Section 2.4

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2.4 Eigenvalue Problems

In this final section of this chapter we shall study eigenvalue problems associated with the operators L and L_h. The results of this discussion will be used frequently in later chapters.

2.4.1 The Continuous Eigenvalue Problem

A real number[9] λ is said to be an *eigenvalue* associated with the boundary value problem (2.1) if

$$Lu = \lambda u \tag{2.36}$$

for a suitable nonzero[10] function $u \in C_0^2((0,1))$. Here, as above, $Lu = -u''$. The function u is referred to as an *eigenfunction*.

[9] In general, eigenvalues are allowed to be complex. However, due to the symmetry property of L given in Lemma 2.2, all eigenvalues will be real in the present case; cf. Exercise 2.28.

[10] The term "a nonzero function" refers to a function that is not identically equal to zero. Thus it is allowed to vanish at certain points, and even on a subinterval, but not for all $x \in [0, 1]$. Sometimes we also use the term "nontrivial" for such functions.

Suppose $\{\lambda_n\}_{n=1}^{\infty}$ is a seq. of e.vals w/ corresponding e.fcns $\{u_n\}_{n=1}^{\infty}$. If $f \in \text{span}\{u_n\}_{n=1}^{\infty}$, then $\exists$ constants $\{c_n\}_{n=1}^{\infty}$ s.t.

$$f = \sum_{n=1}^{\infty} c_n u_n(x)$$

Assuming we know the constants c_n and that $\lambda_n \neq 0 \ \forall n$, then we claim that

$$u(x) = \sum_{n=1}^{\infty} \left(\frac{c_n}{\lambda_n}\right) u_n$$

satisfies $Lu = f$.

"Pf. of claim:"

$$Lu = L\left(\sum_{n=1}^{\infty}\left(\frac{c_n}{\lambda_n}\right)u_n\right)$$

$$Lv = L\left(\sum_{n=1}^{\infty}\left(\frac{c_n}{\lambda_n}\right)u_n\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{c_n}{\lambda_n}\right)Lu_n \quad \leftarrow \text{needs justification}$$

$$= \sum_{n=1}^{\infty}\left(\frac{c_n}{\lambda_n}\right)(\lambda_n u_n)$$

$$= \sum_{n=1}^{\infty} c_n u_n$$

$$= f \qquad \blacksquare$$

The key is to figure out what the coefficients $\{c_n\}_{n=1}^{\infty}$ are for a given f.

We come back to this in Chp. 3.

$$\langle Lu, u\rangle > 0,$$

for all nonzero functions $u \in C_0^2\big((0,1)\big)$. Suppose now that λ and u solve (2.36). Then, upon multiplying both sides of the equation by u and integrating, we obtain

$$\langle Lu, u\rangle = \lambda\langle u, u\rangle.$$

Since the operator L is positive definite and the eigenfunction u is nonzero, it follows that

$$\lambda > 0. \tag{2.37}$$

Given the sign of the eigenvalue, we proceed by finding explicit formulas for both the eigenvalues as well as the eigenfunctions.

Since we know that the eigenvalues are positive, we can define

$$\beta = \sqrt{\lambda},$$

and study the equation

$$u''(x) + \beta^2 u(x) = 0,$$

which has general solutions of the form

$$\Rightarrow -u'' = \beta^2 u = \lambda u$$

$$\Rightarrow -u'' - \beta^2 u = 0$$

$$\Rightarrow u'' + \beta^2 u = 0$$

Char. poly: $r^2 + \beta^2 = 0 \Rightarrow r = \pm\sqrt{-\beta^2}$
$= \pm i\beta$

$\Rightarrow$ F.S.S. $\{\cos \beta x, \sin \beta x\}$

$\Rightarrow$ gen. sol'n: $u(x) = c_1 \cos \beta x + c_2 \sin \beta x$

BCs: $u(0) = 0 = c_1$

$u(1) = 0 = c_2 \sin \beta$

$\sin \beta = 0$ if β is an integer multiple of π

β also has to be positive b/c $\beta = \sqrt{\lambda}$ for some $\lambda > 0$

$\Rightarrow \beta_k = k\pi$, $k = 1, 2, \ldots$

$\Rightarrow$ e.fns are given by $\sin(k\pi x)$

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EXERCISE 2.25 Consider the eigenvalue problem

$$-u'' = \lambda u, \quad x \in (a, b), \quad u(a) = u(b) = 0,$$

where $a < b$ are given real numbers. Find all eigenvalues and eigenvectors.

