21-128 and 15-151 problem sheet 5

Solutions to the following seven exercises and optional bonus problem are to be submitted through Gradescope.

Recall that in class we defined the rectype list over some ground set A:

- single atom nil,
- constructors $\mathsf{prep}[a, L]$, one for each $a \in A$.

The idea is that nil represents the empty list, and $\mathsf{prep}[a, L]$ represents the list L with element a prepended. We also defined append, join and reversal operations. Below we use the informal :: notation from class.

For problems 1 and 2, you can use all the identities proven in the lecture slides, but annotate your argument clearly. If you need additional results (you will), establish them in separate arguments. The argument for commutativity of addition in Dedekind-Peano arithmetic is also a good source for inspiration.

Problem 1

Prove $\mathsf{rev}(\mathsf{rev}(L)) = L$ by induction on lists.

Solution. We proceed by structural induction on L:

Base Case: $L = \mathsf{nil}$.

$$\begin{aligned} \mathsf{rev}(\mathsf{rev}(\mathsf{nil})) &= \mathsf{rev}(\mathsf{nil}) && (\mathsf{rev}_1) \\ &= \mathsf{nil} && (\mathsf{rev}_1) \end{aligned}$$

Inductive Step. $L = a :: L'$. Assume $\mathsf{rev}(\mathsf{rev}(L')) = L'$.

$$\begin{aligned} \mathsf{rev}(\mathsf{rev}(a :: L')) &= \mathsf{rev}(\mathsf{rev}(L') :: a) && (\mathsf{rev}_2) \\ &= a :: \mathsf{rev}(\mathsf{rev}(L')) && (\text{rectypes slide 22}) \\ &= a :: L' && (\text{IH}) \end{aligned}$$

Problem 2

Prove $\mathsf{rev}(L :: K) = \mathsf{rev}(K) :: \mathsf{rev}(L)$ by induction on lists.

Solution.

First we prove the following two lemmas by structural induction on L:

Lemma 1. $L :: \mathsf{nil} = L$.

Base Case. $L = \mathsf{nil}$.

$$\mathsf{nil} :: \mathsf{nil} = \mathsf{nil} \qquad (\mathsf{join}_1)$$

Inductive Step. $L = a :: L'$. Assume $L' :: \mathsf{nil} = L$.

$$\begin{aligned} (a :: L') :: \mathsf{nil} &= a :: (L' :: \mathsf{nil}) && (\mathsf{join}_2) \\ &= a :: L' && (\mathrm{IH}) \end{aligned}$$

Lemma 2. $(L :: K) :: a = L :: (K :: a)$.

Base Case. $L = \mathsf{nil}$.

$$\begin{aligned} (\mathsf{nil} :: K) :: a &= K :: a && (\mathsf{join}_1) \\ &= \mathsf{nil} :: (K :: a) && (\mathsf{join}_1) \end{aligned}$$

Inductive Step. $L = b :: L'$. Assume $(L' :: K) :: a = L' :: (K :: a)$.

$$\begin{aligned} ((b :: L') :: K) :: a &= (b :: (L' :: K)) :: a && (\mathsf{join}_2) \\ &= b :: ((L' :: K) :: a) && (\mathsf{app}_2) \\ &= b :: (L' :: (K :: a)) && (\mathrm{IH}) \\ &= (b :: L') :: (K :: a) && (\mathsf{join}_2) \end{aligned}$$

Now, we show $\mathsf{rev}(L :: K) = \mathsf{rev}(K) :: \mathsf{rev}(L)$ by structural induction on L:

Base Case: $L = \mathsf{nil}$.

$$\begin{aligned} \mathsf{rev}(\mathsf{nil} :: K) &= \mathsf{rev}(K) && (\mathsf{join}_1) \\ &= \mathsf{rev}(K) :: \mathsf{nil} && (\text{Lemma 1}) \\ &= \mathsf{rev}(K) :: \mathsf{rev}(\mathsf{nil}) && (\mathsf{rev}_1) \end{aligned}$$

