# Chapter 1 Worked Exercises

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EXERCISE 1.1 Consider the following differential equations:

$$
\begin{aligned}
(i)\quad & u'(t) = e^t u(t),\\
(ii)\quad & u''(x) = u(x)\sqrt{x},\\
(iii)\ (ii)\quad & u_{xx}(x,y) + u_{yy}(x,y)e^{\sin(x)} = 1,\\
(iv)\quad & u_t(x,t) + u_x(x,t) = u_{xx}(x,t) + u^2(x,t),\\
(v)\quad & (u'(t))^2 + u(t) = e^t.
\end{aligned}
$$

Characterize these equations as:

(a) PDEs or ODEs,

(b) linear or nonlinear,

(c) homogeneous or nonhomogeneous.

(i) ODE, linear, homog.
(ii) ODE, linear, homog
(iii) PDE, linear, inhomog.
(iv) PDE, non-linear, homog
(v) ODE, non-linear, inhomog.

EXERCISE 1.2 Consider

$$
\begin{aligned}
u'(t) &= -\alpha u(t),\\
u(0) &= u_0,
\end{aligned}
\quad \Rightarrow u(t) = u_0 e^{-\alpha t}
$$

for a given $\alpha > 0$. Show that this problem is stable with respect to perturbation in $u_0$.

Let $v' = -\alpha v$, $v(0) = u_0 + \varepsilon \Rightarrow v(t) = (u_0 + \varepsilon)e^{-\alpha t}$

$$\Rightarrow |u(t) - v(t)| = |u_0 e^{-\alpha t} - (u_0+\varepsilon)e^{-\alpha t}|$$

$$= |\varepsilon| e^{-\alpha t}$$

Let $\eta > 0$ and $t > 0$.
Choose $|\varepsilon| < \eta$, then $|\varepsilon| e^{-\alpha t} < |\varepsilon| < \eta$
$\therefore$ $|u(t) - v(t)| < \eta$, so a perturbation can be chosen small enough s.t. at any time $t > 0$ of interest, the difference in $u$ and $v$ is bdd. by $\eta$.
$\Rightarrow$ stable w.r.t. perturbs. in $u_0$. ∎

EXERCISE 1.3 Consider the ordinary differential equation

$$\begin{aligned} u'(t) &= tu(t)(u(t)-2), \\ u(0) &= u_0. \end{aligned} \tag{1.56}$$

(a) Verify that

$$u(t) = \frac{2u_0}{u_0 + (2-u_0)e^{t^2}}$$

solves (1.56).

(b) Show that if $0 \le u_0 \le 2$, then $0 \le u(t) \le 2$ for all $t \ge \cancel{0}\, 0$

(c) Show that if $u_0 > 2$, then $u(t) \to \infty$ as

$$t \to \left(\ln\left(\frac{u_0}{u_0-2}\right)\right)^{1/2}.$$

(d) Suppose we are interested in (1.56) for $u_0$ close to 1, say $u_0 \in [0.9, 1.1]$. Would you say that the problem (1.56) is stable for such data?

(a) $u(t) = 2u_0\left[u_0 + (2-u_0)e^{t^2}\right]^{-1}$

$\Rightarrow u' = -2u_0\left[u_0+(2-u_0)e^{t^2}\right]^{-2}\left[2t(2-u_0)e^{t^2}\right]$

$= \underbrace{\left\{t\, 2u_0\left[u_0+(2-u_0)e^{t^2}\right]^{-1}\right\}}_{=tu} \underbrace{\left\{-2(2-u_0)e^{t^2}\left[u_0+(2-u_0)e^{t^2}\right]^{-1}\right\}}_{}$

$= \left[(2u_0-2u_0) - 2(2-u_0)e^{t^2}\right]\left[u_0+(2-u_0)e^{t^2}\right]^{-1}$

