1. $\ln\left(\frac{p(x)}{q(x)}\right) = \frac{1}{2}\ln\left(\frac{|L|}{|\Sigma|}\right) + \frac{1}{2}(x-m)^T L^{-1}(x-m) - \frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)$

Since $x \sim p(x) = \mathcal{N}(\mu|\Sigma)$

$$\int p(x)dx = 1$$

$$E(x) = \int x\, p(x)\, dx = \mu$$

$$E\left[(x-a)^T A(x-a)\right] = tr(A\Sigma) + (\mu-a)^T A(\mu-a)$$

$$KL = \int \left\{ \frac{1}{2}\ln\frac{|L|}{|\Sigma|} - \frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu) + \frac{1}{2}(x-m)^T L^{-1}(x-m) \right\} p(x)\, dx$$

$$= \frac{1}{2}\ln\frac{|L|}{|\Sigma|} - \frac{1}{2}E\left[(x-\mu)\Sigma^{-1}(x-\mu)^T\right] + \frac{1}{2}E\left[(x-m)^T L^{-1}(x-m)\right]$$

$$= \frac{1}{2}\ln\frac{|L|}{|\Sigma|} - \frac{1}{2}tr\{I_D\} + \frac{1}{2}(\mu-m)^T L^{-1}(\mu-m) + \frac{1}{2}tr(L^{-1}\Sigma)$$

$$= \frac{1}{2}\left[\ln\frac{|L|}{|\Sigma|} - D + tr\{L^{-1}\Sigma\} + (m-\mu)^T L^{-1}(m-\mu)\right]$$

$$2.\ KL(p\|q) = -\int p(x) \ln\left(\frac{q(x)}{p(x)}\right) dx$$

$$= -\int p(x) \ln q(x)\, dx + C$$

$$= -\int p(x) \left[-\frac{D}{2}\ln 2\pi - \frac{1}{2}\ln|\Sigma| - \frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right] dx + C$$

$$= \int p(x) \left[\frac{1}{2}\ln|\Sigma| + \frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right] dx + C$$

$$= \frac{1}{2}\ln|\Sigma| + \int p(x) \left[\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right] dx + C$$

$$= \frac{1}{2}\ln|\Sigma| + \int p(x)\left[x^T\Sigma^{-1}x - 2\mu^T\Sigma^{-1}x + \mu^T\Sigma^{-1}\mu\right] dx + C$$

$$= \frac{1}{2}\ln|\Sigma| + \frac{1}{2}\int p(x)\, Tr\left[\Sigma^{-1}(xx^T)\right] dx - \mu^T\Sigma^{-1}E(x) + \frac{1}{2}\mu^T\Sigma^{-1}\mu + C$$

$$= \frac{1}{2}\ln|\Sigma| + \frac{1}{2}Tr\left[\Sigma^{-1}E(xx^T)\right] - \mu^T\Sigma^{-1}E(x) + \frac{1}{2}\mu^T\Sigma^{-1}\mu + C$$

Set $\frac{\partial KL}{\partial \mu} = -\Sigma^{-1}E(x) + \Sigma^{-1}\mu = 0$

obtain $\mu = E(x)$

$$KL(p\|q) = \frac{1}{2}\ln|\Sigma| + \frac{1}{2}Tr\left[\Sigma^{-1}E(xx^T)\right] - \frac{1}{2}\mu^T\Sigma^{-1}\mu + C$$

Set $\frac{\partial KL}{\partial \Sigma} = \frac{1}{2}\Sigma^{-1} - \frac{1}{2}\Sigma^{-1}E[xx^T]\Sigma^{-1} + \frac{1}{2}\Sigma^{-1}\mu\mu^T\Sigma^{-1} = 0$

$$\frac{\partial a^T X^{-1} b}{\partial X} = -X^{-T}ab^TX^{-T} \quad \text{and} \quad \frac{\partial Tr(AX^{-1}B)}{\partial X} = -X^{-T}A^TB^TX^{-T}$$

