RAY TRACING AND RADIOMETRY 5

CS184: COMPUTER GRAPHICS AND IMAGING

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1 Ray-Triangle Intersection

Given a mesh representation of an object, we would like to render it onto a display. To do so, we need to know which parts of the object are visible, where to put shadows, how to apply the scene's lighting, and more. The simplest idea to handle these problems is to take a ray and intersect it with each triangle in the mesh.

Recall that a ray is defined by its origin $\mathbf{O}$ and a direction vector $\mathbf{D}$ and varies with "time" t for $0 \leq t < \infty$.

$$\mathbf{r}(t) = \mathbf{O} + t\mathbf{D}. \tag{1}$$

A point within a triangle $\mathbf{P_0P_1P_2}$ can be represented as

$$\mathbf{P} = \alpha\mathbf{P}_0 + \beta\mathbf{P}_1 + \gamma\mathbf{P}_2, \tag{2}$$

where $\alpha + \beta + \gamma = 1$. Since α, β and γ are related, we can also write P as

$$\mathbf{P} = (1 - b_1 - b_2)\mathbf{P}_0 + b_1\mathbf{P}_1 + b_2\mathbf{P}_2. \tag{3}$$

1. Let's solve for the intersection of a ray and a triangle. Specifically, if we arrange the unknowns t, b_1 and b_2 into a column vector $\mathbf{x} = [t, b_1, b_2]^T$, can you get a matrix $\mathbf{M}$ and a column vector $\mathbf{b}$ so that $\mathbf{Mx} = \mathbf{b}$?

Solution:

Since the intersection is both along the ray and on the triangle, we have

$$\mathbf{O} + t\mathbf{D} = \mathbf{P_0} + b_1(\mathbf{P_1} - \mathbf{P_0}) + b_2(\mathbf{P_2} - \mathbf{P_0}). \tag{4}$$

Thus,

$$\mathbf{O} - \mathbf{P_0} = -t\mathbf{D} + b_1(\mathbf{P_1} - \mathbf{P_0}) + b_2(\mathbf{P_2} - \mathbf{P_0}). \tag{5}$$

Writing it in matrix form, we have

$$\begin{bmatrix} -\mathbf{D} & \mathbf{P_1} - \mathbf{P_0} & \mathbf{P_2} - \mathbf{P_0} \end{bmatrix} \begin{bmatrix} t \\ b_1 \\ b_2 \end{bmatrix} = \mathbf{O} - \mathbf{P_0} \tag{6}$$

So, $\mathbf{M} = [-\mathbf{D}, \mathbf{P_1} - \mathbf{P_0}, \mathbf{P_2} - \mathbf{P_0}]$, $\mathbf{b} = \mathbf{O} - \mathbf{P_0}$.

2. Now let's derive the **Möller-Trumbore algorithm**!

$$\begin{bmatrix} t \\ b_1 \\ b_2 \end{bmatrix} = \frac{1}{\mathbf{S_1} \cdot \mathbf{E_1}} \begin{bmatrix} \mathbf{S_2} \cdot \mathbf{E_2} \\ \mathbf{S_1} \cdot \mathbf{S} \\ \mathbf{S_2} \cdot \mathbf{D} \end{bmatrix} \tag{7}$$

where $\mathbf{E_1} = \mathbf{P_1} - \mathbf{P_0}$, $\mathbf{E_2} = \mathbf{P_2} - \mathbf{P_0}$, $\mathbf{S} = \mathbf{O} - \mathbf{P_0}$, $\mathbf{S_1} = \mathbf{D} \times \mathbf{E_2}$, $\mathbf{S_2} = \mathbf{S} \times \mathbf{E_1}$.

Hint 1: (Cramer's rule) Linear equations $\mathbf{Mx} = \mathbf{b}$ can be simply solved using determinants of matrices as:

$$\mathbf{x} = \frac{1}{|\mathbf{M}|} \begin{bmatrix} |\mathbf{M_1}| \\ |\mathbf{M_2}| \\ |\mathbf{M_3}| \end{bmatrix}, \tag{8}$$

where $\mathbf{M_i}$ is the matrix $\mathbf{M}$ with its i-th column replaced by $\mathbf{b}$.

