# Hypothetical Analytic Filter for the tracking with bearing and range mixture measurements

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**Abstract:** This paper introduces a hypothetical analytic filter, which is used for the nonlinear tracking problems with noisy bearing and range mixture measurements. The main idea of this filter is to divide the tracking process into two parts, track the target with the bearing information first, and then use the range information to make the tracking process more accurate. This separation is complimentary for that the bearing-correction could overcome the bimodal problem of range-only estimation and the range-only tracking can make the performance more accurate. Compared with other nonlinear filters, it converges very quickly and works well in some scenarios. The lower computation demand and its robust convergence are two main advantages of this filter.

**Key Words:** Analytic filter, Noise before nonlinearity, Target tracking, Bearing and range mixture tracking

## 1 Introduction

In real life, sensors with noisy bearing and range information (like Radar and GPS) are widely used in nonlinear tracking problems, where efficient and robust filters can be applied to estimate the position of the target. The applications of these filters have a significant effect in different cases. Autonomous underwater vehicle (AUV) can be tracked with some filters and these results can be used to compensate the control[1]. In robotics applications, simultaneous localization and navigation need the tracking algorithms to eliminate the effect of noise. The aim of the bearing and range mixture tracking problem is to estimate the position of the target, given noisy bearing and range measurements. The main difficulties of this tracking problem are dealing with the nonlinearity from the measurements and making full use of the bearing and range information[2].

Previous research on bearing and range mixture tracking problems can be separated into two main categories: 'moment matching' class and the 'density computation' class.

'Moment matching' filters attempt to match the first and second moments of the conditional densities of the state. The typical filters in this class: Extended Kalman Filter [3], Pseudo-measurement Filter [4], Unscented Kalman Filter [5] and converted measurement Kalman filter (CMKF) [6]. Although these filters are computationally efficient. The main disadvantage of 'moment matching' filter is that they are not very robust. Sometimes these filters could fail to converge and have worse performance when faced with some challenging tracking situations.

The 'density computation' filter exploits on-line Monte-Carlo methods to construct an empirical distribution to approximate the conditional density of the state. The most important density computation filter is Particle filters [7], which is accurate and practical. But the main disadvantages of particle filters are their heavy computation cost and their sensitivity to the initial parameters.

To overcome the above disadvantages and improve the performance in some scenarios, we propose the hypothetical analytic filter, where we add the 'augmented vector' and apply new noise model. We calculate the expectation and variance of the 'augmented vector' to approximate the conditional density of the state, which makes the results transparent and accurate. This filter is a special 'moment matching filter' in some sense, but we make some changes to improve its performance in some scenarios.

We divide the estimation process into three parts: prediction part, bearing-correction and range-correction parts. By separating the bearing-correction and range-correction, we avoid the co-effect between the nonlinearity of bearing and range measurements. This separation is complimentary for that the bearing-correction could overcome the bimodal problem of range-only estimation and the range-only tracking can make the performance more accurate. Two important themes of this filter are 'nonlinearity before noise' model and its approximation of the conditional densities. Compared with other algorithms used in the tracking problems with the given bearing and range measurements, the 'Hypothetical analytic filter' is more accurate and robust in many practical scenarios.

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In this paper, we firstly list the mathematical model of this tracking problem and then explain why this noise model is suitable under some assumptions. Then we explain the algorithm to estimate the position of the target and prove it. In the last, we run some simulations to compare the performance of this filter with others.

## 2 Problem formulation

The basic part of the standard tracking problem is the calculation of the discrete sequences $x_t$ at the time $t$, given the initial value $x_0$ and measurements value $\gamma_t$, where $t \in (1, 2, .., t-1)$. The k-vector $\gamma_t$ is the measurement information at time $t$. The n-vector $x_t$ is the state of the target in the coordinates at time $t$. ($x_t$ typically includes the position and velocity information of the target in Cartesian coordinates).

