Chapter 2 Worked Exercises

Friday, December 2, 2016 2:45 PM

EXERCISE 2.2 Find the solution of the two-point boundary value problem

$$-u''(x) = f(x), \quad x \in (0,1), \quad u(0) = u(1) = 0.$$

where the right-hand side f is given by

(a) $f(x) = x^2$,

(b) $f(x) = e^x$,

(c) $f(x) = \cos(ax)$, where a is a given real number.

$$G(x,y) = \begin{cases} y(1-x) & \text{if } 0 \le y \le x, \\ x(1-y) & \text{if } x \le y \le 1. \end{cases} \tag{2.8}$$

It follows that the representation (2.7) can be written simply as

$$u(x) = \int_0^1 G(x,y) f(y)\, dy. \tag{2.9}$$

$$\Rightarrow u(x) = x\int_0^1 (1-y)f(y)\,dy - \int_0^x (x-y)f(y)\,dy. \tag{2.7}$$

$$\Rightarrow u(x) = x\int_0^1 (1-y)f(y)\,dy - x\int_0^x f(y)\,dy + \int_0^x y f(y)\,dy$$

(a) $$u(x) = x\int_0^1 (1-y)y^2\,dy - x\int_0^x y^2\,dy + \int_0^x y y^2\,dy$$

$$= x\int_0^1 y^2 - y^3\,dy - x\left[x^3/3 - 0\right] + \left[x^4/4 - 0\right]$$

$$= x\left[\left(\tfrac{1}{3} - \tfrac{1}{4}\right) - (0-0)\right] - \frac{x^4}{12}$$

$$= \boxed{\frac{x}{12} - \frac{x^4}{12}}$$

Verify: BVs: $u(0) = 0$ ✓ $u(1) = \frac{1}{12} - \frac{1}{12} = 0$ ✓

PDE: $u_{xx} = -4 \cdot 3 \cdot x^2/12 = -x^2 \Rightarrow -u_{xx} = x^2$ ✓

(b) Skipped

(c) $$u(x) = x\int_0^1 (1-y)\cos(ay)\,dy - x\int_0^x \cos(ay)\,dy + \int_0^x y\cos(ay)\,dy$$

Aside: $$\int y\cos(ay)\,dy = \left(\tfrac{1}{a}\right)y\sin(ay) - \tfrac{1}{a}\int \sin(ay)\,dy = \left(\tfrac{1}{a^2}\right)\left[ay\sin(ay) + \cos(ay)\right]$$

f dg

$\int f\,dg = fg - \int g\,df$

$f \quad dg \qquad \int f\,dg = fg - \int g\,df$

$$u(x) = \frac{x}{a^2}\Big[a\sin(ay) - ay\sin(ay) - \cos(ay)\Big]_0^1 - \frac{x}{a}\Big[\sin(ax) - 0\Big) + \frac{1}{a}\Big[y\sin(ay) + \cos(ay)\Big]_0^x$$

$$= \frac{x}{a^2}\Big[\big(\underbrace{a\sin(a) - a\sin(a)}_{=0} - \cos(a)\big) - (0-0-1)\Big] - \frac{x}{a}\sin(ax) + \frac{1}{a^2}\Big[\big(ax\sin(ax) + \cos(ax)\big) - (0+1)\Big]$$

cancel, $=0$

$$= \boxed{\frac{x}{a^2}\big(1-\cos(a)\big) + \frac{1}{a^2}\big(\cos(ax) - 1\big)}$$

Verify: BCs: $u(0) = \frac{0}{a^2}\cancel{(x)}^{\,0} + \frac{1}{a^2}\cancel{(1-1)}^{\,0} = 0$

$u(1) = \frac{1}{a^2}\big(1-\cos(a)\big) + \frac{1}{a^2}\big(\cos(a) - 1\big) = 0$ ✓

PDE: $u(x) = \frac{x}{a^2}\big(1-\cos(a)\big) + \frac{1}{a^2}\big(\cos(ax) - 1\big)$

$u_{xx} = -\cos(ax) \Rightarrow -u_{xx} = \cos(ax)$ ✓

EXERCISE 2.6 Consider Poisson's equation with Neumann-type boundary values, i.e.

$$-u''(x) = f(x), \quad x \in (0,1), \quad u'(0) = 0, \quad u'(1) = 0.$$

(a) Show that the condition

$$\int_0^1 f(x)\,dx = 0, \tag{2.49}$$

is necessary in order for this problem to have a solution.

(b) Assume that u is a solution and define $v(x) = u(x) + c$, where c is some given constant. Is v a solution of the problem? Is the solution of this problem unique?

(c) Assume that the condition (2.49) is satisfied. Show that the problem then always has a solution. Furthermore, show that the solution is uniquely determined by the extra condition

$$\int_0^1 u(x)\,dx = 0, \tag{2.50}$$

(a) Pf:

Suppose u is a sol'n. $\Rightarrow u'' = -f$

FTC $\Rightarrow$ $u'(1) - u'(0) = \int_0^1 -f(x)\,dx$

$\Rightarrow$ $0 = \int_0^1 -f(x)\,dx = -\int_0^1 f(x)\,dx$

$\therefore \int_0^1 f(x)\,dx = 0$ ▨

(b) Claim: If u is a sol'n, so is $v = u + c$ $\forall c \in \mathbb{R}$.

Thus, if claim is true and a sol'n exists, then sol'n is not unique.

