1. $E(x) = \int x f_X(x)\, dx$

$= \int x \int f(x,y)\, dy\, dx$

$= \int x \int f(x|y) f(y)\, dy\, dx$

$= \int \left[\int x f(x|y)\, dx\right] f(y)\, dy$

$= \int [E_x(x|y)] f(y)\, dy$

$= E_y[E_x(x|y)]$

Since $Var(X|Y) = E(X^2|Y) - [E(X|Y)]^2$

$E(Var(X|Y)) = E(E(X^2|Y)) - E([E(X|Y)]^2)$

Then $E(Var(X|Y)) = E(X^2) - E(E(X|Y)^2)$ ①

and $Var(E(X|Y)) = E(E(X|Y)^2) - E(X)^2$ ②

From ① and ② $E(Var(X|Y)) + Var(E(X|Y)) = E(X^2) - E(X)^2$

Then $Var(X) = E_y(Var_x[X|Y]) + Var_y[E_x[X|Y]]$

2. $p(x,z) = p(x)\,p(z)$

$$E(y) = E(x+z) = \iint (x+z)\,p(x,z)\,dx\,dz$$

$$= \iint x\,p(x,z)\,dx\,dz + \iint z\,p(x,z)\,dx\,dz$$

$$= \int x\,p(x)\,dx + \int z\,p(z)\,dz$$

$$= E(x) + E(z)$$

$Cov(Y,Y)$

$$= Var(Y) = Var(x+z) = E\left[\left((x+z) - (\mu_x + \mu_z)\right)^2\right]$$

$$= E\left[\left((x-\mu_x) + (z-\mu_z)\right)^2\right]$$

$$= E\left[(x-\mu_x)^2\right] + E\left[(z-\mu_z)^2\right] + 2E\left[(x-\mu_x)(z-\mu_z)\right]$$

$$= Var(x) + Var(z) + 2\underbrace{cov(x,z)}_{0}$$

$$= Var(x) + Var(z)$$

3. $$p(\underset{\sim}{x}|\mu,\Sigma) = \prod_{n=1}^{N} \mathcal{N}(x_n|\mu,\Sigma)$$

$$= \prod_{n=1}^{N} \frac{1}{(2\pi)^{D/2}} \frac{1}{|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(x_n-\mu)^T \Sigma^{-1}(x_n-\mu)\right\}$$

$$= \frac{1}{(2\pi)^{\frac{ND}{2}}} \frac{1}{|\Sigma|^{N/2}} \exp\left\{-\frac{1}{2}\sum_{n=1}^{N}(x_n-\mu)^T\Sigma^{-1}(x_n-\mu)\right\}$$

$$p(\mu) = \mathcal{N}(\mu|\mu_0,\Sigma_0)$$

$$= \frac{1}{(2\pi)^{D/2}} \frac{1}{|\Sigma_0|^{1/2}} e^{-\frac{1}{2}(\mu-\mu_0)^T\Sigma_0^{-1}(\mu-\mu_0)}$$

$$p(\mu|X,\Sigma) = \mathcal{N}(\mu|\mu_{\mu|x},\Sigma_{\mu|x})$$

$$= \frac{1}{(2\pi)^{D/2}} \frac{1}{|\Sigma_{\mu|x}|^{1/2}} \exp\left\{-\frac{1}{2}(\mu-\mu_{\mu|x})^T\Sigma_{\mu|x}^{-1}(\mu-\mu_{\mu|x})\right\}$$

where $\Sigma_{\mu|x} = \Sigma$

$$= (N\Sigma^{-1}+\Sigma_0^{-1})^{-1}$$

$$\mu_{\mu|x} = \Sigma\{A^T L(y-b)+\Lambda\mu\}$$

$$= (N\Sigma^{-1}+\Sigma_0^{-1})^{-1}(N\Sigma^{-1}\bar{x}+\Sigma_0^{-1}\mu_0)$$

4. $P(\mu,\tau|x) = \dfrac{p(x|\mu,\tau)\,p(\mu,\tau)}{p(x)}$

$$\propto p(x|\mu,\tau)\,p(\mu,\tau)$$

$$p(\mu,\tau) \sim \mathcal{N}(x|\mu_0,(\beta\tau)^{-1})\,\mathrm{Gam}(\tau|a,b)$$

$$\propto \tau^{\alpha-1} \exp\{-b\tau\} \left(\frac{\beta\tau}{2\pi}\right)^{\frac{1}{2}} \exp\left\{-\frac{(\beta\tau)}{2}(\mu-\mu_0)^2\right\}$$

$$= \tau^{\alpha-\frac{1}{2}} \exp\left\{-\frac{\tau}{2}\left[\beta\mu^2-2\mu\beta\mu_0+(\beta\mu_0^2+2\beta)\right]\right\} \quad ①$$

$$p(x|\mu,\tau) = \prod_{n=1}^{N} p(x_n|\mu,\tau)$$

$$\propto \prod_{n=1}^{N} \left(\frac{\tau}{2\pi}\right)^{\frac{1}{2}} \exp\left\{-\frac{\tau}{2}(x_n-\mu)^2\right\}$$

$$= \left(\frac{\tau}{2\pi}\right)^{\frac{N}{2}} \exp\left\{-\frac{\tau}{2}\left(\sum_{n=1}^{N} x_n^2 + N\mu^2 - 2\mu\underbrace{\sum_{n=1}^{N} x_n}_{N\mu_{ML}}\right)\right\} \quad ②$$

$$p(\mu,\tau|x) \propto p(x|\mu,\tau)\,p(\mu,\tau)$$

$$= ① \times ②$$

$$= \tau^{\frac{2a+N+1}{2}} \exp\left\{-\frac{\tau}{2}\left[(\beta+N)\mu^2 - 2\mu(\beta\mu_0+N\mu_{ML}) + \left(\beta\mu_0^2+2b+\sum_{n=1}^{N} x_n^2\right)\right]\right\}$$

5. Since independent

$$E(XY) = E(X)\,E(Y)$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)\,E(Y) = 0$$

6.. $p(x) = \mathcal{N}(x|\mu, \sigma^2) \quad q(x) = \mathcal{N}(x|m, s^2)$

$$KL(p, q) = -\int p(x) \log q(x) dx + \int p(x) \log p(x) dx$$

$$= -\int p(x) \log \frac{1}{(2\pi s^2)^{\frac{1}{2}}} e^{-\frac{(x-m)^2}{2s^2}} dx - \frac{1}{2}(1 + \log 2\pi\sigma^2)$$

$$= \frac{1}{2} \log(2\pi s^2) + \frac{\sigma^2 + (\mu - m)^2}{2s^2} - \frac{1}{2}(1 + \log 2\pi\sigma^2)$$

$$= \log \frac{s}{\sigma} + \frac{\sigma^2 + (\mu - m)^2}{2s^2} - \frac{1}{2}$$

7. a. $E(x) = \int x f(x) dx$

$$= \int \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha} e^{-\frac{\beta}{x}} dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int x^{-\alpha} e^{-\frac{\beta}{x}} dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \Gamma(-\alpha - 1, \frac{\beta}{x})$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha - 1)}{\beta^{\alpha - 1}}$$

$$= \frac{\beta}{\alpha - 1}$$

b $f(\underset{\sim}{x}) = \Gamma(\alpha)^{-n} \beta^{n\alpha} \left(\prod_{i=1}^{n} x_i \right)^{-\alpha-1} e^{-\beta \sum_{i=1}^{n} \frac{1}{x_i}}$

