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Synchronizing Traffic Signals for Maximal Bandwidth

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SYNCHRONIZING TRAFFIC SIGNALS FOR MAXIMAL BANDWIDTH

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Traffic signals can be synchronized so that a car, starting at one end of a street and traveling at preassigned speeds, can go to the other end without stopping for a red light. The portion of a signal cycle for which this is possible is called the bandwidth for that direction. Ordinarily the bandwidth in each direction is single, i.e., is not split into two or more intervals within a cycle. We solve two problems for this case: (1) Given an arbitrary number of signals along a street, a common signal period, the green and red times for each signal, and specified vehicle speeds in each direction between adjacent signals, synchronize the signals to produce bandwidths that are equal in each direction and as large as possible. (2) Adjust the synchronization to increase one bandwidth to some specified, feasible value and maintain the other as large as is then possible. The method of calculation has been programmed for a digital computer and results have been used to synchronize signals on a street in Cleveland.

TRAFFIC signals prevent chaos at busy intersections, but nobody likes the frequent stops that often occur on driving down a street with many signals. The number of stops can be held down by proper synchronization of the signals.

Consider a street with a sequence of signals all of which have the same period (cycle length). The *bandwidth* along the street will be defined as that portion of a cycle during which a car could start at one end of the street and, by traveling at preassigned speeds, go to the other end without stopping for a red light. Each direction has its own bandwidth. For example, it is an easy matter to synchronize the signals so that, for one direction, a car that passes the first signal just as it turns green passes all others in the same way. We shall call this a *complete one-way synchronization*. The bandwidth for that direction is as large as possible and equals the shortest of the green times of the signals on the street. Bandwidth in the other direction, however, is likely to be small or zero, unless the distances between signals are particularly fortuitous. Signals synchronized to create a substantial bandwidth are called progression systems.

It is possible to construct examples where the bandwidth in a single cycle in a single direction is split up into two or more intervals separated by very short reds. Since it seems rather unlikely that split bandwidths would often occur in practice, and since the extension of results to cover these cases appears to be rather cumbersome, *we restrict ourselves unless otherwise stated to problems for which the maximal bandwidths are unsplit.*

We shall give procedures for solving two problems: *Problem 1*: Given a common signal period, the green and red times for each signal, and specified travel times in each direction between adjacent signals, synchronize the signals to produce bandwidths that are equal in each direction and as large as possible. *Problem 2:* Resynchronize to favor one direction with a larger bandwidth, if feasible, and give the other direction the largest bandwidth that is then possible.

The objective of maximizing bandwidth has an intuitive appeal and is widely used. A more obvious criterion might be trip delay, but almost any kind of synchronization that treats the street as a whole leads to the concept of a planned speed. Once specified, the planned speed tends to determine trip delay,* unless input flow exceeds street capacity, in which case delay is determined mostly by the amount and duration of the overload. Changes in synchronization tend to produce changes in trip delay that are percentagewise small. The stops themselves may be more irritating than the delay. However, the driver's trade-off between stops and delay does not seem to have been much investigated.

In any case, increases in bandwidth usually tend to decrease both stops and delay. For example, in VON STEIN's[2,3] approach to traffic control, drivers are encouraged by various signalling devices to form compact platoons that travel nonstop through the system at a preset speed. Insofar as this is successful, trip delay is fixed by the speed. The bandwidth determines the maximal platoon size for which stops can be avoided. A stop forces a driver back into the following platoon with a delay of some fraction of a period. Therefore, the objective that will be studied here is that of maximizing main street bandwidths subject to the constraints imposed by service for the cross streets, pedestrian crossings, etc. For further discussion of signal synchronization and for other approaches to the problem, see NEWELL[12, 13] and GRACE AND POTTS.[14]

The literature on bandwidth contains a number of methods, mostly graphical, for solving special cases of Problem 1. MATSON, SMITH, AND HURD[4] consider primarily signals with constant spacing. BRUENING[5] and PETTERMAN[6] approach the problem by trial and error. RAUS[7] treats a limited class of problems algebraically (but his approach will occasionally fail to give the maximal bandwidth—*see* "Examples" section). BOWERS[8] gives a graphical method for maximizing bandwidth when

* For an example, *see* the data in Reference 1.

the green times are all the same and speed is a constant. His standard procedure involves solving the problem for a range of (speed) $\times$ (period) and identifying those values that yield the largest bandwidth as a percentage of period. The synchronization that Bowers' method produces does not necessarily give the maximal bandwidth when green times differ from signal to signal (*see* "Examples").

EVANS[9] presents Bowers' method (but omits Bowers' discussion of how to treat certain special cases). DAVIDSON[10] presents Bowers' method (with the same omission) but redefines the problem slightly. He takes the bandwidth for the main street as given and seeks to maximize the smallest percentage of green assigned to any cross street. This criterion determines green splits for a few critical signals and then the rest are given the largest cross street green, consistent with the specified main street bandwidth. The synchronization that results is the same as Bowers' method would give. Therefore, as with Bowers, if green times are prespecified and unequal, the bandwidth is not necessarily maximal.

Our method solves the above cases and handles two generalizations that have not, to our knowledge, been handled previously in any formal way: (1) arbitrary planned speeds are permitted in either direction between any two adjacent signals. (2) A device is given for apportioning bandwidth between directions on the basis of platoon size. In addition, the method is designed for machine computation and has been programmed for an IBM 1620.