$$\Sigma = E(xx^T) - \mu\mu^T = E(xx^T) - E(x)E(x)^T = cov(x)$$

3. $E(w,\Sigma) = \frac{1}{2}\sum_{n=1}^{N}\{[y(x_n,w)-t_n]^T\Sigma^{-1}[y(w_n,w)-t_n]\} + \frac{N}{2}\ln|\Sigma| + C$ ①

when Σ is fixed,

$$E(w) = \frac{1}{2}\sum_{n=1}^{N}\{[y(x_n,w)-t_n)^T\Sigma^{-1}[y(x_n,w)-t_n]\}+C$$

$$W_{ML} = \min(E(w))$$

If Σ is unknown, since Σ is in the first term on the right of ①, W_{ML} solved will involve Σ.

4. a) $\frac{\partial z}{\partial u} = ye^{x^2y}2x.\frac{1}{2}(uv)^{-\frac{1}{2}}.v$

$$= xvye^{x^2y}(uv)^{-\frac{1}{2}}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= e^{x^2y}.y.2x.\frac{1}{2}(uv)^{-\frac{1}{2}}.u + e^{x^2y}.x^2.(-v^{-2})$$

$$= uxye^{x^2y}(uv)^{-\frac{1}{2}} - x^2e^{x^2y}v^{-2}$$

b) $u = u - \Delta u e$

$$= u - (t-z)\Delta u(-z)$$

$$= u - (z-t)\frac{\partial z}{\partial u}$$

$$= u - (z-t)xvye^{x^2y}(uv)^{-\frac{1}{2}}$$

Similarly, $v = v - (z-t)\frac{\partial z}{\partial v}$

$$= v - (z-t)[uxye^{x^2y}(uv)^{-\frac{1}{2}} - x^2e^{x^2y}v^{-2}]$$

cs539 hw5

Enbo Tian

2022-04-19

#KL Distance

a)

```
library(LaplacesDemon)

## Warning: ³Ì¼°ü'LaplacesDemon'ÊÇÓÃR°æ±¾4.1.3 À´½¨ÔìµÄ

p <- 1/3*dnorm(runif(10),-1,2)+2/3*dnorm(runif(10),1,1)

m <- 1:20/20*2-1 # set 100 m from -1 to 1
s <- 1:10/10+1  # set 100 s from 1 to 2

Dist <- rep(0,200)
count = 1
for(i in 1:20){
  for(j in 1:10){
    q <-  dnorm(runif(10),m[i],s[j])
    Dist[count] <- KLD(p,q)$intrinsic.discrepancy
    count = count+1
  }
}

Dist <- matrix(Dist,20,10)
Dist

##              [,1]        [,2]        [,3]        [,4]        [,5]
[,6]
##  [1,] 0.074264469 0.038137491 0.041601799 0.026631451 0.017080344 0.011848
252
##  [2,] 0.022789293 0.040596545 0.039387350 0.012808086 0.015296266 0.008565
842
##  [3,] 0.034498050 0.013273422 0.013589713 0.017476768 0.009959628 0.007718
538
##  [4,] 0.025155883 0.026360424 0.010264902 0.015939085 0.010605139 0.009498
763
##  [5,] 0.008279208 0.014134710 0.009207988 0.016231977 0.011267477 0.003182
454
##  [6,] 0.016668890 0.006411344 0.012997214 0.010457974 0.007276627 0.005238
825
##  [7,] 0.009451513 0.006448361 0.008092268 0.006755665 0.007200578 0.007050
941
##  [8,] 0.017463857 0.005871399 0.016621786 0.006263301 0.009145212 0.006586
```

```
178
##  [9,] 0.008073879 0.013321547 0.006997762 0.012098170 0.008439370 0.005527
057