$= u - 2$

$= (tu)(2-u)$ ✓

$u(0) = 2u_0\left[u_0+(2-u_0)\right]^{-1} = 2u_0/2 = u_0$ ✓

(b) $u_0 = 0 \Rightarrow u = 0 \quad \forall t \ge 0$

$u_0 = 2 \Rightarrow u = 2 \quad \forall t \ge 0$

$0 < u_0 < 2 \Rightarrow u_0 + (2-u_0)e^{t^2} \ge u_0 \quad \forall t \ge 0$

$\Rightarrow u(t) \le 2u_0/u_0 = 2$

$\therefore \ \forall t \ge 0$ if $0 \le u_0 \le 2$, $0 \le u(t) \le 2$.

(c) $u_0 > 2$: $\quad u_0 + (2-u_0)e^{t^2} \longrightarrow u_0 + (2-u_0)\exp\left[\ln\left(\frac{u_0}{u_0-2}\right)\right]$

$= u_0 + (2-u_0)(u_0/(u_0-2))$

$= u_0 - u_0$

$= 0$

Since @ $t=0$, $u_0 + (2-u_0)e^0 = u_0$ and $e^{t^2}$ monotonically increasing, we see $u(t) \longrightarrow +\infty$ as $t \longrightarrow \left(\ln\left(\frac{u_0}{u_0-2}\right)\right)^{1/2}$

(d) Yes, $u_0 \in [0.9, 1.1]$ implies all perturbs. of $u_0$ are sufficiently bdd. away from 2 and all sol'ns will behave similarly going to zero as $t \to \infty$.

EXERCISE 1.5 Find the exact solution of the following Cauchy problems:

(a)

$$u_t + 2xu_x = 0 \qquad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = e^{-x^2}.$$

See Example 1.2

(b)

$$u_t - xu_x = 0 \qquad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = \sin(87x).$$

(c) HW

$$u_t + xu_x = x \qquad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = \cos(90x).$$

See §1.4.2

(d)

$$u_t + xu_x = x^2 \qquad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = \sin(87x)\cos(90x).$$

(a) $\frac{dx}{dt} = 2x$, $x(0) = x_0 \Rightarrow x(t) = x_0 e^{2t} \Rightarrow x_0 = xe^{-2t}$

$$u(x,t) = e^{-x_0^2} = \exp\left[(-xe^{-2t})^2\right]$$

Verify: (1) IC: $u(x,0) = \exp\left[(-xe^{0})^2\right] = e^{-x^2}$ ✓

(2) PDE: $u_t = 2(-xe^{-2t})(2xe^{-2t})u$

$$2xu_x = 2x\left[2(-xe^{-2t})(-e^{-2t})u\right]$$
$$= -2(-xe^{-2t})(xe^{-2t})u$$

$\therefore u + 2xu_x = 0$ ✓

EXERCISE 1.6 Compute the exact solution of the following Cauchy problem:

$$u_t + u_x = u, \qquad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = \phi(x), \quad x \in \mathbb{R},$$

where $\phi$ is a given smooth function.

let $\frac{dx}{dt} = 1$, $x(0) = x_0 \Rightarrow x(t) = x_0 + t \Rightarrow x_0 = x - t$

$$\frac{d}{dt}(u(x(t),t)) = u_t + u_x \frac{dx}{dt} = u_t + u_x = u$$

$$\Rightarrow u(x(t),t) = u(x(0),0)\, e^t$$
$$= u(x-t,0)\, e^t$$
$$= \boxed{\varphi(x-t)\, e^t}$$

EXERCISE 1.8 Consider the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = \phi(x), \qquad (1.59)$$
$$u_t(x,0) = \psi(x),$$

for a given $c > 0$. Follow the steps used to derive the solution in the case of $c = 1$ and show that

$$u(x,t) = \frac{1}{2}\big(\phi(x+ct) - \phi(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\theta)\,d\theta$$

solves (1.59).