DEFINITIONS AND NOTATION

CONSIDER A two-way street having n traffic signals. Directions on the street will be identified as *outbound* and *inbound*. The signals will be denoted $S_1, S_2, \cdots, S_n$ with the subscript increasing in the outbound direction. Let

C = period, or cycle length, of the signals (seconds).*

r_i = red time of S_i on the street under study (cycles).

$b(\bar{b})$ = outbound (inbound) bandwidth (cycles).

$t_{ij}(\bar{t}_{ij})$ = travel time from S_i to S_j in the outbound (inbound) direction (cycles).

θ_{ij} = relative phase, or offset, of S_i and S_j, measured as the time from the center of a red of S_i to the next center of red of S_j (cycles). By convention $0 \leqq \theta_{ij} < 1$. *See* Fig. 1.

Any time quantity can be expressed in cycles by dividing by C. 'Red time' is used as shorthand for 'unusable time.' A set of $\theta_{ij}, j=1, \cdots, n$ for any i will be called a 'synchronization,' or sometimes a 'phasing' of the signals.

* The units usually used in this paper are indicated in parentheses after the definition.

Travel times between adjacent signals are presumed known and fixed. Then all t_{ij} may be calculated from

$$t_{ij}=\begin{cases}\sum_{k=i}^{j-1} t_{k,k+1}, & (j>i)\\ 0, & (j=i)\\ -\sum_{k=j}^{i-1} t_{k,k+1}, & (j<i)\end{cases}$$

and all $\bar{t}_{ij}$ from corresponding expressions with each t replaced by $\bar{t}$. Although t_{ij} and $\bar{t}_{ij}$ are the basic inputs to the calculation, it is frequently

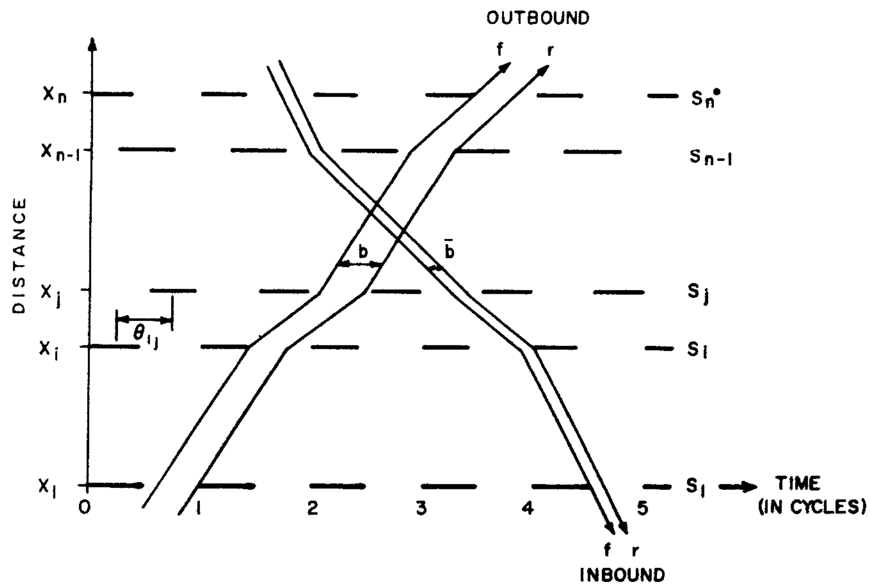

Fig. 1. Space-time diagram showing outbound and inbound green bands. Signals S_1 and S_i are critical signals.

more convenient to think in terms of speeds and distances. Let

x_i = position of S_i on the street (feet),

$v_i(\bar{v}_i)$ = outbound (inbound) speed between S_i and S_{i+1} (feet/sec.).

Then

$$\begin{aligned} t_{i,i+1}&=(x_{i+1}-x_i)/v_iC,\\ \bar{t}_{i,i+1}&=(x_i-x_{i+1})/\bar{v}_iC. \end{aligned} \tag{1}$$

Most previous work has assumed $v_i=\bar{v}_i=v$, in which case $t_{ij}=-\bar{t}_{ij}=(x_j-x_i)/vC$, but our work is not restricted in this way.

Figure 1 shows a space-time diagram for travel on the street. Heavy horizontal lines indicate when the signals are red. The zig-zag lines represent *trajectories* of cars passing unimpeded along the street in the directions indicated. Changes in slope correspond to changes in speed. The set of possible unimpeded trajectories in a given direction forms a *green band* whose horizontal width is the bandwidth for that direction. The trajectory that forms the front edge (earlier in time) of a band and the one that forms the rear edge (later in time) have been marked f and r respectively. Although drawn but once, it is clear that the green bands appear once per cycle in parallel bands across the diagram.

BASIS FOR THE METHOD

THE BASIS for the method will be developed in a sequence of lemmas and theorems. Before starting, let us examine our objectives more carefully. We want to maximize bandwidths but there are bandwidths in each direction, b and $\bar{b}$. We could maximize $b+\bar{b}$, but possibly this will produce an undesirable division of the total between b and $\bar{b}$; for example, one of them might be zero. To unravel the situation, consider the following three problems:

(1) $\max(b+\bar{b})$.

(2) $\max(b+\bar{b})$ subject to $b=\bar{b}$.

(3) $\max(b+\bar{b})$ subject to $b>0$ and $\bar{b}>0$.