## [10,] 0.008245941 0.010639997 0.007920678 0.007978052 0.006520124 0.005523
457
## [11,] 0.040161517 0.040916470 0.020361451 0.018443077 0.012904681 0.009175
163
## [12,] 0.046464729 0.034840081 0.021230236 0.018536609 0.007400951 0.010752
037
## [13,] 0.029235275 0.021210035 0.019985571 0.009061827 0.011104826 0.006848
881
## [14,] 0.028296232 0.009600925 0.015396214 0.014195285 0.010553359 0.010021
208
## [15,] 0.019676134 0.012156798 0.020233373 0.014232360 0.013495992 0.005322
832
## [16,] 0.013788002 0.017514062 0.007804679 0.012463679 0.007873483 0.006841
759
## [17,] 0.013203959 0.011414001 0.011573232 0.010329020 0.004726553 0.007109
610
## [18,] 0.010508522 0.015133242 0.008350040 0.011316486 0.010215789 0.006689
249
## [19,] 0.010322422 0.011515366 0.008444690 0.005927215 0.007591552 0.007368
596
## [20,] 0.015099996 0.006397433 0.013813953 0.007756306 0.010122202 0.005225
046
##              [,7]        [,8]        [,9]       [,10]
##  [1,] 0.007158171 0.006667865 0.006362356 0.009107105
##  [2,] 0.006359808 0.006145424 0.007390584 0.011724314
##  [3,] 0.007832392 0.007546456 0.007267868 0.007375474
##  [4,] 0.007656887 0.006142740 0.006061073 0.011579731
##  [5,] 0.006886548 0.005057920 0.007048503 0.008273023
##  [6,] 0.006592928 0.006798134 0.005326443 0.007263837
##  [7,] 0.005100321 0.005819458 0.007022904 0.008119152
##  [8,] 0.007089890 0.006101534 0.005703961 0.007652606
##  [9,] 0.005866766 0.006393409 0.007286094 0.006268789
## [10,] 0.006413001 0.005833373 0.006175161 0.006582217
## [11,] 0.006911742 0.004902852 0.009293558 0.018269162
## [12,] 0.008706771 0.004197172 0.006982358 0.011665035
## [13,] 0.004929624 0.005889256 0.005178739 0.009101523
## [14,] 0.005917800 0.005589210 0.009652390 0.004275953
## [15,] 0.006250600 0.005982095 0.005803501 0.005973778
## [16,] 0.006730423 0.007081196 0.006331981 0.006262416
## [17,] 0.005794853 0.006195507 0.006417874 0.007060317
## [18,] 0.005951597 0.005989148 0.007120286 0.010489773
## [19,] 0.006626002 0.006505283 0.007015749 0.008444949
## [20,] 0.005896750 0.006735880 0.005582170 0.007559439
```

```
min(Dist)
```

```
## [1] 0.003182454
```

```r
which(Dist== min(Dist), arr.ind = TRUE)

##      row col
## [1,]   5   6

# row = 8, col= 3
8/20-1

## [1] -0.6

3/10+1

## [1] 1.3
```

Then $m_a = -0.6$, $\sigma_m^2 = 1.3$ have the minimum KL distance with p

b)

