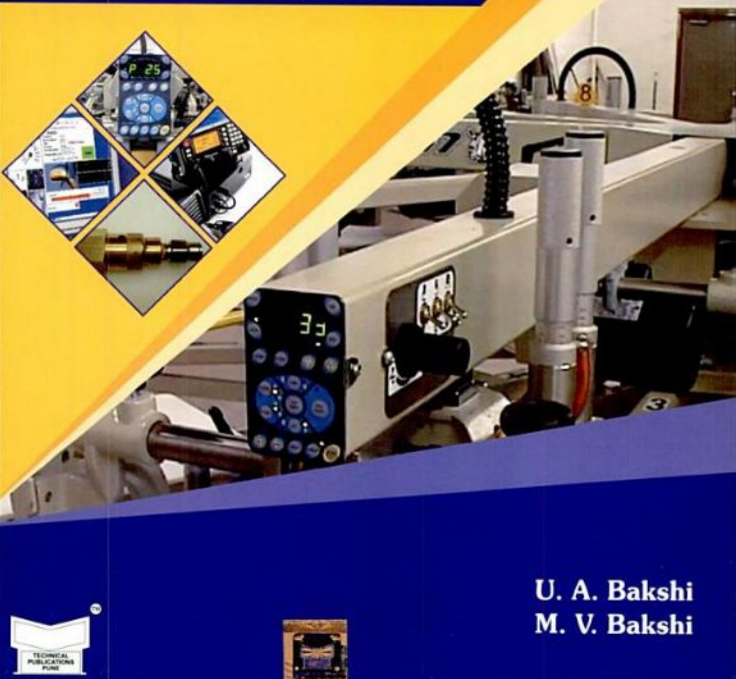
First Edition - 2008

Modern Control Theory



U. A. Bakshi
M. V. Bakshi

Technical Publications Pune™

Table of Contents :

Features of Book

- Use of clear, plain and lucid language making the understanding very easy.
- Excellent theory well supported with the practical examples and illustrations.
- Important concepts are highlighted using key points throughout the book.
- Large number of solved examples.
- Approach of the book resembles classroom teaching.
- Use of informative, self explanatory diagrams, plots and graphs.
- Book provides detailed insight into the subject.
- Stepwise explanation to mathematical derivations for easier understanding.

Modern Control Theory

ISBN 9788184315066

Published by :

Technical Publications Pune®

#1, Amit Residency, 412, Shaniwar Peth, Pune - 411 030, India.

Printer :

Alert DTPrinters
Sr.no. 10/3,Sinhagad Road,
Pune - 411 041

Preface

The importance of **Control Theory** is well known in various engineering fields. Overwhelming response to our books on various subjects inspired us to write this book. The book is structured to cover the key aspects of the subject **Modern Control Theory.**

The book uses plain, lucid language to explain fundamentals of this subject. The book provides logical method of explaining various complicated concepts and stepwise methods to explain the important topics. Each chapter is well supported with necessary illustrations, practical examples and solved problems. All the chapters in the book are arranged in a proper sequence that permits each topic to build upon earlier studies. All care has been taken to make students comfortable in understanding the basic concepts of the subject.

The book not only covers the entire scope of the subject but explains the philosophy of the subject. This makes the understanding of this subject more clear and makes it more interesting. The book will be very useful not only to the students but also to the subject teachers. The students have to omit nothing and possibly have to cover nothing more.

We wish to express our profound thanks to all those who helped in making this book a reality. Much needed moral support and encouragement is provided on numerous occasions by our whole family. We wish to thank the **Publisher** and the entire team of **Technical Publications** who have taken immense pain to get this book in time with quality printing.

Any suggestion for the improvement of the book will be acknowledged and well appreciated.

Authors

U. A. Bakshi

M. V. Bakshi

Dedicated to Varsha, Pradnya and Gururaj

Table of Contents

Properties of Multiplication

1. Commutative law is not valid for the matrix multiplication.

$$AB \neq BA$$

2. The transpose of a product is the product of transposes of individual matrices in the reverse order.

$$(AB)^T = B^T A^T$$

Example 3.3 : *Obtain the matrix multiplication of,*

$$A = \begin{bmatrix} 3 & -1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

Solution : A has order 3 × 2 and B has order 2 × 3 hence C = A × B has order 3 × 3.

$$\therefore \quad C = A \times B = \begin{bmatrix} 3\times1-1\times2 & 3\times0-1\times1 & 3\times-1-1\times0 \\ 0\times1+1\times2 & 0\times0+1\times1 & 0\times-1+1\times0 \\ 2\times1+0\times2 & 2\times0+0\times1 & 2\times-1+0\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -3 \\ 2 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$

3.4 Inverse of a Matrix

In algebraic calculations we write ax = y

i.e. $$x = \frac{y}{a} \text{ and } x = a^{-1}y$$

Similarly in matrix algebra we can write,

$$AX = B$$

i.e. $$X = A^{-1} B$$

where A^{-1} = Inverse of matrix A

The inverse of a matrix exists only under the following **conditions,**

1. The matrix is a square matrix.
2. The matrix is a nonsingular matrix.

Mathematically inverse of a matrix can be calculated by the expression,

$$\boxed{A^{-1} = \frac{[\text{Adjoint of A}]}{|A|}}$$

3.4.1 Properties of Inverse of a Matrix

1. When a matrix is multiplied by its inverse, the result is the identity matrix.

$$\therefore \quad A^{-1}A = AA^{-1} = I$$

2. Inverse of inverse of a matrix is the matrix itself.

$$(A^{-1})^{-1} = A$$

3. Inverse of a product of two matrices is the product of their individual inverses taken in the reverse order.

$$(AB)^{-1} = B^{-1}A^{-1}$$

Example 3.4 : *Find the inverse of* $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

Solution : Check that A is nonsingular.

$$|A| = \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = 2 - 12 = -10$$

$$A^{-1} = \frac{[\text{Adjoint of A}]}{|A|}$$

$$\text{Adj A} = [\text{Cofactor of A}]^T$$

$$= \begin{bmatrix} 1 & -4 \\ -3 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}}{-10} = \begin{bmatrix} -0.1 & +0.3 \\ +0.4 & -0.2 \end{bmatrix}$$

Cross check : $AA^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} -0.1 & +0.3 \\ +0.4 & -0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

> **Important Observation :** For 2 × 2 matrix, the adjoint of the matrix can be directly obtained by changing the positions of main diagonal elements and by changing the signs of the remaining elements.

For example, $A = \begin{bmatrix} 1 & 4 \\ 5 & 8 \end{bmatrix}$, Adj $A = \begin{bmatrix} 8 & -4 \\ -5 & 1 \end{bmatrix}$

This result is directly used hereafter while solving the problems, when matrix has an order 2 × 2. **Note that for any other order of matrix the adjoint must be obtained as transpose of cofactor matrix.**

With this study of matrix algebra, let us discuss the method of obtaining transfer function from the state model.