$$\log f(\underset{\sim}{x}) = -n \log(\Gamma(\alpha)) + n\alpha \ln(\beta) - (\alpha+1) \ln\left(\prod_{i=1}^{n} x_i\right) - \beta \sum_{i=1}^{n} \frac{1}{x_i}$$

$$\frac{\partial \log f(\underset{\sim}{x})}{\partial \beta} = \frac{n\alpha}{\beta} - \sum_{i=1}^{n} \frac{1}{x_i} = 0$$

$$\underline{\beta = \frac{n\alpha}{\sum_{i=1}^{n} \frac{1}{x_i}}}$$

$$\frac{\partial \log f(\underset{\sim}{x})}{\partial \alpha} = -n \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + n \ln(\beta) - \sum_{i=1}^{n} \ln(x_i) = 0$$

$$\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \ln(\beta) + \frac{1}{n} \ln(x_i) = 0$$

$$\psi(\alpha) + \ln(n) + \ln(\alpha) - \ln\left(\sum_{i=1}^{n} x_i\right) + \frac{1}{n} \sum_{i=1}^{n} \ln(x_i) = 0$$

$$\psi(\alpha) + \ln(\alpha) - \ln(\bar{x}) + \frac{1}{n} \sum_{i=1}^{n} \ln(x_i)$$

$$\underline{\alpha = \psi^{-1}\left(-\ln(\alpha_0) + \ln(\bar{x}) - \frac{1}{n} \sum_{i=1}^{n} \ln(x_i)\right)}$$

c. $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp\left(-\frac{\beta}{x}\right)$

$$= \exp\left(-\frac{\beta}{x} + (-\alpha-1) \log x + \log \frac{\beta^\alpha}{\Gamma(\alpha)}\right)$$

Then it is an exponential family.

d. $f(x) = c\,x^{-2} \exp\left(-\frac{c}{x}\right)$

$$= \exp\left(-\frac{c}{x} - 2\log x + \log c\right)$$

$$f(y) = \exp(-cy + 2\log y + \log c)$$

is an exponential family.

e. $f(y \mid c_0) = \exp(-c_0 y + 2\log y + \log c_0)$

$$f(c \mid y) \propto f(y \mid c_0)\, f(y, \theta)$$

$$= \exp(-(c+c_0)y + 4\log y + \log(c+c_0))$$

CS539 hw1

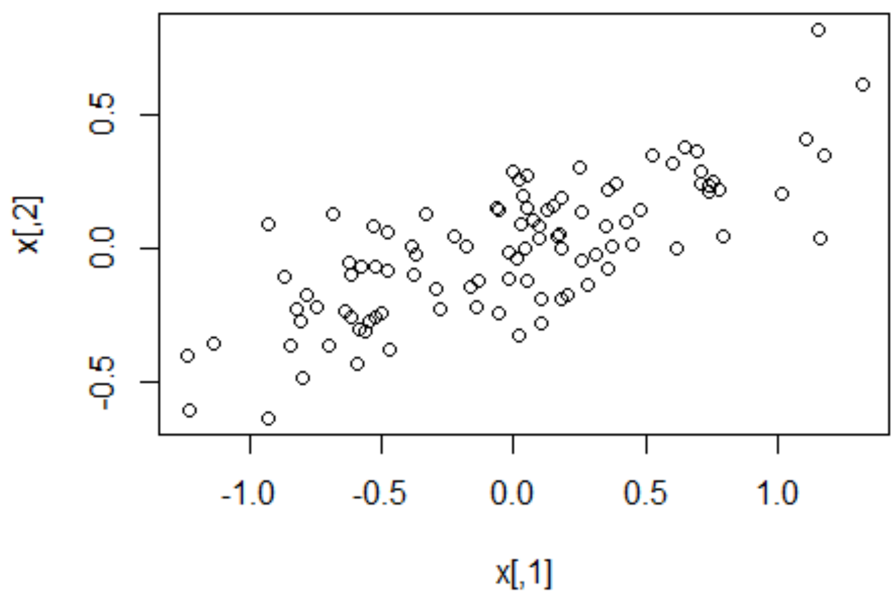
Enbo Tian

2022/1/24

problem 1

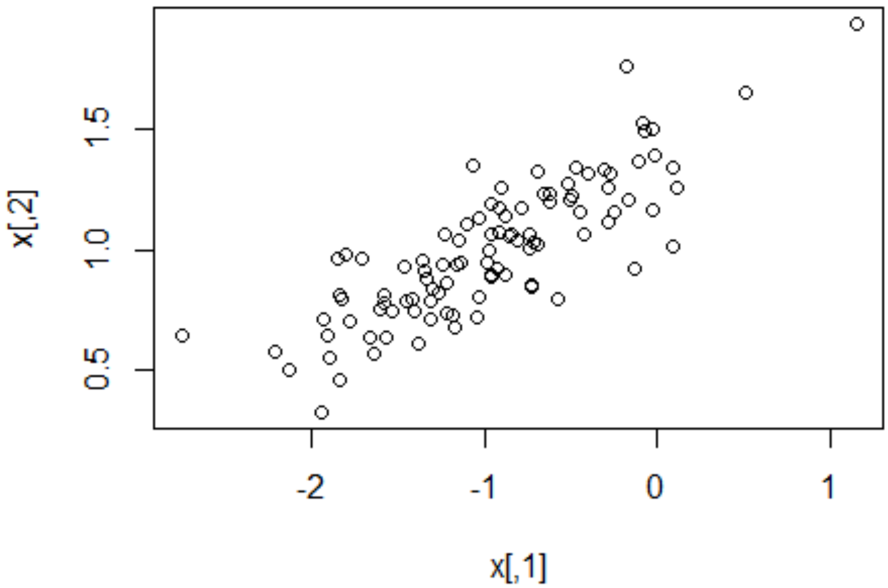
a)

```
X <- matrix(runif(2 * 2), 2, 2)
COV <- crossprod(X)
mu <- rep(0, 2)
library(MASS)
x <- mvrnorm(100, mu, COV)
plot(x)
```



b)

```
mu <- c(-1,1)
x <- mvrnorm(100, mu, COV)
plot(x)
```

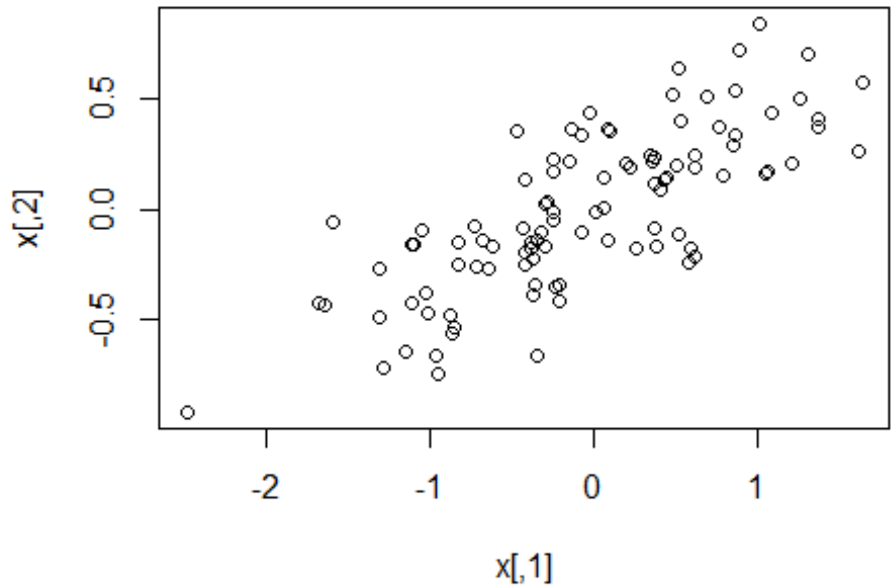



```
mu <- c(0,0)
```

The interval of x1 is moving left by about 1, and the interval of x2 is moving up by about 1.