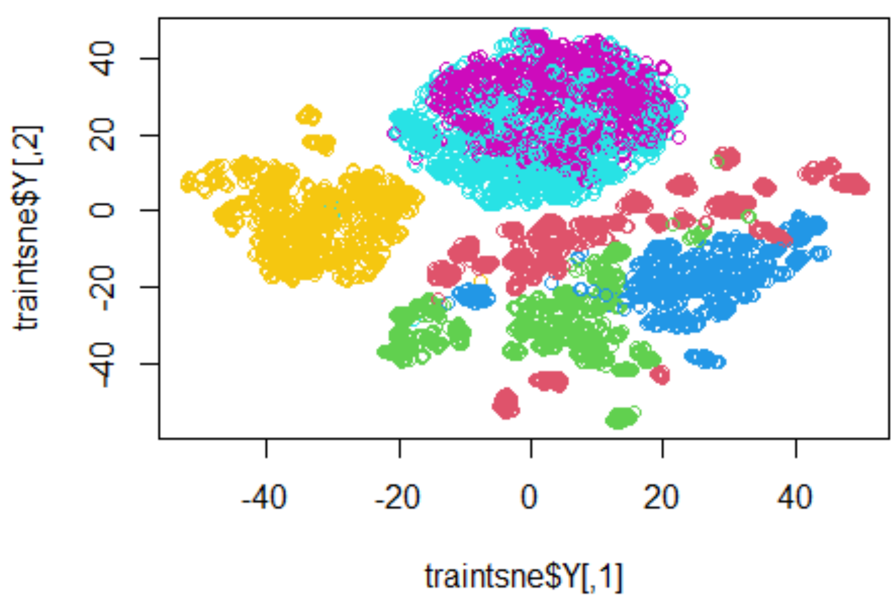
```r
mean(p)

## [1] 0.2695909

var(p)

## [1] 0.0009958066

q_p <- dnorm(runif(10),0.2809131,0.0009584375)

KLD(p,q)

## $KLD.px.py
##  [1] -0.0103197466  0.0165002593  0.0152990167 -0.0079198915  0.0008157114
##  [6] -0.0199717176  0.0039833272 -0.0081637482 -0.0008231197  0.0181593475
##
## $KLD.py.px
##  [1]  0.0115403248 -0.0142666475 -0.0133829129  0.0086094354 -0.0008086586
##  [6]  0.0255801097 -0.0038380996  0.0089108581  0.0008302934 -0.0155083650
##
## $mean.KLD
##  [1] 6.102891e-04 1.116806e-03 9.580519e-04 3.447719e-04 3.526439e-06
##  [6] 2.804196e-03 7.261379e-05 3.735550e-04 3.586853e-06 1.325491e-03
##
## $sum.KLD.px.py
## [1] 0.007559439
##
## $sum.KLD.py.px
## [1] 0.007666338
##
## $mean.sum.KLD
## [1] 0.007612888
##
## $intrinsic.discrepancy
## [1] 0.007559439
```

the KL distanced is 0.004042311, however, it is not the minimum KL distance.

TSNE

a)

```
library(Rtsne)
xtrain <- read.table("X_train.txt")

ytrain <- read.table("y_train.txt")

traintsne <- Rtsne(xtrain)
# Y
# 1   red WALKING
# 2   green WALKING_UPSTAIRS
# 3   blue WALKING_DOWNSTAIRS
# 4   light_blue SITTING
# 5   purple STANDING
# 6   yellow LAYING
plot(traintsne$Y,col= ytrain$V1+1)
```

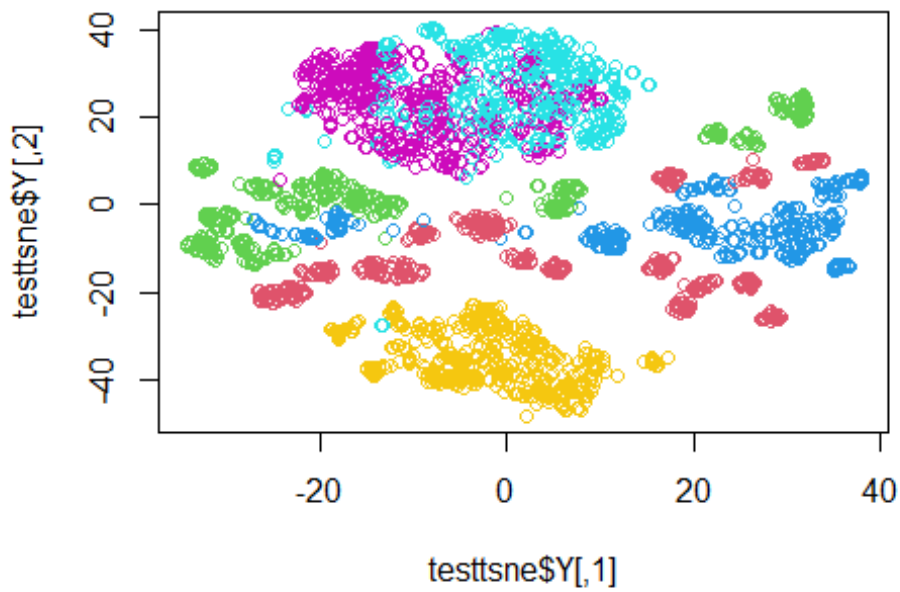


The KL distance for Y=1 is the largest, Y = 7 is the smallest, Y = 4,5 are emed.

b)

```
xtest <- read.table("X_test.txt")
ytest <- read.table("y_test.txt")
```

```
testtsne <- Rtsne(xtest)
plot(testtsne$Y,col= ytest$V1+1)
```



The KL distance for red, green, blue(1, 2, 3) are large, 4,5,6 are small.

c)

The similarities part are the each part distribution of the color groups. both 4 and 5 are neighboring, 1 and 2 are dispersive.

