21-128 and 15-151 problem sheet 4

Solutions to the following seven exercises and optional bonus problem are to be submitted through gradescope by 11pm on

Wednesday, 5th October, 2022.

Problem 1

Let $\langle a_n \rangle_{n \in \mathbb{N}}$ be a sequence satisfying $a_0 = 0$, and

$$a_n = (n+1) \cdot a_{n-1} + n \quad \text{for all } n \geq 1.$$

Use induction to show that $a_n = (n+1)! - 1$ for all $n \in \mathbb{N}$.

Solution. Let $p(n)$:= "$a_n = (n+1)! - 1$". We proceed by induction on n.

- **Basis step.** When $n = 0$, the left-hand side is a_0 which is 0, and the right hand side is $(0+1)! - 1$ which is also 0. Thus, $p(0)$ is true.
- **Induction step.** Fix $n \in \mathbb{N}$ and suppose that $p(n)$ is true, ie $a_n = (n+1)! - 1$. Then

  $$a_{n+1} = (n+2) \cdot a_n + (n+1) = (n+2)[(n+1)! - 1] + (n+1) = (n+2)! - (n+2) + (n+1) = (n+2)! - 1.$$

  The first equality follows from the definition of the sequence and the second equality follows from the induction hypothesis. Thus, $p(n+1)$ is true.

The claim follows by induction. □

Problem 2

Prove that

$$\sum_{i=1}^{n} \frac{1}{(3i-2)(3i+1)} = \frac{n}{3n+1}$$

for all $n \in \mathbb{N}^+$.

Solution. We proceed by induction on n.

- **Basis step.** When $n = 1$, the left-hand side is $\frac{1}{1 \times 4} = \frac{1}{4}$, and the right-hand side is $\frac{1}{4}$. These are equal, so the claim is true when $n = 1$.

- **Induction step.** Fix $n \in \mathbb{N}^+$ and suppose that

$$\sum_{i=1}^{n} \frac{1}{(3i-2)(3i+1)} = \frac{n}{3n+1}$$

We need to prove that

$$\sum_{i=1}^{n+1} \frac{1}{(3i-2)(3i+1)} = \frac{n+1}{3(n+1)+1}$$

Well,

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{1}{(3i-2)(3i+1)} &= \sum_{i=1}^{n} \frac{1}{(3i-2)(3i+1)} + \frac{1}{(3n+1)(3n+4)} && \text{by definition of summation} \\
&= \frac{n}{3n+1} + \frac{1}{(3n+1)(3n+4)} && \text{by the induction hypothesis} \\
&= \frac{n(3n+4)+1}{(3n+1)(3n+4)} && \text{by combining fractions} \\
&= \frac{3n^2+4n+1}{(3n+1)(3n+4)} && \text{by distributivity} \\
&= \frac{(3n+1)(n+1)}{(3n+1)(3n+4)} && \text{by factorisation} \\
&= \frac{n+1}{3n+4} && \text{by cancellation}
\end{aligned}$$

Since $3n+4 = 3(n+1)+1$, the claim is proved.

By induction, we're done. □

Problem 3

For integers $n \geq 2$ find and prove a formula for

$$\prod_{i=2}^{n} (1 - \frac{(-1)^i}{i}).$$

How to approach this problem

This part is meant to help you think about how to find the formula. In particular, this part should only appear on scratch paper but not on your answer sheet.

Usually, when you have a sum or product of lots of things, the natural way to find the closed-form formula is to try some small n's to see how this thing behaves. Therefore,

$$\text{When } n = 2,\ \prod_{i=2}^{2}\left(1 - \frac{(-1)^i}{i}\right) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}$$

$$\text{When } n = 4,\ \prod_{i=2}^{4}\left(1 - \frac{(-1)^i}{i}\right) = \left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) = \frac{1}{2}\cdot\frac{4}{3}\cdot\frac{3}{4} = \frac{1}{2}$$

$$\text{When } n = 5,\ \prod_{i=2}^{5}\left(1 - \frac{(-1)^i}{i}\right) = \left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\left(1 + \frac{1}{5}\right) = \frac{1}{2}\cdot\frac{4}{3}\cdot\frac{3}{4}\cdot\frac{6}{5} = \frac{3}{5}$$

