21-128 and 15-151 C2W Review

Problem 1

By counting in two ways, prove that for $n > 2$

$$n(n-1)(n-2) = \binom{n}{3} \cdot 3 \cdot 2$$

Problem 2

By counting in two ways, prove that for $n > 4$

$$\binom{n}{5} = \sum_{i=2}^{n-3} \sum_{j=i+2}^{n-1} (i-1)(j-i-1)(n-j)$$

Problem 3

By counting in two ways, prove that for $0 < k < n$

$$\binom{n}{k} 2^k = \sum_{i=0}^{k} \binom{n}{i} \binom{n-i}{k-i}$$

Problem 4

By counting in 2 ways, prove that for $k > 1$ and $n > 0$

$$n^k - (n-1)^k = \sum_{i=1}^{k} \binom{k}{i} (n-1)^{k-i}$$

Solution Problem 1

Let the set S be the set of ways to choose a president, treasurer, and social media chair from a group of n people such that no position is held by the same person.

LHS: We can use a 3 step process.

1. Step 1: Choose the president
2. Step 2: Choose the treasurer
3. Step 3: Choose the social media chair

The first step has n choices, the second step has $n-1$ choices and the third step has $n-2$ choices

Using MP we get $n(n-1)(n-2)$

RHS: We can use a 3 step process.

1. Step 1: Choose the three people to have a position
2. Step 2: Choose the president
3. Step 3: Choose the treasurer

The first step has $\binom{n}{2}$ choices, the second step has 3 choices and the third step has 2 choices. Note that we assign the social media chair to the last person of the three chosen in step 1.

Using MP we get $\binom{n}{3} \cdot 3 \cdot 2$.

Solution Problem 2

Let the set S be the number of 5 element subsets of $[n]$. We will count S in two different ways.

LHS: We can simply choose 5 elements of $[n]$ with $\binom{n}{5}$.

RHS: Partition S based on the location of the 2nd lowest element in the subset. This will be from 2 to $n-3$. This partition is exhaustive because every subset must have a 2nd lowest element. Since there must be one element lower than the 2nd lowest element and three elements higher than the 2nd lowest, the range this value can take is between 2 and n-3. Furthermore, this partition is disjoint because a subset can't have two values that are both the 2nd lowest element. We will use i as the location of the 2nd lowest element.

Next, partition the set again based on the location of the 2nd highest element in the subset. This will be from $i+2$ to $n-1$. This partition is exhaustive because every subset must have a 2nd highest element. Since there must be one element higher than the 2nd highest element, the upper bound of this partition is n-1. Given that we know the 2nd lowest element is i and that there must be one element between the 2nd lowest and 2nd highest element in the subset, the lower bound of this partition is $i+2$. Furthermore, this partition is disjoint because a subset can't have two values that are both the 2nd highest element. We will use j as the location of the 2nd highest element.

Since these are both valid partitions, we can use the AP.

From here, we are given that i is the second lowest element in the subset and j is the second highest. We can now create a 3-step process using the MP.

Step 1: Find the lowest element. Since i is the second lowest element, there are $i-1$ options for the lowest element in the subset.

Step 2: Find the third lowest element. Since i is the second lowest element and j is the second highest element, there are $j-i-1$ options for the third lowest element in the subset.

Step 3: Find the highest element. Since j is the second highest element out of n total options, there are $n-j$ options for the highest element in the subset.

This gives us a total of

$$\sum_{i=2}^{n-3} \sum_{j=i+2}^{n-1} (i-1)(j-i-1)(n-j)$$

Solution Problem 3

Set: Let S be the set of ways to color n houses red, yellow, or green such that k houses are not green.

LHS: We can solve this using a (k+1)-step process.

Step 1: Choose which houses are not green. This has $\binom{n}{k}$ choices. We can then color the other houses green.

Step (2)-(k+1): For the remaining k houses that are not green, we can choose weather to color them red or yellow. Thus we will have 2 options for each house.

By MP, we get $\binom{n}{k}2^k$

RHS: We can partition the set based on how many red houses there are. There can be as few as 0 red houses and at most k red houses. This partition is disjoint because the number of red houses can't be two numbers. This partition is exhaustive because each set will have some number of red houses.

Thus, let S_i be the set of colorings with i red houses. We can create a 2-step process to count S_i.

Step 1: Choose the houses to color red. We need to choose i, so we have a total of $\binom{n}{i}$ options for this step.

Step 2: Choose the houses to color yellow. We know that k houses must be red or yellow. So of the remaining $n - i$ houses, we need to choose $k - i$. Thus we have $\binom{n-i}{k-i}$

By MP, we get the size of S_i is $\binom{n}{i}\binom{n-i}{k-i}$

By AP, we get

$$\binom{n}{k}2^k = \sum_{i=1}^{k} \binom{n}{i}\binom{n-i}{k-i}$$

Solution Problem 4

Let S_1 be the set of ways to color k balls with n colors such that there is at least one red ball.

LHS: We can count S using complementary counting. Let S_k be the set of ways to color k balls without restriction. We can partition S_k into S_1 and S_0 where S_0 is the number of ways to color k balls such that there are no red balls.

This partition is exhaustive because there can either be no red balls, or some number of red balls. This partition is disjoint because you can't have both no red balls and some number of red balls.

Thus it suffices to count S_k and S_0 and subtract them by AP.

To count S_k we can create a k-step process. In the ith step, we choose the color of the ith ball. Each step has n choices. So by MP we get n^k.

To count S_0 we can create a k-step process. In the ith step, we choose the color of the ith ball. Each step has n-1 choices since we can't choose red. So by MP we get $(n-1)^k$.

Thus, by complimentary counting, we get $n^k - (n-1)^k$

RHS:

Let us partition S_1 upon how many red balls there are. At a minimum, there must be 1 ball and at a max, all k balls can be red.

This partition is disjoint because the number of red balls is fixed. This partition is exhaustive because there must be some positive number of red balls.

Let S_i be the set of colorings with i red balls.We then create a k-step process. Step 1: Choose the red balls. Step 2: Paint the remaining balls.

Given that there are i balls, we get $\binom{k}{i}$ ways to choose the red balls and $(n-1)^{k-i}$ ways to paint the remaining balls.

Thus, using the addition principle, we get $\sum_{i=1}^{k} \binom{k}{i} (n-1)^{k-i}$ since the number of red balls is disjoint and completely partitions the set.

Note: We say $k > 1$ to avoid the 0^0 scenario.

$$n^k - (n-1)^k = \sum_{i=1}^{k} \binom{k}{i} (n-1)^{k-i}$$