Hint 2: Suppose $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are column vectors, the determinant of the 3×3 matrix $[\mathbf{A}, \mathbf{B}, \mathbf{C}]$ satisfy:

$$|\mathbf{A}, \mathbf{B}, \mathbf{C}| = -(\mathbf{A} \times \mathbf{C}) \cdot \mathbf{B} = -(\mathbf{C} \times \mathbf{B}) \cdot \mathbf{A} = -(\mathbf{B} \times \mathbf{A}) \cdot \mathbf{C}. \tag{9}$$

Solution: Applying Cramer's rule, we immediately have

$$\begin{bmatrix} t \\ u \\ v \end{bmatrix} = \frac{1}{|\mathbf{M}|} \begin{bmatrix} |\mathbf{M_1}| \\ |\mathbf{M_2}| \\ |\mathbf{M_3}| \end{bmatrix} \tag{10}$$

$$= \frac{1}{|\ -\mathbf{D} \quad \mathbf{P_1} - \mathbf{P_0} \quad \mathbf{P_2} - \mathbf{P_0}\ |} \begin{bmatrix} |\ \mathbf{O} - \mathbf{P_0} \quad \mathbf{P_1} - \mathbf{P_0} \quad \mathbf{P_2} - \mathbf{P_0}\ | \\ |\ -\mathbf{D} \quad \mathbf{O} - \mathbf{P_0} \quad \mathbf{P_2} - \mathbf{P_0}\ | \\ |\ -\mathbf{D} \quad \mathbf{P_1} - \mathbf{P_0} \quad \mathbf{O} - \mathbf{P_0}\ | \end{bmatrix} \tag{11}$$

$$= \frac{1}{|\ -\mathbf{D} \quad \mathbf{E_1} \quad \mathbf{E_2}\ |} \begin{bmatrix} |\ \mathbf{S} \quad \mathbf{E_1} \quad \mathbf{E_2}\ | \\ |\ -\mathbf{D} \quad \mathbf{S} \quad \mathbf{E_2}\ | \\ |\ -\mathbf{D} \quad \mathbf{E_1} \quad \mathbf{S}\ | \end{bmatrix} \tag{12}$$

Now let's take a look at these determinants, we have

$$| -\mathbf{D} \quad \mathbf{E_1} \quad \mathbf{E_2} \,| = -\left(-\mathbf{D} \times \mathbf{E_2}\right) \cdot \mathbf{E_1} = \mathbf{S_1} \cdot \mathbf{E_1}, \tag{13}$$

$$| \,\mathbf{S} \quad \mathbf{E_1} \quad \mathbf{E_2} \,| = -\left(\mathbf{E_1} \times \mathbf{S}\right) \cdot \mathbf{E_2} = \mathbf{S_2} \cdot \mathbf{E_2}, \tag{14}$$

$$| -\mathbf{D} \quad \mathbf{S} \quad \mathbf{E_2} \,| = -\left(-\mathbf{D} \times \mathbf{E_2}\right) \cdot \mathbf{S} = \mathbf{S_1} \cdot \mathbf{S}, \tag{15}$$

$$| -\mathbf{D} \quad \mathbf{E_1} \quad \mathbf{S} \,| = -\left(\mathbf{S} \times \mathbf{E_1}\right) \cdot \mathbf{D} = \mathbf{S_2} \cdot \mathbf{D}. \tag{16}$$

3. Once you've solved for t, b_1 and b_2, what conditions must be satisfied so that you have a valid ray-triangle intersection?

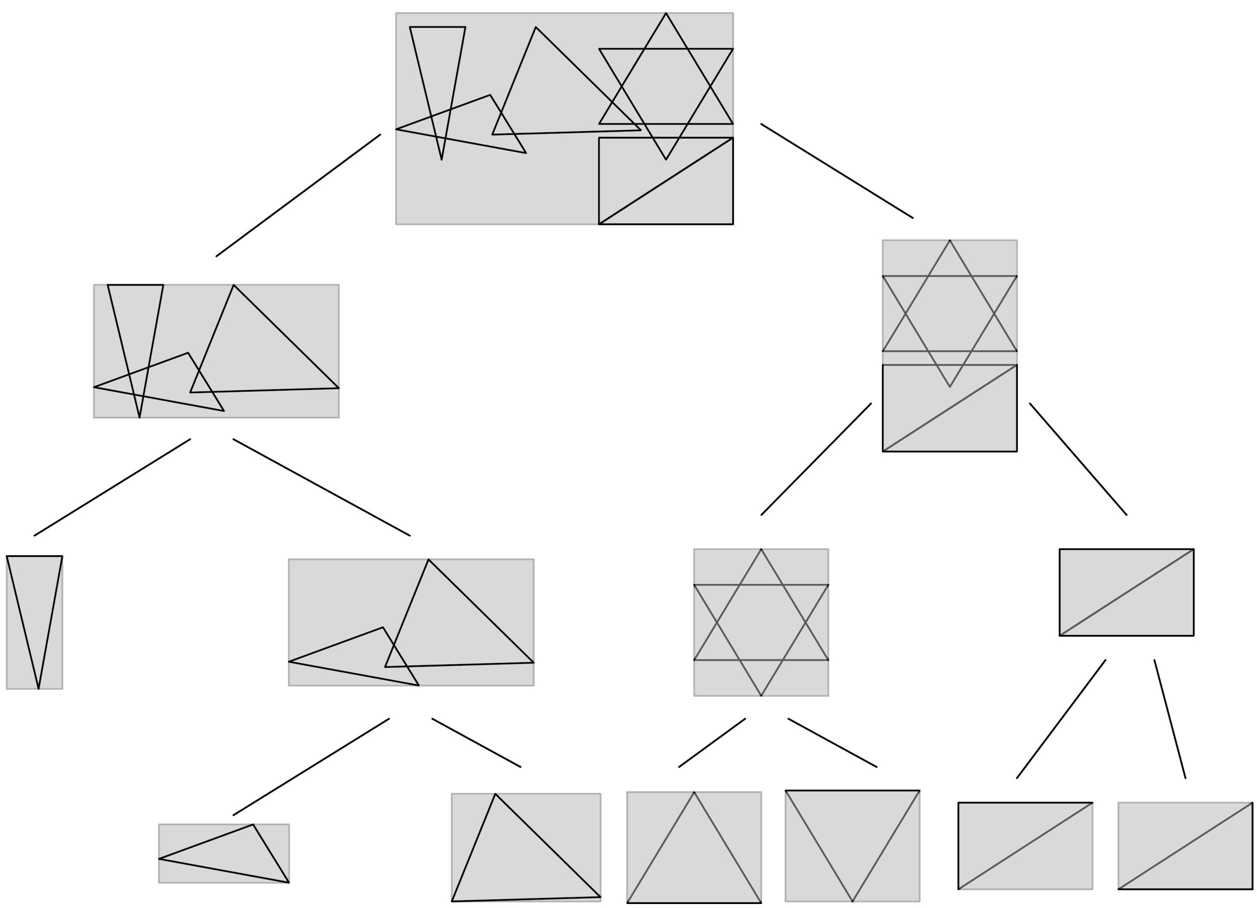
Solution: $t \geq 0$, $0 \leq b_1 \leq 1$, $0 \leq b_2 \leq 1$, $0 \leq 1 - b_1 - b_2 \leq 1$.