3.5 Derivation of Transfer Function from State Model

Consider a standard state model derived for linear time invariant system as,

$$\dot{X}(t) = A\,X(t) + B\,U(t) \quad \text{... (1a)}$$

and

$$Y(t) = C\,X(t) + D\,U(t) \quad \text{... (1b)}$$

Taking Laplace transform of both sides,

$$[s\,X(s) - X(0)] = A\,X(s) + B\,U(s) \quad \text{... (2a)}$$

and

$$Y(s) = C\,X(s) + D\,U(s) \quad \text{... (2b)}$$

Note that as the system is time invariant, the coefficient of matrices A, B, C and D are constants. While the definition of transfer function is based on the assumption of zero initial conditions i.e. $X(0) = 0$.

$$\therefore \quad s\,X(s) = A\,X(s) + B\,U(s)$$

$$\therefore \quad s\,X(s) - A\,X(s) = B\,U(s)$$

Now s is an operator while A is matrix of order n × n hence to match the orders of two terms on left hand side, multiply 's' by identity matrix I of the order n × n.

$$\therefore \quad sI\,X(s) - A\,X(s) = B\,U(s)$$

$$\therefore \quad [sI - A]\,X(s) = B\,U(s)$$

Premultiplying both sides by $[sI - A]^{-1}$,

$$[sI - A]^{-1}\,[sI - A]\,X(s) = [sI - A]^{-1}\,BU(s)$$

Now

$$[sI - A]^{-1}\,[sI - A] = 1$$

$$\therefore \quad X(s) = [sI - A]^{-1}\,BU(s) \quad \text{...(3)}$$

Substituting in the equation (2b),

$$Y(s) = C\,[sI - A]^{-1}\,B\,U(s) + D\,U(s)$$

$$\therefore \quad Y(s) = \{C\,[sI - A]^{-1}\,B + D\}\,U(s)$$

Hence the transfer function is,

$$\boxed{T(s) = \frac{Y(s)}{U(s)} = C\,[sI - A]^{-1}\,B + D} \quad \text{... (4a)}$$

Now $\quad [sI - A]^{-1} = \dfrac{\text{Adj}\,[sI-A]}{|sI-A|}$

$$\therefore \quad \boxed{T(s) = \frac{C\,\text{Adj}\,[sI-A]\,B}{|sI-a|} + D} \quad \text{... (4b)}$$

Key Point: *The state model of a system is not unique, but the transfer function of obtained from any state model is unique. It is independent of the method used to express the system in state model form.*

3.5.1 Characteristic Equation

It is seen from the expression of transfer function that the denominator is $|sI - A|$.

Now the equation obtained by equating denominator of transfer function to zero is called characteristic equation. The roots of this equation are the closed loop poles of the system. Thus the characteristic equation of the system is,

$$\boxed{|sI - A| = 0 \quad \text{... Characteristic equation}}$$

The stability of the system depends on the roots of the characteristic equation. In matrix algebra, the roots of the equation $|sI - A| = 0$ are called **eigen values of matrix A** and these are generally denoted by λ.

➠ **Example 3.5 :** *Consider a system having state model*

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \end{bmatrix} U \quad \text{and } Y = [\,1 \quad 1\,] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

with $D = 0$. Obtain its T.F. **(VTU : Jan./Feb.-2008)**

Solution :

$$\text{T.F.} = C\,[sI - A]^{-1}\,B$$

$$[sI - A]^{-1} = \frac{\text{Adj}\,[sI - A]}{|sI - A|}$$

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & -3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & -3 \\ +4 & 2 \end{bmatrix}$$

$$[sI - A] = \begin{bmatrix} s+2 & 3 \\ -4 & s-2 \end{bmatrix}$$

$$\text{Adj} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} s-2 & 4 \\ -3 & s+2 \end{bmatrix}^T = \begin{bmatrix} s-2 & -3 \\ 4 & s+2 \end{bmatrix}$$

$$|sI - A| = (s+2)(s-2) + 12 = s^2 - 4 + 12 = s^2 + 8$$

$$[sI - A]^{-1} = \frac{\begin{bmatrix} s-2 & -3 \\ 4 & s+2 \end{bmatrix}}{s^2 + 8}$$

$$\therefore \quad \text{T.F.} = \frac{[1 \;\; 1]\begin{bmatrix} s-2 & -3 \\ 4 & s+2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}}{s^2+8} = [\,1 \;\; 1\,]\;\frac{\begin{bmatrix} 3s-21 \\ 5s+22 \end{bmatrix}}{s^2+8}$$

$$= \frac{[8s+1]}{s^2+8}$$

3.5.2 MIMO System

For multiple input multiple output systems, a single transfer function does not exist. There exists a mathematical relationship between each output and all the inputs. Hence for such systems there exists a **transfer matrix** rather than the transfer function. But the method of obtaining transfer matrix remains same as before.

Example 3.6 : *Determine the transfer matrix for MIMO system given by,*

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -2 & -5 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \quad \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Solution : From the given state model,

$$A = \begin{bmatrix} 0 & 3 \\ -2 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad D = [0]$$

$$\text{T.M.} = C\,[sI - A]^{-1}\,B + D$$

$$[sI - A] = \begin{bmatrix} s & -3 \\ 2 & s+5 \end{bmatrix}$$

$$\text{Adj}\,[sI - A] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} s+5 & -2 \\ 3 & s \end{bmatrix}^T = \begin{bmatrix} s+5 & 3 \\ -2 & s \end{bmatrix}$$

$$|sI - A| = s^2 + 5s + 6 = (s+2)(s+3)$$

$$\therefore \quad [sI-A]^{-1} = \frac{\text{Adj } [sI-A]}{|sI-A|} = \frac{\begin{bmatrix} s+5 & 3 \\ -2 & s \end{bmatrix}}{(s+2)(s+3)}$$

$$\therefore \quad \text{T.M.} = C[sI-A]^{-1}B = \frac{\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} s+5 & 3 \\ -2 & s \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}{(s+2)(s+3)}$$

$$= \frac{\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} s+8 & s+8 \\ s-2 & s-2 \end{bmatrix}}{(s+2)(s+3)} = \frac{\begin{bmatrix} 3s+14 & 3s+14 \\ s+8 & s+8 \end{bmatrix}}{(s+2)(s+3)}$$

$$= \begin{bmatrix} \frac{3s+14}{(s+2)(s+3)} & \frac{3s+14}{(s+2)(s+3)} \\ \frac{s+8}{(s+2)(s+3)} & \frac{s+8}{(s+2)(s+3)} \end{bmatrix}$$

i.e. $$Y(s) = \text{T.M. } U(s)$$

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{3s+14}{(s+2)(s+3)} & \frac{3s+14}{(s+2)(s+3)} \\ \frac{s+8}{(s+2)(s+3)} & \frac{s+8}{(s+2)(s+3)} \end{bmatrix}\begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

The above relation indicates that each output depends on both the inputs.