Similar steps w/ $\xi = x + t$ and $\eta = x - t$ changing to

$$\xi = x + ct \text{ and } \eta = x - ct$$

The rest of the changes are shown below

$$v(\xi,\eta) = u(x,t). \checkmark \qquad (1.36)$$

By the chain rule, we get

$$u_x = v_\xi \frac{\partial \xi}{\partial x} + v_\eta \frac{\partial \eta}{\partial x} = v_\xi + v_\eta \checkmark$$

$u_t = c(v_\xi + v_\eta)$

and

$$u_{xx} = v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta}. \checkmark$$

Similarly, we have

$$u_{tt} = v_{\xi\xi} - 2v_{\xi\eta} + v_{\eta\eta}.$$

$u_{tt} = c^2(v_{\xi\xi} - 2v_{\xi\eta} + v_{\eta\eta})$

and thus (1.33) implies that

$0 = u_{tt} - c^2 u_{xx} = -4c^2 v_{\xi\eta}$

$$u_{tt} = v_{\xi\xi} - 2v_{\xi\eta} + v_{\eta\eta},$$

and thus (1.33) implies that

$$0 = u_{tt} - u_{xx} = -4v_{\xi\eta}.$$

$0 = u_{tt} - c^2 u_{xx} = -4c^2 v_{\xi\eta}$

Since

$$v_{\xi\eta} = 0 \tag{1.37}$$

we easily see that

$$v(\xi,\eta) = f(\xi) + g(\eta). \tag{1.38}$$

$u(x,t) = f(x+ct) + g(x-ct)$

$$u(x,t) = f(x+t) + g(x-t) \tag{1.39}$$

solves (1.33) for any smooth $f$ and $g$. This can be verified by direct derivation:

$$\left.\begin{array}{l} u_{tt} = f'' + g'' \\ u_{xx} = f'' + g'' \end{array}\right\} \implies u_{tt} = u_{xx}.$$

$\left.\begin{array}{l} u_{tt} = c^2(f''+g'') \\ u_{xx} = f''+g'' \end{array}\right\} \Rightarrow u_{tt} = c^2 u_{xx}$

Next we turn our attention to the initial data (1.33) and (1.34). We want to determine the functions $f$ and $g$ in (1.39) such that (1.33) and (1.34) are satisfied. Of course, $\phi$ and $\psi$ are supposed to be given functions.

By (1.39) we have

$$u(x,t) = f(x+t) + g(x-t)$$

$u(x,t) = f(x+ct) + g(x-ct)$

and

$$u_t(x,t) = f'(x+t) - g'(x-t).$$

$u_t(x,t) = c\left(f'(x+ct) - g'(x-ct)\right)$

Inserting $t = 0$, (1.34) and (1.35) imply that

$$\phi(x) = f(x) + g(x) \tag{1.40}$$

and

$$\psi(x) = f'(x) - g'(x). \tag{1.41}$$

$\psi(x) = c\left(f'(x) - g'(x)\right)$

By differentiating (1.40) with respect to $x$, we get

$$\phi'(x) = f'(x) + g'(x). \tag{1.42}$$

Combining (1.41) and (1.42) yields

$$f' = \frac{1}{2}(\phi' + \psi)$$

$f' = \frac{1}{2}\left(\varphi' + \frac{1}{c}\psi\right)$

and

$$g' = \frac{1}{2}(\phi' - \psi),$$

$g' = \frac{1}{2}\left(\varphi' - \frac{1}{c}\psi\right)$

and thus, by integration, we have

$f(s) = c_1 + \frac{1}{2}\varphi(s) + \frac{1}{2c}\int_0^s \psi(\theta)d\theta$

$$f(s) = c_1 + \frac{1}{2}\phi(s) + \frac{1}{2}\int_0^s \psi(\theta)d\theta \tag{1.43}$$