2 Bounding Volume Hierarchy

In ray tracing, bounding volumes are used to accelerate ray-triangle intersection tests. If the ray does not intersect a bounding volume, it cannot intersect the triangles contained within, allowing us to perform a batch rejection.

A bounding volume hierarchy (BVH) is simply a tree of bounding volumes. The bounding volume at a given node encloses the bounding volumes of its children. The ray tracing algorithm traverses this hierarchy to determine if the ray intersects an object.

1. Given a set of planar triangles, build a BVH following these rules:
   - Always pick the longest axis to divide.
   - Use barycenters of triangles to decide their relative positions.
   - Keep the BVH as balanced as possible, i.e. try to ensure the same number of triangles for children nodes.



2. Given a box with corners $(-2,-2,-2)$ and $(2,2,2)$. Compute the entry and exit point of this box for a ray that has origin $(-3,4,5)$ and direction $(1,-1,-2)$.

Solution: Intersecting the yz-slabs, we have

$$t_{x,1} = (-2-(-3))/1 = 1, \tag{17}$$
$$t_{x,2} = (2-(-3))/1 = 5. \tag{18}$$

Intersecting the xz-slabs, we have

$$t_{y,1} = (-2-4)/(-1) = 6, \tag{19}$$
$$t_{y,2} = (2-4)/(-1) = 2. \tag{20}$$

Intersecting the xy-slabs, we have

$$t_{z,1} = (-2-5)/(-2) = 3.5, \tag{21}$$
$$t_{z,2} = (2-5/(-2) = 1.5. \tag{22}$$

So we have

$$t_{x,\min} = 1,\ t_{x,max} = 5, \tag{23}$$
$$t_{y,\min} = 2,\ t_{y,max} = 6, \tag{24}$$
$$t_{z,\min} = 1.5,\ t_{z,max} = 3.5. \tag{25}$$

Then

$$t_{\min} = \max\{t_{x,\min}, t_{y,\min}, t_{z,\min}\} = 2, \tag{26}$$
$$t_{\max} = \min\{t_{x,\max}, t_{y,\max}, t_{z,\max}\} = 3.5. \tag{27}$$

Since $t_{\min} <= t_{\max}$ and $t_{\min} > 0$ and $t_{\max} > 0$, we have two intersections. The entry and exit points are at

$$(-3,4,5) + t_{\min}(1,-1,-2) = (-1,2,1) \tag{28}$$

and

$$(-3,4,5) + t_{\max}(1,-1,-2) = (0.5,0.5,-2). \tag{29}$$

3 Radiometry & Photometry

In computer graphics, we study radiometry and photometry to accurately simulate how much light is emitted and received, so that we can generate photo-realistic images.

1. What's the difference between **radiant flux / power** (Φ), **radiant intensity** (I), **irradiance** (E) and **radiance** (L)? How does increasing the distance from the light source affect these values?