It seems that most terms in the middle will cancel out. More specifically, when n is even, only the first term $\frac{1}{2}$ remains, and when n is odd, only the first and last term, namely $\frac{1}{2}$ and $\frac{n+1}{n}$ remains. Therefore, we may guess that:

$$\prod_{i=2}^{n}\left(1 - \frac{(-1)^i}{i}\right) = \begin{cases} \frac{1}{2} & n \text{ is even} \\ \frac{n+1}{2n} & n \text{ is odd} \end{cases}$$

Now, what we are left to do is to formally prove this is the correct formula, which will be presented in the next part. We'll use induction to prove this.

How to how to formally prove this problem

This part is the formal proof, which should be on your answer sheet.

Claim

$$\prod_{i=2}^{n}\left(1 - \frac{(-1)^i}{i}\right) = \begin{cases} \frac{1}{2} & n \text{ is even} \\ \frac{n+1}{2n} & n \text{ is odd} \end{cases}$$

Proof

Because the cases split by parity, we rewrite n as either $2k$ or $2k+1$ to indicate the parity. And we will induct on $k \in \mathbb{N}$ and $k \geqslant 1$.

Base Case When $k = 1$, we have

$$\prod_{i=2}^{2\cdot 1}\left(1 - \frac{(-1)^i}{i}\right) = 1 - \frac{1}{2} = \frac{1}{2}$$

and

$$\prod_{i=2}^{2\cdot 1+1}\left(1-\frac{(-1)^i}{i}\right)=\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)=\frac{1}{2}\cdot\frac{4}{3}=\frac{2}{3}=\frac{3+1}{2\cdot 3}$$

Induction Step Assume for some $k \geqslant 1$ and $k \in \mathbb{N}$,

$$\prod_{i=2}^{2k}\left(1-\frac{(-1)^i}{i}\right)=\frac{1}{2} \text{ and } \prod_{i=2}^{2k+1}\left(1-\frac{(-1)^i}{i}\right)=\frac{2k+2}{2(2k+1)}=\frac{k+1}{2k+1} \text{ (IH)}$$

Then we have

$$\begin{aligned}\prod_{i=2}^{2(k+1)}\left(1-\frac{(-1)^i}{i}\right)&=\left(\prod_{i=2}^{2k+1}\left(1-\frac{(-1)^i}{i}\right)\right)\left(1-\frac{1}{2k+2}\right)\\&=\left(\frac{k+1}{2k+1}\right)\left(1-\frac{1}{2k+2}\right)\\&=\frac{k+1}{2k+1}\cdot\frac{2k+1}{2(k+1)}\\&=\frac{1}{2}\end{aligned}$$

and

$$\begin{aligned}\prod_{i=2}^{2(k+1)+1}\left(1-\frac{(-1)^i}{i}\right)&=\left(\prod_{i=2}^{2(k+1)}\left(1-\frac{(-1)^i}{i}\right)\right)\left(1+\frac{1}{2k+3}\right)\\&=\frac{1}{2}\left(1+\frac{1}{2k+3}\right)\\&=\frac{1}{2}\cdot\frac{2k+4}{2k+3}\\&=\frac{(2k+3)+1}{2(2k+3)}\end{aligned}$$

which also complies with the formula we claimed.

Therefore, by the principle of mathematical induction, we have proved the claim that

$$\prod_{i=2}^{n}\left(1-\frac{(-1)^i}{i}\right)=\begin{cases}\frac{1}{2} & n \text{ is even}\\ \frac{n+1}{2n} & n \text{ is odd}\end{cases}$$

$\square$

Problem 4

Starting with a single stack of 100 coins, two players take alternate turns; in each turn, a player removes 1, 2, 3 or 4 coins from the stack. The player who removes the last coin wins. Prove that the second player has a strategy to win regardless of what strategy the first player uses. (Hint: prove something more general)

Solution. An iteration of the game consists of a move by the first player, followed by a move by the second player. Let $p(n)$ be the statement "Regardless of what strategy the first player has used previously in the game, the second player can leave exactly $100 - 5n$ coins after the n^{th} iteration of the game." We'll prove that $p(k)$ is true for all integers $1 \le k \le 20$, by induction.