$g(s) = c_2 + \frac{1}{2}\varphi(s) - \frac{1}{2c}\int_0^s \psi(\theta)d\theta$

$$g(s) = c_2 + \frac{1}{2}\phi(s) - \frac{1}{2}\int_0^s \psi(\theta)d\theta, \tag{1.44}$$

where $c_1$ and $c_2$ are constants of integration. From (1.40) we note that

$$\phi(x) = f(x) + g(x),$$

and thus by adding (1.43) and (1.44), we observe that

$$c_1 + c_2 = 0.$$

Putting $s = x + t$ in (1.43) and $s = x - t$ in (1.44), it follows from (1.39) that

$s = x + ct$ , $s = x - ct$

$$u(x,t) = \frac{1}{2}(\phi(\underbrace{x+t}_{x+ct}) + \phi(\underbrace{x-t}_{x-ct})) + \frac{1}{\underset{2c}{2}}\int_0^{\overset{x+ct}{x+t}} \psi(\theta)d\theta - \frac{1}{\underset{2c}{2}}\int_0^{\overset{x-ct}{x-t}} \psi(\theta)d\theta,$$

or

$$u(x,t) = \frac{1}{2}(\phi(x+\underset{ct}{t}) + \phi(x-\underset{ct}{t})) + \frac{1}{\underset{2c}{2}}\int_{x-\underset{ct}{t}}^{x+\overset{ct}{t}} \psi(\theta)d\theta. \qquad (1.45)$$

Done.

EXERCISE 1.9 Use the solution derived above to solve the Cauchy problem

$$\begin{aligned} u_{tt} &= 16u_{xx}, & x\in\mathbb{R},\ t>0,\\ u(x,0) &= 6\sin^2(x), & x\in\mathbb{R},\\ u_t(x,0) &= \cos(6x), & x\in\mathbb{R}. \end{aligned}$$

$c = 4$ , $\varphi = 6\sin^2 x$ , $\psi = \cos(6x)$

$$u(x,t) = \frac{1}{2}\left(6\sin^2(x+4t) + 6\sin^2(x-4t)\right) + \frac{1}{8}\int_{x-4t}^{x+4t}\cos(6\theta)\,d\theta$$

$$= \boxed{3\left(\sin^2(x+4t) + \sin^2(x-4t)\right) + \frac{1}{48}\left(\sin(6x+24t) - \sin(6x-24t)\right)}$$

EXERCISE 1.13 Consider the function $u(x,t)$ given by (1.55).

(a) Verify directly that $u$ satisfies the heat equation (1.48) for any $x \in \mathbb{R}$ and $t > 0$.

(b) Let $t > 0$ be fixed. Show that $u(\cdot,t) \in C^\infty(\mathbb{R})$, i.e. $u$ is a $C^\infty$-function with respect to $x$ for any fixed $t > 0$.

(c) Show that
$$u(0,t) = \frac{1}{2} \qquad \text{for all } t > 0.$$

(d) Let $x \neq 0$ be fixed. Show that
$$\lim_{t\to 0+} u(x,t) = H(x).$$

(a)

$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x/2\sqrt{t}} e^{-\theta^2}\,d\theta. \qquad (1.55)$$

$$u_t(x,t) = u_{xx}(x,t), \qquad x\in\mathbb{R},\ t>0, \qquad (1.48)$$

$$u_x = \left[\frac{1}{\sqrt{\pi}}e^{-(x^2/4t)}\right]\frac{1}{2\sqrt{t}} \Rightarrow u_{xx} = \left(\frac{-2x}{4t\sqrt{\pi}}\right)e^{-x^2/4t}\frac{1}{2\sqrt{t}} = \frac{-x}{4\sqrt{\pi}\,t^{3/2}}e^{-x^2/4t}$$

$$u_t = \left[\frac{1}{\sqrt{\pi}}e^{-(x^2/4t)}\right]\frac{d}{dt}\left(\frac{x}{2\sqrt{t}}\right) = \frac{1}{\sqrt{\pi}}e^{-x^2/4t}\left(\frac{x}{2}\right)\left(-\frac{1}{2}t^{-3/2}\right) = \frac{-x}{4\sqrt{\pi}\,t^{3/2}}e^{-x^2/4t}$$

