

SPLINES AND CURVES 3

CS184: COMPUTER GRAPHICS AND IMAGING

Feb, 2022

1 Polynomial interpolation

In polynomial interpolation, our goal is to fit a polynomial given some information about points and derivatives of the desired curve. As seen in lecture, we can solve this problem by formulating it as a system of linear equations in the coefficients of the polynomial, and then finding a solution to these equations.

1. List all degree 2 polynomials satisfying: $f(0) = 1$, $f(1) = 2$, $f(2) = 5$. What about degree 3?

2. Suppose we have a list of constraints:

$$f(0) = p_0, f'(0) = d_0, f(1) = p_1, f'(1) = d_1, \dots, f(k) = p_k, f'(k) = d_k .$$

For a function f , what are the tradeoffs when either

- solving for a single $2k + 1$ degree polynomial, versus
- taking the point and derivative constraints at i and $i - 1$ for $i = 1, \dots, k$ and using them to fit k cubic Hermite splines?

3. A cubic polynomial $f(t) = at^3 + bt^2 + ct + d$ is uniquely determined by specifying both its values and its second derivatives at $t = 0$ and $t = 1$ (as opposed to its values and its first derivatives, as in Hermite interpolation). Write out the system of linear equations given by these constraints on $f(0), f(1), f''(0), f''(1)$.

4. Write the matrix which you would invert and apply to the vector $(f(0), f(1), f''(0), f''(1))^T$ to recover a, b, c , and d in the previous problem.

5. Now, compute the numerical answer for the problem above and use its columns to identify four “basis polynomials” for this problem.

2 de Casteljau’s algorithm

de Casteljau’s algorithm allows us to create a smooth Bézier curve from a series of control points. Though we most commonly apply it to four points to get a cubic Bézier curve, it can be applied to any number of points.

In order to find $f(t)$ on a curve defined for $t \in [0, 1]$, de Casteljau gives us the following iterative step:

- Given $k + 1$ points $\mathbf{p}_0, \dots, \mathbf{p}_k$, create a new set of k points $\mathbf{p}'_0, \dots, \mathbf{p}'_{k-1}$ by computing

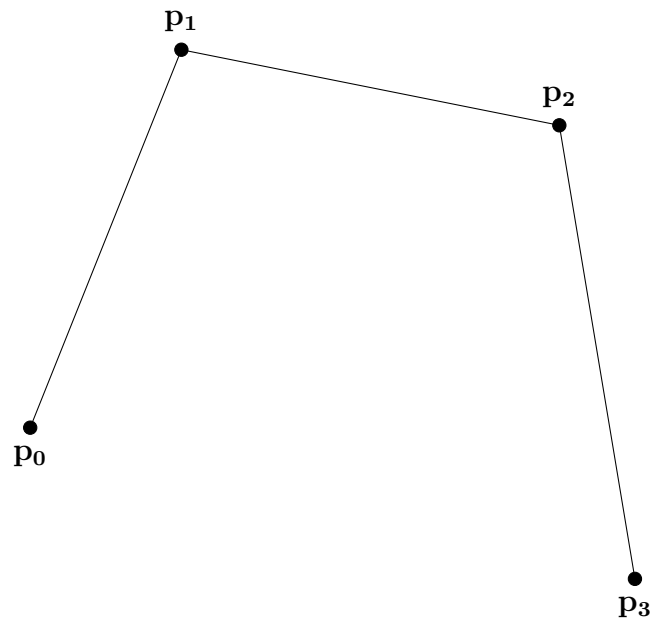
$$\mathbf{p}'_i = \text{lerp}(\mathbf{p}_i, \mathbf{p}_{i+1}, t) ,$$

where $\text{lerp}(\mathbf{p}_i, \mathbf{p}_{i+1}, t) = (1 - t)\mathbf{p}_i + t\mathbf{p}_{i+1}$.

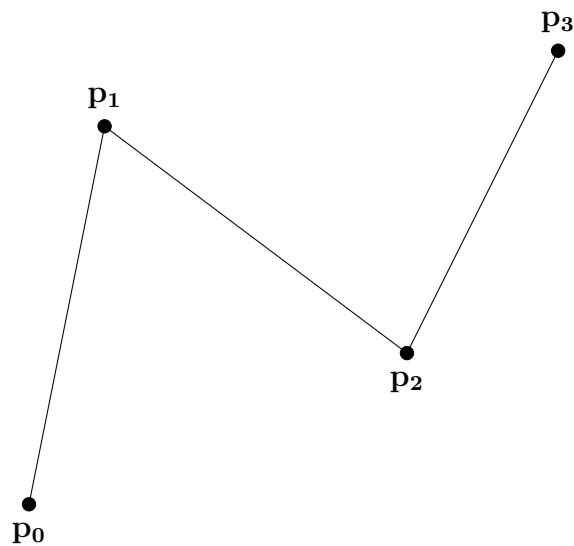
Iteratively applying this step until we are left with a single point yields $f(t)$ for the Bézier curve defined by the initial set of points.

1. For a Bézier curve defined by 3 control points, what is the degree of the polynomial you get from de Casteljau’s algorithm? What about for n points?

2. Use de Casteljau's algorithm to find the point where $t = 1/2$ on the Bézier curve defined by these control points.



3. Use de Casteljau's algorithm to find the point where $t = 1/3$ on the Bézier curve defined by these control points.



4. Show that the point with parameter t on the Bézier curve with control points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ is given by $s^3\mathbf{p}_0 + 3s^2t\mathbf{p}_1 + 3st^2\mathbf{p}_2 + t^3\mathbf{p}_3$, where $s = 1 - t$. (Hint: apply de Casteljau's algorithm algebraically to the control points. With this setup, linear interpolation between two points \mathbf{q}_0 and \mathbf{q}_1 looks like $s\mathbf{q}_0 + t\mathbf{q}_1$.)

5. What is this matrix product? (Hint: *don't* expand it. Instead, think about what each matrix in the product does. How are they related to de Casteljau's algorithm?)

$$(s \ t) \begin{pmatrix} s & t & 0 \\ 0 & s & t \end{pmatrix} \begin{pmatrix} s & t & 0 & 0 \\ 0 & s & t & 0 \\ 0 & 0 & s & t \end{pmatrix}$$