Base case) $p(1)$ is true, because if the first player removes i coins, then the second player can remove $5 - i$ coins (ie $\{5 - i : i \in [4]\} = [4]$).

Induction Step) Now suppose that $p(k)$ is true for some integer $1 \le k \le 19$. By employing the induction hypotheses, the second player can leave $100 - 5k$ coins after the k^{th} iteration of the game. By employing the strategy from the base case, the second player can ensure that a total of 5 coins are removed on iteration $k+1$ of the game, leaving $100 - 5k - 5 = 100 - 5(k+1)$ coins at the end of iteration $k+1$. Hence $p(k+1)$ is true.

Therefore, by the principle of mathematical induction, we have shown that $p(k)$ is true for all integers $1 \le k \le 20$, by induction.

Problem 5

Determine the set of all natural numbers that can be expressed as $3m + 10n$ for natural numbers m and n.

Solution. The numbers that can be expressed in this way are 0, 3, 6, 9, 10, 12, 13, 15, 16 and all $n \in \mathbb{N}$ with $n \ge 18$.

Indeed:

$0 = 0\times 3 \quad 3 = 1\times 3 \quad 6 = 2\times 3 \quad 9 = 3\times 3 \quad 10 = 1\times 10 \quad 12 = 4\times 3 \quad 13 = 1\times 3+1\times 10 \quad 15 = 5\times 3 \quad 16 = 2\times 3+1\times 10$

We'll prove that if $n \ge 18$ then n can be expressed as a nonnegative multiple of 3 plus a nonnegative multiple of 10 by strong induction on n.

- **Basis step.** Note that

$$18 = 6 \times 3 \qquad 19 = 1 \times 10 + 3 \times 3 \qquad 20 = 2 \times 10$$

- **Induction step.** Fix $n \geq 20$ and suppose, for all $k \in \mathbb{N}$ with $18 \leq k \leq n$, that k can be expressed as a nonnegative multiple of 3 plus a nonnegative multiple of 20.

  Since $n \geq 20$ we have $18 \leq n-2 \leq n$. Hence $n-2 = 3a+10b$ for some $a, b \geq 0$. Then

  $$n+1 = 3(a+1) + 10b$$

  so $n+1$ can be expressed as a nonnegative multiple of 3 plus a nonnegative multiple of 10.

It remains to prove that 1, 2, 4, 5, 7, 8, 11 and 14 cannot be written as a nonnegative multiple of 3 plus a nonnegative multiple of 10.

Now, $3a+10b \geq 10$ if $b \geq 1$, so if $3a+10b < 10$ then $b = 0$. Thus the only natural numbers less than 10 which have this property are multiples of 3. This proves that $1, 2, 4, 5, 7, 8$ cannot be written in this form.

Also, $3a+10b \geq 20$ if $b \geq 2$, so the only numbers between 10 and 20 which are of this form are those which are multiples of 3, or those which are of the form $10+3a$. This proves that 11 and 14 cannot be written in the desired form. □

Problem 6

Let $\langle a_n \rangle_{n \in \mathbb{N}}$ be a sequence satisfying

$$a_n = 2a_{n-1} + 3a_{n-2} \text{ for all } n \geq 2$$

(a) Given that a_0, a_1 are odd, prove that a_n is odd for all $n \in \mathbb{N}$.

(b) Given that $a_0 = a_1 = 1$, prove that $a_n = \frac{1}{2}\left(3^n - (-1)^{n+1}\right)$ for all $n \in \mathbb{N}$.

Solution.

(a) Suppose a_0 and a_1 are odd. We'll prove that a_n is odd for all $n \in \mathbb{N}$ by strong induction on n. There is nothing to prove for the basis step, since we're assuming that a_0, a_1 are odd. For the induction step, fix $n \geq 1$ and suppose that a_k is odd for all $k \leq n$. Then, in particular, a_{n-1} and a_n are odd, so $a_{n-1} = 2p-1$ and $a_n = 2q-1$ for some $p, q \in \mathbb{Z}$. Now since $n+1 \geq 2$, we have

$$a_{n+1} = 2a_n + 3a_{n-1} = 2(2q-1) + 3(2p-1) = 2(2q+3p-2) - 1.$$

so a_{n+1} is odd. It follows that a_n is odd for all $n \in \mathbb{N}$

(b) We'll prove that $a_n = \frac{1}{2}\left(3^n - (-1)^{n+1}\right)$ for all $n \in \mathbb{N}$ by strong induction on n.

