A \cup C) \setminus (B \setminus C)) \cup ((B \setminus C) \setminus (A \cup C))$.

Suppose $x \in (A \cup C) \setminus (B \setminus C)$. Then, $x \in A \cup C$ and $x \notin B \setminus C$. Since $x \in A \cup C$, $x \in A$ or $x \in C$. Since $x \notin B \setminus C$, $x \notin B$ or $x \in C$. Case on whether $x \in C$:

(i) Suppose $x \in C$. Then, $x \in (A \Delta B) \cup C$ by the definition of set union.

(ii) Suppose $x \notin C$. Then, $x \in A$ and $x \notin B$. So, $x \in A \setminus B$, so $x \in A \Delta B$. Thus, $x \in (A \Delta B) \cup C$.

Suppose $x \in (B \setminus C) \setminus (A \cup C)$. Then, $x \in B \setminus C$ and $x \notin A \cup C$. So, $x \in B$ and $x \notin C$ and $x \notin A$.

Thus, $x \in B \setminus A$, so $x \in A \Delta B$ and thus $x \in (A \Delta B) \cup C$.

■

Problem 2 *Let* $f : X \to Y$*. Prove that* $f^{-1}[Y \setminus U] = X \setminus f^{-1}[U]$*.*

Solution:
We show both directions of the double containment simultaneously. In particular, we show that
$x \in f^{-1}[Y \setminus U] \iff x \in X \setminus f^{-1}[U]$.

$$
\begin{aligned}
& x \in f^{-1}[Y \setminus U] \\
\iff & f(x) \in Y \setminus U \\
\iff & f(x) \in Y \wedge f(x) \notin U \\
\iff & x \in f^{-1}[Y] \wedge x \notin f^{-1}[U] \\
\iff & x \in X \wedge x \notin f^{-1}[U] \\
\iff & x \in X \setminus f^{-1}[U]
\end{aligned}
$$

$\blacksquare$

Note: *If you are trying to prove a bi-implication, it might be worth stopping for a second and considering how similar your proofs for the two directions are. If they are very similar, it is possible you could just have a single chain of bi-implications rather than writing out the same argument forwards then backwards.*

Problem 3 *Let $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = \frac{x}{1+|x|}$. Find the image of f.*

Solution:
$f[\mathbb{R}] = (-1, 1)$. We show this by double containment.

($\subseteq$):
Let $y \in f[\mathbb{R}]$. We want to show $-1 < x < 1$.
We know $\exists x$ s.t. $\frac{x}{1+|x|} = y$. We consider two cases: $x \geq 0$ and $x < 0$.
If $x \geq 0$, $\frac{x}{1+|x|} = \frac{x}{1+x}$. But $0 \leq x < 1 + x$, so $0 \leq \frac{x}{1+x} < \frac{1+x}{1+x} = 1$. Then $y = \frac{x}{1+x}$ is in $[0, 1)$, and hence $-1 < y < 1$.
If $x \leq 0$, $\frac{x}{1+|x|} = \frac{-|x|}{1+|x|}$. Since $|x| \geq 0$, we can use the above argument, and hence $\frac{x}{1+|x|} \in (-1, 0]$, and hence $y \in (-1, 1)$.

($\supseteq$):
Suppose $y \in (-1, 1)$. We want to find $x \in \mathbb{R}$ such that $f(x) = \frac{x}{1+|x|} = y$.
If $y \geq 0$, consider $x = \frac{y}{1-y}$. Since $0 \leq y < 1$, x is positive. Then $f(x) = \frac{x}{1+|x|} = \frac{x}{1+x} = \frac{y/(1-y)}{1+y/(1-y)} = \frac{y/(1-y)}{1/(1-y)} = y$.
If $y < 0$, consider $x = \frac{y}{1+y}$. Since $-1 < y \leq 0$, this is negative. Then $f(x) = \frac{x}{1-x} = \frac{y/(1+y)}{1-y/(1+y)} = \frac{y/(1+y)}{1/(1+y)} = y$.
Then, $y \in f[\mathbb{R}]$.

$\blacksquare$

Problem 4 *Let $g : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ via $g(x, y) = xy$. Find $g[[\mathbb{Q} \cap (0, 1)]^2]$.*

Solution:
Claim: $g[[\mathbb{Q} \cap (0, 1)]^2] = \mathbb{Q} \cap (0, 1)$. We prove this via double containment.

($\subseteq$):
Suppose $x, y \in \mathbb{Q} \cap (0, 1)$. Then $0 < x, y < 1$ and $x, y \in \mathbb{Q}$. Then $g(x, y) = xy \in \mathbb{Q}$ since the rationals are closed under multiplication. Furthermore, $x < 1 \implies xy < y$, and $y < 1$, so $xy < 1$. It is also clear that $xy > 0$ since $x, y > 0$.