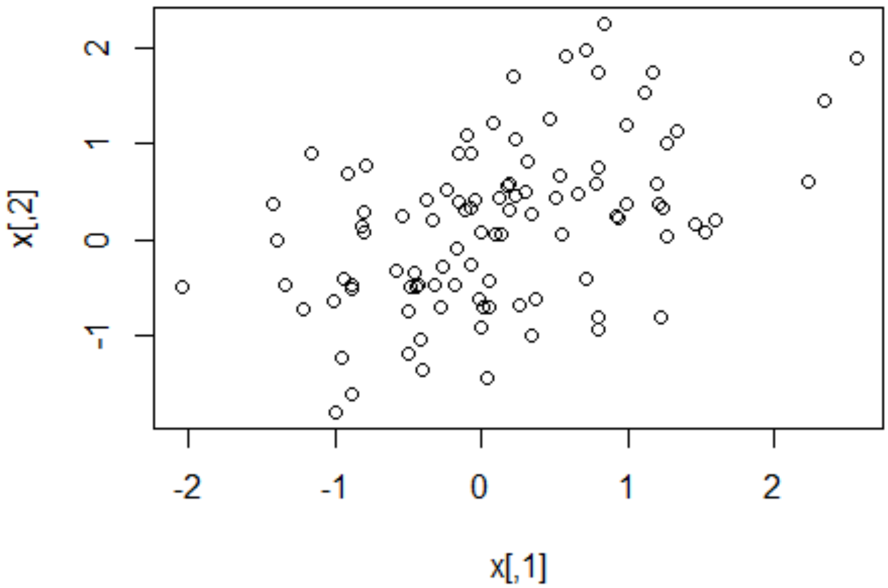
c)

```
COV <- 2*COV
x <- mvrnorm(100, mu, COV)
plot(x)
```



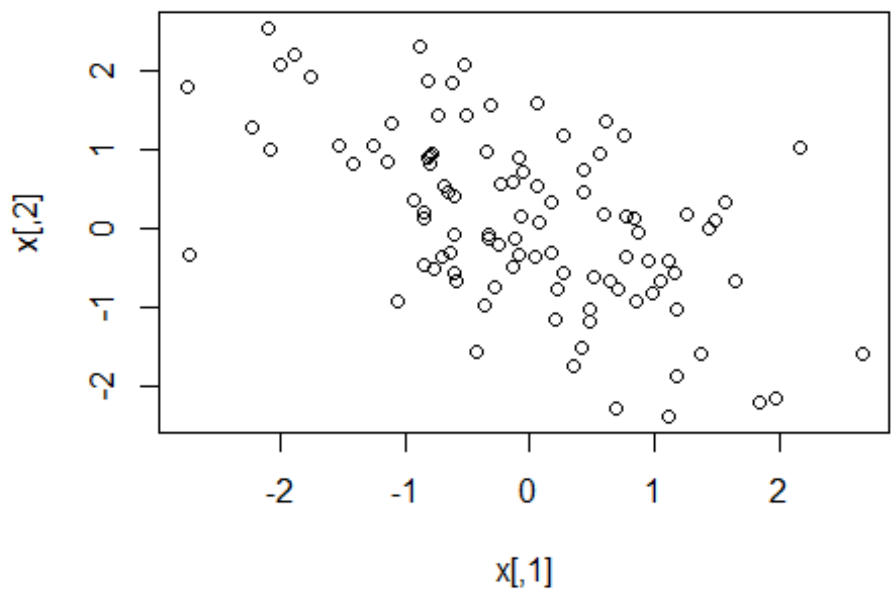
d)

```
COV <- matrix(c(1,0.5,0.5,1), nrow = 2, ncol = 2)
x <- mvrnorm(100, mu, COV)
plot(x)
```



e)

```
COV <- matrix(c(1,-0.5,-0.5,1), nrow = 2, ncol = 2)
x <- mvrnorm(100, mu, COV)
plot(x)
```

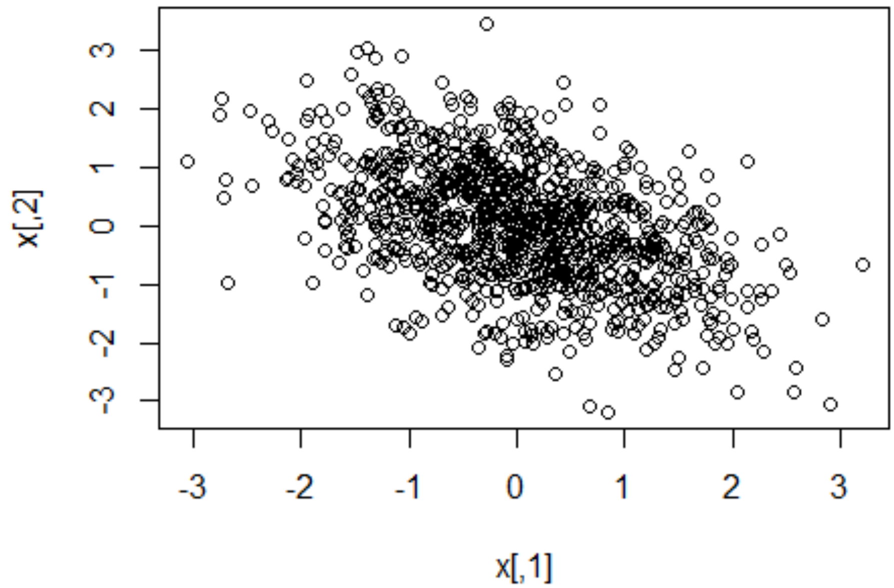


f)

```
X <- matrix(runif(2 * 2), 2, 2)
COV <- crossprod(X)
mu <- rep(0, 2)
x <- mvrnorm(100, mu, COV)
mean(x)

## [1] 0.003945048

cov(x)

##          [,1]     [,2]
## [1,] 1.910007 1.331396
## [2,] 1.331396 0.970151
```

g)

```
COV <- matrix(c(1,-0.5,-0.5,1), nrow = 2, ncol = 2)
x <- mvrnorm(1000, mu, COV)
plot(x)
```



h)

```
mean(x)

## [1] 0.01909849

cov(x)

##            [,1]      [,2]
## [1,]  0.9773492 -0.484074
## [2,] -0.4840740  1.036510
```

i)

```
x <- mvrnorm(10, mu, COV)
mean(x[,1])

## [1] 0.2690736

x <- mvrnorm(100, mu, COV)
mean(x[,1])

## [1] 0.01383696

x <- mvrnorm(1000, mu, COV)
mean(x[,1])

## [1] 0.01326269
```

Mean is tend to 0, as the more samples we have

j)

```r
COV # the initial covariance we use to get the sample
```

```
##      [,1] [,2]
## [1,]  1.0 -0.5
## [2,] -0.5  1.0
```

```r
x <- mvrnorm(10, mu, COV)
cov(x)
```

```
##           [,1]      [,2]
## [1,]  1.196858 -1.169756
## [2,] -1.169756  2.085406
```

```r
x <- mvrnorm(100, mu, COV)
cov(x)
```

```
##           [,1]       [,2]
## [1,]  1.008264 -0.5121600
## [2,] -0.512160  0.9278948
```

```r
x <- mvrnorm(1000, mu, COV)
cov(x)
```

```
##            [,1]       [,2]
## [1,]  0.9841772 -0.4980792
## [2,] -0.4980792  0.9746574
```

covariance is getting closer to the initial covariance, When we have more sample

problem 2

a)