✓

$$u_x = \left[\frac{1}{\sqrt{\pi}} e^{-(x^2/4t)}\right]\frac{1}{2\sqrt{t}} \Rightarrow u_{xx} = \left(-\frac{2x}{4t\sqrt{\pi}}\right) e^{-x^2/4t}\,\frac{1}{2\sqrt{t}} = \frac{-x}{4\sqrt{\pi}\,t^{3/2}}\, e^{-x^2/4t}$$

$$u_t = \left[\frac{1}{\sqrt{\pi}} e^{-(x^2/4t)}\right]\frac{d}{dt}\left(\frac{x}{2\sqrt{t}}\right) = \frac{1}{\sqrt{\pi}} e^{-x^2/4t}\left(\frac{x}{2}\right)\left(-\frac{1}{2}t^{-3/2}\right) = \frac{-x}{4\sqrt{\pi}\,t^{3/2}}\, e^{-x^2/4t}$$ ✓

(b) $u_x(x,t) \in C^\infty(\mathbb{R})\ \forall t>0 \Rightarrow u(x,t) \in C^\infty(\mathbb{R})$ from (a).

(c) $u(0,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{0} e^{-\theta^2}\,d\theta = \frac{1}{2}$ b/c of properties of $N(0,1)$ r.v.'s.
(basic probability)

(d) $x \neq 0$, $\lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+} \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x/2\sqrt{t}} e^{-\theta^2}\,d\theta$

right... take limits...

$$\text{“}=\text{”} \begin{cases} \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\theta^2}\,d\theta, & x>0 \\ \frac{1}{\sqrt{\pi}}\int_{-\infty}^{-\infty} e^{-\theta^2}\,d\theta, & x<0 \end{cases}$$

$$\text{“}=\text{”} \begin{cases} 1, & x>0 \\ 0, & x<0 \end{cases} = H(x).$$ ✓

EXERCISE 1.16 Let $u(x,t)$ be a solution of the heat equation (1.48) with initial data

$$u(x,0) = f(x).$$

(a) Let $a \in \mathbb{R}$ and define a function

$$v(x,t) = u(x-a,t).$$

Show that $v$ solves the heat equation with initial data $v(x,0) = f(x-a)$.

(a) $\left.\begin{array}{l} v_t = u_t \\ v_{xx} = u_{xx} \end{array}\right\} \Rightarrow v_t = v_{xx}.$ ✓

$v(x,0) = u(x-a,0) = f(x-a)$ ✓

(b) Let $k > 0$ be given and define

$$w(x,t) = u(k^{1/2}x, kt).$$

Show that $w$ solves the heat equation with initial data $w(x,0) = f(k^{1/2}x)$.

(b) $\left.\begin{array}{l} w_t = k u_t \\ w_{xx} = k u_{xx} \end{array}\right\} \Rightarrow w_t = w_{xx}$

$w(x,0) = u(k^{1/2}x, 0) = f(k^{1/2}x)$ ✓

(c) Assume that $u^1(x,t), u^2(x,t), \ldots, u^n(x,t)$ are solutions of the heat equation (1.48) with initial functions

$$u^k(x,0) = f^k(x) \quad \text{for } k = 1,2,\ldots,n.$$

Furthermore, let $c_1, c_2, \ldots, c_n \in \mathbb{R}$ and define a new function $u(x,t)$ by

$$u(x,t) = \sum_{k=1}^{n} c_k u^k(x,t).$$

Show that $u$ solves (1.48) with initial data

$$u(x,0) = \sum_{k=1}^{n} c_k f^k(x).$$

(c) $\left.\begin{array}{l} u_t = \sum_{k=1}^{n} c_k u_t^k(x,t) \\ u_{xx} = \sum_{k=1}^{n} c_k u_{xx}^k(x,t) \end{array}\right\} \Rightarrow u_t = u_{xx}$ ✓