($\supseteq$):
This is considerably harder. Say $x \in \mathbb{Q} \cap (0, 1)$. Then, $g\left(\frac{x+1}{2}, \frac{2x}{x+1}\right)$. Note that $\frac{x+1}{2} > \frac{1}{2}$ and $x + 1 < 2 \implies \frac{x+1}{2} < 1$. Then, $\frac{x+1}{2} \in \cap(0, 1)$. Also, $\frac{2x}{x+1} > 0$ clearly since $x > 0$. Also, $x < 1 \implies 2x < x + 1$, so this quantity is < 1.
Furthermore, since $x \neq -1$, both quantities are rationals since dividing a rational by a non-zero rational yields another rational.
Hence, $\frac{x+1}{2}, \frac{2x}{x+1} \in \mathbb{Q} \cap (0, 1)$.

$\blacksquare$

Note: *There are multiple alternate values you could find that map to a given rational. For example, if you assume $x = \frac{p}{q}$, $\left(\frac{pq}{q^2-1}, \frac{q^2-1}{q^2}\right)$ and $\left(\frac{2p}{p+q}, \frac{p+q}{2q}\right)$ both map to x.*
In general, try various examples and try and come up with values that map to your chosen rationals to gain intuition. For example, the intuition for the last example is you want $\left(\frac{p}{something} \frac{something}{q}\right)$. This "something" must also be between p and q, so realistically we want it to be something like $\frac{p+q}{2}$, but this may not be an

integer, so we simply use $\frac{2p}{p+q}$ rather than $\frac{p}{(p+q)/2}$, and similarly for the second term.
If instead you let the "something" be $q - \frac{1}{q}$ and then simplify, you get the first expression I wrote in terms of p, q.

Problem 5 *Let $f : (1, \infty) \to (0, 1)$ be defined via $f(x) = \frac{x-1}{x+1}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined via $g(x) = 2x + 3$.*

(a) *What is $g \circ f$?*

(b) *What are the domain and codomain of $g \circ f$?*

(c) *Prove $g \circ f$ is an injection.*

Solution:

(a) For any $x \in (1, \infty)$:

$$\begin{aligned}(g \circ f)(x) =& g(f(x)) \\ =& g\left(\frac{x-1}{x+1}\right) \\ =& 2\left(\frac{x-1}{x+1}\right) + 3 \\ =& \frac{2(x-1) + 3(x+1)}{x+1} \\ =& \frac{5x+1}{x+1}\end{aligned}$$

(b) $g \circ f : (1, \infty) \to \mathbb{R}$

(c) There are two ways to do this. For convenience, denote $g \circ f$ as h

1) Directly
Suppose $h(x_1) = h(x_2)$. Then $\frac{5x_1+1}{x_1+1} = \frac{5x_2+1}{x_2+1}$. Cross multiplying, you get
$5x_1x_2 + 5x_1 + x_2 + 1 = 5x_1x_2 + 5x_2 + x_1 + 1$.
Simplifying:
$4x_1 = 4x_2$.
Hence $x_1 = x_2$

2) By finding a left inverse
Want to find an h_2 such that $h_2(h(x)) = x \ \forall x \in (1, \infty)$.
The algebra involved is considerably hairier if you use this method, and I should be asleep by now, so if you see this message, that means I never wrote out a full solution.
But, if my math is right, you should get your left inverse to be
$h_2(x) = \begin{cases} \frac{x-1}{5-x} & x \geq 3 \\ 0 \text{ (or anything really)} & \text{otherwise} \end{cases}$

■

Problem 6 *Let $h : [n] \to \mathcal{P}([n])$ be defined via $h(x) = [n] \setminus \{x\}$. Find a left inverse.*

Solution:
Let $f : \mathcal{P}([n]) \to [n]$ be defined via:

$$f(S) = \begin{cases} x & S = [n], \text{for some } x \in [n] \\ \min\{x \notin S\} & \text{otherwise} \end{cases}$$

First, f is well-defined because every subset of $[n]$ is either equal to $[n]$ or not. If $S = [n]$, then $f(S) = x$ and $x \in [n]$, so $f(S) \in [n]$. If $S \subset [n]$, then $f(S) = \min\{x \notin S\}$. Because $S \neq [n]$, such an x must exist since this $x \in [n]$, $f(S) \in [n]$. Finally, $f(S)$ has a unique value for all S because the function species a single element in either case. So, f is well-defined.

