Chapter 3: Exercises

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EXERCISE 3.1 Find the Fourier sine series on the unit interval for the following functions:

(a) $f(x) = 1 + x$,

(b) $f(x) = x^2$,

(c) $f(x) = x(1 - x)$.

$$f(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x)$$

$$c_k = 2 \langle f, \sin(k\pi x) \rangle$$

(a) From Example 3.3,

$f(x) = 1$ in terms of a Fourier sine series. Using (3.29) above,

$$c_k = 2\int_0^1 \sin(k\pi x)\,dx = \frac{2}{k\pi}(1 - \cos(k\pi)).$$

$(-1)^k$

$$\cancel{c_k = \begin{cases} \frac{4}{k\pi} & \text{for } k = 1, 3, 5 \dots, \\ 0 & \text{for } k = 2, 4, 6 \dots, \end{cases}}$$

and

EXAMPLE 3.4 Next we want to compute the Fourier sine series of $\boxed{f(x) = x.}$ Using (3.29), we get

$$c_k = 2\int_0^1 x \sin(k\pi x)\,dx = \left[\frac{2}{(k\pi)^2}\sin(k\pi x) - \frac{2x}{k\pi}\cos(k\pi x)\right]_0^1 = \frac{2}{k\pi}(-1)^{k+1}.$$

Hence the Fourier sine series of $f(x) = x$ on the unit interval is given by

$$x = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin(k\pi x). \qquad (3.31)$$

$\therefore$ we see that for $f(x) = 1 + x$,

$$c_k = 2\int_0^1 (1 + x)\sin(k\pi x)\,dx$$

$$= 2\int_0^1 \sin(k\pi x)\,dx + 2\int_0^1 x \sin(k\pi x)\,dx$$

$$= \frac{2}{k\pi}\Big(1-\underbrace{\cos(k\pi)}_{(-1)^k}\Big) + \frac{2}{k\pi}(-1)^{k+1}$$

$$k = 1,3,5,\ldots \Rightarrow c_k = \frac{4}{k\pi} + \frac{2}{k\pi} = \frac{6}{k\pi}$$

$$k = 2,4,6,\ldots \Rightarrow c_k = 0 - \frac{2}{k\pi} = -\frac{2}{k\pi}$$

Can rewrite this as

$$c_k = \frac{2}{k\pi}\left(1 + 2(-1)^{k+1}\right)$$

$$\therefore \boxed{f(x) = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(1+2(-1)^{k+1})}{k}\sin(k\pi x)}$$

(b) $f(x) = x^2$. Using (3.29),

$$c_k = 2\int_0^1 x^2 \sin(k\pi x)\,dx$$

Tabular integration

$$\begin{array}{c|l}
x^2 & \sin(k\pi x) \\
2x & -\frac{1}{k\pi}\cos(k\pi x) \\
2 & -\left(\frac{1}{k\pi}\right)^2 \sin(k\pi x) \\
0 & \left(\frac{1}{k\pi}\right)^3 \cos(k\pi x)
\end{array}
\qquad
\begin{array}{l}
(+1) \\
(-1) \\
(+1) \\
0
\end{array}$$

$$c_k = \left[\frac{-x^2}{k\pi}\cos(k\pi x) + \frac{2x}{(k\pi)^2}\sin(k\pi x) + \frac{2}{(k\pi)^3}\cos(k\pi x)\right]_{x=0}^{1}$$

$$= \left(\frac{-1}{k\pi}(-1)^k + \frac{2}{(k\pi)^2}\cdot(0) + \frac{2}{(k\pi)^3}(-1)^k\right) - \left(\frac{2}{(k\pi)^3}\right)$$

$$= \frac{1}{k\pi}\left[(-1)^{k+1} + \frac{2}{(k\pi)^3}\left((-1)^k - 1 \right) \right] \longleftarrow$$

$$k = 1, 3, 5, \ldots \Rightarrow c_k = \frac{1}{k\pi}\left(1 - \frac{4}{(k\pi)^3} \right)$$