What our work will show is that there is usually a whole class of synchronizations that solve (3) and, of these, at least one solves (2). Moreover, the $\max(b+\bar{b})$ found in (2) and (3) is a constant that can, within certain limits, be divided arbitrarily between b and $\bar{b}$. However, the constant will, in some cases, be less than the $(b+\bar{b})$ found in (1). The reason is fairly simple. Under sufficiently awkward red times and signal spacings the $\max(b+\bar{b})$ of (2) and (3) can become quite small, even zero. On the other hand, no matter how awkward the spacing, we can always set up a complete one-way synchronization and obtain a $(b+\bar{b})$ at least as large as the smallest green time.

In any case, the work below solves all three problems. The central problem is (2), which will be called the problem of finding maximal equal bandwidths and is solved by Theorem 3. Theorem 4 expresses the solution of all three problems in what seems to be an operationally useful way.

DEFINITION. *A signal S_j is said to be a critical signal if one side of S_j's red touches the green band in one direction and the other side touches the green band in the other direction.* Thus, in Fig. 1, signals S_1 and S_i are critical, but no others are.

LEMMA 1. *If a synchronization maximizes $(b+\bar{b})$ subject to $b>0$ and $\bar{b}>0$, then:*

(a) There exists at least one critical signal.

(b) The red time of any critical signal will touch the front edge of one green band and the rear edge of the other.

(*c*) *All critical signals can be divided into two groups: Group 1 consists of signals whose reds touch the front of outbound and the rear of inbound and Group 2 of signals whose reds touch the front of inbound and the rear of outbound.*

Proof. Consider the set of signals whose reds touch a given side of the green band in one direction. Part (*a*) must be true or else all these signals could be shifted to increase bandwidth in the one direction without reducing it in the other. Part (*b*) is a consequence of the definition of critical signal: Since the right side of red can only touch front edges and the left side only rear edges, a critical signal must touch (at least) one of each.

Part (*c*) follows immediately from (*b*) since there are only two choices.

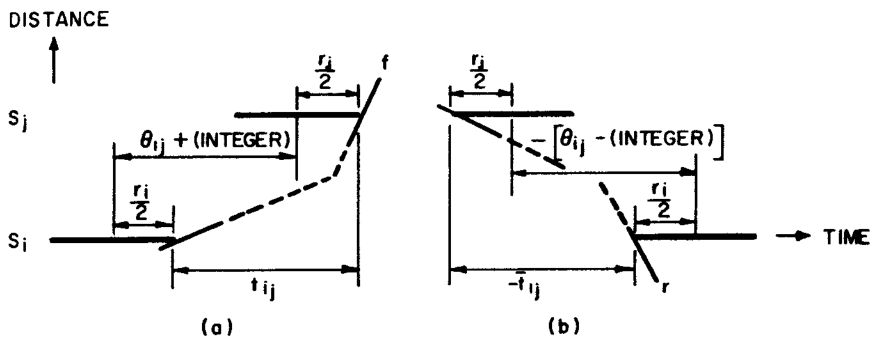

Fig. 2. Geometry when two group 1 signals touch the green bands.

Possibly a signal fits into both groups, in which case it will be considered to be in both. Possibly there is only one critical signal, but then it fits into both groups. This completes the proof.

Suppose two signals, S_i and S_j, are in the same group, say 1. For each signal, the right-hand side of red touches the front of the outbound band and the left-hand side touches the rear of the inbound band. *See* Fig. 2 for the geometry. The quantities are there presented in such a way that, if $j > i$, all the lengths shown are positive. The notation 'integer' is used to indicate that some integer is to be added to an expression to make it valid.

From Fig. 2a:

$$(\tfrac{1}{2})r_i+t_{ij}=(\tfrac{1}{2})r_j+\theta_{ij}+(\text{integer}).$$

From Fig. 2b:

$$(\tfrac{1}{2})r_i-\bar{t}_{ij}=(\tfrac{1}{2})r_j-\theta_{ij}+(\text{integer}).$$

Consequently:

$$\theta_{ij}=(\tfrac{1}{2})(t_{ij}+\bar{t}_{ij})+(\tfrac{1}{2})(\text{integer}). \tag{2}$$

Corresponding arguments lead to the same equation for group 2. By convention, $0 \leqq \theta_{ij} < 1$. Therefore, it may be seen that (2) has two solutions for θ_{ij}, to be found by adding whatever half integers will bring $(\frac{1}{2})(t_{ij}+\bar{t}_{ij})$ into the required range.

A more explicit expression for the two possible values of θ_{ij} can be developed. Let

$$\delta_{ij}=0 \quad \text{or} \quad \tfrac{1}{2},$$

man z = mantissa of z, as obtained by removing the integral part of z and, if the result is negative, adding unity.

Thus, man(5.2) = 0.2, man(−0.2) = 0.8, and in general $0 \leqq \text{man } z < 1$. Now (2) becomes

$$\theta_{ij}=\text{man}[(\tfrac{1}{2})(t_{ij}+\bar{t}_{ij})+\delta_{ij}]. \tag{3}$$

The phasing represented by (3) will be called *half-integer synchronization*. The term can consistently be applied to a collection of signals. In other words, given a set $\delta_{i1}, \delta_{i2}, \cdots, \delta_{im}$, the resulting $\{\theta_{ij}\}$ have the property that $\theta_{ik}=\text{man}(\theta_{ij}+\theta_{jk})$. Furthermore, the same θ_{jk} is obtained by setting $\delta_{ik}=\text{man}(\delta_{ij}+\delta_{jk})$ in (3). The above summarizes into:

LEMMA 2. *Under the conditions of lemma 1, each group of signals has half-integer synchronization.*

The operational meaning of half-integer synchronization is easiest understood in the special case $t_{ij}=-\bar{t}_{ij}$, which occurs, for example, when speeds are the same in each direction. Then (3) gives $\theta_{ij}=0$ or $\frac{1}{2}$ so that any two signals in the same group have the centers of their reds exactly in phase or exactly out of phase.

