21-128 and 15-151 Exam 3 Review Problems

1. Consider the set R of all 6-digit numbers where each digit is non-zero.
a) How many numbers are there in the set R?
b) How many numbers in R have distinct digits?
c) How many numbers in R have 1 as their first digit?
d) How many numbers in R have distinct digits as well as 2 as their first digit and 4 as their last digit?

2. Determine the number of ways of arranging the letters of Mississippi and leave your answer in terms of factorials.

3. In how many ways can one choose 8 people from 18 people and seat them
a) in a row from left to right?
b) in a circle?
c) in a square with 2 on each side?
d) in two rows of 4 facing each other?

4. Consider the experiment of flipping a fair coin 9 times.
a) What is the probability of exactly i heads for $i = 0, 1, 2$?
b) What is the probability of obtaining 8 or more heads?

5. Let $S = \{T \subseteq [n+1] : |T| = k+1\}$ and let $S_i = \{T \subseteq [n+1] : |T| = k+1$, and i is the least element of T$\}$. Show that $\{S_1, S_2, \ldots, S_{n-k+1}\}$ is a partition of S.

6. How many ways can 6 people be partitioned into two groups of 3?

7. Consider the 36 equally likely outcomes when a fair pair of dice is rolled.
a) What is the probability of doubles?
b) What is the probability that the sum is prime?
c) What is the probability that the sum is even or greater than 8?
d) What is the probability that the product is greater than 15?

8. Consider the 16 equally likely outcomes when a fair coin is flipped 4 times.
a) What is the probability of at least one head?
b) What is the probability of exactly 2 heads?
c) What is the probability that no two heads occur consecutively?
d) What is the probability that the first head occurs on the third flip?

9. We wish to choose 9 cards from a usual deck of 52 playing cards.
a) In how many ways can be achieve this?
b) In how many ways can we achieve this if we are required to choose all cards from the same suit?
c) In how many ways can we achieve this if we are required to choose exactly 3 aces and 3 kings?
d) In how many ways can we achieve this if we are required to choose cards of different values (assuming that the 13 cards in each suit are of different values)?

10. Suppose that you are one of 12 candidates for election to a small committee of 3 people. Suppose further that each candidate is equally likely to be elected.
a) What is the probability that you will be successful?
b) Your best friend is also one of the candidates. What is the probability that both of you are successful?

11. We wish to elect 10 members to a committee from 30 candidates, and you and two friends are among the candidates.
a) What is the probability that you and exactly one of your two friends are elected?
b) What is the probability that you and at least one of your two friends are elected?
c) What is the probability that both your friends are elected but you are not?

12. Explain the chairperson identity from the notes by counting chaired committees in two ways.

13. This problem concerns the Chinese Remainder Theorem. Let a, b, and c be integers. Find (in terms of a, b, and c) all integers x which satisfy $x \equiv a \mod 2$, $x \equiv b \mod 3$ and $x \equiv c \mod 7$.

14. Let a and b be real numbers. Show that

$$(a^2+1)(b^2+1) \geq (a+b)^2.$$

15. The 128 and 151 TAs decide to enclose a region for themselves so they can grade your exams in privacy. To do this, they have a 100 meter rope. They use this to create three sides of a rectangle, with the fourth side being one the walls in the hallway of Baker/Porter (i.e. a wall which is very long). What is the maximum area of the rectangle formed by this rope?

16. A line with negative slope passing through the point $(18, 8)$ intersects the x and y axes at $(a, 0)$ and $(0, b)$ respectively.

1. Let the slope of the line be $-m$ (so that $m > 0$). Find a and b in terms of m.
2. (CMIMC 2016) What is the smallest possible value of $a + b$?