3.6 Eigen Values

Consider an equation AX = Y which indicates the transformation of 'n × 1' vector matrix X into 'n × 1' vector matrix Y by 'n × n' matrix operator A.

If there exists such a vector X such that A transforms it to a vector λX then X is called the solution of the equation,

$$AX = \lambda X$$

i.e. $$\lambda X - AX = 0$$

i.e. $$[\lambda I - A]X = 0 \qquad \dots (1)$$

The set of homogeneous equations (1) have a nontrivial solution only under the condition,

$$\boxed{|\lambda I - A| = 0} \qquad \dots (2)$$

The determinant $|\lambda I - A|$ is called **characteristic polynomial** while the equation (2) is called the **characteristic equation**.

After expanding, we get the characteristic equation as,

$$|\lambda I - A| = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_n = 0 \qquad \ldots (3)$$

The 'n' roots of the equation (3) i.e. the values of λ satisfying the above equation (3) are called **eigen values** of the matrix A.

The equation (2) is similar to $|sI - A| = 0$, which is the characteristic equation of the system. Hence values of λ satisfying characteristic equation are the **closed loop poles** of the system. Thus **eigen values are the closed loop poles of the system.**

3.7 Eigen Vectors

Any nonzero vector X_i such that $AX_i = \lambda_i X_i$ is said to be eigen vector associated with eigen value λ_i. Thus let $\lambda = \lambda_i$ satisfies the equation,

$$(\lambda_i I - A) X = 0$$

Then solution of this equation is called eigen vector of A associated with eigen value λ_i and is denoted as M_i.

If the rank of the matrix $[\lambda_i I - A]$ is r, then there are (n – r) independent eigen vectors. Similarly another important point is that if the eigen values of matrix A are all distinct, then the rank r of matrix A is (n – 1) where n is order of the system.

Mathematically, the eigen vector can be calculated **by taking cofactors of matrix $(\lambda_i I - A)$ along any row.**

Thus $$\boxed{M_i = \text{Eigen vector for } \lambda_i = \begin{bmatrix} C_{k1} \\ C_{k2} \\ \vdots \\ C_{kn} \end{bmatrix}}$$ where k = 1, 2, ... n

where C_{ki} is cofactor of matrix $(\lambda_i I - A)$ of k^{th} row.

Key Point: *If the cofactors along a particular row gives null solution i.e. all elements of corresponding eigen vectors are zero then cofactors along any other row must be obtained. Otherwise inverse of modal matrix M cannot exist.*

3.8 Modal Matrix M

Let $\lambda_1, \lambda_2, \ldots \lambda_n$ are the n eigen values of the matrix A while $M_1, M_2, \ldots M_n$ are the eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2 \ldots \lambda_n$ respectively. Each eigen vector is of the order n × 1 and placing all the eigen vectors one after another as the columns of another matrix, n × n matrix can be obtained. Such a matrix obtained by placing all the eigen vectors together is called a **modal matrix** or **diagonalising matrix** of matrix A.

$$\therefore \quad \boxed{M = \text{Modal Matrix} = [M_1 : M_2 : \ldots : M_n]}$$

The important property of the modal matrix is that,

$$AM = A\,[M_1 : M_2 : \ldots : M_n]$$

$$= [AM_1 : AM_2 : \ldots AM_n]$$

$$= [\lambda_1 M_1 : \lambda_2 M_2 : \ldots : \lambda_n M_n]$$

$$\therefore \quad AM = M\,\Lambda \qquad \ldots (1)$$

where
$$\Lambda = \text{Diagonal matrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$$

Thus premultiplying equation (1) by M^{-1}.

$$M^{-1} AM = M^{-1} M\,\Lambda$$

$$\therefore \quad \boxed{M^{-1} AM = \Lambda \text{ (Diagonal matrix)}}$$

It can be noted that,

1. Both A and Λ have same characteristic equation.
2. Both A and Λ have same eigen values.

Hence due to transformation $M^{-1}AM$, eigen values of matrix A remain unchanged and matrix A gets converted to diagonal form.

3.8.1 Vander Monde Matrix

If the state model is obtained using the phase variables then the matrix A is in Bush form or phase variable form as,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

And the characteristic equation i.e. denominator of T(s) is,

$$F(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0$$

In such a case, modal matrix takes a form of a special matrix as,

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & & \cdots & \cdots \\ \lambda_1^{n-1} & \lambda_2^{n-2} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

Such a modal matrix for matrix A which is in phase variable form, is called **Vander Monde matrix** .

Example 3.7 : *Consider a state model with matrix A as,*

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 4 & 0 & 1 \\ -48 & -34 & -9 \end{bmatrix}$$

Determine, a) Characteristic equation, b) Eigen values c) Eigen vectors and d) Modal matrix. Also prove that the transformation $M^{-1}AM$ results a diagonal matrix.

Solution : a) The characteristic equation is $|\lambda I - A| = 0$

$$\therefore \quad \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 2 & 0 \\ 4 & 0 & 1 \\ -48 & -34 & -9 \end{vmatrix} = 0$$

$$\therefore \quad \begin{vmatrix} \lambda & -2 & 0 \\ -4 & \lambda & -1 \\ 48 & 34 & \lambda+9 \end{vmatrix} = 0 \qquad \ldots (1)$$

$$\therefore \lambda^2 (\lambda + 9) + 2 \times 48 + 0 + 0 - 8 (\lambda + 9) + 34 \lambda = 0$$

$$\therefore \quad \lambda^3 + 9 \lambda^2 + 26 \lambda + 24 = 0$$

This is required characteristic equation.

b) To find eigen values, test $\lambda = -2$ for its root.

