21-128 and 15-151 problem sheet 9

Solutions to the following exercises and optional bonus problem are to be submitted through Gradescope.

Problem 1

By counting in two ways, prove that $n^2 = 2\binom{n}{2} + n$ for all $n \geq 0$.

Solution.

Fix $n \geq 0$ and let $X = [n] \times [n]$. We count the number of elements of X by counting in two ways.

- **Procedure 1.** An element of X is an ordered pair (a, b), where $a \in [n]$ and $b \in [n]$. If we first pick a, and then pick b, there are n choices for each, and hence n^2 choices in total by the Multiplication Principle (since we have a multi-step process). Thus $|X| = n^2$.
- **Procedure 2.** Let

  $$D = \{(x, x) : x \in [n]\} \subseteq X \quad \text{and} \quad A = \{(x, y) : x, y \in [n] \text{ and } x \neq y\}$$

  If $(x, y) \in X$ then either $x = y$, in which case $(x, y) \in D$ or $x \neq y$, in which case $(x, y) \in A$. Moreover, if $(x, y) \in D \cap A$ then $x = y$ since $(x, y) \in D$, and $x \neq y$ since $(x, y) \in A$—this is a contradiction, so $D \cap A = \varnothing$. Hence $X = D \cup A$ is a partition of X.

  Now, $|D| = n$ since we can specify a pair $(x, x) \in D$ by choosing $x \in [n]$. Moreover, $|A| = 2\binom{n}{2}$ by the Multiplication Principle (because we have a multi-step process), since we can specify an element $(x, y) \in A$ by first choosing which two (distinct) elements of $[n]$ will appear in the pair ($\binom{n}{2}$ choices), and then choosing which order they appear in ($2! = 2$ choices).

  Hence $|X| = |A| + |D| = 2\binom{n}{2} + n$.

It follows that $n^2 = 2\binom{n}{2} + n$ by the Addition Principle.

Problem 2

By counting in two ways, prove that

$$\binom{n}{j}\binom{n}{k} = \sum_{i=0}^{\min(j,k)} \binom{n}{i}\binom{n-i}{j-i}\binom{n-j}{k-i}$$

for all $n, j, k \in \mathbb{N},\ j, k \leq n$.

Solution. Let $X = \{(A, B) \mid A, B \subseteq [n], |A| = j,\ |B| = k\}$. Any $(A, B) \in X$ can be chosen uniquely by first choosing j elements of $[n]$ to be in A (there are $\binom{n}{j}$ ways to do this) and then choosing k elements of $[n]$ to be in B (there are $\binom{n}{k}$ ways to do this). Since we have a multi-step process, $|X| = \binom{n}{j}\binom{n}{k}$ by the Multiplication Principle.

Another way to uniquely choose $(A, B) \in X$ is to choose $A \cap B$, $A \setminus B$, and $B \setminus A$ (ensuring that the sets are all pairwise disjoint), and then $A = (A \cap B) \sqcup (A \setminus B)$ and $B = (A \cap B) \sqcup (B \setminus A)$. We know $A \cap B$ and $A \setminus B$ are disjoint because $x \in A \cap B \implies x \in B$ but $x \in A \setminus B \implies x \notin B$. Thus, there is a unique way to partition any $A \subseteq [n]$ into the elements of A that are in B and those that are not in B and the cases are exhaustive. The same argument holds for partitioning B. In addition, $A \setminus B$ and $B \setminus A$ are disjoint because $x \in A \setminus B \implies x \in A$ and $B \setminus A \implies x \notin A$. Therefore, choosing $A \cap B$, $A \setminus B$, and $B \setminus A$ results in a unique choice of (A, B).

We now partition on the cardinality of $A \cap B$. It must be that $|A \cap B| \leq \min(j, k)$. AFSOC $|A \cap B| > \min(j, k)$ and WLOG let $\min(j, k) = j$. Then since $A \cap B \subseteq A$, $|A| \geq |A \cap B| > j$, contradiction. We have shown that our partition is exhaustive, and it is mutually exclusive because the size of $A \cap B$ is unique. Then we can let X_i be the set of all $(A, B) \in X$ such that $|A \cap B| = i$.

