Chapter 2, Feb. 13

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In §2.3 and §2.4, the material is focused on the properties of differential/difference operators and sol'ns to 2-pt. BVP

$-u'' = f$, $x \in (0,1)$, $u(0) = 0 = u(1)$. ← Eq. (2.1) in text.

First, some notation is introduced,

2.3.1 Difference and Differential Equations

Let us start by recalling our standard two-point boundary value problem. We let L denote the differential operator

$$(Lu)(x) = -u''(x),$$

so $Lu \in \mathcal{C}((0,1))$

and let $f \in C\big((0,1)\big)$. Then, (2.1) can be written in the following form: Find $u \in C_0^2\big((0,1)\big)$ such that

$$(Lu)(x) = f(x) \quad \text{for all } x \in (0,1). \tag{2.26}$$

Recall here that $u \in C_0^2\big((0,1)\big)$ means that we want the solution to be twice continuously differentiable, and to be zero at the boundaries. Thus, we capture the boundary conditions in the definition of the class where we seek solutions.

So, we think of L as mapping $u \in \mathcal{C}_0^2((0,1))$ to the data $f \in \mathcal{C}((0,1))$. By the same reasoning, we can define, formally, the operator L^{-1} that maps data f to a sol'n of (2.1) given by $u \in \mathcal{C}_0^2((0,1))$.

Now, let us introduce a similar formalism for the discrete case. First, we let D_h be a collection of discrete functions defined at the grid points x_j for $j = 0, \ldots, n+1$. Thus, if $v \in D_h$, it means that $v(x_j)$ is defined for all $j = 0, \ldots, n+1$. Sometimes we will write v_j as an abbreviation for $v(x_j)$. This should cause no confusion. Next, we let $D_{h,0}$ be the subset of D_h containing discrete functions that are defined in each grid point, but with the special property that they are zero at the boundary.

Note that a discrete function $y \in D_h$ has $n+2$ degrees of freedom $y_0, y_1, \ldots, y_{n+1}$. This means that we have to specify $n+2$ real numbers in order to define such a function. A discrete function $z \in D_{h,0}$ has only n degrees of freedom $z_1, \ldots, z_n$, since the boundary values are known.

For a function w we define the operator L_h by

$$(L_h w)(x_j) = -\frac{w(x_{j+1}) - 2w(x_j) + w(x_{j-1})}{h^2},$$

So, for $w \in D_h$, $L_h w \in D_h$

So, $L_h : D_{h,0} \to D_{h,0}$ whereas $L : \mathcal{C}_0^2((0,1)) \to \mathcal{C}((0,1))$.

discrete functions. For two continuous functions u and v, we define the inner product of the functions by

$$\langle u, v\rangle = \int_0^1 u(x)v(x)\,dx. \tag{2.28}$$

"continuous" inner product

Similarly, for two discrete functions, i.e. for u and v in D_h, we define the inner product to be

$$\langle u, v\rangle_h = h\left(\frac{u_0v_0 + u_{n+1}v_{n+1}}{2} + \sum_{j=1}^{n} u_jv_j\right), \tag{2.29}$$

The h indicates a "discrete" inner product.

where we have used the shorthand notation v_j for $v(x_j)$. Clearly, (2.29) is an approximation of (2.28). In the language of numerical integration, this is referred to as the *trapezoidal rule*; you will find more about this in Exercise 2.20.

Having established a suitable notation for the continuous and the discrete problem, we are in position to start deriving some properties.

If (2.29) is like an approximation of (2.28), what does this mean? Well, inner products impart a type of geometric structure (and norm, norm-induced metric, metric-induced topology, etc.) on a space, so this suggests that through analysis of the discrete space $D_{h,0}$ we are somehow approximating what the analysis is for the continuous space and vice versa. We see this type of "mirrored" analysis throughout the rest of the chapter where the steps in analyzing properties of L_h and sol'ns to discrete problem that exist in $D_{h,0}$ essentially "mirror" the steps in analyzing L and sol'ns to the continuous problem.

$\boxed{\text{Property 1: Symmetry of Operators}}$ §2.3.2

Lemma 2.2 *The operator L given in (2.26) is symmetric in the sense that*

$$\langle Lu, v\rangle = \langle u, Lv\rangle \quad \text{for all } u, v \in C_0^2\big((0,1)\big).$$

Proof: The property follows from integration by parts. For $u, v \in C_0^2\big((0,1)\big)$ we have

$$\langle Lu, v\rangle = -\int_0^1 u''(x)v(x)\,dx = -u'(x)v(x)|_0^1 + \int_0^1 u'(x)v'(x)\,dx$$ ← integrate by parts once

Since $v(0) = v(1) = 0$, this implies that

$$\langle Lu, v\rangle = \int_0^1 u'(x)v'(x)\,dx. \tag{2.30}$$ ← obtain useful formula

However, by performing one more integration by parts, we obtain as above that

$$\int_0^1 u'(x)v'(x)\,dx = -\int_0^1 u(x)v''(x)\,dx = \langle u, Lv\rangle,$$ ← integrate by parts again and observe symmetry.

which is the desired result. ■

Mirror 1: An analog to integration by parts

Now, by integrating this identity on the unit interval, we get

$$\underbrace{\int_0^1 u'(x)v(x)\,dx}_{\mathrm{I}a} = \underbrace{[uv]_0^1}_{\mathrm{II}a} - \underbrace{\int_0^1 u(x)v'(x)\,dx}_{\mathrm{III}a}.$$

Then we turn our attention to discrete functions and start by deriving a product rule for differences. Let y and z be two members of D_h, i.e. discrete functions, and observe that

$$y_{j+1}z_{j+1} - y_jz_j = \underbrace{(y_{j+1} - y_j)z_j + (z_{j+1} - z_j)y_{j+1}}_{\text{cancel}}.$$

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By summing this identity from $j = 0$ to $j = n$, we get

$$\underbrace{\sum_{j=0}^{n}(y_{j+1} - y_j)z_j}_{\mathrm{I}b} = \underbrace{y_{n+1}z_{n+1} - y_0z_0}_{\mathrm{II}b} - \underbrace{\sum_{j=0}^{n}(z_{j+1} - z_j)y_{j+1}}_{\mathrm{III}b}. \tag{2.31}$$

Notice that Ia and Ib, IIa and IIb, and IIIa and IIIb are mirroring each other.