THEOREM 1. *There is a half-integer synchronization that gives maximal equal bandwidths.*

Proof (by construction). Suppose we have a set of phases such that $(b+\bar{b})$ is maximal subject to $b>0$ and $\bar{b}>0$. (If none exists, the theorem is trivially true.) Divide the critical signals into groups 1 and 2. Extend the reds of all other signals until they are critical, too, but not so far as to reduce bandwidth. The old reds lie wholly within the new. Move the center of the old red to the center of the new—this cannot extend the new red or change bandwidth. Classify the new critical signals into groups 1 and 2. Change the phases of all group 1 signals by an equal amount in the direction that will decrease the larger of $\bar{b}$ and b. The loss to the larger is just equaled by a gain to the smaller so that $(b+\bar{b})$ stays constant. Choose the amount of change so that $b=\bar{b}$.

Within each group there is half-integer synchronization. It remains to show that there is now half-integer synchronization between signals from different groups. Let S_i be from group 1 and S_j from group 2. Figure 3 shows that:

$$(\tfrac{1}{2})r_i+b+t_{ij}+(\tfrac{1}{2})r_j=\theta_{ij}+(\text{integer}), \tag{4}$$

$$(\tfrac{1}{2})r_i+\bar{b}+\bar{t}_{ij}+(\tfrac{1}{2})r_j=-[\theta_{ij}-(\text{integer})]. \tag{5}$$

But $b=\bar{b}$, whence

$$\theta_{ij}=(\tfrac{1}{2})(t_{ij}+\bar{t}_{ij})+(\tfrac{1}{2})(\text{integer}),$$

which is (2) again and so implies that S_i and S_j have half-integer synchronization.

In the above arguments we have also proved:

COROLLARY 1. *If the maximal equal bandwidths are greater than zero,* $\max(b+\bar{b})$ *subject to* $b>0$ *and* $\bar{b}>0$ *equals* $\max(b+\bar{b})$ *subject to* $b=\bar{b}$.

THEOREM 2. *Under any half-integer synchronization,* $b=\bar{b}$.

Proof. It suffices to consider critical signals. Let S_i be from group 1 and S_j in group 2. Figure 3 applies as do (4) and (5). Substract (5) from (4) and substitute (2). It will be seen that $b=\bar{b}$.

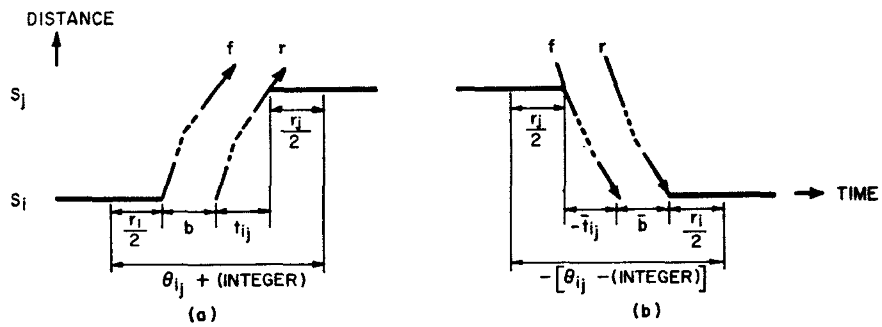

Fig. 3. Geometry when a group 1 and a group 2 signal touch the green bands.

Two special cases of Theorem 1 deserve separate mention. From (1) comes:

COROLLARY 2. *If speeds are the same in each direction at each point of the street, maximal equal bandwidths are achieved by a synchronization in which each* θ_{ij} *is either* 0 *or* $\tfrac{1}{2}$.

Explicit results are possible in the two signal case:

COROLLARY 3. *If there are only two signals,* S_i *and* S_j, *and speeds are the same in each direction, maximal equal bandwidths are achieved by*

$$\theta_{ij}=\begin{cases}0 & 0\leqq \operatorname{man}[(x_j-x_i)/vC]<\tfrac{1}{4},\\ \tfrac{1}{2} & \tfrac{1}{4}\leqq \operatorname{man}[(x_j-x_i)/vC]<\tfrac{3}{4},\\ 0 & \tfrac{3}{4}\leqq \operatorname{man}[(x_j-x_i)/vC]<1.\end{cases}$$

Notice that θ_{ij} does not depend on the green splits of the two signals. This is not always true with more than two sigals. Corollary 3 may be proved by constructing a space-time diagram with S_i at $x=0$ and the start of S_i's green at $t=0$. S_j's position may be varied along an x-axis where $x=(x_j-x_i)/vC$. At each x there are, by Corollary 2, only two possibilities for placing the center of S_j's red and the best one is fairly obvious.

SYNCHRONIZATION FOR MAXIMAL EQUAL BANDWIDTHS

THE SIGNIFICANCE of the results of the previous section is that a synchronization for maximal equal bandwidths can be found by searching through a relatively few cases. By Theorem 1, it suffices to examine half-integer synchronizations. By Theorem 2, it suffices to examine only the outbound direction.