All $(A, B) \in X_i$ can be chosen uniquely by the following multi-step process. First, we choose i elements of $[n]$ to be in $A \cap B$; there are $\binom{n}{i}$ ways to do this. Next, we have to pick $j - i$ elements to be in $A \setminus B$, because then, by the Addition Principle, we can get that $|A| = |A \cap B| + |A \setminus B| = i + (j - i) = j$. These elements cannot be in $A \cap B$, so we have $n - i$ options to choose from. Thus there are $\binom{n-i}{j-i}$ ways to pick $A \setminus B$. Lastly, we pick $k - i$ elements to be in $B \setminus A$, because by the Addition Principle, we can get that $|B| = |A \cap B| + |B \setminus A| = i + (k - i) = k$. We need to exclude the elements that have been chosen to be in A (which will exclude the elements in $A \cap B$ as well), so there are $n - j$ elements to pick from, so there are $\binom{n-j}{k-i}$ ways to choose $B \setminus A$. By the Multiplication Principle, $|X_i| = \binom{n}{i}\binom{n-i}{j-i}\binom{n-j}{k-i}$.

Therefore, $|X| = \sum_{i=0}^{\min(j,k)} \binom{n}{i}\binom{n-i}{j-i}\binom{n-j}{k-i}$ by the Addition Principle, and it follows that the claim is true.

Problem 3

By counting in two ways, prove that $\sum_{i=1}^{n}(i-1)(n-i) = \binom{n}{3}$ for all $n \geq 1$.

Solution. There are $\binom{n}{3}$ subsets of $[n]$ with 3 elements. Note that any such subset can be written uniquely as $\{a, b, c\}$, where $a < b < c$. Since $\{a, b, c\} \subseteq [n]$ we must have $1 \leq b \leq n$, so that the set of subsets of $[n]$ of size 3 is partitioned by the sets $X_1, X_2, \ldots, X_n$, where for each $1 \leq i \leq n$, X_i is the set of 3-element subsets of $[n]$ whose middle element is i.

Hence $\binom{n}{3} = \sum_{i=1}^{n} |X_i|$ by the Addition Principle.

Fix $i \in [n]$. A set $S \in X_i$ can be written uniquely as $\{a, i, b\}$, where $1 \leq a \leq i-1$ and $i+1 \leq b \leq n$. Thus a procedure for choosing such a subset is first to pick a ($i-1$ choices) and then to pick b ($n-i$ choices). Hence $|X_i| = (i-1)(n-i)$ by the Multiplication Principle (since we have a multi-step process).

It follows that $\binom{n}{3} = \sum_{i=1}^{n}(i-1)(n-i)$.

Problem 4

Let x, y, z be nonnegative real numbers such that $y + z \geq 2$. Prove that

$$(x + y + z)^2 \geq 4x + 4yz$$

Solution. Let's make this look a bit more suggestive. Dividing through by 4 yields the equivalent inequality

$$\left(\frac{x + y + z}{2}\right)^2 \geq x + yz$$

Since $y + z \geq 2$ we must have $y \geq 1$ or $z \geq 1$. To see this, suppose $y < 1$ and $z < 1$; then $y + z < 2$, contradicting the assumption that $y + z \geq 2$.

Suppose $z \geq 1$. (If $y \geq 1$, swap the roles of y and z.) The AGM inequality yields

$$\left(\frac{x + y + z}{2}\right)^2 = \left(\frac{(x + y) + z}{2}\right)^2 \geq (x + y)z$$

We know that $z \geq 1$. Since $x \geq 0$, we have $xz \geq x$. Thus

$$\left(\frac{x + y + z}{2}\right)^2 = (x + y)z = xz + yz \geq x + yz$$

Multiplying through by 4 yields the desired inequality:

$$(x + y + z)^2 \geq 4x + 4yz$$

Problem 5

Consider the following system of equations of real numbers:

$$\begin{cases} 3w + 2x + y + z = 14 \\ w^2 + x^2 + y^2 + z^2 = 14 \end{cases}$$

What is the maximum possible value of z?

Solution. Using Cauchy-Schwarz:

$$\begin{aligned} & (3w + 2x + y)^2 \leq (w^2 + x^2 + y^2)(3^2 + 2^2 + 1^2) \\ \implies & (14 - z)^2 \leq (14 - z^2) \cdot 14 \\ \implies & 196 - 28z + z^2 \leq 196 - 14z^2 \\ \implies & 15z^2 - 28z \leq 0 \\ \implies & z(15z - 28) \leq 0 \end{aligned}$$

If $z > \frac{28}{15}$, both parts of the product will be positive. This tells us that $z \leq \frac{28}{15}$.