Inductive Step: $L = a :: L'$. Assume $\mathsf{rev}(L :: K) = \mathsf{rev}(K) :: \mathsf{rev}(L)$.

$$
\begin{aligned}
\mathsf{rev}((a :: L') :: K) &= \mathsf{rev}(a :: (L' :: K)) && (\mathsf{join}_2)\\
&= \mathsf{rev}(L' :: K) :: a && (\mathsf{rev}_2)\\
&= (\mathsf{rev}(K) :: \mathsf{rev}(L')) :: a && \text{(IH)}\\
&= \mathsf{rev}(K) :: (\mathsf{rev}(L') :: a) && \text{(Lemma 2)}\\
&= \mathsf{rev}(K) :: \mathsf{rev}(a :: L') && (\mathsf{rev}_2)
\end{aligned}
$$

Problem 3

For each example below, determine whether the given relation R is an equivalence relation on the given set S:

(a) $S = \mathbb{N} \setminus \{0, 1\}$; $(x, y) \in R$ if and only if $\gcd(x, y) > 1$.

(b) $S = \mathbb{R}$; $(x, y) \in R$ if and only if there exists $n \in \mathbb{Z}$ such that $x = 2^n y$.

Solution.

(a) R is not an equivalence relation since it is not transitive. For example, $(2, 6) \in R$ since $\gcd(2, 6) = 2 > 1$, and $(6, 3) \in R$ since $\gcd(6, 3) = 3 > 1$, but $(2, 3) \notin R$ since $\gcd(2, 3) = 1$.

(b) R is an equivalence relation:

- (R is reflexive.) Let $x \in \mathbb{R}$. Then $x = 2^0 x$, so $(x, x) \in \mathbb{R}$.
- (R is symmetric.) Let $x, y \in \mathbb{R}$ and suppose $(x, y) \in R$. Then $x = 2^n y$ for some $n \in \mathbb{Z}$. But then $y = 2^{-n} x$, and $-n \in \mathbb{Z}$, so $(y, x) \in R$.
- (R is transitive.) Let $x, y, z \in \mathbb{R}$ and suppose $(x, y) \in R$ and $(y, z) \in R$. Then $x = 2^m y$ and $y = 2^n z$ for some $m, n \in \mathbb{Z}$. But then $x = 2^{m+n} z$, and $m + n \in \mathbb{Z}$, so $(x, z) \in R$.

Problem 4

For every $n \in \mathbb{N}$ let $\sim_n$ be the relation on $\mathcal{P}([n])$ specified by $A \sim_n B$ if and only if $A \subseteq B$ or $B \subseteq A$. Determine, with proof, all $n \in \mathbb{N}$ such that $\sim_n$ is an equivalence relation.

Solution. Claim: $\sim_n$ is an equivalence relation if and only if $n = 0$ or $n = 1$.

Proof: First note that if $n = 0$ or $n = 1$, then $\mathcal{P}([n])$ is $\{\emptyset\}$ or $\{\emptyset, \{1\}\}$ and the relation $\sim_n$ is the complete relation (which is an equivalence relation), so $\sim_0$ and $\sim_1$ are equivalence relations.

Now assume that $n \geq 2$. Observe that $\{1\} \sim_n \{1,2\}$ and $\{2\} \sim_n \{1,2\}$, but $\{1\} \not\sim_n \{2\}$. So if $n \geq 2$, then $\sim_n$ is not an equivalence relation.

Problem 5

For each pair below, use the Euclidean algorithm to compute the greatest common divisor, and express the greatest common divisor as an integer combination of the two numbers:

(a) 126 and 224;

(b) 221 and 299.

Solution.

