Exercise 2.29

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EXERCISE 2.29 In this problem we shall derive some properties for finite Fourier series. Such series occur frequently for example in signal processing. Consider finite Fourier series of the form

$$g(x) = \sum_{k=1}^{n} c_k \sin(k\pi x),$$

where $c_1, c_2, \dots, c_n$ are real coefficients. Furthermore, let, as usual, x_j denote the grid points $x_j = j/(n+1)$ for $j = 1, \dots, n$.

(a) Let $z_1, z_2, \dots, z_n$ be arbitrary real numbers. Show that the interpolation conditions

$$g(x_j) = z_j \quad for \quad j = 1, \dots, n$$

are satisfied if and only if

$$c_k = 2h \sum_{j=1}^{n} z_j \sin(k\pi x_j) \quad \text{for} \quad k = 1, \dots, n.$$

$$g(x_j) = z_j = \sum_{k=1}^{n} c_k \sin(k\pi x_j)$$

$$= 2h \sum_{k=1}^{n} \left(\sum_{l=1}^{n} z_l \sin(k\pi x_l) \right) \sin(k\pi x_j)$$

$$= 2h \sum_{k=1}^{n} \left[\left(z_j \sin^2(k\pi x_j) \right) + \underbrace{\sum_{l \neq j} z_l \sin(k\pi x_l) \sin(k\pi x_j)}_{=0} \right]$$

Later, see below

$$\boxed{\text{Claim: } \sum_{k=1}^{n} \sin^2(k\pi x_j) = \frac{1}{2h}}$$

$n = 1 \Rightarrow \sin^2\left(\frac{\pi}{2}\right) = 1$

below

$n=1 \Rightarrow \sin^2\left(\frac{\pi}{2}\right)=1$

$n=2 \Rightarrow x_1=\frac{1}{3},\ x_2=\frac{2}{3},\ h=\frac{1}{3} \qquad \frac{1}{2h}=\frac{3}{2}$ ✓

just checking

ignore this

$\sin^2\left(\frac{\pi}{3}\right)+\sin^2\left(\frac{2\pi}{3}\right)=\frac{3}{4}+\frac{3}{4}=\frac{6}{4}=\frac{3}{2}$ ✓

$\sin^2\left(\frac{2\pi}{3}\right)+\sin^2\left(\frac{4\pi}{3}\right)=\frac{3}{4}+\frac{3}{4}=\frac{3}{2}$ ✓

$\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right)+\sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{4\pi}{3}\right)=0$ ✓

$$\underbrace{\sin(x)=\frac{e^{ix}-e^{-ix}}{2i}}_{\text{key}} \Rightarrow \sum_{k=1}^{n}\sin^2(k\pi x_j)$$

$$=\sum_{k=1}^{n}\left(\frac{e^{ik\pi x_j}-e^{-ik\pi x_j}}{2i}\right)^2$$

$$=-\frac{1}{4}\sum_{k=1}^{n} e^{2ik\pi x_j}-2e^0+e^{-2ik\pi x_j}$$

$$=\frac{n}{2}-\frac{1}{4}\sum_{k=1}^{n} e^{2ik\pi x_j}+e^{-2ik\pi x_j}$$

$$S_n=\sum_{k=1}^{n} r^k \Rightarrow rS_n=\sum_{k=1}^{n} r^{k+1}$$

$$\Rightarrow (1-r)S_n=r-r^{n+1}$$

$$\Rightarrow S_n=\frac{r-r^{n+1}}{1-r}$$

Aside: Partial sums of geom. series.

$$\sum_{k=1}^{n}\left(e^{2i\pi x_j}\right)^k=\frac{e^{i2\pi x_j}-\overbrace{e^{i2\pi(n+1)x_j}}^{1}}{1-e^{i2\pi x_j}}=-1$$

$$\sum_{k=1}^{n}\left(e^{-2i\pi x_j}\right)^k=\frac{e^{-i2\pi x_j}-\overbrace{e^{-i2\pi(n+1)x_j}}^{1}}{1-e^{-i2\pi x_j}}=-1$$

$$\Rightarrow -\frac{1}{4}\sum_{k=1}^{n} e^{2ik\pi x_j} + e^{-2ik\pi x_j} = \frac{1}{2}$$

$$\Rightarrow \sum_{k=1}^{n} \sin^2(k\pi x_j) = \frac{n}{2} + \frac{1}{2} = \frac{n+1}{2} = \frac{1}{2h}$$

This proves the claim.
Still need to show the other terms sum to zero.