Let b_i = greatest outbound bandwidth under half-integer synchronization if S_i's red touches the front of the outbound band.

B = the value of one of the maximal equal bandwidths.

It will be helpful in computations to permit b_i and B to be negative at times—the operational interpretation as a zero bandwidth is clear.

If S_i's red touches the front of the outbound green band, the situation is as shown in Fig. 3a. Take as an origin for measurements the right side of S_i's red. The trajectory (not shown) that touches the right side of S_j's red passes S_i at a time that will be denoted u_{ij}. From Fig. 3a this is seen to be

$$\mathrm{man}[\theta_{ij}+(r_j/2)-(r_i/2)-t_{ij}],$$

except that, when this expression is zero, we shall want $u_{ij}=1$. This may be accomplished by writing

$$u_{ij}=1-\mathrm{man}[-\theta_{ij}-\tfrac{1}{2}\,(r_j)+\tfrac{1}{2}\,(r_i)+t_{ij}].$$

Substituting (3) and making the dependence on δ_{ij} explicit:

$$u_{ij}(\delta_{ij})=1-\mathrm{man}[(\tfrac{1}{2})(r_i-r_j)+(\tfrac{1}{2})(t_{ij}-\bar{t}_{ij})-\delta_{ij}].$$

The trajectory that touches the left side of S_j's red passes S_i at $u_{ij}-r_j$. Therefore, since δ_{ij} is to take on either the value 0 or $\tfrac{1}{2}$ and since S_i's red is to touch the front, the best δ_{ij} is identified by

$$\max[u_{ij}(0)-r_j,u_{ij}(\tfrac{1}{2})-r_j].$$

Therefore,

$$b_i=\min_j\max_{\delta=0,1/2}[u_{ij}(\delta)-r_j],$$

and, finally, $B=\max_i b_i$. Summarizing, we have:

THEOREM 3. *The maximal equal bandwidth is* $\max(0,B)$ *where*

$$B=\max_i\min_j\max_{\delta=0,1/2}[u_{ij}(\delta)-r_j].$$

Let $i=c$ *be a maximizing* i *and* $\delta_{c1},\cdots,\delta_{cn}$ *be the corresponding maximizing* δ*'s. Then, a synchronization for maximal equal bandwidths is* $\theta_{c1},\cdots,\theta_{cn}$ *obtained by substituting the* δ_{cj} *into* (3).

MAXIMAL UNEQUAL BANDWIDTHS

AVERAGE platoon lengths usually differ between the inbound and outbound directions. If the length exceeds bandwidth in one direction and not the other, it may be possible to shift bandwidth from one direction to the other and pass both platoons. We first show how to shift bandwidth and then suggest a method for dividing total bandwidth between directions on the basis of platoon size.

Let $\theta_{c1}, \cdots, \theta_{cn}$ be a maximal equal bandwidth synchronization with S_c a critical signal whose red touches the front of the outbound green band. The corresponding $u_{c1}, \cdots, u_{cn}$ are presumed known as is the maximal equal bandwidth, B. Let

$$\alpha_j = \text{a phase shift for } S_j \text{ (cycles)},$$

$$\theta'_{cj} = \text{man}(\theta_{cj} - \alpha_j) = \text{adjusted phase for } S_j \text{ (cycles)},$$

$$g = \min_i(1 - r_i) = \text{smallest green time (cycles)}.$$

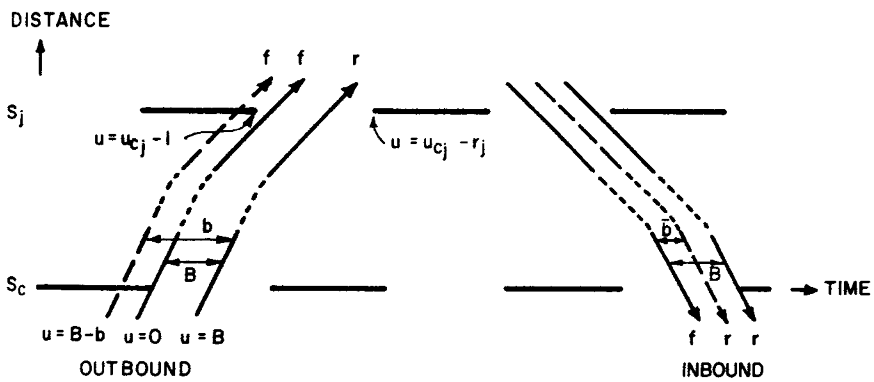

Fig. 4. Widening the outbound green band from B to b.

The shifting procedure is described in

THEOREM 4. *The outbound bandwidth,* b, *can be assigned any value,* $\max[0,B] \leqq b \leqq g$, *by making a phase shift*

$$\alpha_j = \max[u_{cj} - 1 + b - B, 0].$$

Then $\bar{b} = \max[2B - b, 0]$, *and* $\bar{b}$ *is as large as possible for the given* b.

Alternatively, the inbound bandwidth, $\bar{b}$, *can be assigned any value,* $\max[0,B] \leqq \bar{b} \leqq g$, *by making a phase shift*

$$\alpha_j = \max[\bar{b} + r_j - u_{cj}, 0].$$

Then $b = \max[2B - \bar{b}, 0]$, *and* b *is as large as possible for the given* $\bar{b}$.

