21-128 and 15-151 problem sheet 1

Solutions to the following seven exercises and optional bonus problem are to be submitted through gradescope by 11PM on

Wednesday 7th September 2022.

Problem 1

For each statement below, decide whether it is true or false. Prove your claim using only properties of the natural numbers.

(a) If $n \in \mathbb{N}$ and $n^2 + (n+1)^2 = (n+2)^2$, then $n = 3$.

(b) For all $n \in \mathbb{N}$, it is false that $(n-1)^3 + n^3 = (n+1)^3$.

Problem 2

(a) Show that $(p \Rightarrow q) \vee (p \Rightarrow r)$ and $p \Rightarrow (q \vee r)$ are logically equivalent.

(b) Show that $(\forall x \in S\ P(x)) \vee (\forall x \in S\ Q(x))$ and $\forall x \in S\ (P(x) \vee Q(x))$ are not logically equivalent, where $P(x)$ and $Q(x)$ are logical formulae and S is a set.

Problem 3

Let $p(x, y)$ be the predicate '$x + y$ is even', where x and y range over the integers.

(a) Prove that $\forall x \exists y\ p(x, y)$ is true.

(b) Prove that $\exists y \forall x\ p(x, y)$ is false.

Problem 4

You have m indistinguishable marbles, $m \in \mathbb{Z}^+$, and 5 indistinguishable bags.

(a) What is the smallest number of marbles such that you are guaranteed to have 5 marbles in the same bag or 5 different bags with at least one marble? Give proof.

(b) Suppose there are n marbles and k bags, where n and k are positive integers. Prove that there is a unique way to distribute the marbles such that the number of marbles in each bag differs by at most one.

Problem 5

A Pythagorean triple is a triple of positive integers (a, b, c) such that $a^2+b^2 = c^2$. Let (x, y, z) be a Pythagorean triple, and let $P = x+y+z$ and $A = \frac{1}{2}xy$ be the perimeter and area, respectively, of the right-angled triangle whose side lengths are x, y and z.

(a) Find the possible values of (x, y, z) when $P = A$.

(b) Find the possible values of (x, y, z) when $P = 2A$.

Problem 6

(a) Show that the following statement is false:

$$\text{For all } a, x \in \mathbb{R} \text{ there is a unique } y \in \mathbb{R} \text{ such that } x^4y + ay + x = 0$$

(b) Find the set of real numbers a such that the following statement is true:

$$\text{For all } x \in \mathbb{R} \text{ there is a unique } y \in \mathbb{R} \text{ such that } x^4y + ay + x = 0$$

Problem 7

Which of the following numbers are irrational for every choice of numbers r, a and b, such that r is rational and a and b are irrational?

$$a + r \qquad a + b \qquad ar \qquad ab \qquad a^r \qquad r^a \qquad a^b$$

Prove your claims, either by proving that the number must always be irrational or by providing a counterexample. If you claim that a number is irrational, then you should prove it.

Bonus Problem (2 points)

Three brilliant, flawless logicians - Aimee, Brad, and Cindy were blindfolded and each had a hat with a positive integer (possibly different for each) written on it placed on their heads.

Their blindfolds were then removed; they faced each other in a circle and each could see the hats the others were wearing, but not their own hat.

They were told that two of the numbers added up to the third. In order to be generously rewarded they needed to figure out what number was written on their hats.

Here is the conversation that took place:

Aimee: I don't know what my number is.
Brad: I don't know what my number is.
Cindy: I don't know what my number is.
Aimee: Now I know what my number is. It is 50.

(a) What are the other numbers?

(b) What combination(s) of numbers would allow Cindy to solve the problem in round 1?