Solution is to get the estimation function in the form: $\hat{x}_t = f(x_0, \gamma_1, \gamma_2 ... \gamma_t)$, where $\hat{x}_t$ is the estimation of the $x_t$ at time $t$. This estimation is based on the information of the noisy measurements $\gamma_{1...t}$ and the initial state (sometimes the initial state also needs to be approximated). We wish the observer error (the distance between the estimation position and real position) could be very small and could converge to 0 when enough information is available.

In general situation, we can model the tracking problem as below:

$$\begin{cases} x_t = Fx_{t-1} + u^s_{t-1} + v_{t-1} & (v_{t-1} \sim N(0, Q^s)) \\ d_t = Hx_t + u^m_t & \\ \gamma_{\theta t} = \angle(d_t) + w_1 & (w_1 \sim N(0, \sigma^2_\theta)) \\ \gamma_{rt} = |d_t| + w_2 & (w_2 \sim N(0, \sigma^2_r)) \end{cases} \tag{1}$$

| | |
|---|---|
| $x_t(n \times 1)$ | the state of target at the time $t$ |
| $F(n \times n)$ | the matrix to describe the motion of the state from the time $t-1$ to the time $t$ |
| $u^s_t(n \times 1)$ $u^m_t(r \times 1)$ | additional deterministic sequences at the time $t$, which can be used to model some complex motion. |
| $v_t(n \times 1)$ | the system noise process at time $t$, with covariances $Q^s(n \times n)$, |
| $d_t(r \times 1)$ | the relative displacement vector between the target and the observer platform |
| $H(r \times n)$ | the output matrix used to produce the relative displacement vector $d_t$. |
| $r$ | the dimension of space (r=2 or 3). |
| $\gamma_{\theta t}, \gamma_{rt}$ | the bearing and the range measurements of the target (we also write it as $\theta$ and $r_t$ for simplification). |
| $w_1, w_2$ | the measurement noise process at the time $t$. They all are sequences of independent Gaussian random variables with covariances $\sigma^2_\theta$ and $\sigma^2_r$ respectively. |

Table 1: Parameters in the equation (1)

To make the efficient calculation of the conditional density of $x_t$ in this algorithm, we introduce a new noise model in the form of equation below. Compared with the general model, we add the augmented vector $z_t$ and use this vector to produce the measurements:

$$\begin{cases} x_t = Fx_{t-1} + u^s + v_{t-1} & (v_{t-1} \sim N(0, Q^s)) \\ d_t = Hx_t + u^m & \\ z_t = d_t + w_t & (w_t \sim N(0, Q^m)) \\ b_t = \prod(z_t) & \\ r_t = |z_t| & \end{cases} \tag{2}$$

| | |
|---|---|
| $z_t(r \times 1)$ | the augmented vector, which may not exist in the real world. And $z_t$ is the sum of the relative displacement vector $d_t$ and the hypothetical noise $w_t$. |
| $w_t(r \times 1)$ | the hypothetical noise on the $d_t$, and it is a Gaussian process with covariance $Q^m(r \times r)$ |
| $b_t(r \times 1)$ | the bearing vector, and the $\prod$ notes the projection of the r-vectors of the relative displacement onto the unit circle (r=2) or the unit sphere (r=3). When $r = 2$, $b_t = (\cos\theta, \sin\theta)^T$. When $r = 3$, $b_t = (\cos\theta\cos\psi, \sin\theta\cos\psi, \sin\psi)^T$, where $\theta$ is the azimuth and $\psi$ is the elevation of the target position. |
| $r_t$ | the range measurement which is calculated from the augmented vector. |

Table 2: Parameters in the equation (2)

## 3 Noise Model

In the derivation of equation (2), the bearing vector $b_t$ and the range measurement $r_t$ are both modeled as the nonlinear functions of the 'augmented vector' $z_t$. It has been proven that this 'noise before nonlinearity' model (1) is indistinguishable with the standard noise model (2) in the 'density' sense under some assumption in the paper[8][9].

For the noisy bearing model: If standard covariance of measurements ($\sigma_\theta$) is less than the radians value, $\sigma_\theta < \theta$, the density of 'noise before nonlinearity' bearing model ($b_t = \prod(z_t)$) is indistinguishable with the density of standard noisy bearing model ($\gamma_{\theta t}$).

