21-128 and 15-151 problem sheet 10

Solutions to the following exercises and optional bonus problem are to be submitted through Gradescope.

Problem 1

Prove that the set of all natural numbers, the set of all even natural numbers, and the set of all odd natural numbers all have the same cardinality.

Solution. Let E be the set of all even natural numbers and let O be the set of all odd natural numbers. Define functions $f : \mathbb{N} \to O$ and $g : O \to E$ by

$$\begin{aligned} f(n) &= 2n+1 \text{ for all } n \in \mathbb{N} \\ g(n) &= n-1 \quad \text{for all } n \in O \end{aligned}$$

f is well defined because $2n+1$ odd by definition, and there is a unique way to represent any odd natural number in this form. g is well defined because given an odd natural number $2k+1$, subtracting 1 from it will give $2k$, which is an even natural number by definition, and there is a unique way to represent any even natural number in this form.

f and g are injective:

$$f(a) = f(b) \implies 2a+1 = 2b+1 \implies 2a = 2b \implies a = b$$

and

$$g(a) = g(b) \implies a-1 = b-1 \implies a = b$$

.

f is surjective. Let $y \in O, y = 2n+1$, $n \in \mathbb{N}$. Consider $x = \frac{y-1}{2}$. We know $x \in \mathbb{N}$ because $\frac{y-1}{2} = \frac{(2n+1)-1}{2} = \frac{2n}{2} = n \in \mathbb{N}$. Then $f(x) = 2(\frac{2y-1}{2}) + 1 = (y-1)+1 = y$. Similarly, g is surjective. Let $y \in E, y = 2n, n \in \mathbb{N}$. Consider $x = y+1$. We know $x \in O$ because $x = y+1 = 2n+1 \in O$ by defintion. Then $g(x) = (y+1)-1 = y$.

Since f, g injective and surjective, f and g are bijections. So $\mathbb{N}$ has the same cardinality of O and O has the same cardinality as E, and $\mathbb{N}$ has the same cardinality as E by transitivity.

Problem 2

A function $f : \mathbb{Z} \to \mathbb{Z}$ is *periodic* if there exists a positive integer k such that $f(x+k) = f(x)$ for all $x \in \mathbb{Z}$. Prove that the set of all periodic functions $\mathbb{Z} \to \mathbb{Z}$ is countable.

Solution. For each positive integer k, let S_k be the set of all k-periodic functions $\mathbb{Z} \to \mathbb{Z}$. Define $g_k : S_k \to \mathbb{Z}^k$ via $g_k(f) = (f(1), f(2), \cdots, f(k))$. This is an injection because there exists a well-defined left-inverse $g_k^{-1}(x_1, x_2, \cdots, x_k) = f$, where $f(i) = i$ for all $i \in [k]$ and $f(x+k) = f(x)$ for all $x \in \mathbb{Z}$. This function f is well-defined; we can express any $x \in \mathbb{Z}$ uniquely as $x = qk + i$, where $i \in [k]$ (by Division Theorem). Then $f(x) = f(i) = x_i$ for all $x \in \mathbb{Z}$. Our inverse g_k^{-1} is well-defined because we consider all tuples in $\mathbb{Z}^k$ and there is a unique way to map i to x_i for all $i \in [k]$. This function is an injection because $g_k(f) = g_k(f') \implies (f(1), \cdots, f(k)) = (f'(1), \cdots, f'(k))$. So $\forall i \in [k],\ f(x) = f'(x)$ and we know $\forall x \in \mathbb{Z},\ f(x+k) = f(x) = f'(x) = f'(x+k)$, so $f = f'$.

Since the cartesian product of finitely many countable sets is countable, we know that $\mathbb{Z}^k$ is countable, and hence so is S_k since it is in bijective correspondence with $\mathbb{Z}^k$.

The set of all periodic functions $\mathbb{Z} \to \mathbb{Z}$ is the union over all positive integers k, of S_k. Since the set of positive integers is countable and S_k is countable for each positive integer k, we conclude that the set of all periodic functions $\mathbb{Z} \to \mathbb{Z}$ is the countable union of countable sets, and is therefore countable.

Problem 3

For each set, determine, with proof, whether it is countable or uncountable.

(a) $F = \{f : \mathbb{N} \to \mathbb{N} \mid (\forall x > 0, x \in \mathbb{N})(\exists y \in \mathbb{N})(y < x \land f(x) = f(y))\}$

(b) $G = \{g : \mathbb{N} \to \mathbb{N} \mid (\forall x > 1, x \in \mathbb{N})(\exists y \in \mathbb{N})(y < x \land g(x) = g(y))\}$

Solution.