- **Basis step.** When $n = 0$ and $n = 1$ we just plug in values:

$$\frac{1}{2}\left(3^0 - (-1)^1\right) = \frac{1+1}{2} = 1 = a_0$$

$$\frac{1}{2}\left(3^1 - (-1)^2\right) = \frac{3-1}{2} = 1 = a_1$$

- **Induction step.** Fix $n > 1$ and suppose that $a_i = \frac{1}{2}\left(3^i - (-1)^{i+1}\right)$ for all $0 \le i < n$. Since $n > 1$ it follows that $n \ge 2$, so we can apply the formula to obtain

$$a_n = 2a_{n-1} + 3a_{n-2}$$

Since $0 \le n-1 < n$ and $0 \le n-2 < n$, the induction hypothesis yields

$$\begin{aligned} a_n &= 2 \cdot \frac{1}{2}\left(3^{n-1} - (-1)^n\right) + 3 \cdot \frac{1}{2}\left(3^{n-2} - (-1)^{n-1}\right) \\ &= \frac{1}{2}\left(2 \cdot 3^{n-1} + 3 \cdot 3^{n-2} - 2 \cdot (-1)^n - 3 \cdot (-1)^{n-1}\right) \\ &= \frac{1}{2}\left(2 \cdot 3^{n-1} + 3^{n-1} + 2 \cdot (-1)^{n-1} - 3 \cdot (-1)^{n-1}\right) \\ &= \frac{1}{2}\left(3 \cdot 3^{n-1} - (-1)^{n-1}\right) \\ &= \frac{1}{2}\left(3^n - (-1)^{n+1}\right) \end{aligned}$$

This is what we needed to prove.

By strong induction, it follows that $a_n = \frac{1}{2}\left(3^n - (-1)^{n+1}\right)$ for all $n \in \mathbb{N}$. □

Problem 7

Let $\langle f_n \rangle_{n \in \mathbb{N}}$ be the sequence satisfying $f_0 = 0$, $f_1 = 1$ and

$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \ge 2.$$

Show, by induction, that $f_n^2 + f_{n+1}^2 = f_{2n+1}$ and $f_{n+2}^2 - f_n^2 = f_{2n+2}$ for every $n \in \mathbb{N}$.

Solution.
Let propositions P, Q, and R be defined as follows

$$P(n) : f_n^2 + f_{n+1}^2 = f_{2n+1}$$

$$Q(n) : f_{n+2}^2 - f_n^2 = f_{2n+2}$$

$$R(n) : P(n) \wedge Q(n)$$

We will show $\forall n \in \mathbb{N} R(n)$ by induction on n

Base case: $n = 0$

$P(0)$
$LHS = f_0^2 + f_1^2 = 0 + 1 = 1$
$RHS = f_{2(0)+1} = f_1 = 1$
$LHS = RHS$

$Q(0)$
$LHS = f_2^2 - f_0^2 = 1 - 0 = 1$
$RHS = f_{2(0)+2} = f_2 = 1$
$LHS = RHS$

We have shown that $R(0)$ is true

Inductive hypothesis:

Assume $R(n)$ is true for some $n \in \mathbb{N}$

$$P(n) : f_n^2 + f_{n+1}^2 = f_{2n+1}$$

$$Q(n) : f_{n+2}^2 - f_n^2 = f_{2n+2}$$

Want to show: $R(n+1)$ is true

$$P(n+1) : f_{n+1}^2 + f_{n+2}^2 = f_{2n+3}$$

$$Q(n+1) : f_{n+3}^2 - f_{n+1}^2 = f_{2n+4}$$

Inductive step:

$P(n+1)$

$$\begin{aligned}
& LHS = f_{n+1}^2 + f_{n+2}^2 \\
\Rightarrow & LHS = f_{2n+1} - f_n^2 + f_{2n+2} + f_n^2 && (f_n^2 + f_{n+1}^2 = f_{2n+1} \text{ and } f_{n+2}^2 - f_n^2 = f_{2n+2}) \\
\Rightarrow & LHS = f_{2n+1} + f_{2n+2} = f_{2n+3} = RHS && (f_n = f_{n-1} + f_{n-2} \forall n \geq 2.)
\end{aligned}$$

Therefore $P(n+1)$ is true

$Q(n+1)$

$$\begin{aligned}
LHS &= f_{n+3}^2 - f_{n+1}^2 \\
&= (f_{n+2} + f_{n+1})^2 - f_{n+1}^2 && (f_n = f_{n-1} + f_{n-2} \forall n \geq 2) \\
&= f_{n+2}^2 + 2f_{n+1}f_{n+2} \\
&= f_{2n+2} + f_n^2 + 2f_{n+1}f_{n+2} && (f_n^2 + f_{n+1}^2 = f_{2n+1} \text{ by inductive hypothesis}) \\
&= f_{2n+2} + (f_{n+2} - f_{n+1})^2 + 2f_{n+1}f_{n+2} && (f_n = f_{n-1} + f_{n-2} \forall n \geq 2) \\
&= f_{2n+2} + f_{n+2}^2 + f_{n+1}^2 \\
&= f_{2n+2} + f_{2n+3} && (P(n+1), \text{ which we just proved}) \\
&= f_{2n+4} = RHS
\end{aligned}$$

Therefore $Q(n+1)$ is also true.

Thus $R(n+1)$ is true

Therefore, by principle of mathematical induction, $\forall n \in \mathbb{N} R(n)$ holds

□

Bonus Problem (2 points)

On the island of Perfect Reasoning, $k > 0$ people dress unfashionably. Clive Newstead arrives on a boat, somewhat distressed at the stereotypical unfashionability of the logically adept, and announces, "At least one of you is dressed unfashionably. No unfashionably dressed person realizes that they are dressed unfashionably, but everyone else on the island realizes it. In n days from now if you have concluded that your apparel is unfashionable, you will come to the beach at noon that day to denounce your apparel." The islanders gather at noon every day thereafter. When will the unfashionable apparel be denounced, and how will the wearers reason?

Solution. First consider the case when $k = 1$. Then, this person will immediately know that they are dressed unfashionably, and the unfashionable apparel will be denounced on day 1.

Although not strictly necessary, investigating the $k = 2$ case is enlightening. Say the unfashionable people on the island are p_1, p_2. From the perspective of p_1, if he is not unfashionable, then p_2 will know that p_1 is not unfashionable and p_2 will denounce his fashion on day 1. Thus, if p_2 does not denounce his fashion on day 1, p_1 knows that he must be unfashionable, and so he will denounce his fashion on day 2. A key aspect is that p_2 will think exactly the same way as p_1, and so p_2 will do the same thing, and p_1, p_2 will *both* denounce their fashion on day 2.

Now, let $P(k)$ be the proposition that for $k > 0$ unfashionable people, it will take k days for the unfashionable apparel to be denounced; furthermore, all k unfashionable people on the island will denounce their unfashionable apparel on the k^{th} day. We have established that the base case $P(1)$ is true. We proceed via induction. Consider the case where we have $k+1$ unfashionable people, say $p_1, p_2, \ldots, p_{k+1}$. From the perspective of p_{k+1}, if he is not unfashionable, then by the inductive hypothesis $p_1, p_2, \ldots, p_k$ will all denounce their unfashionable apparel on day k. Therefore, if p_{k+1} does not see anybody denounce their fashion on day k, then he will conclude that he must be unfashionable and announce this on day $k+1$. Every unfashionable person p_i on the island will reason the same way, and so all $k+1$ unfashionable people will denounce their fashion on the $(k+1)^{\text{th}}$ day. As $P(k) \Rightarrow P(k+1)$, and $P(1)$ is true, we have by the principle of induction that $P(k)$ is true for every $k \geq 1$.