– 2	1	9	26	24
		– 2	– 14	– 24
	1	7	12	$\boxed{0}$

$$\therefore \quad (\lambda + 2)(\lambda^2 + 7\lambda + 12) = 0$$

$$\text{i.e.} \quad (\lambda + 2)(\lambda + 3)(\lambda + 4) = 0$$

$$\text{i.e.} \quad \lambda_1 = -2, \lambda_2 = -3 \quad \text{and} \quad \lambda_3 = -4$$

These are the eigen values of matrix A.

c) To find eigen vectors, obtain matrix $[\lambda_i \, I - A]$ for each eigen value by substituting value of λ in equation (1).

$$\text{For } \lambda_1 = -2, \quad [\lambda_i \, I - A] = \begin{bmatrix} -2 & -2 & 0 \\ -4 & -2 & -1 \\ 48 & 34 & 7 \end{bmatrix}$$

$$\therefore \quad M_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} \text{ where } C_{11}, C_{12}, C_{13} \text{ cofactors of row 1.}$$

$$\therefore \quad M_1 = \begin{bmatrix} 20 \\ -20 \\ -40 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \quad \text{... The common factor can be taken out.}$$

$$\text{For } \lambda_2 = -3, \quad [\lambda_i \, I - A] = \begin{bmatrix} -3 & -2 & 0 \\ -4 & -3 & -1 \\ 48 & 34 & 6 \end{bmatrix}$$

$$\therefore \quad M_2 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 16 \\ -24 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$$\text{For } \lambda_3 = -4, \quad [\lambda_i \, I - A] = \begin{bmatrix} -4 & -2 & 0 \\ -4 & -4 & -1 \\ 48 & 34 & 5 \end{bmatrix}$$

$$\therefore \quad M_3 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 14 \\ -28 \\ 56 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

M_1, M_2 and M_3 are the eigen vectors corresponding to the eigen values λ_1, λ_2 and λ_3.

d) The modal matrix is,

$$M = [M_1 : M_2 : M_3] = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -3 & -2 \\ -2 & 1 & 4 \end{bmatrix}$$

Let us prove $M^{-1}AM$ is a diagonal matrix.

$$M^{-1} = \frac{\text{Adj}\,[M]}{|M|} = \frac{[\text{Cofactor of } M]^T}{|M|}$$

$$\text{Adj}\,[M] = \begin{bmatrix} -10 & +8 & -7 \\ -7 & 6 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} -10 & -7 & -1 \\ 8 & 6 & 1 \\ -7 & -5 & -1 \end{bmatrix}$$

$$|M| = \begin{vmatrix} 1 & 2 & 1 \\ -1 & -3 & -2 \\ -2 & 1 & 4 \end{vmatrix} = -1$$

$$\therefore \quad M^{-1} = \frac{\text{Adj}\,[M]}{|M|} = \begin{bmatrix} 10 & 7 & 1 \\ -8 & -6 & -1 \\ 7 & 5 & 1 \end{bmatrix}$$

$$AM = \begin{bmatrix} 0 & 2 & 0 \\ 4 & 0 & 1 \\ -48 & -34 & -9 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & -3 & -2 \\ -2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -6 & -4 \\ 2 & 9 & 8 \\ 4 & -3 & -16 \end{bmatrix}$$

$$M^{-1}AM = \begin{bmatrix} 10 & 7 & 1 \\ -8 & -6 & -1 \\ 7 & 5 & 1 \end{bmatrix} \begin{bmatrix} -2 & -6 & -4 \\ 2 & 9 & 8 \\ 4 & -3 & -16 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \Lambda$$

Thus $M^{-1}AM$ is a diagonal matrix.

Example 3.8 : *Obtain the Eigen values, Eigen vectors and Modal matrix for the matrix*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

Solution : Eigen values are roots of $|\lambda I - A| = 0$

$$|\lambda I - A| = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{bmatrix} = 0$$

i.e. $$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

$\therefore$ $$(\lambda+1)(\lambda+2)(\lambda+3) = 0$$

$\lambda_1 = -1,\ \ \lambda_2 = -2,\ \ \lambda_3 = -3$ are eigen values.

To find Eigen vector,

Let $$\lambda = \lambda_1 = -1$$

$\therefore$ $$[\lambda_1 I - A] = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 6 & 11 & 5 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix}_{K=2} \quad \text{where } C = \text{Cofactor}$$

$$M_1 = \begin{bmatrix} 6 \\ -6 \\ 6 \end{bmatrix} \quad \text{i.e.} \quad M_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = -2$

$$[\lambda_2 I - A] = \begin{bmatrix} -2 & -1 & 0 \\ 0 & -2 & -1 \\ 6 & 11 & 4 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix}_{\text{for } K=3} = \begin{bmatrix} 3 \\ -6 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

For $\lambda_3 = -3$

$$[\lambda_3 I - A] = \begin{bmatrix} -3 & -1 & 0 \\ 0 & -3 & -1 \\ 6 & 11 & 3 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} C_{31} \\ C_{32} \\ C_{33} \end{bmatrix}_{\text{for } K=3} = \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}$$

Key Point: *The cofactors about any row can be obtained.*

$$\therefore \text{ Modal matrix } M = [M_1 : M_2 : M_3]$$

$$M = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

It can be easily checked that,

$$M^{-1} AM = \Lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

3.9 Diagonalisation

The diagonal matrix plays an important role in the matrix algebra. The eigen values and inverse of a diagonal matrix can be very easily obtained just by an inspection. Thus in state variable analysis, if the matrix A is diagonalised then it is very easy to handle mathematically.

When matrix A is diagonalised, then the elements along its principle diagonal are the eigen values. The eigen values are the closed loop poles of the system, from which stability of the system can be analysed.

The diagonal matrix A indicates **noninteraction** of various state variables. The derivative of one state variable is dependent on the corresponding state variable alone and other state variables do not interact in it, when A is diagonal matrix. This is practically important from controlling point of view.

Due to all these reasons, the diagonalisation of matrix A is always motivated. The techniques used to transform the general state model into diagonal form i.e. canonical form are called **diagonalisation techniques**.

Consider n^{th} order state model in which matrix A is not diagonal.

$$\dot{X}(t) = A\,X(t) + B\,U(t) \qquad \ldots (1)$$

$$Y(t) = C\,X(t) + D\,U(t) \qquad \ldots (2)$$

Let Z(t) is new state vector such that the transformation is,

$$X(t) = M\,Z(t) \qquad \ldots (3)$$

Here M = Modal matrix of A

$$\therefore \quad \dot{X}(t) = M\,\dot{Z}(t) \qquad \ldots (4)$$

Using equations (3) and (4) into equations (1) and (2),

$$M\,\dot{Z}(t) = AM\,Z(t) + B\,U(t) \qquad \ldots (5)$$

and

$$Y(t) = CM\,Z(t) + DU(t) \qquad \ldots (6)$$

Premultiplying equation (5) by M^{-1} on both sides,

$$M^{-1}M\,\dot{Z}(t) = M^{-1}AM\,Z(t) + M^{-1}B\,U(t)$$

$$\therefore \quad \dot{Z}(t) = M^{-1}AM\,Z(t) + M^{-1}B\,U(t) \qquad \ldots (7)$$

and

$$Y(t) = CM\,Z(t) + D\,U(t) \qquad \ldots (8)$$

The equations (7) and (8) gives the **canonical state model** in which $M^{-1}AM$ is a **diagonal matrix** denoted as Λ.