For the noisy range model: If the value of range is relatively great ($|d_t|/\sigma_r > 10$), the density of 'noise before nonlinearity' range model ($r_t = |z_t|$) is indistinguishable with the density of standard noisy bearing model ($\gamma_{rt}$).

### 3.1 Noise model for the bearing measurements

The standard model for the bearing measurement:

$$\theta = \angle d + n = \arctan(d_y/d_x) + w_1 \tag{3}$$

where $w_1$ is the measurement noise and $w_1 \sim N(0, \sigma^2_\theta)$, $w_1$ is independent with $x_t$ and $d_t$.
The 'noise before nonlinearity' model:

$$\begin{cases} z_t = d_t + w_t \\ b_t = \Pi z_t \end{cases} \tag{4}$$

where $w_t$ is the independent hypothetical noise with covariance $Q^m_t = \sigma^2_\theta E[|d_t|^2 | b_{1:t-1}] I_{k\times k} + Q^{tr}$. Here the $Q^{tr}$ is the covariance of the 'translational' noise and is used to take account of the noisy perturbation. In this paper, we set $r = 2$, $Q^{tr} = 0_{2\times 2}$ for the fair comparison.

From the paper [8], it has been proven that if standard covariance of measurements $\sigma_\theta$ is relatively small (at least less than the radians value), the 'noise before nonlinearity' bearing model (4) is indistinguishable with the standard noisy bearing model (3). In the practice, the $\sigma_\theta$ is around $0.01 - 0.5$, which is obviously satisfy this constraint. We can use this noise formation to model the bearing part in our mixture bearing and range tracking.

### 3.2 Noise model for the range measurements

The standard noise model of the range measurement:

$$\begin{cases} d_t = Hx_t + u^m \\ \gamma_{rt} = |d_t| + w_2 \end{cases} \tag{5}$$

where $w_2 \sim N(0, \sigma_r^2)$
The new noise model on the range measurement:

$$\begin{cases} d_t = Hx_t + u^m \\ z_t = d_t + w_t \\ r_t = |z_t| \end{cases} \tag{6}$$

where $w_t$ is the hypothetical noise, $w_t \sim N(0, \sigma_r^2 I_{2\times 2})$.

From the paper [9], we can know that if the value of range is great, the two noise models are indistinguishable. Actually, when $|d_t|/\sigma_r > 10$, the density of two noise model are very close. And when the $|d_t|/\sigma_r$ is bigger, the noise approximation is better. In the practice, the $|d_t|$ is usually around 1000 and $\sigma_r$ is around $0.01 - 1$, which is obviously satisfy this constraint $|d_t|/\sigma_r > 10$. So we can use this noise formation to model the range part in our mixture bearing and range tracking.

We can conclude that the 'noise before nonlinearity' model is very suitable for the mixture range and bearing tracking for that these assumptions are all easily satisfied. The method to construct the augmented vector and to approximate the noise is a good choice. And it provides the possibility to do the accurate computation of the conditional density with the 'Hypothetical Analytic Filter'.

## 4 General Algorithm

In this part, we propose a general class of the 'Hypothetical Analytic Filter' to approximate the first moment and the second moment of the $p(x_t|\gamma_{1:t})$ recursively.

The algorithm uses the bearing information to correct the estimation firstly, and then uses the range information to correct the estimation. In each correction step, we use the conditional mean of the relative displacement to replace the nonlinear part.

Firstly, we used the model introduced before:

$$\begin{cases} x_t = Fx_{t-1} + u^s + v_t & (v_t \sim N(0, Q^s)) \\ d_t = Hx_t + u^m & \\ z_t = d_t + w_t & (w_t \sim N(0, Q^m)) \\ b_t = \angle(z_t) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} & \\ r_t = |z_t| & \end{cases} \tag{7}$$

This algorithm uses the Kalman Filter construction and makes a little change and extension. We use the two dimension situation as the example and divide the algorithm into three steps: prediction step, bearing-correction step and range-correction step.

