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Assignment #1 - Exercise 1

Macro Exercise 1

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EXERCISE 1

$$U = \ln(C_1) + \ln(C_2) + \ln(1-L)$$

POINT A

period 1 B.C. $\rightarrow$ $\boxed{C_1 + S_1 = wL}$

period 2 B.C. $\rightarrow$ $C_2 + S_2 = (1+R)S_1$

Terminal condition $S_2 = 0$

$$\boxed{C_2 = (1+R)S_1}$$

Express $S_1 = \frac{C_2}{1+R}$ and plug it into period 1 BC:

$$\boxed{C_1 + \frac{C_2}{1+R} = wL}$$

LIFETIME IBC

The present discounted value of the stream of consumption (LHS) must equal the present discounted value of the stream of income (RHS).

POINT B

It's useful to express C2 as ==> $C_2 = (wL - C_1)(1+R)$ and plug it into the utility function and then maximize with respect to the variables we're interested in.

$$U = \ln(C_1) + \ln\left((wL - C_1)(1+R)\right) + \ln(1-L)$$

$$\frac{\partial U}{\partial C_1} = \frac{1}{C_1} - \frac{1}{wL - C_1} = 0$$

Solving for C_1 it gets $\boxed{C_1 = \frac{1}{2} wL}$

Solving for c_1 it gets $\boxed{c_1 = \frac{1}{2} wL}$

Plugging this result into the expression of c_2

$$c_2 = \left(wL - \frac{1}{2} wL\right)(1+r)$$

$$\boxed{c_2^* = \frac{1}{2} wL(1+r)}$$

$$\frac{\partial U}{\partial L} = \frac{w}{wL - c_1} - \frac{1}{1-L} = 0$$

Solving for L it gets $\boxed{L = \frac{w + c_1}{2w}}$

Plug the value of c_1 into the expression for L

$$L = \frac{w}{2w} + \frac{1}{2} wL \cdot \frac{1}{2w}$$

Solving for L, $\boxed{L^* = \frac{2}{3}}$

Use $L^* = \frac{2}{3}$ to find c_1^*, c_2^* as follows

$$c_1^* = \frac{1}{2} w \frac{2}{3} \rightarrow \boxed{c_1^* = \frac{1}{3} w}$$

$$c_2^* = \frac{1}{2} w \frac{2}{3} (1+r) \rightarrow \boxed{c_2^* = \frac{1}{3} w(1+r)}$$

COMMENT ON THE FINDINGS OF POINT B:

In total the individual earns 2/3w, since L=2/3 and the unitary salary is w. Thus the consumption in period 1 is the exact half of the overall income. This allows us to understand that the individual perfectly chooses the half of the income for current consumption, keeping the remaining for the future.

Consumption in period 2 is given by the resources saved in period 1, plus the "appreciation" coming from the interest rate.

At last, labor supply is constant. This means that it doesn't depend on the level of wage.

These conclusions is a consequence of the structure of the utility function, which gives equal weight to consumption1, consumption2 and leisure.

POINT C

period 1 BC $\rightarrow$ $\boxed{C_1 + S_1 = wL - T}$

period 2 BC $\rightarrow$ $C_2 + S_2 = (1+R)S_1 + T$

TERMINAL CONDITION $S_2 = 0$ $\boxed{C_2 = (1+R)S_1 + T}$

Express $S_1 = \dfrac{C_2 - T}{1+R}$ and find the IBC

$$\boxed{C_1 + \frac{C_2}{1+R} = wL + \frac{T}{1+R} - T}$$

"NEW" IBC

The present discounted value of the stream of consumption (LHS) must equal the present discounted value of the stream of income (RHS), which is given by the working income + the (actualized)pension - the tax paid nowadays.

To study the impact of the tax, it is necessary to study agent's behavior with respect to consumption (both periods) and labor.

This time it will be done by the Lagrangian method. (I tried the substitution method, but in this case the Lagrangian one was less cumbersome)

From "NEW" IBC, retrieve the constraint as

$$\boxed{C_2 = (wL - C_1 - T)(1+R) + T}$$

The Lagrangian Function will be:

$$\mathcal{L} = \ln(C_1) + \ln(C_2) + \ln(1-L) - \lambda\left[(1+R)(wL - C_1 - T) + T - C_2\right]$$

$$\frac{\partial \mathcal{L}}{\partial C_1} = \frac{1}{C_1} + \lambda(1+R) = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial C_2} = \frac{1}{C_2} + \lambda = 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial L} = -\frac{1}{1-L} - \lambda(1+R)w = 0 \qquad (3)$$