We will now show that f is a left inverse. Consider $x \in [n]$ and $f(h(x))$. Then, $f(h(x)) = f([n] \setminus \{x\})$ by the definition of $h(x)$. Then, clearly $[n] \setminus \{x\} \neq [n]$, so $f([n] \setminus \{x\}) = min\{x \notin ([n] \setminus \{x\})\} = min\{x \in \{x\}\} = x$. ■

Problem 7 *Show that the binary expansion of every decimal number is unique.*

Solution:
Firstly, using the base conversion algorithms from class, it can be shown that n must have a binary representation.

Let n be a decimal number and suppose it has two different binary representations. So, $n = b_k b_{k-1} ... b_0$ and $n = c_j c_{j-1} ... c_0$ for some $k, j \in \mathbb{N}$. WLOG, $k > j$. Then, since the binary representations are different, let i be the first index at which they differ.

So, using the base-b expansion, we have $b_k 2^k + b_{k-1} 2^{k-1} + ... + b_i 2^i + b_{i-1} 2^{i-1} + ... + b_0 = c_j 2^j + c_{j-1} 2^{j-1} + ... + c_i 2^i + c_{i-1} 2^{i-1} + ... + c_0$. Thus, since all the digits before index i are the same, we have that $b_k 2^k + b_{k-1} 2^{k-1} + ... + b_i 2^i = c_j 2^j + c_{j-1} 2^{j-1} + ... + c_i 2^i$. Without loss of generality, $b_i = 1$ and $c_i = 0$. Then, $b_k 2^k + b_{k-1} 2^{k-1} + ... + 2^i = c_j 2^j + c_{j-1} 2^{j-1} + ... + c_{i+1} 2^{i+1}$. Then $c_j 2^j + c_{j-1} 2^{j-1} + ... + c_{i+1} 2^{i+1}$ is divisible by 2^{i+1} but $b_k 2^k + b_{k-1} 2^{k-1} + ... + 2^i$ is not divisible by 2^{i+1} because it has a remainder 2^i. So, we have a contradiction since the two numbers are supposed to be equal. ■

Problem 8 *If x and y are distinct real numbers, then $(x+1)^2 = (y+1)^2$ iff $x + y = -2$.*

Solution:
($\Rightarrow$):
Suppose x, y are distinct real numbers such that $(x+1)^2 = (y+1)^2$. Then:

$$\begin{array}{rrl} & (x+1)^2 - (y+1)^2 & = 0 \\ \iff & ((x+1) - (y+1))((x+1) + (y+1)) & = 0 \\ \iff & (x-y)(x+y+2) & = 0 \end{array}$$

This is true if $x = y$ or $x + y = -2$. However, since x and y are assumed to be distinct, $x \neq y$. Thus, $x + y = -2$.

($\Leftarrow$):
Suppose $x + y = -2$. Then $x = -y - 2$. Then:

$$\begin{aligned} (x+1)^2 &= (-y-2+1)^2 \\ &= (-y-1)^2 \\ &= (-1)^2(y+1)^2 \\ &= (y+1)^2 \end{aligned}$$

■

Problem 9 *If you have a total of $2n+1$ items placed in 2 boxes, show that at least one box must have at least $n+1$ items.*

Solution:
AFSOC that every box has less than $n+1$ items. So, each of the two boxes has at most n items. Then, the total number of items is at most $2n$. However, this contradicts the assumption that there was a total of $2n+1$ items.

$\blacksquare$

Note: *This is called the pigeonhole principle (when generalized to more than 2 boxes)*

Problem 10 *If $a, b, c \in \mathbb{Z}$ such that $a^2 + b^2 = c^2$, either a or b is even.*

Solution:
AFSOC neither a nor b are even. Then, $a = 2k+1$ for some $k \in \mathbb{Z}$ and $b = 2j+1$ for some $j \in \mathbb{Z}$. So, $a^2 + b^2 = 4j^2 + 4j + 1 + 4k^2 + 4k + 1 = 4j^2 + 4k^2 + 4j + 4k + 2 = c^2$.

There are two cases:
c is even:
Then, $c = 2m$ for some $m \in \mathbb{Z}$, so $c^2 = 4m^2$. Thus, c^2 is divisible by 4, but this is a contradiction since $c^2 = 4j^2 + 4k^2 + 4j + 4k + 2$.

c is odd:
Then, $c = 2m+1$ for some $m \in \mathbb{Z}$, so $c^2 = 4m^2 + 4m + 1$. Thus, c^2 has a remainder of 1 when divided by 4, but this is a contradiction since c^2 has a remainder of 2.

$\blacksquare$

Note: *In general, whenever dealing with squares (especially when the problem relates to whether a number is even or odd), it is useful to consider what various numbers leave as remainder when divided by 4.*