Notation: we write the $(x_{t|t}, P_{t|t})$ in the form of $(x_t, P_t)$ for simplification. The $\hat{x}$ is the estimation of the state at the time $t$. The $\hat{x}_{t|t-1}$ is the estimation of the state $x_t$ at time $t$ given the measurements $\gamma_{1:t-1}$ from the time 1 to $t-1$. The matrix $P_{t|t-1}$ means the covariance of the estimation of $x_t$. The $S_{bt}$, $K_{bt}$ also mean that the $S$ and $K$ matrix at time $t$ in the bearing-correction step. The $S_{rt}$, $K_{rt}$ also mean that the $S$ and $K$ matrix at time $t$ in the range-correction step.

Step 1. Prediction step:

$$\begin{cases} x_{t|t-1} = F\hat{x}_{t-1} + u^s \\ d_{t|t-1} = Hx_{t|t-1} + u^m \\ z_t = d_{t|t-1} + w_t \\ P_{t|t-1} = FP_{t-1}F^T + Q^s \end{cases} \tag{8}$$

Step 2. Bearing-correction step:

$$\begin{cases} S_{bt} = HP_{t|t-1}H^T + Q_{bt}^m \\ K_{bt} = P_{t|t-1}H^T S_{bt}^{-1} \\ \zeta_{bt} = (b_t^T S_{bt}^{-1} b_t)^{-1/2} \rho(z_t) b_t \\ \Gamma_{bt} = \delta_t(z_t) b_t b_t^T \\ \hat{x}_{bt} = x_{t|t-1} + K_{bt}(\zeta_{bt} - d_{t|t-1}) \\ P_{bt} = (I - K_{bt}H)P_{t|t-1} + K_{bt}\Gamma_{bt}K_{bt}^T \end{cases} \tag{9}$$

Here we introduce the function $z_t$, $\rho(z_t)$, $\delta_t(z_t)$ and $Q_{bt}^m$ to compute $\zeta_t$ and $\Gamma_t$. $F_{normal}$ is the cumulative distribution function for the random variable $N(0,1)$. $\zeta_t = E[z_t|b_t]$, $\Gamma_t = Cov[z_t|b_t]$, $z_t \sim N(d_{t|t-1}, S_{bt})$. We introduce these parameters to estimate the conditional density $p(x_t|b_{1:t})$ and compute the first and second moment of the $p(x_t|b_{1:t})$. This calculation method can also be seen in paper [8].

$$\begin{cases} Q_{bt}^m = \sigma_\theta^2 E[|d_t|^2|b_{1:t-1}] I_{k\times k} + Q^{tr} \\ z_t = (b_t^T S_{bt}^{-1} b_t)^{-1/2} b_t^T S_{bt}^{-1} (H\hat{x}_{t|t-1} + u_t^m) \\ \rho(z_t) = \frac{z_t e^{-z_t^2/2} + \sqrt{2\pi}(z_t^2+1)F_{normal}(z_t)}{e^{-z_t^2/2} + \sqrt{2\pi}(z_t)F_{normal}(z_t)} \\ \delta_t(z_t) = (b_t^T S_t^{-1} b_t)^{-1}(2 + z_t\rho(z_t) - \rho^2(z_t)) \end{cases} \tag{10}$$

In this algorithm, $Q_{bt}^m$ is the covariance of hypothetical noise $w_t$. $K_{bt}$ is 'Kalman parameter' used to correct the results in bearing part. The $\zeta_{bt}$ is the conditional mean of the $d_t$ given the bearing information, and $\Gamma_{bt}$ is the covariance of the $d_t$ given the bearing information. In some sense, we change the measurements from the $\theta_t$ to the augmented displacement $z_t$. In this process, we do some approximation of the range information indirectly. So this step contains 'more' information than reality. So we use the equation: $K_{bt}\Gamma_{bt}K_{bt}^T$ to compensate for the uncertainty of the algorithm.

We separate this correction step out of the range-correction step to avoid the co-effect between the nonlinearity of bearing and range measurements. We assume that there is small time period $\Delta t$ and in this period the state is almost unchanged. Then we use the output of the bearing-correction step (the estimation of state $\hat{x}_{bt}$ and its uncertainty $P_{bt}$) as the initial value to further compute the range-correction step. This separation is complementary for that the bearing-correction could overcome the bimodal problem of range-only estimation in matrix $P_{bt}$ and the range-only tracking can make the performance more accurate.