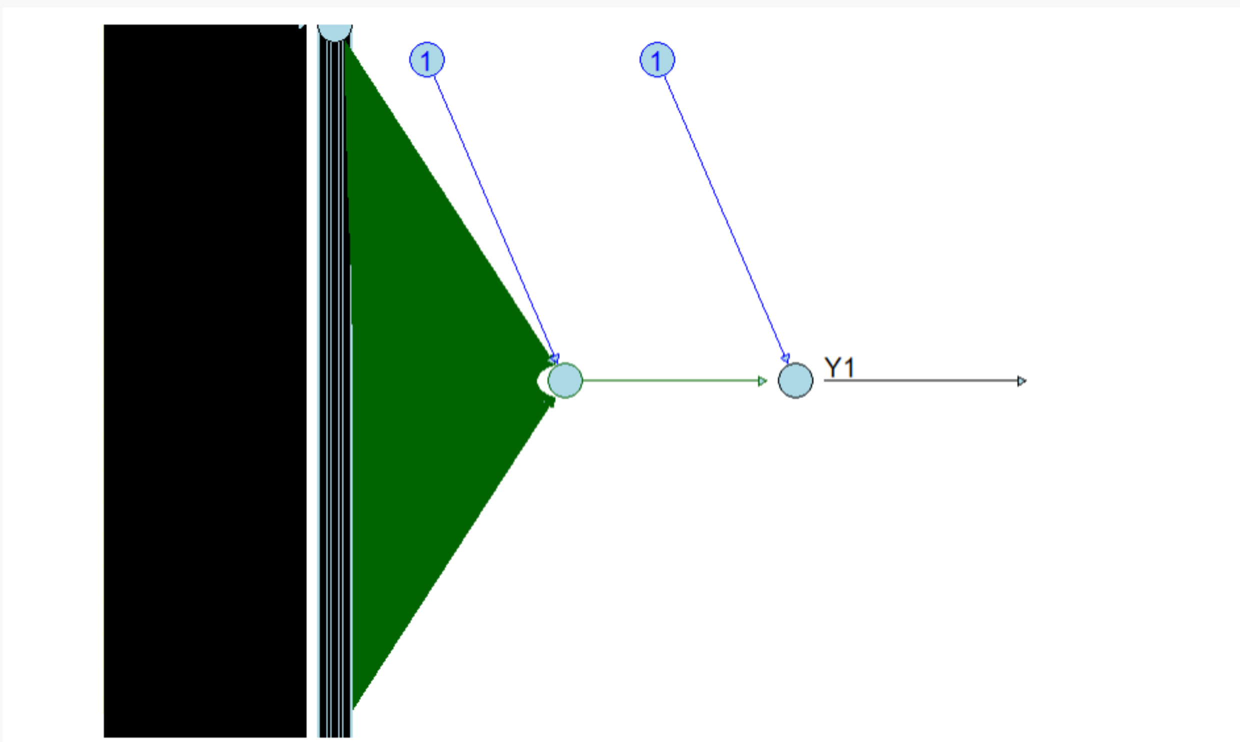
Neural Networks

a)

```
library(neuralnet)

## Warning: ³Ì¼°ü'neuralnet'ÊÇÓÃR°æ±¾4.1.3 À´½¨ÔìµÄ

Y1 <- ytrain$V1
training <- data.frame(Y1,xtrain)
TrainNN <- neuralnet(Y1 ~ .,data = training)
summary(TrainNN)

##                        Length  Class      Mode
## call                         3 -none-     call
## response                  7352 -none-     numeric
## covariate              4124472 -none-     numeric
```

```
## model.list                   2 -none-      list
## err.fct                      1 -none-      function
## act.fct                      1 -none-      function
## linear.output                1 -none-      logical
## data                       562 data.frame list
## exclude                      0 -none-      NULL
## net.result                   1 -none-      list
## weights                      1 -none-      list
## generalized.weights          1 -none-      list
## startweights                 1 -none-      list
## result.matrix              567 -none-      numeric
```

```