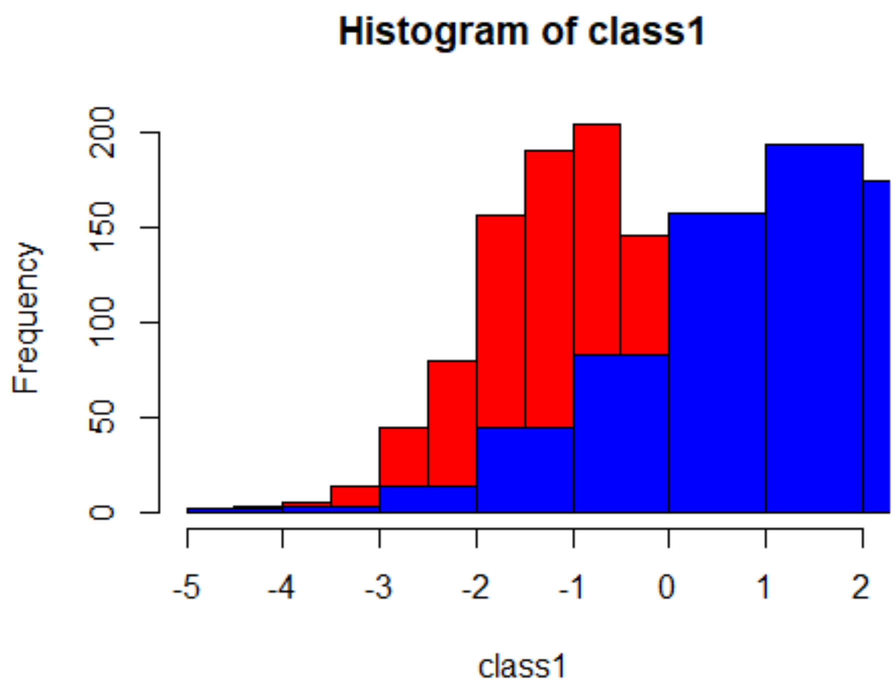
```r
class1 <- rnorm(1000,-1,1)
```

b)

```r
class2 <- rnorm(1000,2,2)
```

c)

```r
hist(class1, col='red')
hist(class2, col='blue', add=TRUE)
```



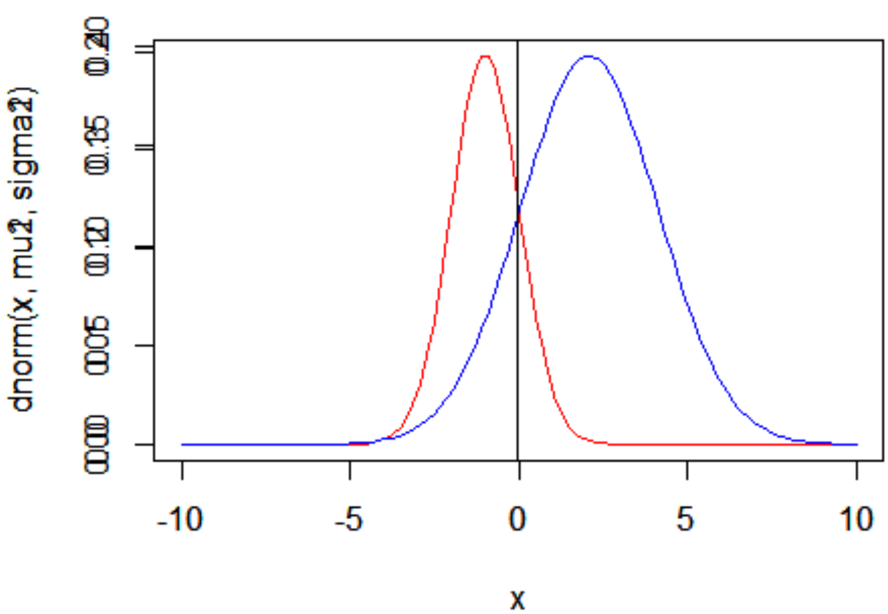
d)

```
library(stats4)
library(methods)
options(warn = -1)
LL1 <- function(mu,sigma){
  -sum(log(dnorm(class1,mu,sigma)))
}
m1<-mle(LL1,start = list(mu=0,sigma=1))
m1

##
## Call:
## mle(minuslogl = LL1, start = list(mu = 0, sigma = 1))
##
## Coefficients:
##         mu      sigma
## -0.9980152  1.0034457

LL2 <- function(mu,sigma){
  R = dnorm(class2,mu,sigma)
  -sum(log(R))
}
m2<-mle(LL2,start = list(mu=0,sigma=1))
m2

##
## Call:
```

```
## mle(minuslogl = LL2, start = list(mu = 0, sigma = 1))
##
## Coefficients:
##       mu    sigma
## 2.082832 2.043154

options(warn = getOption("warn"))
```

e)

```
mu1 <- m1@coef[1]
sigma1<- m1@coef[2]
mu2<-m2@coef[1]
sigma2<-m2@coef[2]

x <- seq(-10, 10, length=100)
plot(x,dnorm(x,mu1,sigma1), type = "l",col = "red")
par(new=TRUE)
plot(x,dnorm(x,mu2,sigma2), type = "l",col="blue")
i = -2
while(round(dnorm(i,mu1,sigma1),5)!=round(dnorm(i,mu2,sigma2),5)){
  i=i+0.00001
}
par(new=TRUE)
abline(v=-0.014)
```

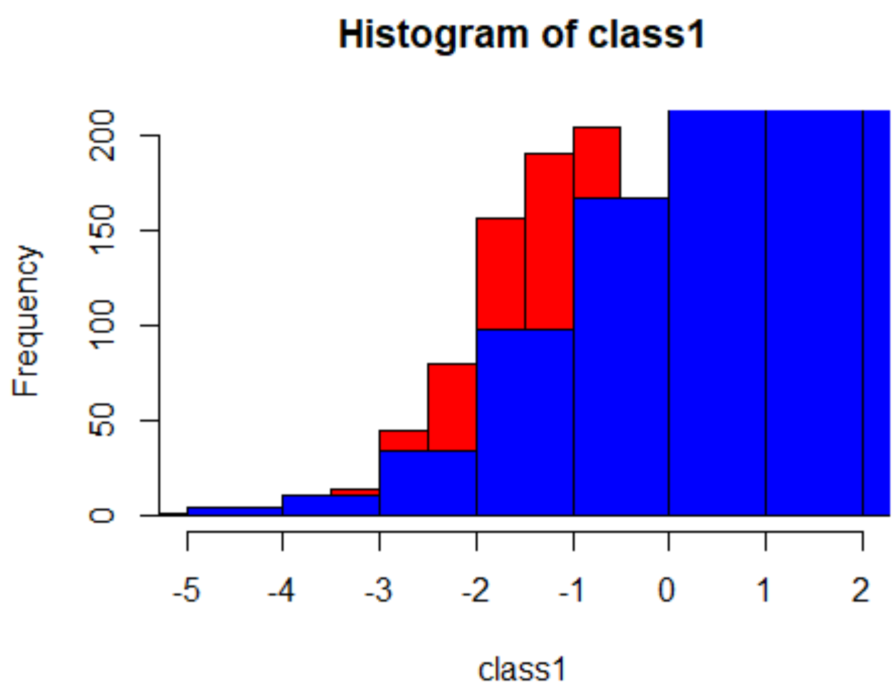


f)

both of the decision boundary of pdf and histogram are about 0

g)

```
class2 <- rnorm(2000,2,2)
#c
hist(class1, col='red')
hist(class2, col='blue', add=TRUE)
```

