Chapter 3: Energy Exercises

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EXERCISE 3.16 Assume that $u(x,t)$ is a solution of the Neumann problem (3.37). Use energy arguments to show that

$$\int_0^1 u^2(x,t)dx \le \int_0^1 f^2(x)dx, \qquad t \ge 0.$$

For each $t \ge 0$ let

$$E(t) = \int_0^1 u^2(x,t)dx.$$

[4]Maximum principles are another set of properties that can be derived without analytical formulas for the solution. This is studied in Chapter 6.

We now consider how $E(t)$, which is a scalar variable, evolves in time. We consider

$$E'(t) \equiv \frac{d}{dt}\int_0^1 u^2(x,t)dx.$$

For smooth functions u we can interchange the order of differentiation and integration such that for $t > 0$

$$E'(t) = \int_0^1 \frac{\partial}{\partial t} u^2(x,t)dx. \qquad (3.59)$$

In this case we then derive from equations (3.56)–(3.57) and integration by parts that

$$\begin{aligned} E'(t) &= 2\int_0^1 u(x,t)u_t(x,t)dx \\ &= 2\int_0^1 u(x,t)u_{xx}(x,t)dx \\ 0 = &= 2[u(x,t)u_x(x,t)]_0^1 - 2\int_0^1 (u_x(x,t))^2\,dx \\ &= -2\int_0^1 (u_x(x,t))^2\,dx \le 0. \end{aligned}$$

$u_x(0,t) = u_x(1,t) = 0$

Hence, $E(t)$ is a nonincreasing function, i.e.

$$E(t) \le E(0).$$

$$E(0) = \int_0^1 u^2(x,0)\,dx$$

$E(0) = \int_0^1 u^2(x,0)\,dx$

$$= \int_0^1 f^2(x)\,dx$$

$$\therefore \int_0^1 u^2(x,t)\,dx \le \int_0^1 f^2(x)\,dx. \quad \blacksquare$$

EXERCISE 3.17 Let $g = g(u)$ be a function u such that $ug(u) \le 0$ for all u. Use energy arguments to show that any solution of the (possibly nonlinear) problem

$$\begin{aligned} u_t &= u_{xx} + g(u) \quad \text{for} \quad x \in (0,1), \quad t > 0, \\ u(0,t) &= u(1,t) = 0, \\ u(x,0) &= f(x). \end{aligned}$$

satisfies the estimate

$$\int_0^1 u^2(x,t)dx \le \int_0^1 f^2(x)dx, \qquad t \ge 0.$$

For each $t \ge 0$, let

$$E(t) = \int_0^1 u^2(x,t)\,dx$$

As before,

$$E'(t) = 2\int_0^1 u(x,t)\,u_t(x,t)\,dx$$

But, now, $u_t = u_{xx} + g(u)$, so

$$E'(t) = 2\int_0^1 u\,(u_{xx} + g(u))\,dx$$

$$= \underbrace{2\int_0^1 u\,u_{xx}\,dx}_{\substack{\le 0 \text{ for} \\ \text{same reasons}}} + \underbrace{2\int_0^1 u\,g(u)\,dx}_{\substack{\le 0 \text{ b/c} \\ ug(u) \le 0}}$$

as in §3.7 $\forall u$.

$\therefore \; E'(t) \le 0$, so $E(t) \le E(0)$

$$\Rightarrow \int_0^1 u^2(x,t)\,dx \le \int_0^1 f^2(x)\,dx. \quad \blacksquare$$