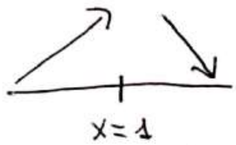
23 Calcula la imagen de las funciones $F, G: \mathbb{R}^+ \to \mathbb{R}$ definidas por:

a) $F(x) = \int_1^x \frac{1-t}{t(t+1)(t^2+1)}\, dt$

Defino $f(x) = \frac{1-x}{x(x+1)(x^2+1)}$. Como f es continua e integrable en $\mathbb{R}^+$, por el Teorema Fundamental del Cálculo, $F(x)$ será derivable en $\mathbb{R}^+$, con derivada $F'(x) = f(x) = \frac{1-x}{x(x+1)(x^2+1)}$

Ahora que sabemos que F es derivable, podemos estudiar su monotonía para calcular su imagen.

$$F'(x) = 0 \iff \frac{1-x}{x(x+1)(x^2+1)} = 0 \iff x = 1$$



$$\left.\begin{array}{l} f'(\tfrac{1}{2}) > 0 \\ f'(2) < 0 \end{array}\right\} \Rightarrow x = 1 \text{ es máximo}$$

Estudio el límite en 0:

- $\lim_{x\to 0} F(x) = \lim_{x\to 0} \int_1^x \frac{1-t}{t(t+1)(t^2+1)}\, dt$

Calculo $\int_1^x \frac{1-t}{t(t+1)(t^2+1)}\, dt = \int_1^x \frac{A}{t} + \frac{B}{t+1} + \frac{C}{t^2+1}\, dt =$

$$= \int_1^x \frac{A(t^2+1)(t+1) + Bt(t^2+1) + Ct(t+1)}{t(t+1)(t^2+1)}\, dt$$

$$1-t = A(t^3+t^2+t+1) + B(t^3+t) + C(t^2+t)$$
$$1-t = (A+B)t^3 + (A+C)t^2 + (A+B+C)t + A$$

Escaneado con CamScanner

Resuelvo

$$\left.\begin{array}{l} A = 1 \\ A+B+C = 1 \\ A + C = 0 \\ A+B = 0 \end{array}\right\} \Rightarrow \begin{array}{l} A = 1 \\ B = -1 \\ C = -1 \end{array}$$

Luego $\displaystyle\int_1^x \frac{1-t}{t(t+1)(t^2+1)}dt = \int_1^x \left(\frac{1}{t} - \frac{1}{t+1} - \frac{1}{t^2+1}\right) dt =$

$$= \Big[\ln(t) - \ln(t+1) - \operatorname{arctg}(t) \Big]_1^x =$$

$$= \ln(x) - \ln(x+1) - \operatorname{arctg}(x) - \cancel{\ln(1)} + \ln(2) + \operatorname{arctg}(1) =$$

$$= \ln(x) - \ln(x+1) - \operatorname{arctg}(x) + \ln(2) + \frac{\pi}{4}$$

$$\lim_{x\to 0} \int_1^x \frac{1-t}{t(t+1)(t^2+1)} dt = \lim_{x\to 0} \ln(x) - \ln(x+1) - \operatorname{arctg}(x) + \ln(2) + \frac{\pi}{4} =$$

$$= -\infty$$

Concluimos que $\boxed{F(\mathbb{R}^+) =]-\infty, 0]}$.

Escaneado con CamScanner

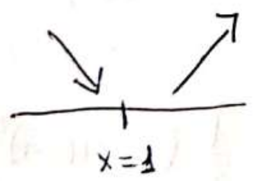
b) $G(x) = \int_1^{1+(x-1)^2} \frac{\operatorname{arctg}(t)}{t^2}\,dt$

Al igual que en el caso anterior, por el TFC, sabemos que $G(x)$ es derivable. Para calcular su derivada aplicamos el corolario del TFC.

Defino $f(t) = \frac{\operatorname{arctg}(t)}{t^2}$ y $\begin{cases} h(x) = 1+(x-1)^2 \\ g(x) = 1 \end{cases}$

$$G'(x) = f(h(x))\,h'(x) - \cancel{f(g(x))\,g'(x)} =$$
$$= f\left(1+(x-1)^2\right) 2(x-1) = \frac{\operatorname{arctg}\left(1+(x-1)^2\right)}{\left(1+(x-1)^2\right)^2}\,2(x-1)$$

$$G'(x) = 0 \iff 2(x-1)\operatorname{arctg}\left(1+(x-1)^2\right) = 0 \iff$$
$$\iff \begin{cases} 2(x-1) = 0 \iff x = 1 \\ \operatorname{arctg}(1+(x-1)^2) = 0 \iff 1+(x-1)^2 = 0 \end{cases} \rightarrow \text{No tiene sol. en } \mathbb{R}$$



$\left.\begin{array}{l} f'(2) > 0 \\ f'(-1) < 0 \end{array}\right\} \Rightarrow x=1$ es mínimo.