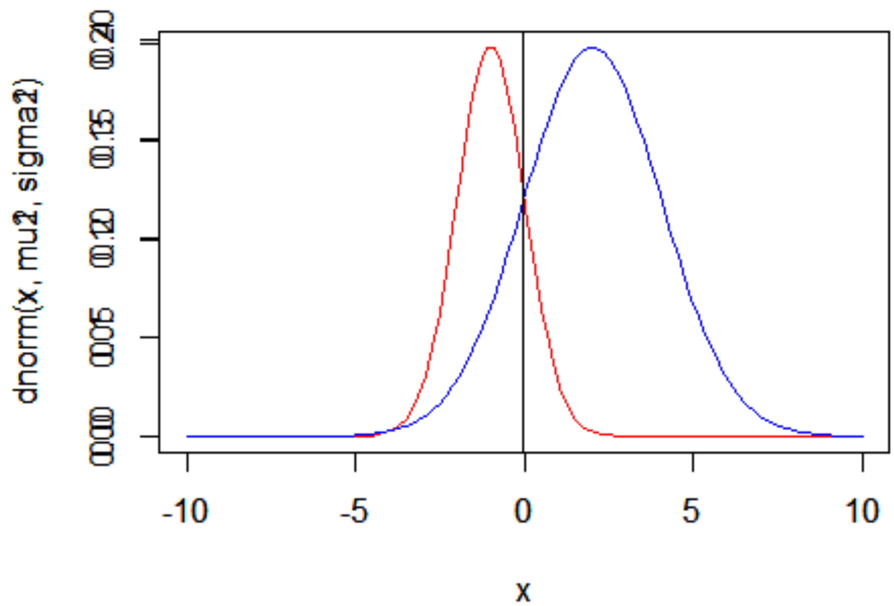



```
#d
options(warn = -1)
LL1 <- function(mu,sigma){
  -sum(log(dnorm(class1,mu,sigma)))
}
m1<-mle(LL1,start = list(mu=0,sigma=1))
m1

##
## Call:
## mle(minuslogl = LL1, start = list(mu = 0, sigma = 1))
##
## Coefficients:
##         mu      sigma
## -0.9980152  1.0034457
```

```r
LL2 <- function(mu,sigma){
  R = dnorm(class2,mu,sigma)
  -sum(log(R))
}
m2<-mle(LL2,start = list(mu=0,sigma=1))
m2
```

```
##
## Call:
## mle(minuslogl = LL2, start = list(mu = 0, sigma = 1))
##
## Coefficients:
##       mu    sigma
## 2.014727 2.036637
```

```r
options(warn = getOption("warn"))
#e
mu1 <- m1@coef[1]
sigma1<- m1@coef[2]
mu2<-m2@coef[1]
sigma2<-m2@coef[2]

x <- seq(-10, 10, length=100)
plot(x,dnorm(x,mu1,sigma1), type = "l",col = "red")
par(new=TRUE)
plot(x,dnorm(x,mu2,sigma2), type = "l",col="blue")
i = -2
while(round(dnorm(i,mu1,sigma1),5)!=round(dnorm(i,mu2,sigma2),5)){
  i=i+0.0001
}
par(new=TRUE)
abline(v=-0.016)
```




```
#f
```

Since there are more samples in class2, the decision boundary of histogram comes to -1, but the decision boundary of pdf does not change.

h)

```
library(MASS)
fitdistr(class1, densfun="normal")

##       mean           sd
##   -0.99793273    1.00348621
##  ( 0.03173302) ( 0.02243863)

class2 <- rnorm(1000,2,2)
fitdistr(class2, densfun="normal")

##       mean          sd
##   1.97822236   1.94124094
##  (0.06138743) (0.04340747)
```

the error rate are on the second line.

```
library(MASS)
fitdistr(class1, densfun="normal")

##       mean           sd
##   -0.99793273    1.00348621
##  ( 0.03173302) ( 0.02243863)
```

```r
class2 <- rnorm(2000,2,2)
fitdistr(class2, densfun="normal")
```

```
##       mean          sd
##   1.95616385   2.03949020
##  (0.04560439) (0.03224717)
```

i)

```r
options(warn = -1)
df <- data.frame(class1,class2)
library(pROC)
```

```
## Type 'citation("pROC")' for a citation.
```

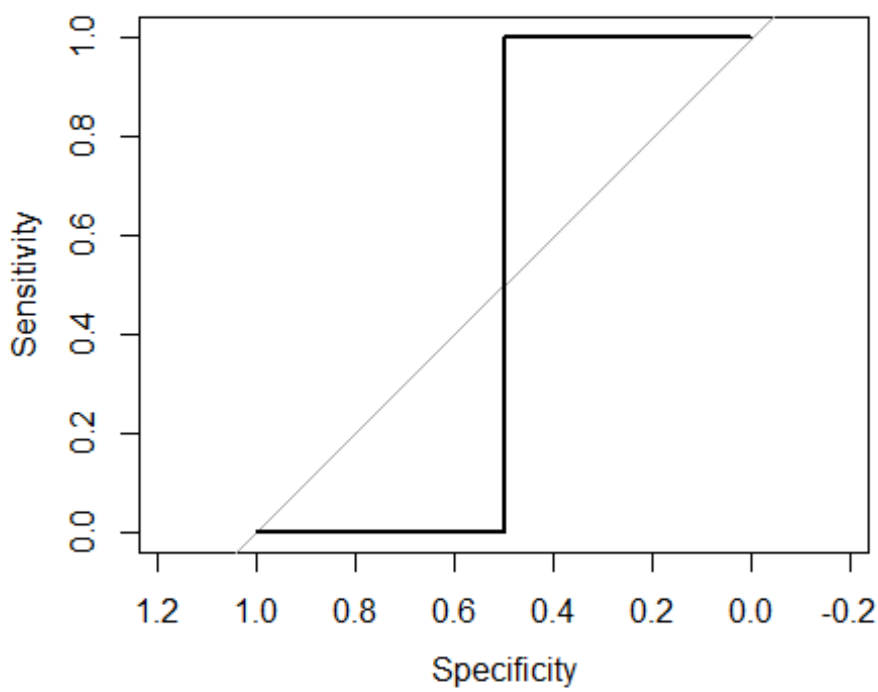
```
##
## 载入程辑包：'pROC'
```

```
## The following objects are masked from 'package:stats':
##
##     cov, smooth, var
```

```r
roc(df$class1,df$class2,plot=TRUE)
```

```
## Setting levels: control = -4.6552738797062, case = -4.31843317823687
```

```
## Setting direction: controls < cases
```



```
##
## Call:
## roc.default(response = df$class1, predictor = df$class2, plot = TRUE)
##
## Data: df$class2 in 2 controls (df$class1 -4.6552738797062) < 2 cases
 (df$class1 -4.31843317823687).
## Area under the curve: 0.5
```

```
options(warn = getOption("warn"))
```