The matrix M which transforms A into diagonal form is called **diagonalising matrix** or **modal matrix.**

The new canonical state model is represented as,

$$\dot{Z}(t) = \Lambda\,Z(t) + \tilde{B}\,U(t) \qquad \ldots (9)$$

$$Y(t) = \tilde{C}\,Z(t) + D\,U(t) \qquad \ldots (10)$$

where

$$\Lambda = M^{-1}AM = \text{Diagonal matrix}$$

$$\tilde{B} = M^{-1}B$$

$$\tilde{C} = CM$$

The method of obtaining modal matrix M is already discussed in the last section. The transformation $M^{-1}AM$ is called **diagonalisation** of matrix A.

Example 3.9 : *Reduce the given state model into its canonical form by diagonalising matrix A,*

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U(t)$$

and $$Y(t) = [1 \quad 0 \quad 0]\,X(t)$$

Solution : From the given state model,

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = [1 \;\; 0 \;\; 0]$$

Let us find eigen values, eigen vectors and modal matrix of A.

$$|\lambda I - A| = \left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix} \right| = 0$$

$$\therefore \quad \begin{vmatrix} \lambda & -1 & 1 \\ 6 & \lambda+11 & -6 \\ 6 & 11 & \lambda-5 \end{vmatrix} = 0$$

$$\therefore \; \lambda(\lambda + 11)(\lambda - 5) + 36 + 66 - 6(\lambda + 11) + 6(\lambda - 5) + 66\lambda = 0$$

$$\therefore \quad \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

$$\therefore \quad (\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$$

Thus eigen values are, $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$

To find eigen vectors,

For $\lambda_1 = -1$,

$$[\lambda_1 I - A] = \begin{bmatrix} -1 & -1 & 1 \\ 6 & 10 & -6 \\ 6 & 11 & -6 \end{bmatrix}$$

$$\because \quad M_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda_2 = -2$,

$$[\lambda_2 I - A] = \begin{bmatrix} -2 & -1 & 1 \\ 6 & 9 & -6 \\ 6 & 11 & -7 \end{bmatrix}$$

$$\because \quad M_2 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} +3 \\ +6 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

For $\lambda_3 = -3$,

$$[\lambda_3 I - A] = \begin{bmatrix} -3 & -1 & 1 \\ 6 & 8 & -6 \\ 6 & 11 & -8 \end{bmatrix}$$

$$\because \quad M_3 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix}$$

$$\because \quad M = [M_1 \ : \ M_2 : \ M_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 1 & 4 & 9 \end{bmatrix} = \text{Modal matrix}$$

$$\because \quad M^{-1}AM = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \Lambda$$

While $\tilde{B} = M^{-1}B$

$$M^{-1} = \frac{\text{Adj}[M]}{|M|} = \frac{\begin{bmatrix} -6 & 6 & -2 \\ -5 & 8 & -3 \\ 4 & -6 & 2 \end{bmatrix}^T}{-14} = \frac{\begin{bmatrix} -6 & -5 & 4 \\ 6 & 8 & -6 \\ -2 & -3 & 2 \end{bmatrix}}{-14}$$

$$= \begin{bmatrix} 0.4285 & 0.3571 & -0.2857 \\ -0.4285 & -0.5714 & +0.4285 \\ 0.1428 & 0.2142 & -0.1428 \end{bmatrix}$$

$$\therefore \quad \tilde{B} = M^{-1}B = \begin{bmatrix} -0.2857 \\ +0.4285 \\ -0.1428 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} \quad \text{... Multiplying by 7}$$

and $\tilde{C} = CM = [1 \ \ 1 \ \ 1]$

Thus canonical state model is,

$$\dot{Z}(t) = \Lambda Z(t) + \tilde{B}U(t)$$

and $Y(t) = \tilde{C}Z(t)$ where $X(t) = M Z(t)$

3.10 Generalised Eigen Vectors

The generalised eigen vectors are the eigen vectors corresponding to repeated eigen values. Uptill now, it is assumed that the eigen values are distinct but if an eigen value λ_1 is repeated for r times then the corresponding eigen vectors are called generalised eigen vectors.

If a particular eigen value λ_1 is repeated r times then the rank of n × n matrix $[\lambda_1 I - A]$ is n – 1 and there is only one independent eigen vector associated with λ_1 given by,

$$M_1 = \begin{bmatrix} C_{k1} \\ C_{k2} \\ \vdots \\ C_{kn} \end{bmatrix} \qquad \ldots C_{kn} \text{ are cofactors about } k^{th} \text{ row}$$

Important Note : If cofactors along a particular row gives all values zero i.e. eigen vector as null matrix, then cofactors along other row must be obtained.

For remaining (r – 1) eigen vectors for r – 1 times repeated value λ_1, generalised eigen vectors must be used as,

$$M_2 = \begin{bmatrix} \frac{1}{1!} \frac{\partial C_{k1}}{\partial \lambda_1} \\ \frac{1}{1!} \frac{\partial C_{k2}}{\partial \lambda_1} \\ \vdots \\ \frac{1}{1!} \frac{\partial C_{kn}}{\partial \lambda_1} \end{bmatrix}, \quad M_3 = \begin{bmatrix} \frac{1}{2!} \frac{\partial^2 C_{k1}}{\partial \lambda_1^2} \\ \frac{1}{2!} \frac{\partial^2 C_{k2}}{\partial \lambda_1^2} \\ \vdots \quad \vdots \\ \frac{1}{2!} \frac{\partial^2 C_{kn}}{\partial \lambda_1^2} \end{bmatrix}$$

$$M_r = \begin{bmatrix} \frac{1}{(r-1)!} \frac{\partial^{r-1}(C_{k1})}{\partial \lambda_1^{r-1}} \\ \frac{1}{(r-1)!} \frac{\partial^{r-1}(C_{k2})}{d\lambda_1^{r-1}} \\ \vdots \\ \frac{1}{(r-1)!} \frac{\partial^{r-1}(C_{kn})}{\partial \lambda_1^{r-1}} \end{bmatrix}$$

The modal matrix M then can be obtained by placing all the eigen vectors one after the other.

Note that in such a case, $M^{-1}AM$ matrix is not diagonal but consists of a r × r Jordan block in it where r is the order of repetition of a particular eigen value.