From (2) express $\boxed{-\lambda = \frac{1}{C_2}}$

Plug $-\lambda$ into (1) & (3) and find:

(1) $$\frac{1}{C_1} = -\lambda(1+R)$$

$$\frac{1}{C_1} = \frac{1}{C_2}(1+R) \longrightarrow \boxed{C_1 = \frac{C_2}{1+R}}$$

(3) $$\frac{1}{1-L} = -\lambda(1+R)w$$

$$\frac{1}{1-L} = \frac{1}{C_2}(1+R)w$$

$$1-L = \frac{C_2}{(1+R)w} \longrightarrow \boxed{L = 1 - \frac{C_2}{(1+R)w}}$$

Since C1 and L are expressed in terms of C2, we can plug them into the expression of C2.

$$C_2 = (wL - C_1 - T)(1+R) + T$$

$$c_2 = wL(1+R) - c_1(1+R) - T(1+R) + T$$

$$c_2 = w\left(1 - \frac{c_2}{(1+R)w}\right)(1+R) - \frac{c_2}{\cancel{1+R}}\cancel{(1+R)} - T(1+R) + T$$

$$c_2 = w(1+R) - w\cancel{(1+R)}\frac{c_2}{\cancel{(1+R)}w} - c_2 - \cancel{T} - TR + \cancel{T}$$

$$C_2 = w(1+R) - c_2 - c_2 - TR$$

$$\boxed{C_2^* = \frac{w(1+R) - TR}{3}}$$

Exploit c_2 into 1 & 3

1 $$C_1 = \frac{c_2}{1+R}$$

$$C_1 = \frac{1}{1+R} \cdot \left(\frac{w(1+R) - TR}{3}\right)$$

$$C_1 = \frac{1}{\cancel{1+R}} \frac{w\cancel{(1+R)}}{3} - \frac{1}{1+R} \cdot \frac{TR}{3}$$

$$c_1^* = \frac{1}{3}w - \frac{TR}{3(1+R)}$$

$$\boxed{C_1^* = \frac{1}{3}\left(w - \frac{TR}{1+R}\right)}$$

3 $$L = 1 - \frac{c_2}{(1+R)w}$$

$$L = 1 - \frac{1}{(1+R)w} \cdot \left(\frac{w(1+R) - TR}{3}\right)$$

$$L = 1 - \frac{1}{(1+R)w} \cdot \left(\frac{w(1+R) - TR}{3} \right)$$

$$L = 1 - \frac{1}{\cancel{(1+R)w}} \cdot \frac{\cancel{w(1+R)}}{3} + \frac{1}{(1+R)w} \cdot \frac{TR}{3}$$

$$L = 1 - \frac{1}{3} + \frac{TR}{3(1+R)w}$$

$$\boxed{L^* = \frac{2}{3} + \frac{1}{3} \frac{TR}{(1+R)w}}$$

COMMENT ON FINDINGS OF POINT C:

- Labor supply is not constant now. Moreover, for fixed values of R and w, it can be higher than in the scenario without taxation, since after the first element (2/3) there is a second element which is determined by T, R and w.

  Holding other factors constant:
  The higher the tax T, the higher will be the labor supply (up to physical limits, of course). This represents the fact that the worker would like to compensate for the "loss" caused by the tax.
  In this context, a higher wage w faster compensates for the "loss" of the tax, for this reason the equation suggests that when w increases, the second element of the labor supply function will decrease (ceteris paribus).

- As regards consumption in period 1, in this taxation scenario it is lower because the tax reduces the available income in the "working age".
  Another difference is that in this scenario the interest rate R plays a role, while in the simpler scenario at point B it was not present in the expression for C1.

- As regards consumption in period 2, TR is negative. This represents the "missing gain" the individual would have realized in period2 by saving extra income in period1 and investing it at rate R, in the case the tax wouldn't have existed.

Question: What about the old when the pension is introduced? Give intuition for your answers.
Since it's not clear (to me) how to interpret this sentence, I'll provide two scenarios.

Scenario 1:
When the pension is intrduced, old man won't receive anything. The tax paid by the young will be enjoyed by the same young when he will be old.