problem 3

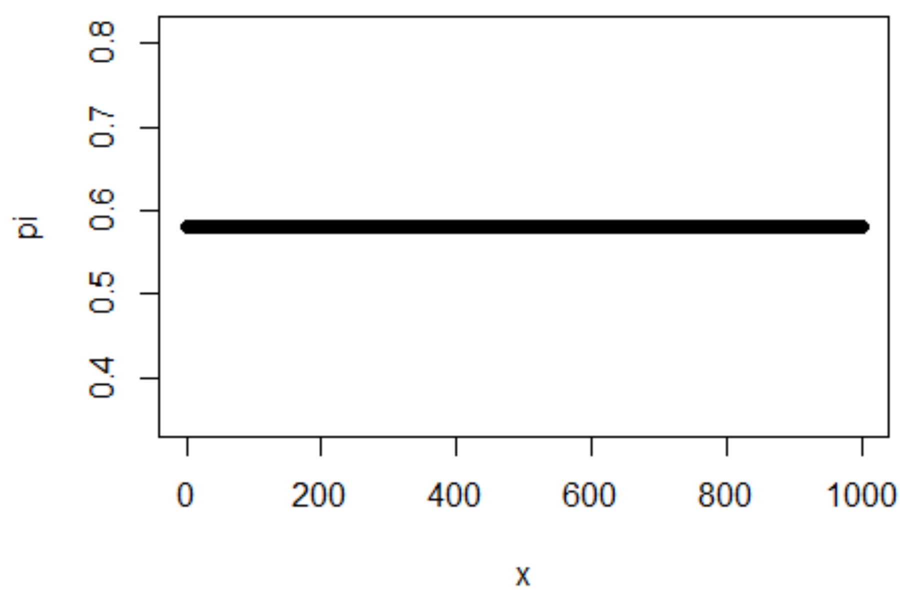
a)

```
library("Rlab")

## Rlab 2.15.1 attached.

##
## 载入程辑包：'Rlab'

## The following object is masked from 'package:MASS':
##
##     michelson

## The following objects are masked from 'package:stats':
##
##     dexp, dgamma, dweibull, pexp, pgamma, pweibull, qexp, qgamma,
##     qweibull, rexp, rgamma, rweibull

## The following object is masked from 'package:datasets':
##
##     precip

coin <- rbern(1000, 0.6)
```

##b)

```
options(warn = -1)
LL1 <- function(p){
  -sum(log(dbern(coin,p)))
}
m1<-mle(LL1,start = list(p=0.01))
m1@coef

##         p
## 0.5809974

LL <- function(n,p){
  -sum(log(dbern(rbern(n, 0.6),p)))
}
```

```r
for(n in 1:1000){
  ll <- function(p){
    LL(n,p)
  }
  m1<-mle(LL1,start = list(p=0.01))
  pi[n]<-m1@coef
}

x<-seq(0, 1000, length=1000)
plot(x,pi)
```

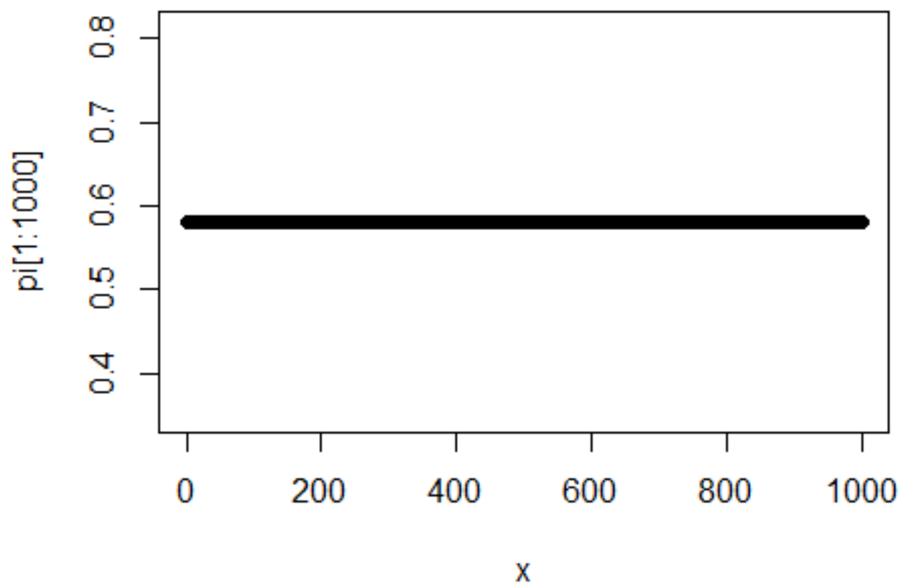



```r
options(warn = getOption("warn"))
```

c)

```r
coin2 <- rbern(1000, 0.6)
LL2 <- function(p){
  -sum(log(dbern(coin2,p)))
}
m2<-mle(LL2,start = list(p=0.01))
m2@coef

##         p
## 0.5999919

LL <- function(n,p){
  -sum(log(dbern(rbern(n, 0.6),p)))
}
```

```
for(n in 1:100){
  ll <- function(p){
    LL(n,p)
  }
  m1<-mle(LL1,start = list(p=0.01))
  pi[n]<-m1@coef
}
x<-seq(0, 1000, length=1000)
plot(x,pi[1:1000])
```

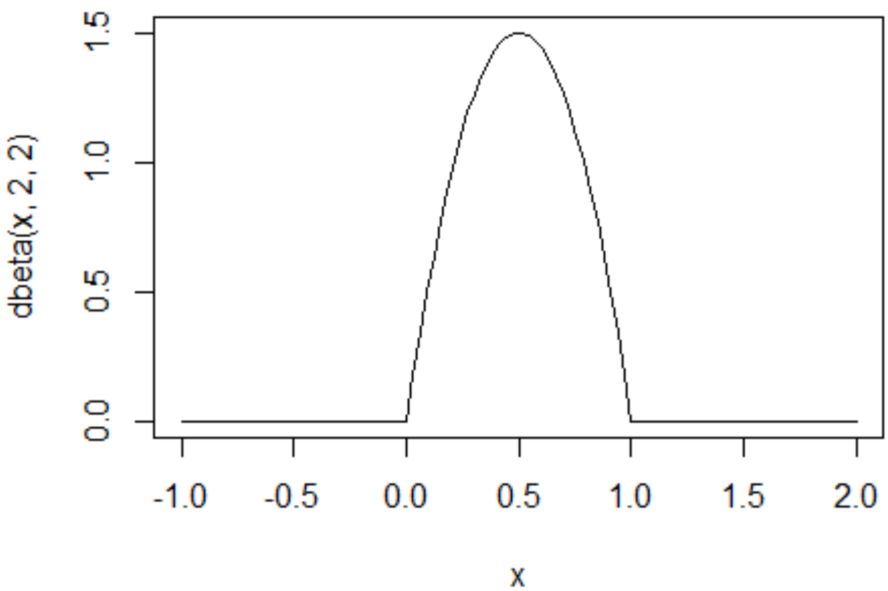



```
options(warn = getOption("warn"))
```