   Solution: The radiant flux (power) Φ is the energy emitted, reflected, transmitted or received, per unit time.
   The radiant intensity I is the power per unit solid angle emitted by a point light source.
   The irradiance E is the power per unit area incident on a surface point.
   The radiance (L) is the power emitted, reflected, transmitted or received by a surface, per unit solid angle, per unit projected area.

   Their relations:

   $$I = \frac{\mathrm{d}\Phi}{\mathrm{d}\omega}, \tag{30}$$

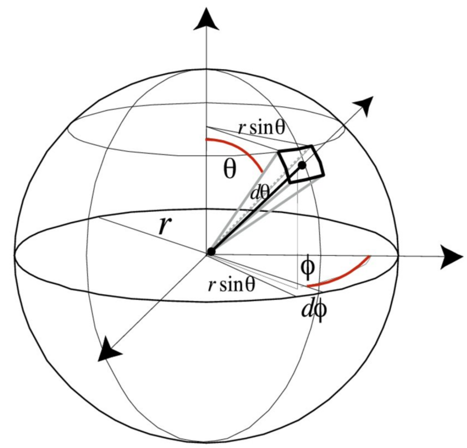
   $$E = \frac{\mathrm{d}\Phi}{\mathrm{d}A}, \tag{31}$$

   $$L = \frac{\mathrm{d}I}{\mathrm{d}A\cos\theta} = \frac{\mathrm{d}^2\Phi}{\mathrm{d}\omega\mathrm{d}A\cos\theta}. \tag{32}$$

   For all of these values, only irradiance changes as we get further away from the light source. Imagine a sphere surrounding the light source, where the radius is equal to the distance from the light source. This light source always emits a fixed amount of radiant flux.

   This sphere will always have 4π steradians, but its surface area will increase as the distance increases. Since we spread the same amount of flux over a greater surface area, irradiance will decreases. As we spread it over the same number of steradians, radiant intensity will not decrease. Radiance along each ray also doesn't decrease over distance, we just integrate over fewer rays when calculating irradiance as the light source is further away.

2. Supppose we use (θ, ϕ)-parameterization of directions. Recall that the solid angle represents the ratio of the subtended area on a sphere to the radius squared, $\Omega = \frac{A}{r^2}$. Estimate the solid angle subtended by a patch that covers $\theta \in [\pi/6 - \pi/12, \pi/6 + \pi/12]$ and $\phi \in [\pi/5 - \pi/24, \pi/5 + \pi/24]$? (Hint: you may assume that the patch is small enough. Recall or derive the differential solid angle $d\omega$, then use the values given.)



Solution: Under (θ, ϕ)-parameterization, we know that the differential solid angle is $\mathrm{d}\omega = \sin\theta\,\mathrm{d}\theta\mathrm{d}\phi$. When a patch is small enough, we can use this to approximate its solid angle as

$$\Delta\omega = \sin\theta\,\Delta\theta\Delta\phi, \tag{33}$$

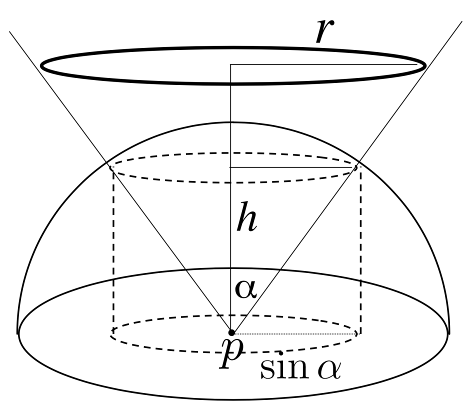
where ϕ is the azimuth angle, and θ the elevation angle, at the center of the patch.

In our specific case, the solid angle subtended by the patch is now

$$\Delta\omega \approx \sin\frac{\pi}{6} \cdot \left[\frac{\pi}{12} - \left(-\frac{\pi}{12}\right)\right] \cdot \left[\frac{\pi}{24} - \left(-\frac{\pi}{24}\right)\right] \tag{34}$$

$$= \frac{\pi^2}{144}. \tag{35}$$

3. Calculate the irradiance at point p from a disk area light overhead with uniform radiance L. (Hint: irradiance is an integral of incoming radiance over the hemisphere: $E(p) = \int_{H^2} L_i(p,\omega)\cos\theta\,\mathrm{d}\omega$.)



Solution:

$$\begin{aligned}
E(p) &= \int_{H^2} L_i(p,\omega)\cos\theta\,\mathrm{d}\omega && (36)\\
&= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\alpha} L\cos\theta\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi && (37)\\
&= 2\pi L\left.\frac{\sin^2\theta}{2}\right|_0^{\alpha} && (38)\\
&= \pi L\sin^2\alpha && (39)\\
&= \frac{\pi L r^2}{r^2+h^2} && (40)
\end{aligned}$$