The shifting procedure may be developed as follows: Suppose it is desired to increase outbound bandwidth to $b > B$, or, if B is negative, to $b > 0$. The trajectory at the front edge of the outbound band is moved to the left, pushing before it any reds that start to touch it. *See* Fig. 4, or the change from Fig. 5a to 5b. During the movement, the critical signals will cut down $\bar{b}$ just as much as b is increased, except that, if $\bar{b}$ reaches zero, no further decrease can occur. Thus, by Corollary 1, $\bar{b}$ is as large as possible for the given b. There is a limit to the increase that can be made in b—eventually the pushing of a red to the left will bring the next red of that signal in from the right to cut into the rear of the outbound band. Then that signal limits both front and rear of the band. The signal must be one with the smallest value of green time; therefore $b=g$. From this argument we conclude that b can be increased from max$[0,B]$ to any value less than or equal to g and that $\bar{b}$ is then max$[B-(b-B),0]$. Analogous remarks apply to increasing $\bar{b}$.

The algebra of the shift may be worked out with the help of Fig. 4. Define a u-axis which measures right and left from the front edge of the outbound green band under the given maximal equal bandwidth synchronization. The rear of the band is then at $u=B$ and the right side of a red of S_j is at $u=u_{cj}$.

Consider first the shift to obtain b. The front of the outbound band is pushed left to the position $u=B-b$. (*See* the dashed line in the outbound portion of Fig. 4.) This will require moving some reds but no more will be moved than necessary and those moved will be moved as little as possible. The next S_j red to the left of the old front edge is met at $u=u_{cj}-1$. Therefore, the appropriate phase shift for S_j is to the left by an amount:

$$\alpha_j=\max[(u_{cj}-1)-(B-b),0].$$

For the case of shifting to obtain $\bar{b}$, it is first observed that under the given synchronization, as under any half-integer synchronization, the distance from the front of the inbound green band to the next S_j red on the left is the same as the distance from the rear of the outbound green band to the next S_j red on the right. (Otherwise we could contradict Theorem 2 by enlarging r_j until S_j started to reduce one green band and not the other.) Consequently we can calculate how much to shift S_j by seeing what is required to move the rear of the outbound band to the right and make it $\bar{b}$ wide. Getting help from Fig. 4, we find that the magnitude of the shift should be

$$\alpha_j=\max[\bar{b}-(u_{cj}-r_j),0].$$

To move the front of inbound to the left, these shifts are made to the left. This concludes the proof of Theorem 4. We note for completeness that, if $g>2B$, $\max(b+\bar{b})=g$; otherwise $\max(b+\bar{b})$, $\max(b+\bar{b})$ subject to $b=\bar{b}$, and $\max(b+\bar{b})$ subject to $b>0$ and $\bar{b}>0$ all equal $2B$.

Finally, we give a way to apportion total bandwidth between directions on the basis of the length (in time) of the platoons. Let

$P(\bar{P})=$platoon length in the outbound (inbound) direction (cycles).