$U(x,0) = \sum_{k=1}^{n} c_k u^k(x,0) = \sum_{k=1}^{n} c_k f^k(x)$ ✓

EXERCISE 1.17 Consider the function $S(x,t)$ given by

$$S(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \qquad \text{for } x \in \mathbb{R}, \quad t > 0.$$

This function is well known in probability theory. It corresponds to the density function for the normal distribution with variance $2t$. As we shall see below, this function also appears naturally in the analysis of the Cauchy problem for the heat equation. In the context of differential equations the function $S$ is therefore frequently referred to as the Gaussian kernel function or the fundamental solution of the heat equation.

(a) Use the result of Exercise 1.11 to show that

$$\int_{\mathbb{R}} S(x,t)\, dx = 1 \qquad \text{for any } t > 0.$$ Calculus

(b) Consider the solution (1.55) of the heat equation (1.48) with the Heaviside function $H$ as a initial function. Show that $u(x,t)$ can be expressed as

$$u(x,t) = \int_{\mathbb{R}} S(x-y,t) H(y)\, dy.$$

(b) $u(x,t) = \int_{\mathbb{R}} S(x-y,t) H(y)\, dy$

$= \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-(x-y)^2/4t}\, dy$

Let $\theta = \frac{x-y}{2\sqrt{t}}$

$\Rightarrow y = x - 2\sqrt{t}\,\theta$

$\Rightarrow dy = -2\sqrt{t}\, d\theta = -\sqrt{4t}\, d\theta$

$\theta(0) = x/2\sqrt{t}$

forgive me $\rightarrow \theta(\infty) = -\infty$

(c) Let $a \in \mathbb{R}$ be given and define

$$v(x,t) = \int_{\mathbb{R}} S(x-y,t) H(y-a)\, dy.$$

Use the result of Exercise 1.16 $(a)$ to show that $v$ solves (1.48) with initial condition

$$u(x,0) = H(x-a).$$ skipped, easy

(d) Let $a, b \in \mathbb{R}$, $a < b$, be given and define

$$u(x,0) = H(x-a).$$ skip

forgive me $\longrightarrow \theta(\infty) = -\infty$

$$\therefore u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/2\sqrt{t}} e^{-\theta^2} d\theta \checkmark$$

(d) Let $a, b \in \mathbb{R}$, $a < b$, be given and define

$$\chi_{a,b}(x) = \begin{cases} 1 & \text{for } x \in [a,b], \\ 0 & \text{otherwise.} \end{cases}$$

Show that the function

$$u(x,t) = \int_{\mathbb{R}} S(x-y,t)\chi_{a,b}(y)\,dy$$

solves (1.48) with initial condition

$$u(x,0) = \chi_{a,b}(x).$$

Hint: Observe that $\chi_{a,b}(x) = H(x-a) - H(x-b)$ and use Exercise 1.16 (c).

yup, easy, skipped

(e) Let $f(x)$ be a step function of the form

$$f(x) = \begin{cases} 0 & \text{for } x \le a_0, \\ c_1 & \text{for } x \in [a_0, a_1], \\ \vdots & \\ c_n & \text{for } x \in [a_{n-1}, a_n], \\ 0 & \text{for } x > a_n, \end{cases}$$

where $c_1, c_2, \ldots, c_n$ and $a_0 < a_1 < \cdots < a_n$ are real numbers. Show that the function $u(x,t)$ given by

$$u(x,t) = \int_{\mathbb{R}} S(x-y,t) f(y)\,dy \tag{1.61}$$

solves the heat equation (1.48) with initial condition

$$u(x,0) = f(x).$$

Same idea over and over ... gets boring...

In fact, the solution formula (1.61) is not restricted to piecewise constant initial functions $f$. This formula is true for general initial functions $f$, as long as $f$ satisfies some weak smoothness requirements. We will return to a further discussion of the formula (1.61) in Chapter 12.

sure we will