21-128 and 15-151 Midterm 1
September 28, 2020

1. Prove that if x is an odd integer, then $8 \mid (x^2 - 1)$.

2. Supply proofs or counterexamples (with explanation) for each of the following statements:

(i) $\forall x \in \mathbb{R}^+ \ \exists y \in \mathbb{R} \ \ [(x = y^2) \ \wedge \ (y - |y| \neq 0)]$

(ii) $\forall x \in \mathbb{R} \ \exists y \in \mathbb{R} \ \ [x^2 - y^2 > 0]$

3. Prove that for all sets A, B, and C

$$(A \cup B) \setminus C \ \subseteq \ [A \setminus (B \cup C)] \ \cup \ [(B \setminus (A \cap C)].$$

4. Let $f : \mathbb{A} \to \mathbb{B}$ be a function. Show that for all $S, T \subseteq \mathbb{B}$,

$$f^{-1}[S \cup T] \ = \ f^{-1}[S] \cup f^{-1}[T].$$

5. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Z} \\ 1 - x & \text{if } x \notin \mathbb{Z} \end{cases}$$

Determine, with proof, whether or not f is a bijection.

Bonus. Assume that on the show Love Island, each contestant must always tell the truth or always lie. If I am watching the show and three contestants **A**, **B**, and **C** make the following statements, which ones (if any) should I believe? Briefly justify your answer.

A: "All three of us are liars."

B: "Exactly two of us are liars."

C: "**A** and **B** are both liars."