Step 3. Range-correction step:

$$\begin{cases} S_{rt} = HP_{bt}H^T + Q_{rt}^m \\ K_{rt} = P_{bt}H^T S_{rt}^{-1} \\ \zeta_{rt} = r_t g_t \\ \Gamma_{rt} = r_t^2 G_t \\ \hat{x}_t = \hat{x}_{bt} + K_{rt}(\zeta_{rt} - d_{t|t-1}) \\ P_t = (I - K_{rt}H)P_{bt} + K_{rt}\Gamma_{rt}K_{rt}^T \end{cases} \tag{11}$$

To complete the computation, we need to introduce the $Q_{rt}^m$ and parameters $V_t$, $g_t$, $G_t$ to compute $\zeta_{rt}$ and $\Gamma_{rt}$. $\zeta_{rt} = E[z_t|r_t]$, $\Gamma_{rt} = Cov[z_t|r_t]$, $z_t \sim N(d_{t|t-1}, S_{rt})$. We introduce these parameters to estimate the conditional density $p(x_t|r_{1:t})$ and compute the first and second moment of the $p(x_t|r_{1:t})$. This calculation method can also be seen in paper [9].

$$\begin{cases} Q_{rt}^m = \sigma_r^2 I_{2\times 2} \\ g_t = \int_0^{2\pi} \begin{bmatrix} cos(\theta_t) \\ sin(\theta_t) \end{bmatrix} p(\theta_t|r_{1:t})d\theta_t \\ G_t = \int_0^{2\pi} (\begin{bmatrix} cos(\theta_t) \\ sin(\theta_t) \end{bmatrix} - g_t)(\begin{bmatrix} cos(\theta_t) \\ sin(\theta_t) \end{bmatrix} \\ \quad -g_t)^T \times p(\theta_t|r_{1:t})d\theta_t \end{cases} \tag{12}$$

We define $V_t$, $M_t$ to compute the $p(\theta|r_{1:t})$:

$$\begin{array}{ll} p(\theta_t|r_{1:t}) = c^{-1}e^{(h_t(\theta_t))} & \\ V_t = S_t^{-1}\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} & M_t = \begin{bmatrix} \frac{(v_{11}-v_{22})}{2} \\ v_{12} \end{bmatrix} \\ a_1 = |V_t d_{t|t-1}| & a_2 = |M_t| \\ b_1 = \angle(V_t d_{t|t-1}) & b_2 = \angle(M_t) \end{array} \tag{13}$$

where $c$ is the constant to make the $\int_{-\infty}^{\infty} p(\theta_t|r_{1:t}) = 1$.

$$h_t(\theta_t) = a_1 r_t cos(\theta_t - b_1) - \frac{1}{2}a_2 r_t^2 cos(2\theta_t - b_2) \tag{14}$$

In this algorithm, $Q_{rt}^m$ is the covariance of hypothetical noise $w_t$. $K_{rt}$ is 'Kalman parameter' used to correct the results. The $\zeta_{rt}$ is the conditional mean of the $d_t$ given the range measurements $r_t$, and $\Gamma_{rt}$ is the covariance of the $d_t$ given the range measurements $r_t$. In some sense, we change the measurements from the $r_t$ to the augmented displacement $z_t$. In this process, we do some approximation of the bearing information based on the bearing-correction step indirectly. So this step contains 'more' information than reality. So we use the equation: $K_{rt}\Gamma_{rt}K_{rt}^T$ to compensate for the uncertainty of the algorithm.

We separate this correction step out of the bearing-correction step to avoid the co-effect between the nonlinearity of bearing and range measurements. We assume that there is small time period $\Delta t$ and in this period the state is almost unchanged. Then we use the output of the bearing-correction step (the estimation of state $\hat{x}_{bt}$ and its uncertainty $P_{bt}$) as the initial value to further compute the range-correction step. In this process, we use the range-correction step to make the performance better and make the $P_t$ matrix smaller.

