21-128 and 15-151 problem sheet 9

Solutions to the following seven exercises and optional bonus problem are to be submitted through gradescope by 11PM on

Tuesday 22th November 2022.

Problem 1

By counting in two ways, prove that $n^2 = 2\binom{n}{2} + n$ for all $n \geq 0$.

Problem 2

By counting in two ways, prove that

$$\binom{n}{j}\binom{n}{k} = \sum_{i=0}^{\min(j,k)} \binom{n}{i}\binom{n-i}{j-i}\binom{n-j}{k-i}$$

for all $n, j, k \in \mathbb{N},\ j, k \leq n$.

Problem 3

By counting in two ways, prove that $\sum_{i=1}^{n}(i-1)(n-i) = \binom{n}{3}$ for all $n \geq 1$.

Problem 4

Let x, y, z be nonnegative real numbers such that $y + z \geq 2$. Prove that

$$(x+y+z)^2 \geq 4x + 4yz$$

Problem 5

Consider the following system of equations of real numbers:

$$\begin{cases} 3w + 2x + y + z = 14 \\ w^2 + x^2 + y^2 + z^2 = 14 \end{cases}$$

What is the maximum possible value of z?

Problem 6

The standard way to define ordered fields is to start with a strict order on $\mathbb{F}$ and then axiomatize the properties that make it compatible with arithmetic:

$$(O1) \qquad x < y \implies x + z < y + z$$

$$(O2) \qquad 0 < x, y \implies 0 < x * y$$

Alternatively, we can introduce positive sets $P \subseteq \mathbb{F}$ and use them to define order:

$$(P1) \qquad x, y \in P \implies x + y \in P$$

$$(P2) \qquad x, y \in P \implies x * y \in P$$

$$(P3) \qquad x \in P \vee x = 0 \vee -x \in P$$

In $(P3)$, exactly one of the cases is supposed to hold. Given $<$ we can define $P_< = \{x \in \mathbb{F} \mid 0 < x\}$ and, conversely, $x <_P y \iff y - x \in P$.

(a) Show that $P_<$ is a positive set in any ordered field.

(b) Show that for any positive set P, the order $<_P$ produces an ordered field.

Bonus

Show by counting in two ways that:

$$2^{(n^2)} = \sum_{i=0}^{n} \binom{n}{i} (2^n - 1)^i$$