Scratch Work

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EXERCISE 2.18 We want to solve the problem (2.26) numerically for a right-hand side satisfying

$$||f''||_\infty \leq 1705,$$

and we want an approximation for which the error measured in absolute values at the grid points is less than 1/80545.

(a) How large do we have to choose n in order to be sure that the approximate solution defined by (2.27) is sufficiently accurate?

(b) It turns out that we are only interested in the solution at $x = 1/10$ and at $x = 9/10$. Is this information of any help? Can we reduce the number of grid points computed above?

Theorem 2.2 *Assume that $f \in C^2([0,1])$ is given. Let u and v be the corresponding solutions of (2.26) and (2.27), respectively. Then*

$$||e||_{h,\infty} = ||u - v||_{h,\infty} \leq \frac{||f''||_\infty}{96} h^2.$$

$$(Lu)(x) = f(x) \quad \text{for all } x \in (0,1). \qquad (2.26) \Rightarrow u \text{ is exact sol'n}$$

$$(L_h v)(x_j) = f(x_j) \quad \text{for all } j = 1, \ldots, n. \qquad (2.27) \Rightarrow v \text{ is approx sol'n}$$

(a) Goal: choose $h = \frac{1}{n}$ sufficiently small s.t.

$$||e||_{h,\infty} < \frac{1}{80545}$$

Thm 2.2 $\Rightarrow$ choose h s.t. $\frac{||f''||_\infty}{96} h^2 \leq \frac{1705}{96} h^2 < \frac{1}{80545}$

$$\Rightarrow h < 8.39 \times 10^{-4}$$
$$\Rightarrow \frac{1}{n} < 8.39 \times 10^{-4}$$
$$\Rightarrow n > 1196$$

(b) Maybe use non-uniform mesh. The analysis we currently have will <u>not</u> apply b/c base on $\|\cdot\|_\infty$ norms, so the error analysis is a global, not local, result.

Alternative: use adjoint based analysis.

EXERCISE 2.19 Consider the differential equation

$$-u''(x) + u(x) = f(x)$$

and the difference approximation

$$-\frac{v_{j-1} - 2v_j + v_{j+1}}{h^2} + v_j = f(x_j).$$

(a) Identify the differential operator L and the difference operator L_h.

(b) Define and compute the truncation error τ_h.

(c) Show that the scheme is consistent provided that the solution u is sufficiently smooth.

(a) $L = -\frac{d^2}{dx^2} + 1$, $L_h = -\frac{\delta_{j-1,i} - 2\delta_{ji} + \delta_{j+1,i}}{h^2} + \delta_{ji}$

Here, we are using the Kronecker delta fns, where

$$\delta_{ji} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

(b) **Definition 2.2** *Let $f \in C((0,1))$, and let $u \in C_0^2((0,1))$ be the solution of (2.26). Then we define the discrete vector τ_h, called the* truncation error, *by*

$$\tau_h(x_j) = (L_h u)(x_j) - f(x_j) \quad \text{for all} \quad j = 1, \ldots, n.$$

$$\tau_h(x_j) := (L_h u)(x_j) - f(x_j)$$

$$= -\left(\frac{u_{j-1} - 2u_j + u_{j+1}}{h^2}\right) + u_j - f(x_j)$$

$$= -\left(u_j'' + \frac{u_j^{(4)}(\xi_1)}{24}h^2 + \frac{u_j^{(4)}(\xi_2)}{24}h^2\right) + u_j - f(x_j)$$

$$\xi_1 \in (x_j, x_{j+1}),\ \xi_2 \in (x_{j-1}, x_j)$$

$$= -\left(\frac{u_j^{(4)}(\xi_1)}{24} + \frac{u_j^{(4)}(\xi_2)}{24}\right)h^2$$

(c) *We say that the finite difference scheme (2.27) is* consistent *with the differential equation (2.26) if*

$$\lim_{h\to 0} ||\tau_h||_{h,\infty} = 0.$$

$$||\tau_h||_{h,\infty} \le \frac{1}{12}||u^{(4)}||_\infty h^2$$

$$-u'' + u = f \Rightarrow -u'' = f - u$$

$$\Rightarrow -u^{(4)} = f'' - u'' = f'' + f - u$$

$\Rightarrow ||\tau_h||_{h,\infty} \le \frac{1}{12}\left(||f''||_\infty + ||f||_\infty + ||u||_\infty\right)h^2$ by Δ-ineq.

$\exists$ Green's fcn for this problem, $G(x,y)$, s.t.

$$u(x) = \int_0^1 G(x,y) f(y)\,dy$$

$$\Rightarrow ||u||_\infty \le C||f||_\infty,\quad C = \sup_{x\in[0,1]} \sup_{y\in[0,1]} |G(x,y)|$$

$\therefore \ \|\tau_h\|_{h,\infty} \leq \frac{1}{12}\left(\|f''\|_\infty + (c+1)\|f\|_\infty\right)h^2 \longrightarrow 0$ as $h \to 0$