PROBABILITY AND MONTE CARLO ESTIMATORS 6

CS184: COMPUTER GRAPHICS AND IMAGING

March 1, 2022

1 Quick Terminology

Expectation: probability-weighted average of all possible values. In the discrete case, given by

$$E[X] = \sum_i x_i p_i$$

In the continuous case, given by

$$E[X] = \int xp(x)dx$$

.

Variance: the expected value of the squared deviation from the mean, or how spread apart values are from the mean. Given by

$$\begin{aligned} Var(x) &= E[(X - E[X])^2] &(1)\\ &= E[X^2] - E[X]^2 &(2)\end{aligned}$$

Cumulative Distribution Function (CDF): probability that a sample from distribution X will take a value less than or equal to x.

Lagrange Multipliers: a method to find the maxima or minima of a function $f(x)$ with constraints $g(x) = 0$. We create a function

$$L(x, \lambda) = f(x) + \lambda g(x)$$

and look for critical points where the gradient of L is 0. The critical points of L are the maxima/minima of f.

2 Inversion Method

Recall the inversion method from class. Given a uniform random variable U in the interval $[0, 1]$, we can generate a random variable from any other one dimensional distribution if we have access to the inverse of its cumulative distribution function, $F^{-1}(x)$. We simply have to return $X = F^{-1}(U)$. This is how we choose sample points when running a ray tracing algorithm.

1. What function of U will return a sample from the exponential distribution (with parameter λ)? This distribution has density

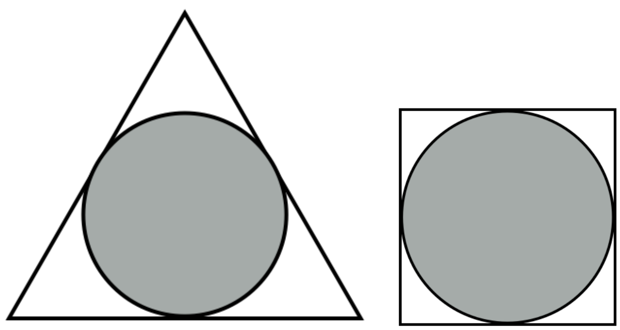
$$p_\lambda(x) = \lambda e^{-\lambda x}$$

   and is defined for $x \geq 0$.

3 Rejection Sampling

Recall that rejection sampling is one way of using the Monte Carlo method to sample from a probability distribution. We repeatedly sample values from a proposed distribution, then accept or reject that sample based on whether or not it falls within a probability density function that we know how to sample from. If the sample falls outside of the PDF, then we reject that sample. The remaining samples should be uniformly distributed within our target probability function.

The figure below shows two methods for estimating π using rejection sampling. Method (a) generates random points uniformly in an equilateral triangle. Method (b) generates random points uniformly in a square. Both methods estimate π using the ratio of the number of points that lie within the shape's tangent circle to the total number of points sampled.



(a) Triangle (b) Square

1. In method (a), suppose there are k points out of all n points sampled that lie within the triangle's tangent circle. Then, a formula for the estimated value of π is:

2. In method (b), suppose there are k points out of all n points sampled that lie within the square's tangent circle. Then, a formula for the estimated value of π is:

3. Which method ((a) or (b)) has lower variance with the same number of samples?

4. What are some downsides to using uniform random sampling?

5. What are some downsides to using rejection sampling for high-dimensional spaces?

4 Unbiased Monte Carlo Estimator

The Monte Carlo method can also be used to estimate an integral. When this method is used properly, the expectation of an **unbiased** Monte Carlo estimator is equal to the true value of the integral.

1. You have two random variables X and Y, which are drawn uniformly from $[-2, 2]$. What is an **unbiased** Monto Carlo estimator for the given integral?

$$F = \int_{-2}^{2} \int_{-2}^{2} f(x, y) dx dy$$

(Hint: Your answer should include a summation over N samples.)