both P_{ML} from b) and c) are approximate and get close to 0.6,

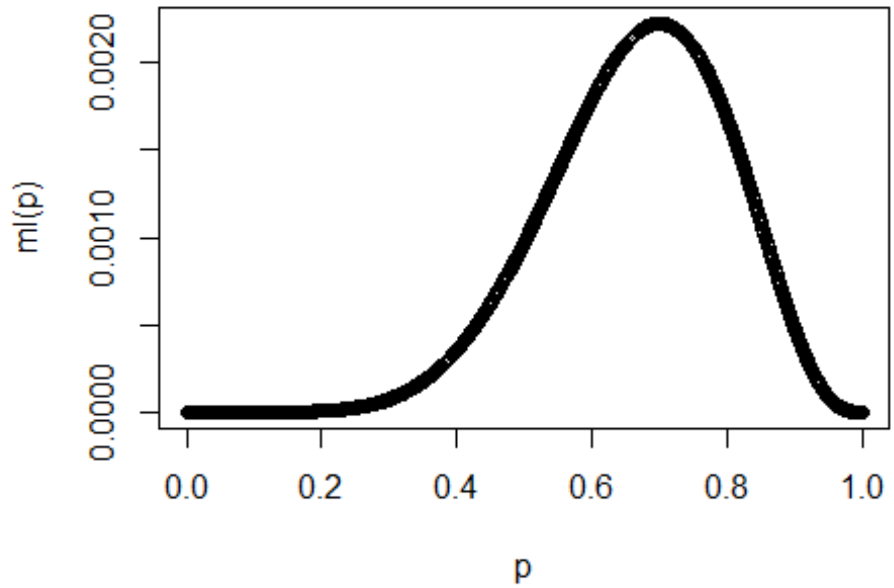
d)

```
x <- seq(-1, 2, length=100)
plot(x,dbeta(x, 2, 2), type = "l")
```



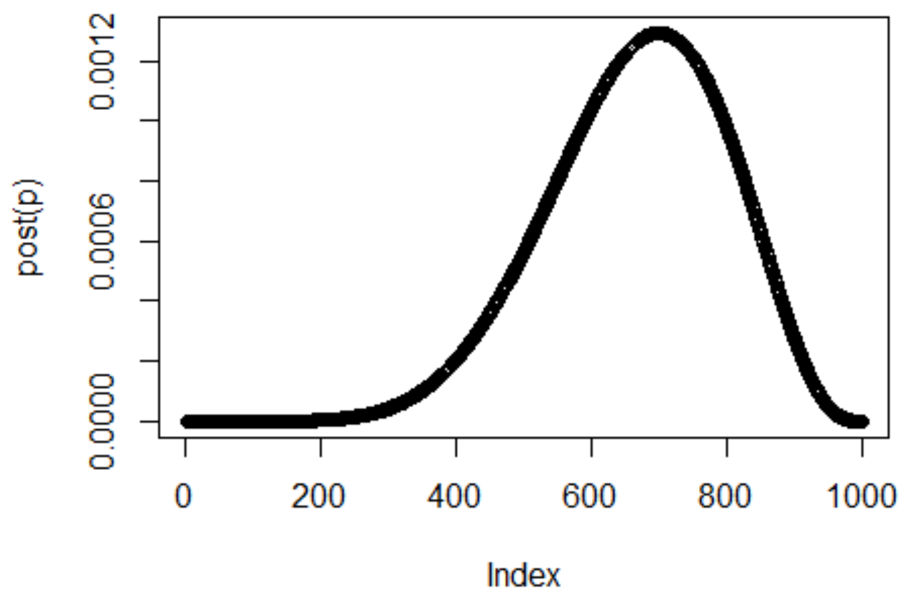
e)

```
p <- seq(0, 1, length=1000)
ml <- function(p){
  mult = 1
  for(i in 1:10){
    mult <- mult*p^coin[i]*(1-p)^(1-coin[i])
  }
  mult
}
plot(p,ml(p))
```



f)

```
post <- function(p){
  ml(p)*pi
}
plot(post(p))
```



Since the posterior is proportion to prior and likelihood, the curve is not change too much.

g)

```
max(post(p))

## [1] 0.001291883
```

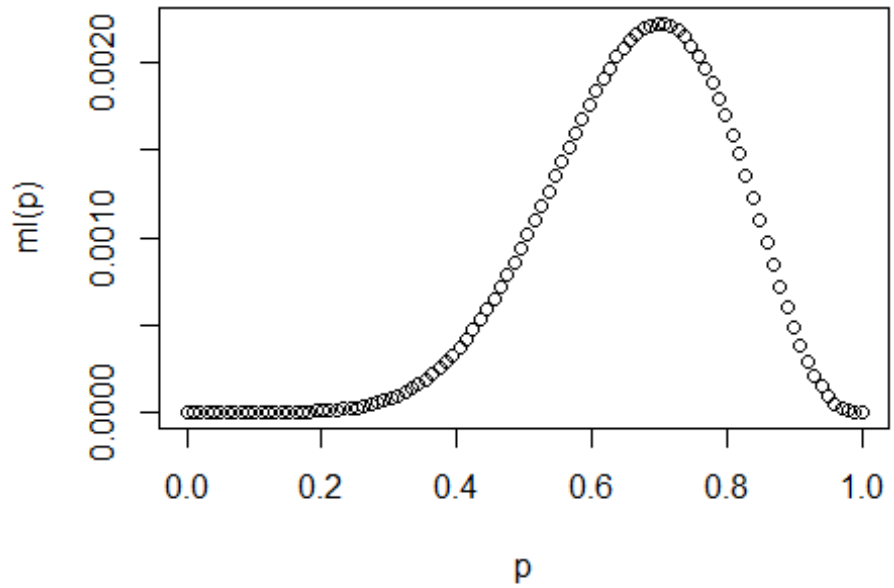
MAP is 6.915e-04

h)

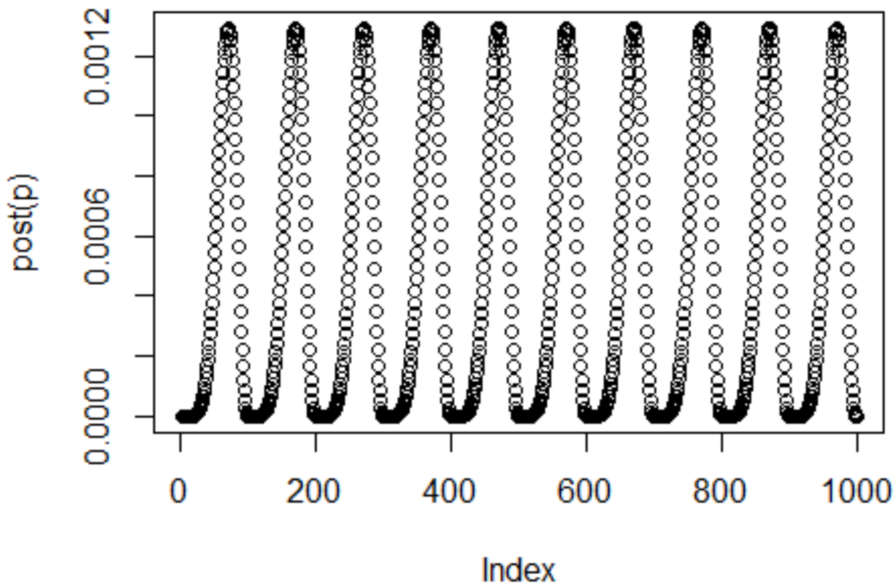
```
var(post(p))

## [1] 2.211729e-07
```

i)

```
p <- seq(0, 1, length=100)
ml <- function(p){
  mult = 1
  for(i in 1:10){
    mult <- mult*p^coin[i]*(1-p)^(1-coin[i])
  }
  mult
}
plot(p,ml(p))
```