Observe that

$$\sum_{k=1}^{n} \sin(k\pi x_\ell)\sin(k\pi x_j) = \sum_{k=1}^{n}\left(\frac{e^{ik\pi x_\ell} - e^{-ik\pi x_\ell}}{2i}\right)\left(\frac{e^{ik\pi x_j} - e^{-ik\pi x_j}}{2i}\right)$$

$$= -\frac{1}{4}\sum_{k=1}^{n}\Big[e^{ik\pi(x_\ell+x_j)} - e^{ik\pi(x_\ell-x_j)} - e^{-ik\pi(x_\ell-x_j)} + e^{-ik\pi(x_\ell+x_j)}\Big]$$

$\ell+j$ even $\Rightarrow \sum_{k=1}^{n}$ (any of terms) $= 0$ for same reason as above.

$$\ell+j \text{ odd} \Rightarrow e^{ik\pi(n+1)(x_\ell+x_j)} = e^{ik\pi(n+1)(x_\ell-x_j)} = e^{-ik\pi(n+1)(x_\ell-x_j)} = e^{-ik\pi(n+1)(x_\ell+x_j)} = -1$$

hmmm....

Then, if $\ell+j$ odd,

$$\sum_{k=1}^{n} e^{ik\pi(x_\ell+x_j)} + e^{-ik\pi(x_\ell+x_j)}$$

$$= \frac{e^{i\pi(x_\ell+x_j)}+1}{1-e^{i\pi(x_\ell+x_j)}} + \underbrace{\frac{e^{-i\pi(x_\ell+x_j)}+1}{1-e^{-i\pi(x_\ell+x_j)}}}_{\text{multiply by } e^{i\pi(x_\ell+x_j)}}$$

$$\underbrace{\qquad\qquad}_{\text{multiply by } \frac{e^{i\pi(x_\ell+x_j)}}{e^{i\pi(x_\ell+x_j)}}}$$

$$= \frac{e^{i\pi(x_\ell+x_j)}+1}{1-e^{i\pi(x_\ell+x_j)}} + \frac{1+e^{i\pi(x_\ell+x_j)}}{\underbrace{e^{i\pi(x_\ell+x_j)}-1}_{=\text{ negative of this}}}$$

$$= 0$$

Similarly, $\displaystyle\sum_{k=1}^{n} e^{ik\pi(x_\ell-x_j)} + e^{-ik\pi(x_\ell-x_j)} = 0$ when $\ell+j$ odd

Thus, finally, $g(x_j) = z_j$.

Now, we observe that

$$g(x_1) = z_1$$
$$g(x_2) = z_2$$
$$\vdots$$
$$g(x_n) = z_n$$

can be rewritten in matrix-vector form

$$T\underset{\sim}{c} = \underset{\sim}{z}$$

where $T = [t_{jk}] \in \mathbb{R}^{n\times n}$, $t_{j,k} = \sin(k\pi x_j)$,

$\underset{\sim}{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$ and $\underset{\sim}{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{R}^n$.

$$\underset{\sim}{c} = \begin{bmatrix} \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n \quad \text{and} \quad \underset{\sim}{z} = \begin{bmatrix} z_2 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{R}^n .$$

Since we show above that $\forall \underset{\sim}{z} \in \mathbb{R}^n \ \exists \underset{\sim}{c} \in \mathbb{R}^n$

s.t. $T\underset{\sim}{c} = \underset{\sim}{z}$, this implies T is nonsingular.

Also, $t_{j,k} = \sin(k\pi x_j) = \sin\left(k\pi\left(\frac{j}{n+1}\right)\right)$
$$= \sin\left(j\pi\left(\frac{k}{n+1}\right)\right)$$
$$= t_{k,j}$$

Thus, $T = T^T \Rightarrow T$ is symmetric.

All of the above proves (a) and (b).

$T\underset{\sim}{c} = \underset{\sim}{z}$, and, from (a), we see that

$\underset{\sim}{c} = 2hT\underset{\sim}{z}$, so $T(2hT\underset{\sim}{z}) = \underset{\sim}{z}$

$$\Rightarrow 2hT^2\underset{\sim}{z} = \underset{\sim}{z}$$

This is true $\forall \underset{\sim}{z} \in \mathbb{R}^n$, so $2hT^2 = I$
$$\Rightarrow T^2 = \frac{1}{2h} I .$$

This proves (c).