This identity is referred to as *summation by parts*, and it is exactly the tool we need to prove that L_h is symmetric.

Mirror 2: An analog to Lemma 2.2

Lemma 2.3 *The operator L_h is symmetric in the sense that*

$$\langle L_h u, v\rangle_h = \langle u, L_h v\rangle_h \quad \text{for all } u, v \in D_{h,0}.$$

Proof: Note that $u_0 = v_0 = u_{n+1} = v_{n+1} = 0$, and define also $u_{-1} = v_{-1} = 0$. Then, using summation by parts twice, we get

$$\begin{aligned}
\langle L_h u, v\rangle_h &= -h^{-1}\sum_{j=0}^{n}((u_{j+1}-u_j)-(u_j-u_{j-1}))v_j \\
&= h^{-1}\sum_{j=0}^{n}(u_{j+1}-u_j)(v_{j+1}-v_j) \\
&= -h^{-1}\sum_{j=0}^{n}((v_{j+1}-v_j)-(v_j-v_{j-1}))u_j \\
&= \langle u, L_h v\rangle_h.
\end{aligned}$$

Apply (2.31) to $\sum (u_{j+1}-u_j)v_j$ and $\sum (u_j-u_{j-1})v_j$, i.e. sum by parts once

Analog to (2.30)

sum by parts again and observe symmetry

■

Mirror 3: L and L_h are positive definite

Lemma 2.4 *The operators L and L_h are positive definite in the following sense:*

(i) For any $u \in C_0^2\big((0,1)\big)$ we have

$$\langle Lu, u\rangle \geq 0,$$

with equality only if $u \equiv 0$.

(ii) For any $v \in D_{h,0}$ we have

$$\langle L_h v, v\rangle_h \geq 0,$$

with equality only if $v \equiv 0$.

Proofs of (i) and (ii) are "mirror" images of each other using (2.30) for (i) and the analog to (2.30) to prove (ii).

Mirror 4: Uniqueness of Sol'ns.

2.3.3 *Uniqueness*

We have already seen that the continuous problem (2.26) and the discrete problem (2.27) have unique solutions. This is stated in Theorem 2.1, page 44, and Corollary 2.1, page 55, respectively. In this section, we shall use the results on positive definiteness derived above to give an alternative proof of these facts.

Lemma 2.5 *The solution u of (2.26) and the solution v of (2.27) are unique solutions of the continuous and the discrete problems, respectively.*

Proof: Let $f \in C\big((0,1)\big)$ be given and assume that $u^1, u^2 \in C_0^2\big((0,1)\big)$ are two solutions of (2.26), thus

$$Lu^1 = f \quad \text{and} \quad Lu^2 = f.$$

In order to show that $u^1 \equiv u^2$, we let $e = u^1 - u^2$. Then

$$Le = L(u^1 - u^2) = Lu^1 - Lu^2 = 0.$$

Hence, by multiplying this identity by the error e and integrating over the unit interval, we get

$$\langle Le, e \rangle = 0.$$

By Lemma 2.4 we therefore derive that $e(x) \equiv 0$, and thus $u^1 \equiv u^2$. A similar argument can be given in the discrete case. ■

↳ Try writing it out by just "mirroring" the above steps.

Remaining Mirrors in §2.3.4: This subsection simply mirrors the analysis presented in §2.1.

From §2.1:

Proposition 2.1 *Assume that $f \in C\big((0,1)\big)$ is a nonnegative function. Then the corresponding solution u of (2.1) is also nonnegative.*

Proof: Since $G(x,y) \geq 0$ for all $x, y \in [0,1]$, this follows directly from (2.9). ■

Analog in §2.3.4:

Proposition 2.5 *Assume that $f(x) \geq 0$ for all $x \in [0,1]$, and let $v \in D_{h,0}$ be the solution of (2.27). Then $v(x_j) \geq 0$ for all $j = 1, \dots, n$.*

Proof: Since $G(x,y) \geq 0$ this follows directly from (2.33). ■

From §2.1:

Proposition 2.2 *Assume that $f \in C\big([0,1]\big)$ and let u be the unique solution of (2.1). Then*

$$||u||_\infty \leq (1/8)||f||_\infty.$$

Analog in §2.3.4:

2.2 for the continuous problem.

Proposition 2.6 *The solution $v \in D_{h,0}$ of (2.27) satisfies*

$$||v||_{h,\infty} \leq (1/8)||f||_{h,\infty}.$$

§ 2.3.5 is very important b/c it proves a priori error bounds on $v \in D_{h,0}$ (Thm 2.2) that depends on a constant $C = \|f''\|_\infty / 96$ (that is independent of the mesh and only depends on known data) multiplied by h^2. In other words, Ch^2 bounds the error and we see that $\lim_{h \to 0} Ch^2 = C \lim_{h \to 0} h^2 = 0$.

possible b/c C does not depend on h.

Thus, convergence is clear.