Scenario 2:
when the pension is introduced, the old enjoys the tax paid by the young and this is extra income because he didn't pay the tax during his working age. The young will receive his pension in the future and it will be funded by some other young that will work and pay the tax. This scenario may lead to unsustainability of the system if the n° of young (workers) becomes lower than the n° of old

POINT D

The pension is now funded by an income tax
Will the behaviour of the young change in the case where R = 0?

period 1 BC $\longrightarrow$ $C_1 + S_1 = wL - \tau wL$

$$\boxed{C_1 + S_1 = wL(1-\tau)}$$

period 2 BC $\longrightarrow$ $C_2 + S_2 = (1+R)S_1 + T$

TERMINAL CONDITION : $S_2 = 0$

$$\boxed{C_2 = (1+R)\, S_1 + \tau wL}$$

Express $S_1 = \dfrac{C_2 - \tau wL}{1+R}$ and build the IBC

$$C_1 + \frac{C_2 - \tau wL}{1+R} = wL(1-\tau)$$

$$\boxed{C_1 + \frac{C_2}{1+R} = wL(1-\tau) + \frac{\tau wL}{1+R}}$$

The present discounted value of the stream of consumption (LHS) must equal the present discounted value of the stream of income (RHS), which is given by the net working income + the (actualized) pension.

From the NEW IBC, express C_2 as

$$\boxed{C_2 = \left[wL(1-\tau) - C_1\right](1+R) + \tau wL}$$

The Lagrangian Function is :

$$\mathcal{L} = \ln(C_1) + \ln(C_2) + \ln(1-L)$$
$$- \lambda \left[wL(1-\tau)(1+R) - C_1(1+R) + \tau wL - C_2 \right]$$

Proceed as before:

$$\frac{\partial \mathcal{L}}{\partial C_1} = \frac{1}{C_1} + \lambda(1+R) = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{C_2} = \frac{1}{C_2} + \lambda = 0 \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial L} = -\frac{1}{1-L} - \lambda\left[W(1+R)(1-\tau) + \tau W\right] = 0$$

$$-\frac{1}{1-L} - \lambda\left[W(1+R-R\tau\right] = 0 \quad (3)$$

From (2), write $-\lambda = \frac{1}{C_2}$ and use it into the other F.O.C.s

$$(1) \quad \frac{1}{C_1} = -\lambda(1+R)$$

$$\frac{1}{C_1} = \frac{1}{C_2}(1+R) \Rightarrow \boxed{C_1 = \frac{C_2}{1+R}}$$

$$(3) \quad \frac{1}{1-L} = -\lambda\left[W(1+R-\tau R\right]$$

$$\frac{1}{1-L} = \frac{1}{C_2}\left[W(1+R-\tau R\right]$$

$$1-L = \frac{1}{W(1+R-\tau R)} \cdot C_2$$

$$\boxed{L = 1 - \frac{C_2}{W(1+R-\tau R)}}$$

Since C1 and L are expressed in terms of C2, we can plug them into the expression of C2.

$$C_2 = \left[wL(1-\tau) - c_1 \right](1+R) + \tau wL$$

$$C_2 = wL(1-\tau)(1+R) - c_1(1+R) + \tau wL$$

$$C_2 = w(1-\tau)(1+R)\left(1 - \frac{c_2}{w(1+R-\tau R)}\right) - \frac{c_2}{\cancel{1+R}} \cdot \cancel{1+R} + \tau wL$$

Solving for c_2 ...

$$\boxed{C_2^* = \frac{1}{3} w(1+R-\tau R)}$$

Exploit C_2^* into 1 & 3

1 $c_1 = \dfrac{c_2}{1+R}$

$$C_1 = \frac{1}{1+R}\left[\frac{1}{3} w(1+R-\tau R)\right]$$

Solving for C_1 ...

$$\boxed{c_1^* = \frac{1}{3}\left(w - \frac{w\tau R}{1+R}\right)}$$

3

$$L = 1 - \frac{C_2}{w(1+R-\tau R)}$$

$$L = 1 - \frac{1}{3} \cancel{w}\cancel{(1+R-\tau R)} \cdot \frac{1}{\cancel{w}\cancel{(1+R-\tau R)}}$$

$$L = 1 - \frac{1}{3}$$

$$\boxed{L^* = \frac{2}{3}}$$

COMMENT ON FINDINGS OF POINT D:

In this point the tax was no longer a lump-sum tax: it was a distortionary tax.

Rather surprisingly, the result suggests that labor supply is constant (as in the case of no taxation). Probably this arises from the fact that the worker incorporates the tax in his choices about labor supply.

Question: Will the behaviour of the young change in the case where R = 0?
Talking about consumption, an element needs to be analyzed: $w\tau r$, which represents the potential decrease in the overall income due to the fact that the worker pays the tax, and thus a portion of the income is transferred from period 1 (tax) to period 2 (pension), but on this quantity doesn't produce interests (same discussion as the one related to TR in point C). But when the interest rate R=0, then this element is null! As a result, when R=0 consumption is perfectly smoothed across the two periods (C1=C2).