plot(TrainNN,col.hidden = 'darkgreen',
col.hidden.synapse = 'darkgreen',
     show.weights = F,
     information = F,
     fill = 'lightblue')
```

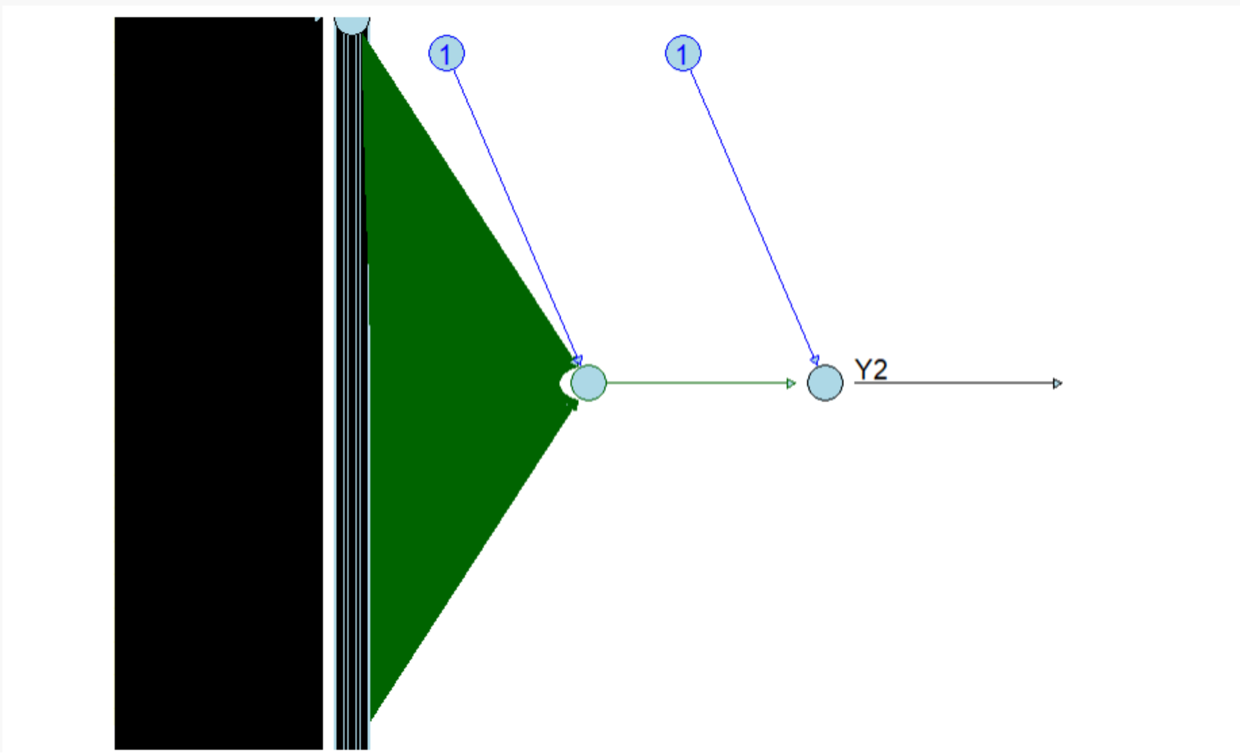



```
Y2 <- ytest$V1
testing <- data.frame(Y2,xtest)
TestNN <- neuralnet(Y2 ~ .,data = testing)
summary(TestNN)
```

```
##                        Length  Class      Mode
## call                         3 -none-     call
## response                  2947 -none-     numeric
## covariate              1653267 -none-     numeric
## model.list                   2 -none-     list
```

```
## err.fct                     1 -none-     function
## act.fct                     1 -none-     function
## linear.output               1 -none-     logical
## data                      562 data.frame list
## exclude                     0 -none-     NULL
## net.result                  1 -none-     list
## weights                     1 -none-     list
## generalized.weights         1 -none-     list
## startweights                1 -none-     list
## result.matrix             567 -none-     numeric
```