(a) We first prove the following lemma: $\forall f \in F, x \in \mathbb{N},\ f(x) = f(0)$.
Let $f \in F$.
Base Case. $f(0) = f(0)$ by reflexivity.
Inductive Step. Let $n \in \mathbb{N}$. Assume that $f(k) = f(0)$ for all $0 \leq k \leq n$. Since $f \in F$, $\exists y \in \mathbb{N}$ such that $y < n+1 \land f(n+1) = f(y)$. Then $f(n+1) = f(y)$ and $f(y) = f(0)$ by IH, so $f(n+1) = f(0)$ by transitivity.

Let $h : F \to \mathbb{N}$ via $h(f) = f(0)$. This function h is clearly well-defined because we consider all $f \in F$ and since f is well-defined, $f(0)$ is a unique natural number. We know h is an injection because for all $f, f' \in F$, $h(f) = h(f') \implies f(0) = f'(0) \implies f = f'$, where the last step is true by the lemma. Therefore, since we have found an injection from F to $\mathbb{N}$ and $\mathbb{N}$ is countable, we know that F is countable.

(b) Let $h : \{0,1\}^\infty \to G$ via $h(b) = g$ where

$$g(x) = \begin{cases} 0 & x = 0 \\ 1 & x = 1 \\ b_{x-2} & x \geq 2 \end{cases}$$

where b_i is the ith bit of b. The cases clearly partition the naturals and the outputs exist (since $x - 2 \geq 0$ when $x \geq 2$) and are unique. Our function h is well-defined, totality and uniqueness are clear, so we justify existence by showing $g \in G$. For all $x \in \mathbb{N}$, we know that $g(x) \in \{0,1\}$ since the output is always explicitly 0, 1, or a bit in a binary string. For all $x > 1$, if $g(x) = 0$, then we can let $y = 0$ and then $0 < x \wedge f(x) = 0 = f(0)$. Similarly, if $g(x) = 1$, then we can let $y = 1$ and then $1 < x \wedge f(x) = 1 = f(1)$.

Now we show that h is injective. Let $b, b' \in G$. Then $h(b), h(b')$ yield functions g, g' respectively, where

$$g(x) = \begin{cases} 0 & x = 0 \\ 1 & x = 1 \\ b_{x-2} & x \geq 2 \end{cases} \qquad \text{and} \qquad g'(x) = \begin{cases} 0 & x = 0 \\ 1 & x = 1 \\ b'_{x-2} & x \geq 2 \end{cases}$$

So $h(b) = h(b') \implies g = g'$. Then $\forall x \geq 2, b_{x-2} = b'_{x-2}$, so $\forall x \in \mathbb{N},\ b_x = b'_x \implies b = b'$, as desired. Since we have an injection from $\{0,1\}^\infty$ to G and $\{0,1\}^\infty$ uncountable, then G is uncountable.

Problem 4

Maxwell puts 2 grape and 2 strawberry Hi-Chew candies in a bag. Suppose Jennifer draws two candies out of the bag without replacement and then Brad draws the other two candies. All the candies feel the same. So when two candies are drawn from the bag, all possibilities have equal probability.

(a) Define a sample space Ω describing the possible outcomes of this experiment.

(b) Let A and B be the events that Jennifer gets a matching pair of candies and that Brad gets a matching pair of candies, respectively. Determine the probabilities $\Pr(A)$ and $\Pr(B)$.

(c) Are the events A and B independent?

Solution.

(a) Let us index the candies so we have strawberry #1 (denoted as $S1$), strawberry #2 (denoted as $S2$), grape #1 (denoted as $G1$), and grape #2 (denoted as $G2$). Then the sample space can be represented as

$$\begin{aligned}\Omega = \{&(S1S2, G1G2),\\ &(S1G1, S2G2), (S1G2, S2G1), (S2G1, S1G2), (S2G2, S1G1),\\ &(G1G2, S1S2)\},\end{aligned} \tag{1}$$

where the first entry in the pair denotes the candies Jennifer draws and the second entry denotes the candies Brad draws.

(b) We can write the events A and B as follows:

$$A = \{(S1S2, G1G2), (G1G2, S1S2)\}, \tag{2}$$
$$B = \{(S1S2, G1G2), (G1G2, S1S2)\}. \tag{3}$$

We can see that they are actually the same event. Under the Ω we specify in (a), each outcome has the same probability since all the candies feel the same. So

$$\Pr(A) = \Pr(B) = \frac{|A|}{|\Omega|} = \frac{1}{3}. \tag{4}$$

(c) We have seen that $A = B$, so $A \cap B = A$. Then $\Pr(A \cap B) = \Pr(A) = \frac{1}{3}$. But $\Pr(A)\Pr(B) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$. So $\Pr(A \cap B) \neq \Pr(A)\Pr(B)$, and thus A and B are not independent.