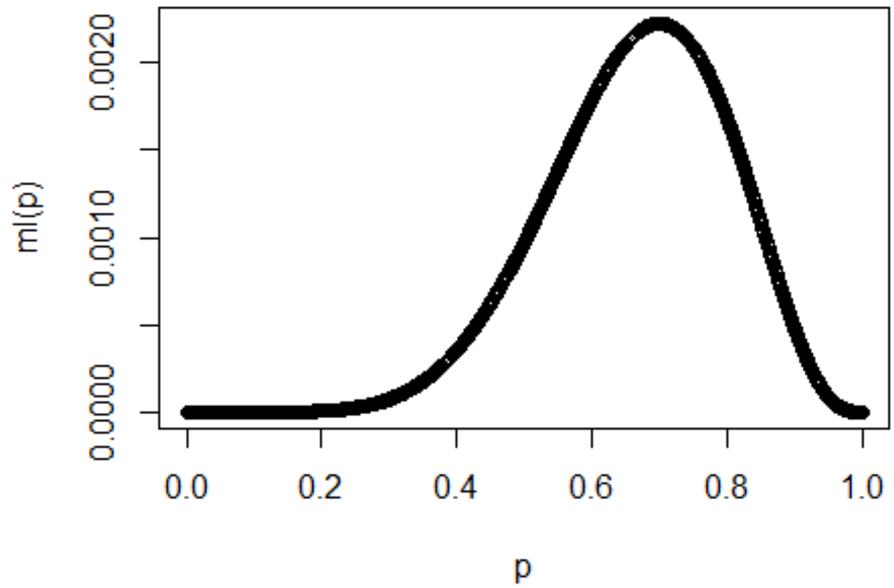

```
post <- function(p){
  ml(p)*pi
}
plot(post(p))
```



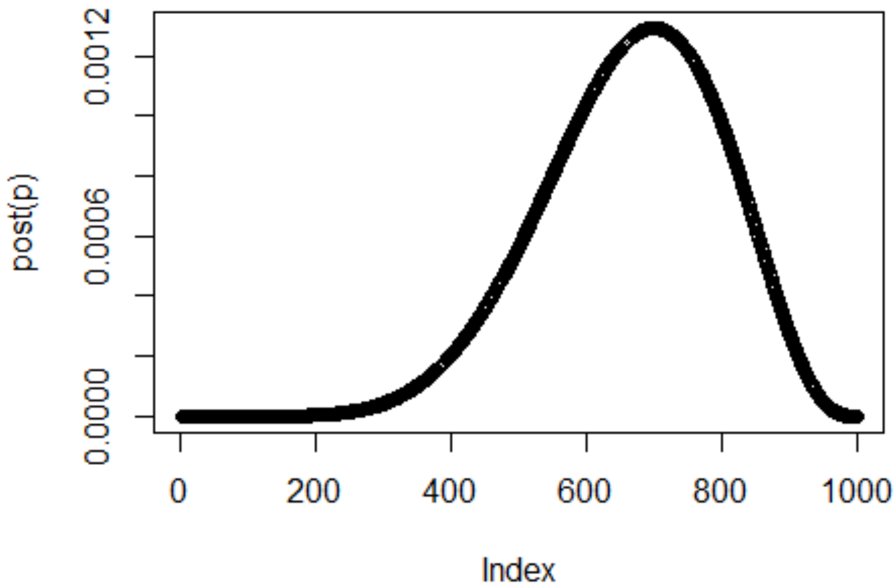

```
max(post(p))

## [1] 0.001291605

var(post(p))

## [1] 2.209082e-07
```

j)

```
p <- seq(0, 1, length=1000)
ml <- function(p){
  mult = 1
  for(i in 1:10){
    mult <- mult*p^coin[i]*(1-p)^(1-coin[i])
  }
  mult
}
plot(p,ml(p))
```




```
post <- function(p){
  ml(p)*pi
}
plot(post(p))
```




```
max(post(p))

## [1] 0.001291883

var(post(p))

## [1] 2.211729e-07
```

problem 4

a)

```
library(MASS)
pi1 <- 0.1
mu1 <- c(3, 2)
COV1 <- matrix(c(1,0,0,1), nrow = 2, ncol = 2)

pi2 <- 0.6
mu2 <- c(-5, -3)
COV2 <- matrix(c(2,-1,-1,3), nrow = 2, ncol = 2)

pi3<-0.3
COV3 <- matrix(c(6,3,3,3), nrow = 2, ncol = 2)
mu3 <- c(4, 2)
```

```
p <- pi1*mvrnorm(1000, mu1, COV1)+pi2*mvrnorm(1000, mu2, COV2)+pi3*mvrn
orm(1000, mu3, COV3)
```

b)

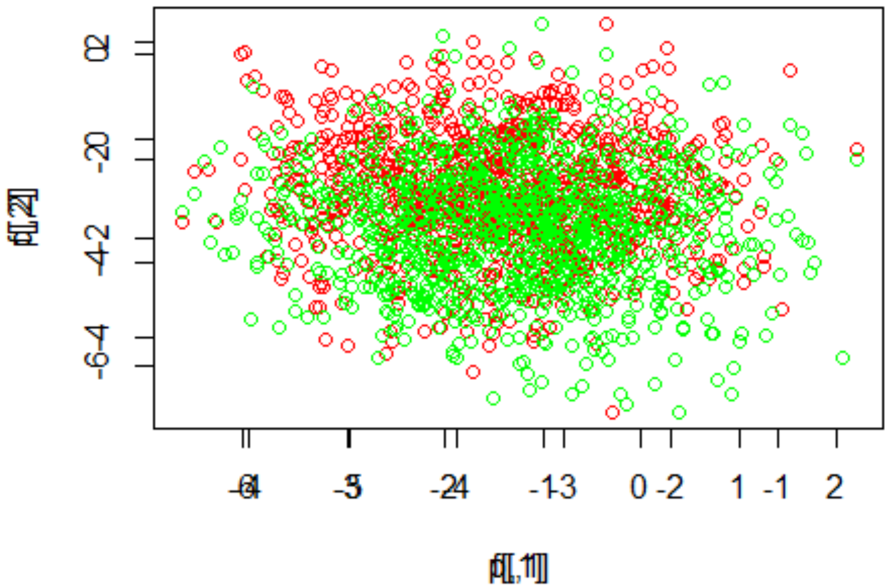
```
mean(p[1])

## [1] -3.523997

mean(p[2])

## [1] -3.288717

cov(p)

##             [,1]      [,2]
## [1,]  1.298926 -0.109365
## [2,] -0.109365  1.385695
```

c)

```
mu <- c(mean(p[1]),mean(p[2]))
COV <- cov(p)
f <- mvrnorm(1000, mu, COV)
```

d)

```
plot(p,col="red")
par(new=TRUE)
plot(f,col="green")
```



the mixture model have the same concentration area with multivariate normal distribution. The difference is that the mixture model have a more range.

e)

```
K3<-kmeans(p, centers = 3, nstart = 25)
str(K3)

## List of 9
##  $ cluster     : int [1:1000] 2 2 2 1 2 2 1 3 2 1 ...
##  $ centers     : num [1:3, 1:2] -2.239 -1.715 -0.166 -0.147 -2.185
 ...
##   ..- attr(*, "dimnames")=List of 2
##   .. ..$ : chr [1:3] "1" "2" "3"
##   .. ..$ : NULL
##  $ totss       : num 2682
##  $ withinss    : num [1:3] 466 422 324
##  $ tot.withinss: num 1212
##  $ betweenss   : num 1470
##  $ size        : int [1:3] 368 348 284
##  $ iter        : int 3
##  $ ifault      : int 0
##  - attr(*, "class")= chr "kmeans"
```