Para calcular la imagen de G necesito evaluar en $x=1$ y calcular los límites en 0 y $+\infty$. Para ello resuelvo la integral.

Escaneado con CamScanner

$$\int \frac{\operatorname{arctg}(t)}{t^2}\,dt = \begin{bmatrix} u = \operatorname{arctg}(t), & du = \frac{1}{t^2+1}\,dt \\ dv = \frac{1}{t^2}\,dt, & v = -\frac{1}{t} \end{bmatrix} =$$

$$= -\frac{\operatorname{arctg}(t)}{t} - \int -\frac{1}{t(t^2+1)}\,dt$$

$$\cancel{\int \frac{1}{t(t^2+1)}\,dt = \int \frac{A}{t} + \frac{B}{t^2+1}\,dt = \int \frac{A(t^2+1)+Bt}{t(t^2+1)}}\,dt$$

$$\int \frac{1}{t(t^2+1)}\,dt = \int \frac{1+t^2-t^2}{t(t^2+1)}\,dt = \int \frac{\cancel{1+t^2}}{t\cancel{(t^2+1)}} - \frac{t^{\cancel{2}}}{\cancel{t}(t^2+1)}\,dt =$$

$$= \int \frac{1}{t} - \frac{t}{t^2+1}\,dt = \ln|t| - \frac{1}{2}\ln|t^2+1|$$

Luego $\displaystyle\int \frac{\operatorname{arctg}(t)}{t^2}\,dt = -\frac{\operatorname{arctg}(t)}{t} + \ln|t| - \frac{1}{2}\ln|t^2+1|$

Entonces $\displaystyle G(x) = \int_1^{1+(x-1)^2} \frac{\operatorname{arctg}(t)}{t^2}\,dt =$

$$= \left[-\frac{\operatorname{arctg}(t)}{t} + \ln|t| - \frac{1}{2}\ln|t^2+1|\right]_1^{1+(x-1)^2} =$$

$$= -\frac{\operatorname{arctg}(1+(x-1)^2)}{1+(x-1)^2} + \ln|1+(x-1)^2| - \frac{1}{2}\ln|(1+(x-1)^2)^2+1| -$$

$$-\left[-\frac{\pi}{4} + \cancel{\ln|1|} - \frac{1}{2}\ln|2|\right]$$

- $G(1) = -\operatorname{arctg}(1) + \cancel{\ln|1|} - \frac{1}{2}\ln|2| - \left[-\frac{\pi}{4} - \frac{1}{2}\ln|2|\right] =$

$$= \cancel{-\frac{\pi}{4}} \cancel{-\frac{1}{2}\ln|2|} \cancel{+\frac{\pi}{4}} \cancel{+\frac{1}{2}\ln|2|} = \boxed{0}$$

Escaneado con CamScanner

- $\lim\limits_{x\to 0^+} G(x) = -\dfrac{\operatorname{arctg}(2)}{2} + \ln|2| - \dfrac{1}{2}\ln|5| + \dfrac{\pi}{4} + \dfrac{1}{2}\ln|2| =$

$$= \boxed{0.467}$$

- $\lim\limits_{x\to +\infty} G(x) = \lim\limits_{x\to+\infty}\left(\dfrac{-\operatorname{arctg}(1+(x-1)^2)}{1+(x-1)^2}\right) + \lim\limits_{x\to+\infty} \ln\left|\dfrac{1+(x-1)^2}{\sqrt{(1+(x-1)^2)^2+1}}\right| +$

$$+\frac{\pi}{4} + \frac{1}{2}\ln|2|$$

$\rightarrow \lim\limits_{x\to+\infty} \dfrac{-\operatorname{arctg}(1+(x-1)^2)}{1+(x-1)^2} = \dfrac{-\operatorname{arctg}(+\infty)}{+\infty} = \dfrac{-\frac{\pi}{2}}{+\infty} = 0$

$\rightarrow \lim\limits_{x\to+\infty} \ln\left|\dfrac{1+(x-1)^2}{\sqrt{(1+(x-1)^2)^2+1}}\right| = \ln|1| = 0$

Entonces $\lim\limits_{x\to+\infty} G(x) = \dfrac{\pi}{4} + \dfrac{1}{2}\ln|2| = \boxed{1,13197}$

Por tanto, $\boxed{G(\mathbb{R}^+) = [0, 1.13197[}$

Escaneado con CamScanner