Remark. The sample space Ω can be defined in different ways. For example, we can imagine that Jennifer draws her pair of candies one by one, and then define $(S1S2, G1G2)$ and $(S2S1, G1G2)$ to be two different outcomes. A sample space is a valid one to use as long as it describes the fact that "when two candies are drawn from the bag, all possibilities have equal probability" and allows you to calculate the probabilities of A and B. Under different sample spaces, the answers for problems (b) and (c) should be the same.

Problem 5

Ryan, AJ, and Vianna go to the 151/128 store and will buy either a plushie or a bottle of Diet Coke. The three choices are mutually independent and each TA buys a plushie with probability p, with $0 < p < 1$. Let M be the event that the majority of TAs buy plushies, and let R be the event that Ryan buys a plushie.

(a) Express $\mathbb{P}(M)$ in terms of p.

(b) Express $\mathbb{P}(M \mid R)$ in terms of p.

(c) Express $\mathbb{P}(R \mid M)$ in terms of p.

Solution.

(a) We can write the sample space as $\Omega = \{(x_1, x_2, x_3) \mid x_i \in \{0,1\}\ \forall i \in [3]\}$, where $x_i = 0$ denotes that the ith TA buys coke and $x_i = 1$ denotes that the ith TA buys a plushie. Since the three choices are mutually independent, we have that for each $(x_1, x_2, x_3) \in \Omega$,

$$\begin{aligned}\mathbb{P}(\{(x_1,x_2,x_3)\}) &= p^{x_1}(1-p)^{1-x_1}p^{x_2}(1-p)^{1-x_2}p^{x_3}(1-p)^{1-x_3}\\ &= p^{x_1+x_2+x_3}(1-p)^{3-(x_1+x_2+x_3)}.\end{aligned}$$

Let M_2 and M_3 be the events that 2 students buy plushies and 3 students buy plushies, respectively. Then $M = M_2 \cup M_3$ and $M_2 \cap M_3 = \emptyset$. Since

$$\mathbb{P}(M_2) = \sum_{(x_1,x_2,x_3)\in M_2} \mathbb{P}(\{(x_1,x_2,x_3)\}) = p^2(1-p)|M_2| = 3p^2(1-p),$$

and

$$\mathbb{P}(M_3) = \sum_{(x_1,x_2,x_3)\in M_3} \mathbb{P}(\{(x_1,x_2,x_3)\}) = p^3,$$

by additivity of probability we have

$$\mathbb{P}(M) = \mathbb{P}(M_2) + \mathbb{P}(M_3) = 3p^2(1-p) + p^3.$$

(b) It is easy to see that $\mathbb{P}(R) = p$. We first compute $\mathbb{P}(M \cap R)$. Since $M \cap R = \{(1,1,0),(1,0,1),(1,1,1)\}$, we have $\mathbb{P}(M \cap R) = 2p^2(1-p) + p^3$. Then $\mathbb{P}(M \mid R) = \frac{\mathbb{P}(M\cap R)}{\mathbb{P}(R)} = 2p(1-p) + p^2$.

(c) By definition, $\mathbb{P}(R \mid M) = \frac{\mathbb{P}(R\cap M)}{\mathbb{P}(M)} = \frac{2p^2(1-p)+p^3}{3p^2(1-p)+p^3} = \frac{2-p}{3-2p} = \frac{1}{2} + \frac{1}{6-4p}$.

Problem 6

The Michelles want to conduct a two-step experiment. First, they flip a biased coin which shows heads with probability 0.7 and tails with probability 0.3. If it shows heads, in the second step, they flip the same coin for 10 more times; otherwise, i.e., if the coin in the first step shows tails, in the second step they flip a fair coin for 10 times. Let A be the event that the coin in the first step shows heads, and let B be the event that the coin flips in the second step show exactly 2 heads.

(a) Find $\mathbb{P}(B)$.

(b) Find $\mathbb{P}(A \mid B)$, i.e., the probability that the first coin shows heads given that you see exactly 2 heads in the second step. Compare it with $\mathbb{P}(A)$.

Solution.