Pf of claim:

Assume u is a sol'n, let $c \in \mathbb{R}$, and $v = u + c$.

$\Rightarrow v' = u' \Rightarrow v$ satisfies Neumann BCs.

Also, $v'' = u'' \Rightarrow v$ satisfies D.E.

$\therefore v$ is also a sol'n. ▨

(c) Assume $f \in C([0,1])$ and $\int_0^1 f(x)\,dx = 0$.

Suppose $u \in \mathcal{C}^2([0,1])$ w/ $u'(0)=u'(1)=0$.

$$\text{FTC} \Rightarrow \quad u(x) = u(0) + \int_0^x u'(y)\,dy$$

$$u'(y) = \cancelto{0}{u'(0)} + \int_0^y u''(z)\,dz = \int_0^y u''(z)\,dz$$

If $-u''=f \Rightarrow u''=-f$

$$\Rightarrow \quad u'(y) = \int_0^y -f(z)\,dz$$

$$\int_0^x \int_0^y u'(y) = \int_0^x (x-y)\,f(y)\,dy \qquad \text{(follow steps on p. 41)}$$

$$\therefore \quad u(x) = u(0) - \int_0^x (x-y)\,f(y)\,dy$$

<u>Claim:</u> If $f \in \mathcal{C}([0,1])$ and $\int_0^1 f(x)\,dx = 0$, then

$u(x) = -\int_0^x (x-y)\,f(y)\,dy$ solves the Neumann BVP.

<u>Pf:</u>

$$u = -\int_0^x (x-y)\,f(y)\,dy = -x\int_0^x f(y)\,dy + \int_0^x y\,f(y)\,dy$$

$$\Rightarrow u' = -\int_0^x f(y)\,dy - x f(x) + x f(x)$$

$$= -\int_0^x f(y)\,dy$$

$$0 = -\int_0^0 f(y)\,dy = u'(0)$$

and $0 = -\int_0^1 f(y)\,dy = u'(1) \Rightarrow$ Neumann BC satisfied.

Finally,

$$u'' = -f(x) \Rightarrow -u'' = f \Rightarrow \text{D.E. satisfied.} \ \square$$

<u>Claim:</u> If in addition $\int_0^1 u(x)\,dx = 0$ is required, then sol'n unique.

Claim: If in addition $\int_0 u(x)\,dx = 0$ is required, then sol'n unique.

Pf: By above claim and (b), $v = c - \int_0^x (x-y) f(y)\,dy$ is a sol'n w/o the constraint for any $c \in \mathbb{R}$.

w/ the constraint,

$$0 = \int_0^1 v(x)\,dx = \underbrace{\int_0^1 c\,dx}_{=c} - \int_0^1 \int_0^x (x-y) f(y)\,dy\,dx$$

$$\Longrightarrow \quad c = \int_0^1 \int_0^x (x-y) f(y)\,dy\,dx$$

$\therefore$ c is uniquely determined. $\blacksquare$

EXERCISE 2.8 Consider the boundary value problem

$$-(u''(x) + u(x)) = f(x), \quad x \in (0,1), \quad u(0) = u(1) = 0.$$

Show that the solution of this problem can be written in the form (2.9) where the Green's function is given by

$$G(x,y) = \begin{cases} c\sin(y)\sin(1-x) & \text{if } 0 \le y \le x, \\ c\sin(x)\sin(1-y) & \text{if } x \le y \le 1, \end{cases}$$

where $c = 1/\sin(1)$.

Verification that $u(x) = \int_0^1 G(x,y) f(y)\,dy$ solves BVP.

$$u(x) = \int_0^x c\sin(y)\sin(1-x) f(y)\,dy + \int_x^1 c\sin(x)\sin(1-y) f(y)\,dy$$

$$= c\sin(1-x)\int_0^x \sin(y) f(y)\,dy + c\sin(x)\int_x^1 \sin(1-y) f(y)\,dy$$

BCs? $u(0) = \underbrace{c\sin(0)}_{=0} \int_0^1 \sin(1-y) f(y)\,dy = 0$ ✓

$u(1) = \underbrace{c\sin(1-1)}_{=0} \int_0^1 \sin(y) f(y)\,dy = 0$ ✓

$\therefore$ BCs ✓

DE? $\frac{1}{c}u' = -\cos(1-x)\int_0^x \sin(y) f(y)\,dy + \cos x \int_x^1 \sin(1-y) f(y)\,dy$

$\frac{1}{c}u'' = \Big[\quad \Big) \int^x \quad \Big]$

$$\cos(1-x)\int_0 \sin(y) f(y)\,dy + \cos x \int_x \sin(1-y) f(y)\,dy$$

$$\frac{1}{c}u'' = \left[-\sin(1-x)\int_0^x \sin(y) f(y)\,dy - \cos(1-x)\sin x\, f(x)\right]$$

$$+ \left[-\sin x \int_x^1 \sin(1-y) f(y)\,dy - \cos x \sin(1-x) f(x)\right]$$

$$= -\frac{1}{c}u - f(x)\Big[\underbrace{\cos(1-x)\sin x + \cos x \sin(1-x)}_{= \sin(x+1-x) = \sin 1}\Big]$$

$\Rightarrow u'' = -u - f$ (since $c = \frac{1}{\sin 1}$)

$\Rightarrow -u'' - u = f$ ✓

$\therefore$ DE ✓

$\therefore$ u is a sol'n to BVP ∎