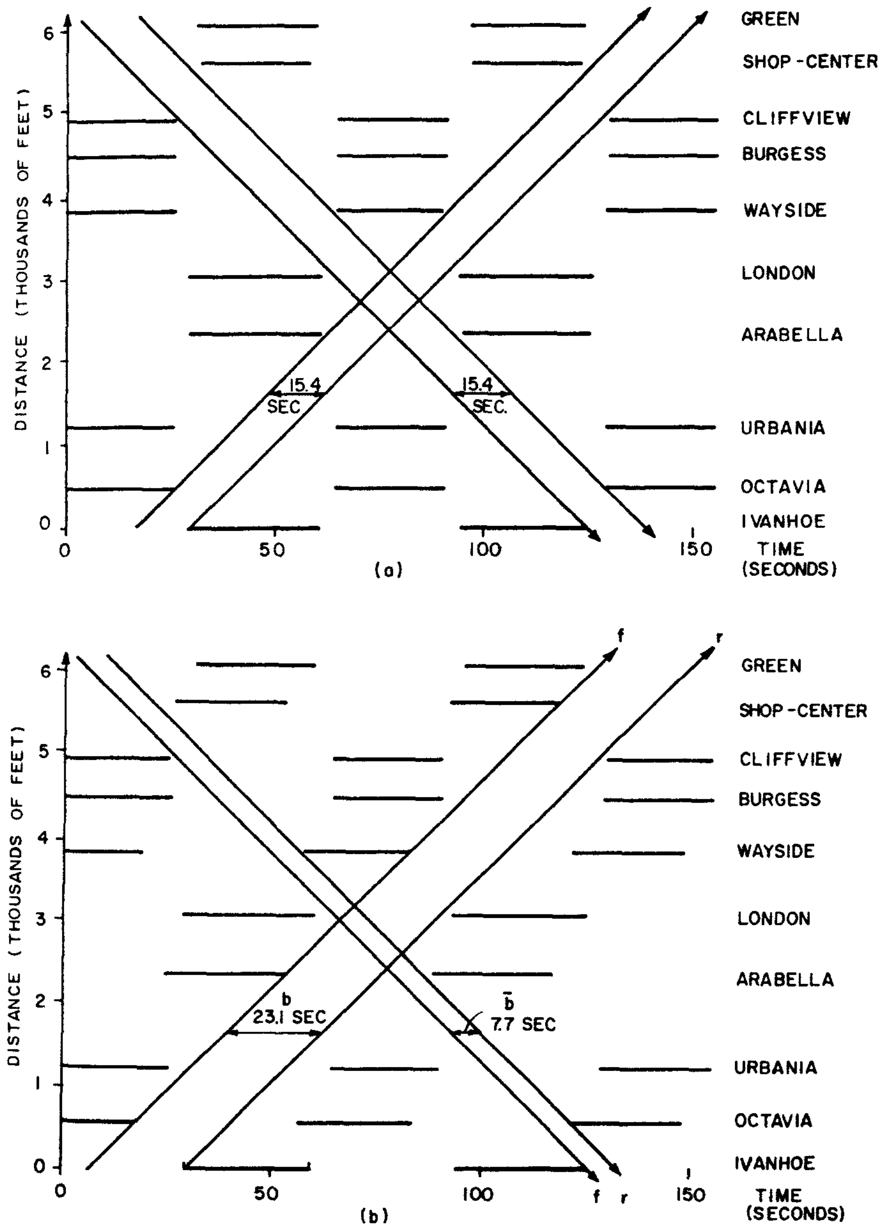

Fig. 5. Space-time diagram for part of Euclid Avenue in Cleveland. $C = 65$ seconds, $v = \bar{v} = 50$ feet/second. (a) Maximum equal bandwidths. (b) Setting for platoon sizes $P = 0.3$ cycles. $\bar{P} = 0.1$ cycles.

Whenever $P=\bar{P}$ maximal equal bandwidths are proposed. Otherwise we proceed as follows: If $P+\bar{P}\leqq 2B$, there may be enough bandwidth to accommodate both platoons. The bandwidth is made proportional to platoon length if possible. Thus, if $P>\bar{P}$:

$$b=\min[2BP/(P+\bar{P}),g],$$
$$\bar{b}=\max[2B-b,0].$$

If $P+\bar{P}>2B$, the larger platoon is accommodated, if possible, and the remainder, if any, is given to the smaller. Thus, if $P>\bar{P}$,

$$b=\min(P,g),$$
$$\bar{b}=\max(2B-b,0),$$

except that if $\bar{b}=0$, b is set to g. Appropriate interchanges apply if $\bar{P}>P$.

SUMMARY OF THE METHOD

To SYNCHRONIZE signals for maximal equal bandwidths, first number the signals in order of distance along the street, say, $i=1, 2, \cdots, n$. The direction of increasing i will be called outbound. Next specify the following data: the signal period, C, in seconds; the red times, $r_1, \cdots, r_n$, in fractions of a cycle; the signal positions, $x_1, \cdots, x_n$; in feet, the outbound speeds between signals, $v_1, v_2, \cdots, v_{n-1}$, in feet/second; and the inbound speeds between signals $\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_{n-1}$ in feet/second.

The computation proceeds in the following steps:

1. Calculate $y_1, \cdots, y_n$ from

$$y_1=0,$$
$$y_i=y_{i-1}-(\tfrac{1}{2})(r_i-r_{i-1})+(x_i-x_{i-1})(1/2C)[1/v_{i-1})+(1/\bar{v}_{i-1})].$$

2. Calculate $z_1, \cdots, z_n$ from

$$z_1=0,$$
$$z_i=z_{i-1}+(x_i-x_{i-1})(1/2C)[(1/v_{i-1})-(1/\bar{v}_{i-1})].$$

3. Calculate

$$B=\max_i \min_j \max_{\delta=0,1/2}[u_{ij}(\delta)-r_j],$$

where

$$u_{ij}(\delta)=1-\mathrm{man}(y_j-y_i-\delta)$$

and the operation 'man' is as defined earlier. Consider a specific i. As the max over δ is performed, the maximizing value (one for each j) may be recorded in a temporary table, $\delta_{i1}, \cdots, \delta_{in}$. As the max over i is performed, the maximizing i, say $i=c$, identifies the best set, $\delta_{c1}, \cdots, \delta_{cn}$, which is saved. For computations below, it is necessary to save the set, $u_{c1}, \cdots, u_{cn}$, corresponding to the $\delta_{c1}, \cdots, \delta_{cn}$. This means saving the value of u_{ij} whenever a value of δ_{ij} is saved.

4. A synchronization $\theta_{c1}, \cdots, \theta_{cn}$, for maximal equal bandwidths is calculated from

$$\theta_{cj}=\text{man}[z_j-z_c+\delta_{cj}].$$

The bandwidth in each direction is $\max(0,B)$.

To adjust the synchronization for platoon lengths of P, outbound, and $\bar{P}$, inbound, specify P and $\bar{P}$, perform the above calculations and continue as follows.

5. Calculate $g=\min(1-r_i)$.
6. If $P=\bar{P}$ accept equal bandwidth solution.
7. If $\bar{P}>P$, go to Step 11, otherwise continue.
8. If $P+\bar{P}\leqq 2B$, set $b=\min[g,2BP/(P+\bar{P})]$. Otherwise, set $b=\min(P,g)$; unless $P\geqq 2B$, in which case, set $b=g$.
9. Calculate $\alpha_1, \cdots, \alpha_n$ from $\alpha_j=\max(u_{cj}-1+b-B,0)$.
10. Calculate $\bar{b}=\max(2B-b,0)$. Go to Step 14.
11. If $P+\bar{P}\leqq 2B$, set $\bar{b}=\min[g,2B\bar{P}/(P+\bar{P})]$. Otherwise, set $\bar{b}=\min(\bar{P},g)$, unless $\bar{P}\geqq 2B$, in which case, set $\bar{b}=g$.
12. Calculate $\alpha_1, \cdots, \alpha_n$ from $\alpha_j=\max(\bar{b}+r_j-u_{cj},0)$.
13. Calculate $b=\max(2B-\bar{b},0)$.
14. The adjusted synchronization, $\theta'_{c1}, \cdots, \theta'_{cn}$, is calculated from

$$\theta'_{cj}=\text{man}(\theta_{cj}-\alpha_j)$$

and the bandwidths are b, outbound and $\bar{b}$, inbound as previously determined.