$$k = 2, 4, 6, \ldots \Rightarrow c_k = \frac{1}{k\pi}$$

(c) $f(x) = x(1-x) = x - x^2$

$$\Rightarrow c_k = 2\int_0^1 (x - x^2)\sin(k\pi x)\,dx$$

$$= 2\int_0^1 x\sin(k\pi x)\,dx - 2\int_0^1 x^2 \sin(k\pi x)\,dx$$

From above,

$$c_k = \frac{2}{k\pi}(-1)^{k+1} - \frac{1}{k\pi}\left[(-1)^{k+1} + \frac{2}{(k\pi)^3}\left((-1)^k - 1 \right) \right]$$

$$= \frac{1}{k\pi}(-1)^{k+1} - \frac{2}{(k\pi)^3}\left((-1)^k - 1 \right) \longleftarrow$$

$$k = 1, 3, 5, \ldots \Rightarrow c_k = \frac{1}{k\pi} + \frac{4}{(k\pi)^3}$$

$$k = 2, 4, 6, \ldots \Rightarrow c_k = \frac{1}{k\pi}$$

EXERCISE 3.4 Find the formal solution of the problem

$$\begin{aligned} u_t &= u_{xx} \quad \text{for} \quad x \in (0,1), \quad t > 0 \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= f(x), \end{aligned}$$

for the initial functions

(a) $f(x) = \sin(14\pi x)$,

(b) $f(x) = x(1-x)$,

(c) $f(x) = \sin^3(\pi x)$.

(a) Trivial, recall from text

same as above

$$\begin{aligned} u_t &= u_{xx} \quad \text{for} \quad x \in (0,1), \quad t > 0, \\ u(0,t) &= u(1,t) = 0, \\ u(x,0) &= f(x). \end{aligned} \tag{3.20}$$

Suppose first that the initial function f can be written as a finite linear combination of the eigenfunctions $\{\sin(k\pi x)\}$. Thus, there exist constants $\{c_k\}_{k=1}^N$ such that

$$f(x) = \sum_{k=1}^{N} c_k \sin(k\pi x). \tag{3.21}$$

Then, by linearity, it follows that the solution of (3.20) is given by

$$u(x,t) = \sum_{k=1}^{N} c_k e^{-(k\pi)^2 t} \sin(k\pi x). \tag{3.22}$$

You can easily check that this is a solution by explicit differentiation.

EXAMPLE 3.1 Let us look at one simple example showing some typical features of a solution of the heat equation. Suppose

$$f(x) = 3\sin(\pi x) + 5\sin(4\pi x);$$

then the solution of (3.20) is given by

$$u(x,t) = 3e^{-\pi^2 t} \sin(\pi x) + 5e^{-16\pi^2 t} \sin(4\pi x).$$

$$\therefore \boxed{u(x,t) = e^{-(14\pi)^2 t} \sin(14\pi x)}.$$

(b) $f(x) = x(1-x)$.

From Exercise 3.1(c), $f(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x)$,

with c_k given in 3.1(c) sol'n $\forall\ k = 1, 2, \ldots$

Then, from text,

$$f(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x). \tag{3.25}$$

By letting N tend to infinity in (3.22), we obtain the corresponding formal solution of the problem (3.20),

$$u(x,t) = \sum_{k=1}^{\infty} c_k e^{-(k\pi)^2 t} \sin(k\pi x). \tag{3.26}$$

$\therefore$ $\underline{u(x,t) = \sum_{k=1}^{\infty} c_k e^{-(k\pi)^2 t} \sin(k\pi x)}$ w/ $\underline{c_k}$

<u>given in sol'n to 3.1(c) above.</u>

(c) $f(x) = \sin^3(\pi x)$

$$c_k = 2\int_0^1 f(x)\sin(k\pi x)\,dx$$

$$= 2\int_0^1 \sin^3(\pi x)\sin(k\pi x)\,dx$$

$$\int \sin^3 ax\,dx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a}$$ ⟵ using this in integration-by-parts

$$c_k = 2\left\{\left[\left(-\frac{3}{4\pi}\cos(\pi x) + \frac{1}{12\pi}\cos(3\pi x)\right)\cancelto{0}{\sin(k\pi x)}\right]_{x=0}^{1}\right.$$

$$-\int_0^1 \left(-\frac{3}{4\pi}\cos(\pi x) + \frac{1}{12\pi}\cos(3\pi x)\right) k\pi\cos(k\pi x)\,dx \Big\}$$

$$= \begin{cases} \frac{3}{8}, & k=1 \\ -\frac{1}{8}, & k=3 \end{cases} \qquad \underline{(\text{used } (3.54))}$$

$$\Rightarrow f(x) = \sin^3(\pi x) = \frac{3}{8}\sin(\pi x) - \frac{1}{8}\sin(3\pi x)$$

$$\therefore \boxed{u(x,t) = \frac{3}{8}e^{-(\pi)^2 t} - \frac{1}{8}e^{-(3\pi)^2 t}\sin(3\pi x)}$$