## 5 Analysis and Justification

We can see that both bearing-correction and range-correction share the construction below:

$$\begin{cases} x_{t|t-1} = F\hat{x}_{t-1} + u^s \\ d_{t|t-1} = Hx_{t|t-1} + u^m \\ z_t = d_{t|t-1} + w_t \\ P_{t|t-1} = FP_{t-1}F^T + Q^s \end{cases} \tag{15}$$

$$\begin{cases} S_t = HP_tH^T + Q_t^m \\ K_t = P_tH^T S_t^{-1} \\ \hat{x}_t = \hat{x}_{t|t-1} + K_t(\zeta_t - d_{t|t-1}) \\ P_t = (I - K_tH)P_t + K_t\Gamma_tK_t^T \end{cases} \tag{16}$$

where $\zeta_t = E[z_t|\gamma_t]$, $\Gamma_t = Cov[z_t|\gamma_t]$, $z_t \sim N(d_{t|t-1}, S_t)$. $\gamma_t$ is the bearing or range measurement. If we can prove this construction is accurate, then this algorithm can be proven. For this purpose, we introduce the Lemma 1.

**Proposition 1** If the initial conditional density in recursive equation (15-16) satisfies: $p(x_{t-1}|\gamma_{1:t-1}) = N(\hat{x}_{t-1}, P_{t-1})$. Then we can get the results:

$$E[x_t|\gamma_{1:t}] = \hat{x}_t \quad Cov(x_t|\gamma_{1:t}) = P_t \tag{17}$$

**Proof:** As we know the enhanced vector:

$$z_t = d_t + w_t, \quad (w_t \sim N(0, Q_t^m)).$$

We can decompose the equation of $x_t$ into:

$$x_t = e_t + (I - K_tH)\hat{x}_{t|t-1} - K_tu^m + K_tz_t \tag{18}$$

Here, we introduce independent variable $e_t \sim N(0, (I - K_tH)P_{t|t-1})$, which can be seen as estimation error. The equation shows that $x_t$ can be expressed as the sum of $\hat{x}_{t|t-1}$, 'enhanced vector' $z_t$ and 'estimation error' $e_t$.

To compute $E[x_t|\gamma_{1:t}]$. We use the information that $e_t$ is zero mean independent process, $E[e_t|\gamma_{1:t}] = 0$:

$$\begin{aligned} &E[x_t|\gamma_{1:t}] \\ &= E[(e_t + (I - K_tH)\hat{x}_{t|t-1} - K_tu^m + K_tz_t)|\gamma_{1:t}] \\ &= E[e_t] + E[(I - K_tH)\hat{x}_{t|t-1}] - E[K_tu^m] + E[K_tz_t|\gamma_{1:t}] \\ &= (I - K_tH)\hat{x}_{t|t-1} - K_tu^m + K_tE[z_t|\gamma_{1:t}] \end{aligned} \tag{19}$$

From definition:
$E[z_t|\gamma_{1:t}] = E[z_{t_{z_t \sim N(d_{t|t-1}, S_t)}}|\gamma_t] = \zeta_t$
So we have proved that:
$E[x_t|\gamma_{1:t}] = x_{t|t-1} + K_t(\zeta_t - d_{t|t-1}) = \hat{x}_t$

To compute $Cov(x_t|\gamma_{1:t})$. As we know that $e_t$ is zero mean independent process, $Cov[e_t|\gamma_{1:t}] = (I - K_tH)P_{t|t-1}$.

$$\begin{aligned} &Cov[x_t|\gamma_{1:t}] = Cov[\zeta_t] + K_tCov[z_t|\gamma_{1:t}]K_t^T \\ &= (I - K_tH)P_{t|t-1} + K_tCov[z_t|\gamma_{1:t}]K_t^T \end{aligned} \tag{20}$$

From the definition:
$Cov[z_t|\gamma_{1:t}] = Cov[z_{t_{z_t \sim N(d_{t|t-1}, S_t)}}|\gamma_t] = \Gamma_t$
So we have proved that
$Cov(x_t|\gamma_{1:t}) = (I - K_tH)P_{t|t-1} + K_t\Gamma_tK_t^T = P_t$

In conclusion, we have proved this proposition:
$E[\hat{x}_t|\gamma_{1:t}] = \hat{x}_t, Cov(x_t|\gamma_{1:t}) = P_t$
So the unified construction generates the accurate conditional density in moment matching sense for the bearing-correction step and range-correction step.

