MA 550 – Homework 4

Due Wed, 11:30am, November 4, 2020

Homework Assignment Policy and Guidelines

(a) Homework assignment should be submitted electronically through `Canvas`, and your submission should be combined into ONE PDF file. Credit will not be given for homework turned in late.

(b) Homework assignments should be well organized. It is required that you show your work in order to receive credit.

(c) It's highly recommended to use R Markdown to write up your homework solutions, especially for problems that involve data analysis. But this is not required.

(d) When writing up solutions via R Markdown, please refrain from showing lengthy results/output for data analysis problems. Please display only relevant results.

(e) If you are not to use R Markdown, please keep the R code and/or R output for all the relevant problems in a *well organized appendix*, and please refrain from copying and pasting R code and/or R output directly into your answers.

(f) For problems that involve technical writing, strive to be clear, concise, and cogent. Organize figures and tables in an efficient manner while maintaining clarity.

(g) For problems that involve data analysis, always interpret the results in the context of the study. This may include comments on whether the results are meaningful or not. Think of your answers as a mini technical report of your findings and follow the guidelines on technical writing.

(h) You may discuss most homework problems with others including your peers and instructor, but you must write up your homework solutions by yourself in order to receive credit. Similarly, you must write your own computer code and obtain your computer output independently.

1. Consider the model equation for a process $\{Y_t\}$

$$(1 - 1.2B + 0.7B^2)Y_t = (1 + 0.7B)\epsilon_t, \quad \epsilon_t \sim WN(0, 6).$$

(a) Verify that $\{Y_t\}$ is a causal and invertible ARMA(2,1).

(b) Compute the coefficients $\psi_1, \psi_2, \psi_3, \psi_4$ in the infinite moving average form for the process Y_t

$$Y_t = \epsilon_t + \psi_1 \epsilon_{t-1} + \psi_2 \epsilon_{t-2} + \cdots$$

(c) Compute the coefficients $\pi_1, \pi_2, \pi_3, \pi_4$ in the infinite autoregressive form for the process Y_t

$$Y_t = \sum_{j=1}^{\infty} \pi_j Y_{t-j} + \epsilon_t$$

(d) State the generalized Yule-Walker equations satisfied by ACF $\rho(k)$ of the stationary process Y_t.

(e) Determine the values of $\gamma(0) = Var(Y_t)$ and $\rho(1), \rho(2), \rho(3)$ for $\{Y_t\}$.

(f) Determine the first three values $\phi_{11}, \phi_{22}, \phi_{33}$ of the PACF of $\{Y_t\}$.

(g) Verify your results in (e) and (f) using `R` function `ARMAacf`.

(h) Give the explicit form of the forecast function $\hat{Y}_t(l)$, as a function of the lead time l, for this model.

(i) ~~Compute the spectral density function $f(\lambda)$ of Y_t. Make a sketch of $f(\lambda)$ versus λ. Find the frequency at which its spectral density achieves its maximum value.~~

2. Consider Slides #11, page 5. Please verify that $\sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+k} z^k = \psi(z)\psi(z^{-1})$.

3. Let $\{Y_t, t \in \mathbb{Z}\}$ be the stationary solution of the non-causal and non-invertible ARMA(1,1) equation

$$(1 - 2B)Y_t = (1 - 1.25B)\epsilon_t, \quad \epsilon_t \sim WN(0, \sigma^2).$$

Show that $\{Y_t, t \in \mathbb{Z}\}$ also satisfies a causal and invertible ARMA(1,1) equation:

$$(1 - .5B)Y_t = (1 - .8B)\epsilon_t^*, \quad \epsilon_t^* \sim WN(0, \tilde{\sigma}^2),$$

for a suitable chosen white noise process ϵ_t^*. Determine $\tilde{\sigma}^2$.

4. Consider Slides #12, page 37. Verify that, for $h > 0$, the best linear backcast of Y_{1-h} in terms of $\{Y_1, \ldots, Y_t\}$ is $\hat{Y}_{1-h} = \beta_{t,1}Y_1 + \ldots + \beta_{t,t}Y_t$ where $\boldsymbol{\beta}_t = (\beta_{t,1}, \ldots, \beta_{t,t})'$ satisfies $\Gamma_t \boldsymbol{\beta}_t = \gamma_t^{(h)}$.

5. Continue to work on Problem 4 in Homework 2.

(a) Find the best linear prediction of Y_3 in terms of Y_2 and Y_1. That is, by supposing that $\hat{Y}_2(1) = \beta_1 Y_2 + \beta_2 Y_1$, find $\boldsymbol{\beta} = (\beta_1, \beta_2)'$ that satisfies the prediction equation:

$$\langle Y_3 - \hat{Y}_2(1), Y_k \rangle = 0, \quad k = 1, 2. \tag{1}$$

(b) Evaluate the mean squared error $E(Y_3 - \hat{Y}_2(1))^2$.

6. Suppose a series of T observations follows the AR(2) model

$$Y_t = 5 + 1.2Y_{t-1} - 0.7Y_{t-2} + \epsilon_t$$

and the last 3 values of the series are $Y_T = 14.4$, $Y_{T-1} = 13.9$, and $Y_{T-2} = 14.2$.

(a) Compute forecasts $\hat{Y}_T(l)$ for $l = 1, 2, 3, 4$ for the next 4 time periods $T+1, T+2, T+3, T+4$.

(b) Give the explicit form of the forecast function $\hat{Y}_T(l)$, as a function of the lead time l, for this model.