Now we need to prove that there indeed exists a solution where $z = \frac{28}{15}$. Equality for Cauchy-Schwarz holds when $\frac{w}{3} = \frac{x}{2} = y = k$, or when $w = 9k, x = 4k, y = k$. So we need

$$\begin{cases} 14k^2 + (\frac{28}{15})^2 = 14 \\ 14k + (\frac{28}{15}) = 14 \end{cases}$$

$14k = 14 - \frac{28}{15} \implies 14k^2 = k(14 - \frac{28}{15}) = 14 - (\frac{28}{15})^2 \implies k = \frac{14-(\frac{28}{15})^2}{14-\frac{28}{15}} = \frac{13}{15}$

$k = \frac{13}{15}$ is a solution, so we've proven that the maximum possible value of z is $\frac{28}{15}$.

Problem 6

The standard way to define ordered fields is to start with a strict order on $\mathbb{F}$ and then axiomatize the properties that make it compatible with arithmetic:

$$(O1) \qquad x < y \implies x + z < y + z$$

$$(O2) \qquad 0 < x, y \implies 0 < x * y$$

Alternatively, we can introduce positive sets $P \subseteq \mathbb{F}$ and use them to define order:

$$(P1) \qquad x, y \in P \implies x + y \in P$$

$$(P2) \qquad x, y \in P \implies x * y \in P$$

$$(P3) \qquad x \in P \lor x = 0 \lor -x \in P$$

In $(P3)$, exactly one of the cases is supposed to hold. Given $<$ we can define $P_< = \{x \in \mathbb{F} \mid 0 < x\}$ and, conversely, $x <_P y \iff y - x \in P$.

(a) Show that $P_<$ is a positive set in any ordered field.

(b) Show that for any positive set P, the order $<_P$ produces an ordered field.

Solution.

(a) Let $P = P_<$.

$(P1)$ Let $x, y \in P$. Then $0 < x, y$ so by $(O1)$ $y < x + y$ and by transitivity $0 < x + y$.

$(P2)$ Entirely similar, but use $(O2)$.

$(P3)$ $(O1)$ plus trichotomy of $<$ produces trichotomy for P.

(b) Let $< = <_P$.

Clearly $<$ is irreflexive and asymmetric. Transitivity follows from $(P1)$ and trichotomy for P.

$(O1)$ Let $x < y$ so that $y - x \in P$. Hence, by $(P1)$, $(y + z) - (x + z) \in P$ and thus $x + z < y + z$.

$(O2)$ Suppose $0 < x, y$ whence $x, y \in P$. By $(P2)$, $x * y \in P$ and thus $0 < x * y$.

Bonus

Show by counting in two ways that:

$$2^{(n^2)} = \sum_{i=0}^{n} \binom{n}{i} (2^n - 1)^i$$

Solution. Let S be the set of $n \times n$ matrices made up of only zeros and ones.

LHS: We define an n^2 step process.

1) Choose 0 or 1 for position (0,0): 2 choices

2) Choose 0 or 1 for position (0,1): 2 choices

...

n^2) Choose 0 or 1 for position (n, n): 2 choices

We have a valid k step process, so by MP, there are $2^{(n^2)}$ different possibilities.

RHS: We next partition our set based on the number of rows that are not all 0. The partition is disjoint because for any matrix, there is a unique number of non-all 0 rows, and it is exhaustive because there is some number of rows between 0 and n that are not all 0.

We now have that this is a valid partition, so by AP, we know that

$$|S| = \sum_{i=0}^{n} |S_i|$$

where S_i is the number of matrices with i all zero rows.

Next we count S_i.

First, I define a process for picking values for a single row such that not all entries are 0. I do so by complementary counting. First, I pick 0 or 1 for position 1, then position 2, up until position n. By MP, there are 2^n options. Exactly 1 of them (all zeros) is not allowed, so the overall number of ways to pick values in a row such that they are not all 0 is $2^n - 1$.

Now that I know this quantity, I can count the number of matrices with i all zero rows:

1) Choose the i non-all zero rows: $\binom{n}{i}$ choices

2) Choose the 0s and 1's for the first non-all zero row: $(2^n - 1)$ choices

...

$i + 1$) Choose the 0s and 1's for last non-all zero row: $(2^n - 1)$ choices

This is a valid k step process, so by MP, $|S_i| = \binom{n}{i}(2^n - 1)^i$. This yields a final count of:

$$\sum_{i=0}^{n} \binom{n}{i} (2^n - 1)^i$$

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