(a) We know that $\mathbb{P}(B) = \mathbb{P}(B \mid A)\mathbb{P}(A) + \mathbb{P}(B \mid A^c)\mathbb{P}(A^c)$. Given A, the conditional probability of B is the probability of getting exactly 2 heads out of 10 independent flips of the biased coin. So $\mathbb{P}(B \mid A) = \binom{10}{2}(0.7)^2(0.3)^8$. Given A^c, the conditional probability of B is the probability of getting exactly 2 heads out of 10 independent flips of a fair coin. So $\mathbb{P}(B \mid A) = \binom{10}{2}(0.5)^{10}$. We also know that $\mathbb{P}(A) = 0.7$ and $\mathbb{P}(A^c) = 0.3$. Thus

$$\begin{aligned}\mathbb{P}(B) &= \binom{10}{2}(0.7)^2(0.3)^8 \cdot 0.7 + \binom{10}{2}(0.5)^{10} \cdot 0.3 \\ &= \binom{10}{2}\left((0.7)^3(0.3)^8 + (0.5)^{10} \cdot 0.3\right).\end{aligned}$$

(b) Using Bayes Theorem, we have

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{(0.7)^3(0.3)^8}{(0.7)^3(0.3)^8 + (0.5)^{10} \cdot 0.3} \approx 0.0713,$$

which is much smaller than $\mathbb{P}(A) = 0.7$. So this observation of getting exactly 2 heads in the second step greatly changes our perception of the result of the first step.

Bonus (a)

A subset A of $\mathbb{R}$ has the property that, for every $\varepsilon > 0$ and $x \in \mathbb{R}$, there exist $a, b \in \mathbb{R}$ such that ($a \in A$ and $b \notin A$ and $|x - a| < \varepsilon$ and $|x - b| < \varepsilon$). Can A be countable? Can A be uncountable? You may assume that between any two distinct real numbers, one can always find a rational number and an irrational number.

Solution. Per the last line of the problem statement, we will use the following fact:

Theorem. $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are dense in $\mathbb{R}$. That is, given $x, y \in \mathbb{R}$ with $x < y$, there exist $q \in \mathbb{Q}$ and $r \in \mathbb{R} - \mathbb{Q}$ with $x < q < y$ and $x < r < y$.

Fix $\varepsilon > 0$ and $x \in \mathbb{R}$. By the Theorem, there exists $q \in \mathbb{Q}$ and $r \in \mathbb{R} - \mathbb{Q}$, such that $x < q < x + \varepsilon$ and $x < r < x + \varepsilon$. Hence $|x - q| < \varepsilon$ and $|x - r| < \varepsilon$.

Letting $A = \mathbb{Q}$, this example proves that A can be countable. Letting $A = \mathbb{R} - \mathbb{Q}$, this example proves that A can be uncountable: indeed, if $\mathbb{R} - \mathbb{Q}$ were countable, then $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} - \mathbb{Q})$ would be the union of two countable sets (hence countable), contradicting the fact that $\mathbb{R}$ is uncountable.

Bonus (b)

A subset B of $\mathbb{R}$ has the property that, for every $b \in B$, there exists $\varepsilon > 0$ such that for every $x \in \mathbb{R}$, $0 < |b - x| < \varepsilon$ implies $x \notin B$. Is B countable? You may assume that between any two distinct real numbers, one can always find a rational number and an irrational number.

Solution. B must be countable. We prove this by defining an injection $f : B \to \mathbb{Q}$ as follows:

Given $b \in B$, let $\varepsilon > 0$ be such that if $x \in \mathbb{R}$ with $0 < |x - b| < \varepsilon$, then $x \notin B$. By the last line of the problem statement, there exists $q_b \in \mathbb{Q}$ with $b < q_b < b + \varepsilon$. Define $f(b) = q_b$ for such a (fixed) q_b. Note: We are using the axiom of choice when we do this for all $b \in B$.

Then f is certainly a well-defined function, since by definition $q_b \in \mathbb{Q}$ for all $b \in B$. Moreover, f is injective. Suppose $b, b' \in B$ and $f(b) = f(b')$. Then $q_b = q_{b'}$. Let $\varepsilon, \varepsilon' > 0$ be such that:

- If $x \in \mathbb{R}$ and $0 < |x - b| < \varepsilon$ then $x \notin B$; and
- If $x \in \mathbb{R}$ and $0 < |x - b'| < \varepsilon'$ then $x \notin B$.

We may assume that $\varepsilon' < \varepsilon$, otherwise rename the variables appropriately. Now, by definition of q_b and $q_{b'}$, we have $b < q_b < b + \varepsilon$, and $b' < q_{b'} = q_b < b' + \varepsilon'$.

Hence

$$0 < q_b - b < \varepsilon \qquad \text{and} \qquad -\varepsilon < -\varepsilon' < b' - q_b < 0$$

Adding these together yields $-\varepsilon < b' - b < \varepsilon$, and hence $|b' - b| < \varepsilon$. Since $b' \in B$, we must have $|b' - b| = 0$, and so $b = b'$. So f is injective.

[*Source: Cambridge Mathematical Tripos Part IA 2010 Exam Paper 4 Question 8.*]