3.10.1 Vander Monde Matrix for Generalised Eigen Vectors

If matrix A is in phase variable form and an eigen value λ_1 is repeated for r times, then n eigen values are $\lambda_1, \lambda_1 \ldots \lambda_1, \lambda_{r+1} \ldots, \lambda_n$. The corresponding modal matrix is Vander Monde matrix in modified form as,

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 1 & \cdots & 1 \\ \lambda_1 & 1 & 0 & 0 & \cdots & \cdots & \lambda_{r+1} & \cdots & \lambda_n \\ \lambda_1^2 & 2\lambda_1 & 1 & 0 & \cdots & \cdots & \lambda_{r+1}^2 & \cdots & \lambda_n^2 \\ \lambda_1^3 & 3\lambda_1^2 & 3\lambda_1 & 1 & \cdots & \cdots & \lambda_{r+1}^3 & \cdots & \lambda_n^3 \\ \vdots & & & & & & & & \\ \lambda_1^{n-1} & \dfrac{d\,\lambda_1^{n-1}}{d\lambda_1} & \dfrac{1}{2!}\dfrac{d^2\,\lambda_1^{n-1}}{d\lambda_1^2} & & & \cdots & \lambda_{r+1}^{n-1} & & \lambda_n^{n-1} \end{bmatrix}$$

Example 3.10 : *Find the eigen values, eigen vectors and modal matrix for,*

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 8 & 2 & -5 \end{bmatrix}$$

Solution : Find eigen values which are roots of $|\lambda I - A| = 0$

$$\therefore \quad \left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ -8 & -2 & -5 \end{bmatrix} \right| = 0$$

$$\therefore \quad \begin{vmatrix} \lambda & 0 & -1 \\ -2 & \lambda & 0 \\ +8 & 2 & \lambda+5 \end{vmatrix} = 0$$

$$\therefore \quad \lambda^2(\lambda+5) + 4 + 8\lambda = 0$$

$$\therefore \quad \lambda^3 + 5\lambda^2 + 8\lambda + 4 = 0$$

Try $\lambda = -1$

$$\begin{array}{r|rrrr} -1 & 1 & 5 & 8 & 4 \\ & & -1 & -4 & -4 \\ \hline & 1 & 4 & 4 & \boxed{0} \end{array}$$

$$\therefore \quad (\lambda+1)(\lambda^2 + 4\lambda + 4) = 0$$

$$\therefore \quad (\lambda+1)(\lambda+2)^2 = 0$$

$$\therefore \qquad \lambda_1 = -1, \lambda_2 = -2, \lambda_3 = -2$$

Thus '– 2' is repeated twice hence it is necessary to use generalised eigen vectors.

For $\lambda_1 = -1$, $[\lambda_1 I - A] = \begin{bmatrix} -1 & 0 & -1 \\ -2 & -1 & 0 \\ 8 & 2 & 4 \end{bmatrix}$

$$\therefore \qquad M_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Now $\lambda_2 = -2$ is repeated twice.

$$\therefore \qquad [\lambda_2 I - A] = \begin{bmatrix} \lambda_2 & 0 & -1 \\ -2 & \lambda_2 & 0 \\ 8 & 2 & \lambda_2 + 5 \end{bmatrix}$$

$$\therefore \qquad M_2 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix}_{\lambda_2 = -2} = \begin{bmatrix} \lambda_2^2 + 5\lambda_2 \\ 2\lambda_2 + 10 \\ -4 - 8\lambda_2 \end{bmatrix}_{\lambda_2 = -2}$$

$$= \begin{bmatrix} -6 \\ 6 \\ 12 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

and
$$M_3 = \begin{bmatrix} \frac{1}{1!} \frac{d C_{11}}{d\lambda_2} \\ \frac{1}{1!} \frac{d C_{12}}{d\lambda_2} \\ \frac{1}{1!} \frac{d C_{13}}{d\lambda_2} \end{bmatrix} = \begin{bmatrix} 2\lambda_2 + 5 \\ 2 \\ -8 \end{bmatrix}_{\lambda_2 = -2} = \begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix}$$

Thus M_1, M_2 and M_3 are the required eigen vectors.

$$\therefore \text{ Modal matrix } M = [M_1 : M_2 : M_3] = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -8 \end{bmatrix}$$

Examples with Solutions

Example 3.11 : *Find the T.F. of the system having state model,*

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U \quad \text{and} \quad Y = [1 \;\; 0]\, X$$

Solution : From the given model,

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = [1 \;\; 0], \quad D = 0$$

$$\text{T.F.} = \frac{Y(s)}{U(s)} = C\,[sI - A]^{-1}\, B + D$$

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\therefore \quad \text{Adj}\,[sI - A] = \begin{bmatrix} s+3 & +1 \\ -2 & s \end{bmatrix}$$

... Change diagonal elements and change signs of other elements.

$$|sI - A| = s\,(s+3) + 2 = s^2 + 3s + 2 = (s+1)\,(s+2)$$

$$\therefore \quad \text{T.F.} = \frac{C\,\text{Adj}\,[sI-A]\,B}{|sI-A|} = \frac{[1 \;\; 0]\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{(s+1)(s+2)}$$

$$= \frac{(s+3)}{(s+1)\,(s+2)}$$

Example 3.12 : *Find the T.F. of the system having state model,*

$$\dot{X} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U \quad \text{and} \quad Y = [1 \;\; 0]\; X$$

Solution : From the given model,

$$A = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad 0], \; D = 0$$

$$\text{T.F.} = \frac{T(s)}{U(s)} = C\,[sI - A]^{-1}\, B + D$$

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix}$$

$$\text{Adj}\,[sI - A] = \begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}$$

$$|sI - A| = s(s+3) + 2 = s^2 + 3s + 2 = (s+1)(s+2)$$

$$\text{T.F.} = \frac{C\,\text{Adj}\,[sI-A]\,B}{|sI-A|} = \frac{[1\ \ 0]\begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{(s+1)\,(s+2)}$$

$$= \frac{1}{(s+1)\,(s+2)}$$

Example 3.13 : *Obtain the transfer function matrix for the MIMO system having state model,*

$$\dot{X} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} X + \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} U$$

and $$Y = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} X$$

Solution : From the given state model,

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad D = 0$$

$$\text{Transfer matrix} = C\,[sI - A]^{-1}\,B + D = \frac{C\,\text{Adj}\,[sI-A]\,B}{|sI-A|}$$

$$[sI - A] = s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s-2 & 1 & 0 \\ -1 & s-1 & -2 \\ 1 & 0 & s-1 \end{bmatrix}$$

$$\text{Adj}\,[sI - A] = \begin{bmatrix} (s-1)^2 & s-3 & s-1 \\ -s+1 & (s-2)(s-1) & 1 \\ -2 & 2(s-2) & s^2-3s+3 \end{bmatrix}^T$$

$$= \begin{bmatrix} s^2-2s+1 & -s+1 & -2 \\ s-3 & s^2-3s+2 & 2s-4 \\ s-1 & 1 & s^2-3s+3 \end{bmatrix}$$

$$|sI - A| = (s-2)(s-1)^2 - 2 + s - 1$$

$$= s^3 - 4s^2 + 6s - 5$$

$$\text{Adj } [sI - A]B = \begin{bmatrix} s^2-2s+1 & -s+1 & -2 \\ s-3 & s^2-3s+2 & 2s-4 \\ s-1 & 1 & s^2-3s+3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -s^2+5 & -4 \\ s^2-4s+5 & 4s-8 \\ -s+2 & 2s^2-6s+6 \end{bmatrix}$$

$$\therefore \quad C \text{ Adj } [sI - A] B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s^2+5 & -4 \\ s^2-4s+5 & 4s-8 \\ -s+2 & 2s^2-6s+6 \end{bmatrix}$$

$$= \begin{bmatrix} -3s+5 & 4s-12 \\ -s^2+2 & 2s^2-6s+2 \end{bmatrix}$$

$$\therefore \quad \text{Transfer matrix} = \frac{\begin{bmatrix} -3s+5 & 4s-12 \\ -s^2+2 & 2s^2-6s+2 \end{bmatrix}}{s^3-4s^2+6s-5}$$