The whole algorithm can be proved in this way:
Firstly, for the initial input at each recursive step $p(x_{t-1}|\gamma_{1:t-1}) = N(\hat{x}_{t-1}, P_{t-1})$. From the proposition 1, we can get the results from the bearing-step: $E[x_{bt}|b_{1:t}] = \hat{x}_{bt}, Cov(x_{bt}|b_{1:t}) = P_{bt}$.
Secondly, choose $p(x_{bt}|b_{1:t}) = N(\hat{x}_{bt}, P_{bt})$ as input . From proposition 1, we can compute the results from the range-step as $E[x_t|r_{1:t}] = \hat{x}_t, Cov(x_{rt}|r_{1:t}) = P_t$

So we have proved that if $p(x_{t-1}|\gamma_{1:t-1}) = N(\hat{x}_{t-1}, P_{t-1})$. Then we can get the results from the whole algorithm: $E[x_t|\gamma_{1:t}] = \hat{x}_t, Cov(x_t|\gamma_{1:t}) = P_t$

## 6 Simulation

As we introduced in the problem formulation part:

$$x_t = Fx_{t-1} + u^s + v_{t-1} \quad (v_{t-1} \sim N(0, Q^s)) \tag{21}$$

We set $x_t$ with the position information and velocity information in two-dimension:

$$x_t = \begin{bmatrix} x \\ V_x \\ y \\ V_y \end{bmatrix} \tag{22}$$

We choose the scenario where the target moves in one direction with constant velocity and observer platform (ship) moves in another direction with constant velocity. We use this filter to estimate the position of the target, and use the observer error (the distance between the estimation position and the real position) to evaluate the performance of the filter.

So the $u^s = 0$ and $F$ is in the form below:

$$x_t = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix} x_{t-1} + \begin{bmatrix} T^2/2 \\ T \\ T^2/2 \\ T \end{bmatrix} v_{t-1} \tag{23}$$

We can set different initial values of the $V_x$, $V_y$ and $x_0$, $y_0$ to run the simulations. And $v_t$ is a Gaussian noise $N(0, 0.01)$, the sample time of the discrete-time system is $T = 1s$.

For the fair comparison, we add the noise after nonlinearity. We write the vector of the target and observer platform as $(x_{target}, y_{target})$, $(x_{observer}, y_{observer})$.
So we make the measurements in the form of:

$$\begin{aligned} &\theta_t = \arctan(\tfrac{y_{target} - y_{observer}}{x_{target} - x_{observer}}) + w_1 \\ &w_1 \sim N(0, \sigma^2), \sigma = 0.05 \end{aligned} \tag{24}$$

$$\begin{aligned} &\gamma_t = \sqrt{(y_{tar} - y_{ob})^2 + (x_{tar} - x_{ob})^2} + w_2 \\ &w_2 \sim N(0, \sigma^2), \sigma = 0.1 \end{aligned} \tag{25}$$

Approximate the initial value of the state and its covariance to run the algorithm (we set different initial value, and for different situation we calculate the error of the estimation position and the true position $e_t = \sqrt{x_t - \hat{x}_t)^2 + (y_t - \hat{y}_t)^2}$ at time $t$):

1.Good initial value:

$$x_{0|0} = \begin{bmatrix} 120 \\ 10 \\ -20 \\ 10 \end{bmatrix}, P_{0|0} = 10 \begin{bmatrix} 20 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{26}$$

2.Bad initial value:

$$x_{0|0} = \begin{bmatrix} 120 \\ 0 \\ -20 \\ 10 \end{bmatrix}, P_{0|0} = 10 \begin{bmatrix} 20 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{27}$$