(a) What follows is, if you like, the 'output' of the extended Euclidean algorithm, working down the left-hand column and then up the right-hand column:

$$\begin{array}{rclcrll}
224 &=& 1 \times 126 + 98 & \Rightarrow & 14 &= 4 \times (224 - 1 \times 126) - 3 \times 126 & = 4 \times 224 - 7 \times 126 \\
126 &=& 1 \times 98 + 28 & \Rightarrow & 14 &= 98 - 3 \times (126 - 98) & = 4 \times 98 - 3 \times 126 \\
98 &=& 3 \times 28 + 14 & \Rightarrow & 14 &= 98 - 3 \times 28 & \\
28 &=& 2 \times 14 + 0 & & & &
\end{array}$$

So $\gcd(224, 126) = 14$ and $4 \times 224 + (-7) \times 126 = 14$.

(b) We proceed as in (a):

$$\begin{array}{rclcrll}
299 &=& 1 \times 221 + 78 & \Rightarrow & 13 &= 3 \times (299 - 1 \times 221) - 1 \times 221 & = 3 \times 299 - 4 \times 221 \\
221 &=& 2 \times 78 + 65 & \Rightarrow & 13 &= 78 - 1 \times (221 - 2 \times 78) & = 3 \times 78 - 1 \times 221 \\
78 &=& 1 \times 65 + 13 & \Rightarrow & 13 &= 78 - 1 \times 65 & \\
65 &=& 1 \times 13 + 0 & & & &
\end{array}$$

So $\gcd(299, 221) = 13$ and $3 \times 299 + (-4) \times 221 = 13$.

Problem 6

Suppose that $\gcd(a, b) = 1$. Prove that $\gcd(na, nb) = n$ without using the fact that integers 2 and larger factor uniquely into primes..

Solution. Clearly $n \mid na$ and $n \mid nb$. Now suppose $d \in \mathbb{N}$ with $d \mid na$ and $d \mid nb$. In order to show that n is the greatest common divisor of na and nb, we need to show that $d \mid n$.

Since $\gcd(a, b) = 1$, the extended Euclidean algorithm yields integers s and t such that $as+bt = 1$. Thus, $nas + nbt = n$. Since $d \mid na$ and $d \mid nb$, $d \mid nas$ and $d \mid nbt$, and hence $d \mid nas + nbt$ $(= n)$, as desired.

Problem 7

Fiona, Jennifer, and Twain go to the 151/128 store after teaching recitation. The store is selling bottles of diet coke for 25 cents and hedgehog plushies for 10 cents. The TAs buy the same number of plushies and bottles of diet coke. What is the minimum number of items they have to buy such that the total cost is a nonzero whole number of dollars?

Solution. Let n be the total cost of the items, and let x be the number of plushies, which is also the number of bottles of diet coke. Then $10x + 25x = 100n$, that is $35x = 100n$.

It follows that $7x = 20n$. Since 7 and 20 are relatively prime, we must have $20 \mid x$. Thus the least value of x for which the total cost is a nonzero whole number of dollars is 20. Since there are x plushies and x bottles of diet coke, there are 40 items in total, with a value of \$7.

Bonus Problem (2 points)

Meher has two jars of jelly beans, one with x beans and the other with y beans. Each jar has a lever. When a jar has at least 2 beans, pressing its lever will give Meher one bean from it and move one bean from it to the other jar (if there are 1 or 0 beans in the jar, then pressing the lever has no effect). Determine necessary and sufficient conditions on x and y, so that Meher can extract all but one of the jelly beans.

Solution. Let $p(n)$:= "When there are a total of n jellybeans, then Meher can win if and only if the difference between the number of beans in the jars is not divisible by 3." Note that $p(1)$ and $p(2)$ are true and now assume that $n \geq 2$ and that $p(n)$ is true.

Assume that there are a total of $n + 1$ jellybeans in the jars. Since $n + 1 > 2$, Meher can and must press a lever. After Meher presses a lever there will be a total of n jellybeans in the jars, and since one jar will lose 2 jellybeans while the other gains 1 jellybean, the difference between the number of jellybeans in the jars has changed by 3, and the original difference is divisible by 3 if and only if the new difference is divisible by 3. By the truth of $p(n)$, Meher can win if and only if the new difference is divisible by 3, which happens if and only if the original difference is divisible by 3.

By induction, $p(n)$ is true for every positive integer n.