Example 3.14 : *Obtain the eigen values, eigen vectors and modal matrix for,*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix}$$

and prove that $M^{-1}AM = \Lambda$ = Diagonal matrix. **(VTU: July/Aug.-2007,Jan./Feb.-2008)**

Solution : For eigen values,

$$|\lambda I - A| = 0$$

$$\begin{vmatrix} \lambda & -1 & 0 \\ -3 & \lambda & -2 \\ 12 & 7 & \lambda+6 \end{vmatrix} = 0$$

$$\therefore \quad \lambda^2(\lambda+6)+24-3\lambda-18+14\lambda = 0$$

$$\therefore \quad \lambda^3+6\lambda^2+11\lambda+6 = 0$$

Test $\lambda = -1$,

$$\begin{array}{r|rrrr} -1 & 1 & 6 & 11 & 6 \\ & 0 & -1 & -5 & -6 \\ \hline & 1 & 5 & 6 & \boxed{0} \end{array}$$

$$\therefore \quad (\lambda+1)(\lambda^2+5\lambda+6) = 0$$

i.e. $$(\lambda+1)(\lambda+2)(\lambda+3) = 0$$

Hence eigen values are, $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$

For the eigen vectors,

$$\lambda_1 = -1, \quad [\lambda_1 I - A] = \begin{bmatrix} -1 & -1 & 0 \\ -3 & -1 & -2 \\ 12 & 7 & 5 \end{bmatrix}$$

$$\therefore \quad M_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 9 \\ -9 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = -2, \quad [\lambda_2 I - A] = \begin{bmatrix} -2 & -1 & 0 \\ -3 & -2 & -2 \\ 12 & 7 & 4 \end{bmatrix}$$

$$\therefore \quad M_2 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 6 \\ -12 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$$

$$\lambda_3 = -3, \quad [\lambda_3 I - A] = \begin{bmatrix} -3 & -1 & 0 \\ -3 & -3 & -2 \\ 12 & 7 & 3 \end{bmatrix}$$

$$\therefore \quad M_3 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 5 \\ -15 \\ 15 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$$

$$\therefore \quad M = \text{Modal matrix} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -4 & -3 \\ -1 & 1 & 3 \end{bmatrix}$$

$$M^{-1} = \frac{\text{Adj}[M]}{|M|} = \frac{\begin{bmatrix} -9 & 6 & -5 \\ -5 & 4 & -3 \\ -2 & 2 & -2 \end{bmatrix}^T}{-2} = \frac{\begin{bmatrix} -9 & -5 & -2 \\ 6 & 4 & 2 \\ -2 & -3 & -2 \end{bmatrix}}{-2}$$

$$= \begin{bmatrix} 4.5 & 2.5 & 1 \\ -3 & -2 & -1 \\ 2.5 & 1.5 & 1 \end{bmatrix}$$

$$AM = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & -4 & -3 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -4 & -3 \\ 1 & 8 & 9 \\ 1 & -2 & -9 \end{bmatrix}$$

$$\therefore \quad M^{-1}AM = \begin{bmatrix} 4.5 & 2.5 & 1 \\ -3 & -2 & -1 \\ 2.5 & 1.5 & 1 \end{bmatrix} \begin{bmatrix} -1 & -4 & -3 \\ 1 & 8 & 9 \\ 1 & -2 & -9 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Thus, $M^{-1}AM = \Lambda$ = Diagonal matrix

Example 3.15 : *Find the eigen values, eigen vectors and modal matrix for,*

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}$$

(VTU: Jan./Feb.-2005)

Solution : For eigen values,

$$|\lambda I - A| = 0$$

$$\therefore \begin{vmatrix} \lambda-4 & -1 & 2 \\ -1 & \lambda & -2 \\ -1 & 1 & \lambda-3 \end{vmatrix} = 0$$

$$\therefore \quad \lambda(\lambda-4)(\lambda-3) - 2 - 2 + 2\lambda - \lambda + 3 + 2\lambda - 8 = 0$$

$$\therefore \quad \lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$$

$$\therefore \quad (\lambda-1)(\lambda^2 - 6\lambda + 9) = 0$$

$$\therefore \quad (\lambda-1)(\lambda-3)^2 = 0$$

$$\therefore \quad \lambda_1 = 1, \lambda_2 = 3, \quad \lambda_3 = 3$$

Now $\lambda_2 = 3$ is repeated twice hence for $\lambda_2 = 3$, the generalised eigen vectors must be used.

For $\lambda_1 = 1$, $$[\lambda_1 I - A] = \begin{bmatrix} -3 & -1 & 2 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{bmatrix}$$

$\therefore$ $$M_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As it is a null matrix, calculate cofactors about other row.

$\therefore$ $$M_1 = \begin{bmatrix} C_{21} \\ C_{22} \\ C_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix}$$

Now for $\lambda_2 = \lambda_3 = 3$, use

$$[\lambda_2 I - A] = \begin{bmatrix} \lambda_2 - 4 & -1 & 2 \\ -1 & \lambda_2 & -2 \\ -1 & 1 & \lambda_2 - 3 \end{bmatrix}$$

$\therefore$ $$M_2 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix}_{\lambda_2 = 3} = \begin{bmatrix} \lambda_2^2 - 3\lambda_2 + 2 \\ \lambda_2 - 1 \\ -1 + \lambda_2 \end{bmatrix}_{\lambda_2 = 3} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

For $\lambda_3 = 3$, $$M_3 = \begin{bmatrix} \frac{1}{1!} \frac{dC_{11}}{d\lambda_2} \\ \frac{1}{1!} \frac{dC_{12}}{d\lambda_2} \\ \frac{1}{1!} \frac{dC_{13}}{d\lambda_2} \end{bmatrix}_{\lambda_2 = 3} = \begin{bmatrix} 2\lambda_2 - 3 \\ 1 \\ 1 \end{bmatrix}_{\lambda_2 = 3} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

Hence the modal matrix M is,

$$M = \begin{bmatrix} 0 & 2 & 3 \\ 8 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Example 3.16 : *The Fig. 3.1 shows the block diagram of a control system using state variable feedback and integral control. The state model of plant is,*

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$$

$$Y = [0 \quad 1] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

i) Derive the state model of the entire system.

ii) Derive the transfer function Y(s)/U(s).