All the preparations are completed for the filter:

$$u_t^s = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, u_t^m = \begin{bmatrix} -2t \\ -3t \end{bmatrix} \tag{28}$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, F = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

$$Q^s = 0.01 \begin{bmatrix} 0.25 & 0.5 & 0 & 0 \\ 0.5 & 0.1 & 0 & 0 \\ 0 & 0 & 0.25 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix} \tag{30}$$

$$Q_t^{m_\theta} = Q^{tr} + \sigma_\theta^2(|H\hat{x}_{t|t-1} + u_t^m|^2 + tr(HP_{t|t-1}H^T))I_{2\times 2} \quad (31)$$

$$Q^{m_r} = \sigma_r^2 I_{2\times 2} \quad (32)$$

To avoid the random factor, we run the algorithm 100 times. We compare the tracking performance and the observer error with different initial values. Then we plot the Figure 3 and 4. The 'True' in the figure means true position of the target, and the 'ship' means the position of the observer platform. The 'Est' means the estimation from the filter. The observer error means the distance between the estimation and true position.

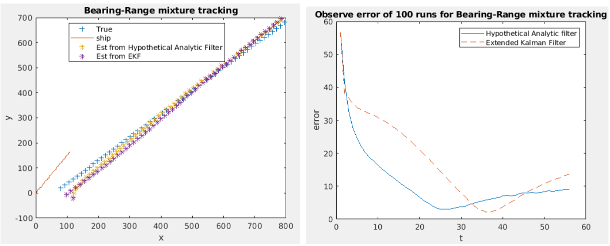


Figure 1: The tracking performance and observer error of 100 runs in the good initial value

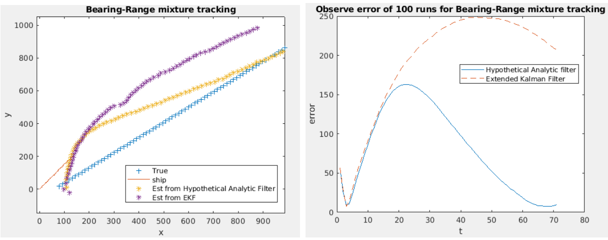


Figure 2: The tracking performance and observer error of 100 runs in the bad initial value

In this section, we compare the performance of hypothetical analytic filter with Extended Kalman Filter (EKF) for different initial value. We don't choose Shifted Rayleigh Filter(SRF) or Gaussian mixture filter(GMF) for comparison as hypothetical analytic filter has more information than bearing-only or range-only and EKF is a representing nonlinear filter in 'moment matching' class. We plot the tracking figure and 'observe error' to demonstrate the performance of filters.

From Figure 1, both filters can converge with good initial values. It is because of its full information of the target (bearing and range) and convergence of the algorithm. We can see that hypothetical analytic filter works better than EKF filter with faster and more accurate performance in this scenario, where it can converge in 20 steps. Figure 2 shows that when the initial value is not very good, the analytic filter can converge but the EKF might lose the target. It means that the analytic filter has fewer requirements with the initial value than the EKF. Also, 'Hypothetical analytic filter' can save lots of computation cost for that we need to do the partial differential computation in EKF, but the analytic filter can do the faster matrix computation (all calculations are trigonometric and deterministic matrices).

## 7 Conclusion

In this paper, we have introduced the 'Hypothetical analytic filter'. Compared with common filter (EKF), it can converge quickly and save lots of computation cost with less requirements of initial values.

For the future work: As 'Hypothetical filter' is inspired from two filters the Shifted Rayleigh Filter [8] and Gaussian mixture filter[9], we can make full use of the benefits of these two filter to design the scenario. And we assume that this is a very small period $\Delta t$ after SRF filter process and then we use this time to do the 'range filter process' to make the estimation more accurate. This algorithm is designed to improve the double-peaks phenomena of the conditional density from the range-only tracking problems. And we use this way to improve the tracking performance, and wish it could make full use of the benefits from two filters. The scenarios we designed are all very straightforward. And these two filters can also do good performances in some complex scenario [8]. We could find some special or more complex scenarios where the analytic filter can also work well.

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