For plotting space-time diagrams it is helpful to know where the edges of the green bands are. Take as a reference point the center of a red of S_c. The left side of an outbound band is at $\frac{1}{2}\, r_c$, the right side at $\frac{1}{2}\,(r_c)+b$. An inbound band has its left side at $1-\frac{1}{2}\,(r_c)-\bar{b}$ and its right side at $1-\frac{1}{2}\,(r_c)$. The edges of the same outbound band at S_j are found by adding t_{cj} to the edges at S_c. For inbound, add $\bar{t}_{cj}$.

EXAMPLES

THE METHOD has been used to synchronize the signals on a stretch of Euclid Avenue in Cleveland under off-rush hour conditions. (During rush hours a complete one-way synchronization is used.) Signals are at 0, 550, 1250, 2350, 3050, 3850, 4500, 4900, 5600, 6050 feet. Corresponding red times are 0.47, 0.40, 0.40, 0.47, 0.48, 0.42, 0.40, 0.40, 0.42 cycles. $T=65$ seconds, $v=50$ feet/second in both directions. Figure 5a shows the space-time diagram for maximal equal bandwidths. $B=0.237$ or 15.4 seconds. Figure 5b shows the case: $P=0.30$ cycles, $\bar{P}=0.10$ cycles.

Turning to the literature, we have worked out certain problems to see whether our method would give different answers. One case which turned out differently was a problem of RAUS[7] involving 9 signals spaced one every 500 feet from 0 to 4000 feet, a period of 80 seconds, a speed of 40 feet/second, and red times of 32 seconds for each signal. Raus finds a bandwidth of 16.2 seconds. The method here gives a bandwidth of 18 seconds using the phases (listed in order of increasing signal distance), 0, 0, 0, ½, ½, ½, 0, 0, 0.

Bowers' example (*see* references 8 or 9) has been examined. His optimum at 17.5 mph with a 50-second period has the interesting property that each signal is pairwise optimal with every other according to the rules of Corollary 2. Therefore, the synchronization will be optimal for all possible green splits. However, in the optimum at 23.4 mph with a 50 second period, the situation is different. With all the reds at 25 seconds, as he uses, our method gives his synchronization, 0, ½, 0, ½, ½, ½, 0, ½ (signals in order of increasing distance), and his bandwidth, 8.36 seconds. If signals C, E, and H are cut to a red time of 15 seconds, bandwidth will be unchanged under his synchronization. However, our method will give a bandwidth of 10.44 seconds under a synchronization: 0, ½, ½, 0, ½, ½, 0, ½. Thus, the best synchronization depends on the mix of green splits, a situation that Bowers' method does not attempt to handle.

DISCUSSION

THE TIME to solve a 10-signal problem is about a minute on a 20K IBM 1620. A listing of the program is found in reference 11. Since computing time is small, it is a reasonable task to explore a range of vC (if v is a constant) looking for particularly large bandwidths or to make sensitivity tests on other parameters of the system.

The ability to handle different platoon speeds between different signals makes it possible to adjust the synchronization for the presence of a queue waiting at a signal. Such a queue might arise because turning traffic is entering the main street at the previous intersection or might be the tail of a platoon that does not fit through the green band. Let τ be the time length of the queue waiting when the next platoon tries to come through. Unless the queue is released early, the arriving platoon will have to stop or slow down. Let v be the normal platoon speed and x the distance from the preceding signal. If the speed, $v' = v/[1-(\tau/x)v]$, is used in the computation, the synchronization will permit the platoon to travel at v and not stop (but note that the departing platoon is longer than the arriving platoon). Similarly, an allowance can be made for cars leaving the platoon, although, unless the cars always leave from the head, it may be more reasonable to expect (or guide) the lead car to maintain a planned peed and encourage subsequent cars to adjust speed to close the gaps. A egative value of v' is permitted by our calculation—this would imply a ackward movement of the green wave.

Although we have ruled out problems for which the maximal bandidth is split in one or both directions, our results extend to one aspect of hese cases. Denote the largest unsplit segment of a bandwidth as the *rimary bandwidth* and the corresponding green band as the *primary green and.* The substitution of these terms for bandwidth and green band proughout our analysis will make the results hold for any problem. In nost problems our method is likely to maximize total bandwidth even if plit, but it is possible to construct examples where this does not happen.

The method may be useful in connection with traffic control by on-line omputer. The α_j adjustment could be made to follow actual flow. nother possibility is to vary C. Suppose that the green split restrictions re expressed in cycles and remain valid for a range of C. From (1) it may e seen that, if each speed is multiplied by a constant and C is divided by iat constant, the travel times (expressed in cycles) between signals are nchanged. Therefore the synchronization for maximum bandwidth is nchanged. Suppose, then, that in real operations the traffic speed emporarily decreases from the planned speed because of weather, ineased vehicle density, or the like. Normally, the synchronization beomes invalid, but this will not happen if the signal period is correspondigly lengthened, as could easily be done in real time control. Such engthening would probably increase capacity slightly.

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