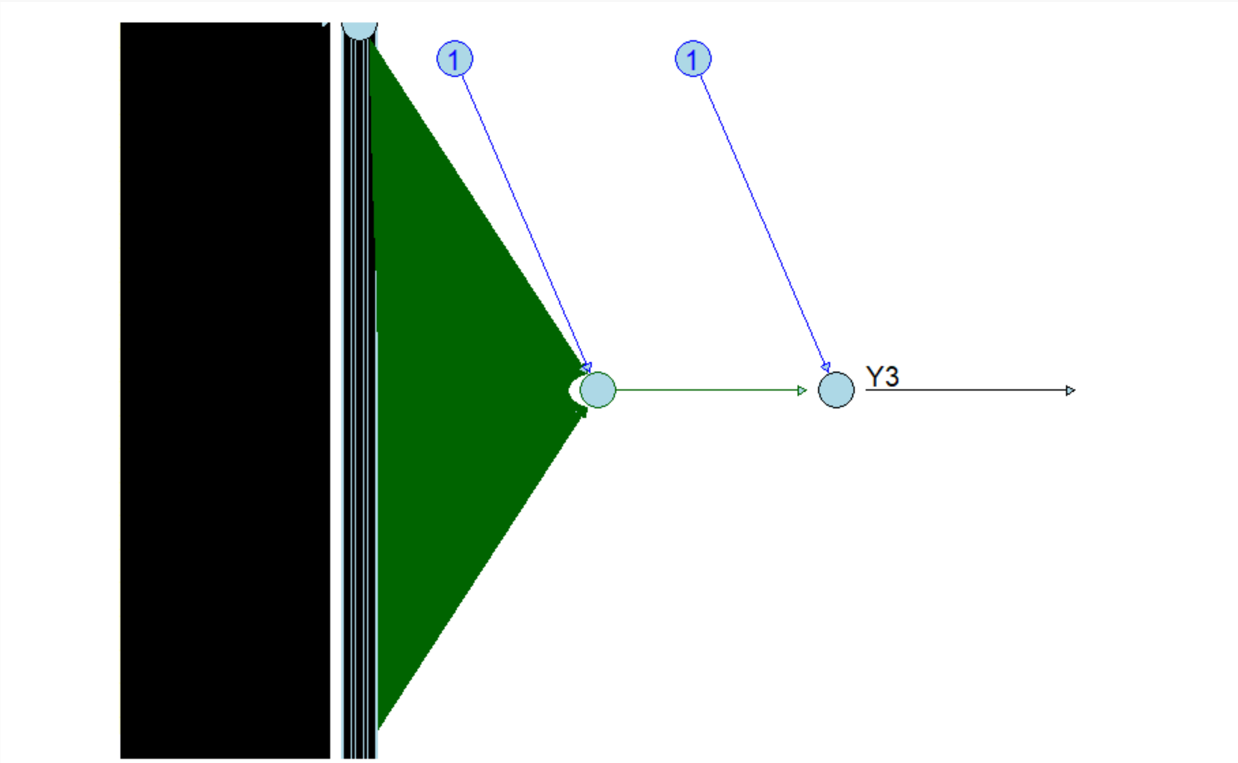
```
plot(TestNN,col.hidden = 'darkgreen',
col.hidden.synapse = 'darkgreen',
     show.weights = F,
     information = F,
     fill = 'lightblue')
```



