21-128 and 15-151 problem sheet 4

Solutions to the following seven exercises and optional bonus problem are to be submitted through gradescope by 11pm on

Wednesday, 5th October, 2022.

Problem 1

Let $\langle a_n \rangle_{n \in \mathbb{N}}$ be a sequence satisfying $a_0 = 0$, and

$$a_n = (n+1) \cdot a_{n-1} + n \quad \text{for all } n \geq 1.$$

Use induction to show that $a_n = (n+1)! - 1$ for all $n \in \mathbb{N}$.

Problem 2

Prove that

$$\sum_{i=1}^{n} \frac{1}{(3i-2)(3i+1)} = \frac{n}{3n+1}$$

for all $n \in \mathbb{N}^+$.

Problem 3

For integers $n \geq 2$ find and prove a formula for

$$\prod_{i=2}^{n} (1 - \frac{(-1)^i}{i}).$$

Problem 4

Starting with a single stack of 100 coins, two players take alternate turns; in each turn, a player removes 1, 2, 3 or 4 coins from the stack. The player who removes the last coin wins. Prove that the second player has a strategy to win regardless of what strategy the first player uses. (Hint: prove something more general)

Problem 5

Determine the set of all natural numbers that can be expressed as $3m + 10n$ for natural numbers m and n.

Problem 6

Let $\langle a_n \rangle_{n \in \mathbb{N}}$ be a sequence satisfying

$$a_n = 2a_{n-1} + 3a_{n-2} \text{ for all } n \geq 2$$

(a) Given that a_0, a_1 are odd, prove that a_n is odd for all $n \in \mathbb{N}$.

(b) Given that $a_0 = a_1 = 1$, prove that $a_n = \frac{1}{2} \left(3^n - (-1)^{n+1} \right)$ for all $n \in \mathbb{N}$.

Problem 7

Let $\langle f_n \rangle_{n \in \mathbb{N}}$ be the sequence satisfying $f_0 = 0$, $f_1 = 1$ and

$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2.$$

Show, by induction, that $f_n^2 + f_{n+1}^2 = f_{2n+1}$ and $f_{n+2}^2 - f_n^2 = f_{2n+2}$ for every $n \in \mathbb{N}$.

Bonus Problem (2 points)

On the island of Perfect Reasoning, $k > 0$ people dress unfashionably. Clive Newstead arrives on a boat, somewhat distressed at the stereotypical unfashionability of the logically adept, and announces, "At least one of you is dressed unfashionably. No unfashionably dressed person realizes that they are dressed unfashionably, but everyone else on the island realizes it. In n days from now if you have concluded that your apparel is unfashionable, you will come to the beach at noon that day to denounce your apparel." The islanders gather at noon every day thereafter. When will the unfashionable apparel be denounced, and how will the wearers reason?