EXERCISE 3.8 Find a family of particular solutions to the following problem:

$$u_t = u_{xx} - u \quad \text{for} \quad x \in (0,1), \quad t > 0,$$
$$u(0,t) = u(1,t) = 0.$$

Assume $u_k = T_k(t) X_k(x)$, so PDE $\Rightarrow$

$$T_k'(t) X_k(x) = T_k(t)\left[X_k''(x) - X_k(x)\right]$$

$$\Rightarrow \frac{T_k'(t)}{T_k(t)} = \frac{X_k''(x) - X_k(x)}{X_k(x)} = -\lambda_k \in \mathbb{R}$$

$\Rightarrow T_k'(t) + \lambda_k T_k(t) = 0 \longleftarrow$ same as before

and

$$\begin{cases} X_k''(x) + \underline{(\lambda_k - 1)}\, X_k(x) = 0 \\ X_k(0) = X_k(1) = 0 \end{cases}$$

$\longrightarrow$ same form as (3.12) and (3.14) w/ $\lambda_k - 1$

instead of λ_k, so

$$\lambda_k - 1 = (k\pi)^2, \quad k=1,2,\ldots$$
$$\Rightarrow \lambda_k = (k\pi)^2+1, \quad k=1,2,\ldots, \text{ and}$$
$$X_k(x) = \sin(k\pi x)$$

Check! $X_k'' + (\lambda_k - 1)X_k = X_k'' + (k\pi)^2 X_k$
$= -(k\pi)^2 \sin(k\pi x) + (k\pi)^2 \sin(k\pi x)$
$= 0$, so PDE ✓
IC clearly ✓

$$T_k(x) = e^{-\lambda_k t}$$
$$= e^{-[(k\pi)^2+1]t} \quad \leftarrow \text{not same as before b/c } \lambda_k \text{ different.}$$

$$\therefore \boxed{u_k(x,t) = e^{-[(k\pi)^2+1]t} \sin(k\pi x)}$$

EXERCISE 3.12 Find a formal solution of the following problem:

$$\begin{aligned} u_t &= u_{xx} + 2x \quad \text{for} \quad x \in (0,1), \quad t > 0, \\ u(0,t) &= 0, \quad u(1,t) = 0, \\ u(x,0) &= f(x). \end{aligned} \tag{3.69}$$

Here, you may find it helpful to introduce $v(x,t) = u(x,t) + w(x)$ for a suitable w which is only a function x.

Let u be a sol'n to (3.69) and suppose $v(x,t) = u(x,t) + w(x)$.

Then,

$v_t = u_t$, and $v_{xx} = u_{xx} + w_{xx} \Rightarrow u_{xx} = v_{xx} - w_{xx}$

$u_t = u_{xx} + 2x \Rightarrow v_t = v_{xx} - w_{xx} + 2x$

If $-w_{xx} + 2x = 0$, $w(0) = w(1) = 0$, then v solves

$$v_t = v_{xx}, \qquad x \in (0,1), \ t > 0,$$
$$v(0,t) = v(1,t) = 0,$$
$$v(x,0) = f(x) - w(x)$$

Goal: Find such a w!

$w = \frac{1}{3}x(x^2-1)$ is such a w (used direct integration)

Since $v = \sum_{k=1}^{\infty} c_k e^{-(k\pi)^2 t} \sin(k\pi x)$, $c_k = 2\langle f - w, \sin(k\pi x)\rangle$

$$\boxed{u = \sum_{k=1}^{\infty} c_k e^{-(k\pi)^2 t} \sin(k\pi x) - \frac{1}{3}x(x^2-1)}$$