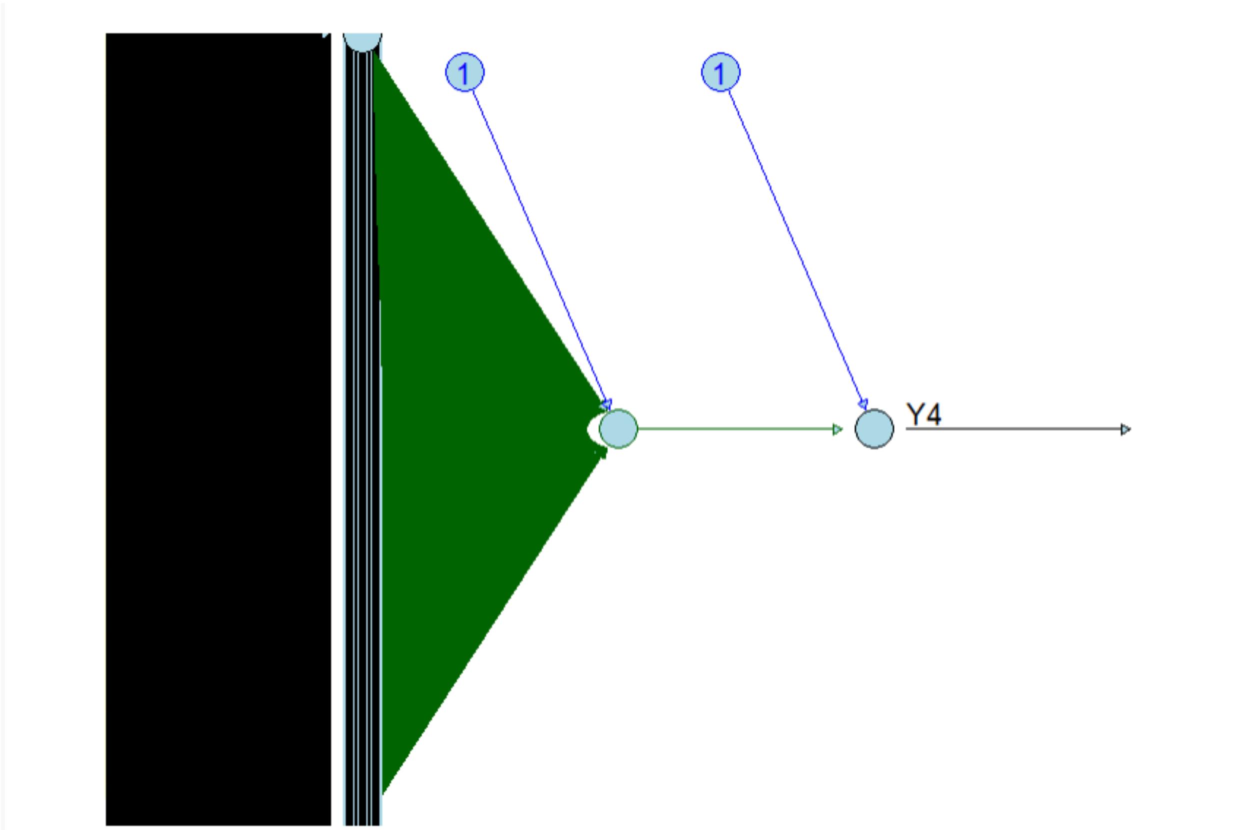
b)

```
trainpartci<-read.table("subject_train.txt")
testpartci <- read.table("subject_test.txt")

Y3 <- trainpartci$V1
training <- data.frame(Y3,xtrain)
TrainNN <- neuralnet(Y3 ~ .,data = training)
plot(TrainNN,col.hidden = 'darkgreen',
col.hidden.synapse = 'darkgreen',
     show.weights = F,
```

```
     information = F,
     fill = 'lightblue')
```




```
Y4 <- testpartci$V1
testing <- data.frame(Y4,xtest)
TestNN <- neuralnet(Y4 ~ .,data = testing)
plot(TestNN,col.hidden = 'darkgreen',
col.hidden.synapse = 'darkgreen',
     show.weights = F,
     information = F,
     fill = 'lightblue')
```


1
1
Y4