If we use $\{\cos \beta x, \sin \beta x\}$ as the F.S.S. and look for evals β that give e.fns $c_1 \cos \beta x + c_2 \sin \beta x$ for some c_1 and c_2, then we end up w/

$$\beta_k = \frac{k\pi}{b-a}, \quad k = 1, 2, \ldots$$

and $u_k(x) = \underbrace{-\tan\left(\frac{k\pi}{b-a}\right)}_{=c_1} \cos\left(\frac{k\pi}{b-a} x\right) + \sin\left(\frac{k\pi}{b-a} x\right)$

We could instead define F.S.S. $\{\cos(\beta(x-a)), \sin(\beta(x-a))\}$

then, following the same steps, we get

$$\beta_k = \frac{k\pi}{b-a}, \quad k=1,2,\ldots$$

$$u_k(x) = \sin\left(\frac{k\pi}{b-a}(x-a)\right)$$

Check: (1) BCs: $u_k(a) = \sin\left(\frac{k\pi}{b-a}\cdot 0\right) = \sin 0 = 0$ ✓

$$u_k(b) = \sin\left(\frac{k\pi}{b-a}(b-a)\right) = \sin k\pi = 0 \checkmark$$

(2) $u_k' = \frac{k\pi}{b-a}\cos\left(\frac{k\pi}{b-a}(x-a)\right)$

$$\Rightarrow u_k'' = -\left(\frac{k\pi}{b-a}\right)^2 \sin\left(\frac{k\pi}{b-a}(x-a)\right)$$

$$= -\beta_k^2 u_k \quad \checkmark$$

EXERCISE 2.28 The purpose of this exercise is to show that all eigennvalues of the problem (2.36) are real. Assume more generally that $Lu = \lambda u$, where

$$u(x) = v(x) + iw(x) \quad \text{and} \quad \lambda = \alpha + i\beta.$$

Here $i = \sqrt{-1}$, $v, w \in C_0^2\big((0,1)\big)$ and $\alpha, \beta \in \mathbb{R}$. In addition u should not be the zero function.

(a) Show that

$$Lv = \alpha v - \beta w \quad \text{and} \quad Lw = \beta v + \alpha w.$$

(b) Use the symmetry of the operator L (see Lemma 2.2) to show that

$$\beta\left(\langle v, v\rangle + \langle w, w\rangle\right) = 0.$$

(c) Explain why $\beta = 0$ and why the real eigenvalue $\lambda = \alpha$ has a real eigenfunction.

eigenfunction.

(a) Assume $Lu = \lambda u$

$\Rightarrow L(v+iw) = (\alpha + i\beta)(v+iw)$

$\Rightarrow Lv + iLw = (\alpha+i\beta)(v+iw)$ (by linearity of L)

$\Rightarrow Lv + iLw = (\alpha v - \beta w) + i(\beta v + \alpha w)$

$\therefore$ by collecting like terms (the real and imag.)

we see that $Lv = \alpha v - \beta w$ and $Lw = \beta v + \alpha w$

as desired. ▨

(b) Lemma 2.2 $\Rightarrow \langle Lv, w\rangle = \langle v, Lw\rangle$

$\Rightarrow \langle \alpha v - \beta w, w\rangle = \langle v, \beta v + \alpha w\rangle$

$\Rightarrow \alpha\langle v,w\rangle - \beta\langle w,w\rangle = \beta\langle v,v\rangle + \alpha\langle v,w\rangle$

cancel

$\Rightarrow -\beta\langle w,w\rangle = \beta\langle v,v\rangle$

$\Rightarrow \beta(\langle v,v\rangle + \langle w,w\rangle) = 0$ ▨

(c) If u is an efn, then u is non-zero at least somewhere in $(0,1)$, so either v or w is at least non-zero somewhere in $(0,1)$.

$\therefore \ \langle v,v\rangle = \int_0^1 v^2\,dx > 0$ or $\langle w,w\rangle = \int_0^1 w^2\,dx > 0$
(and at worst one of these could be zero), so
$\beta(\langle v,v\rangle + \langle w,w\rangle) = 0 \iff \beta = 0.$

If $u = v + iw$ and $\lambda = \alpha \in \mathbb{R}$, then

$$Lu = \lambda u = \alpha u = \alpha v + i\alpha w$$

and

$$Lu = L(v+iw) = Lv + iLw$$

$$\Rightarrow Lv = \alpha v \quad \text{and} \quad Lw = \alpha w$$

Both $v, w \in \mathcal{C}_0^2((0,1))$ by assumption, so
the e.fcns. of α are real.