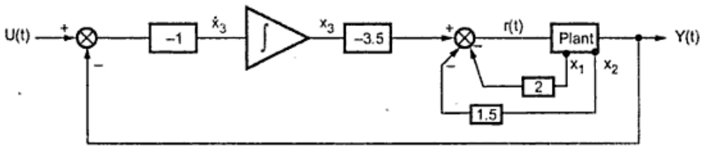


Fig. 3.1

Solution : i) The input of integrator shown is $\dot{X}_3$.

$$\therefore \quad \dot{X}_3 = -1 \times [U(t) - Y(t)] = Y(t) - U(t) \quad \text{... (1)}$$

$$\text{and} \quad r(t) = -3.5\, X_3 - 2X_1 - 1.5\, X_2 \quad \text{... (2)}$$

From the plant model given,

$$\dot{X}_1 = -3X_1 + 2X_2 + r(t) \quad \text{... (3a)}$$

$$\dot{X}_2 = 4\, X_1 - 5\, X_2 \quad \text{...(3b)}$$

$$\text{and} \quad Y(t) = X_2 \quad \text{... (3c)}$$

Using equation (2) in equation (3a) we get,

$$\dot{X}_1 = -3X_1 + 2X_2 - 3.5\, X_3 - 2X_1 - 1.5\, X_2$$

$$\therefore \quad \dot{X}_1 = -5X_1 + 0.5X_2 - 3.5\, X_3 \quad \text{... (4a)}$$

$$\therefore \quad \dot{X}_2 = 4X_1 - 5X_2 \quad \text{... (4b)}$$

Substituting equation (3c) in equation (1),

$$\dot{X}_3 = X_2 - U(t) \quad \text{... (4c)}$$

and $Y(t) = X_2$ ... (4d)

Thus equations (4a) to (4d) give state model of the entire system.

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} -5 & 0.5 & -3.5 \\ 4 & -5 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} U(t)$$

and $$Y(t) = [0 \ \ 1 \ \ 0] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

i.e. $\dot{X} = AX + BU$ and $Y = CX$ with $D = 0$

ii) The transfer function Y(s)/U(s) is,

$$\frac{Y(s)}{U(s)} = C\,[sI - A]^{-1} B + D$$

$$[sI - A] = s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & 0.5 & -3.5 \\ 4 & -5 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} s+5 & -0.5 & 3.5 \\ -4 & s+5 & 0 \\ 0 & -1 & s \end{bmatrix}$$

$$\therefore \quad \text{Adj}\,[sI - A] = \begin{bmatrix} s^2+5s & \vdots & 4s & \vdots & 4 \\ 0.5s-3.5 & \vdots & s^2+5s & \vdots & s+5 \\ -3.5(s+5) & \vdots & -14 & \vdots & (s+5)^2-2 \end{bmatrix}^T$$

$$= \begin{bmatrix} s^2+5s & 0.5s-3.5 & -3.5s-17.5 \\ 4s & s^2+5s & -14 \\ 4 & s+5 & s^2+10s+23 \end{bmatrix}$$

$$|sI - A| = s\,(s+5)^2 + 14 - 2s = s^3 + 10s^2 + 23s + 14$$

$$\therefore \quad \frac{Y(s)}{U(s)} = \frac{C\,\text{Adj}\,[sI-A]\,B}{|sI-A|} \quad \text{and} \quad D = 0$$

$$C\,\text{Adj}\,[sI - A]\,B = [0 \ \ 1 \ \ 0] \begin{bmatrix} s^2+5s & 0.5s-3.5 & -3.5s-17.5 \\ 4s & s^2+5s & -14 \\ 4 & s+5 & s^2+10s+23 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$= 14$$

$$\therefore \quad \frac{Y(s)}{U(s)} = \frac{14}{|sI-A|} = \frac{14}{s^3+10s^2+23s+14}$$

$$= \frac{14}{(s+1)(s+2)(s+7)}$$

This is the required transfer function.

Example 3.17 : *For the matrix* $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}$

Find the eigen values, eigen vectors and modal matrix M.

Solution : For eigen values,

$$|\lambda I - A| = 0$$

$$\therefore \quad \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 2 & 4 & \lambda+3 \end{vmatrix} = 0$$

$$\therefore \quad \lambda^2 (\lambda + 3) + 2 + 4\lambda = 0$$

$$\therefore \quad \lambda^3 + 3\lambda^2 + 4\lambda + 2 = 0$$

Test $\lambda = -1$,

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 4 & 2 \\ & & -1 & -2 & -2 \\ \hline & 1 & 2 & 2 & \boxed{0} \end{array}$$

$$\therefore \quad (\lambda + 1)(\lambda^2 + 2\lambda + 2) = 0$$

$$\therefore \quad (\lambda + 1)(\lambda + 1 + j1)(\lambda + 1 - j1) = 0$$

$$\therefore \quad \lambda_1 = -1, \quad \lambda_2 = -1 - j1, \quad \lambda_3 = -1 + j1$$

Calculate eigen vectors.

For $\lambda_1 = -1$,

$$[\lambda_1 I - A] = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 2 & 4 & 2 \end{bmatrix}$$

$$\therefore \quad M_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = -1 - j1, \; [\lambda_2 I - A] = \begin{bmatrix} -1-j1 & -1 & 0 \\ 0 & -1-j1 & 1 \\ 2 & 4 & 2-j1 \end{bmatrix}$

$$\therefore \quad M_2 = \begin{bmatrix} C_{31} \\ C_{32} \\ C_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ -1-j \\ j2 \end{bmatrix}$$

... Choosing 3^{rd} row to calculate cofactors.

For $\lambda_3 = -1 + j1, \; [\lambda_3 I - A] = \begin{bmatrix} -1+j1 & -1 & 0 \\ 0 & -1+j1 & -1 \\ 2 & 4 & 2+j1 \end{bmatrix}$

$$\therefore \quad M_3 = \begin{bmatrix} C_{31} \\ C_{32} \\ C_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ -1+j \\ -j2 \end{bmatrix}$$

$$\therefore \quad M = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1-j & -1+j \\ 1 & j2 & -j2 \end{bmatrix}$$

Example 3.18 : *For a system with state model matrices.*

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}; \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \; C = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}^T$$

Obtain the system transfer function. **(VTU: March-2001)**

Solution : The T.F. is given by,

$$\text{T.F.} = C[sI-A]^{-1}B$$

$$[sI - A] = \begin{bmatrix} s+1 & 0 & -1 \\ -1 & s+2 & 0 \\ 0 & 0 & s-3 \end{bmatrix}$$

$$\text{Adj}[sI-A] = [\text{Cofactor } sI-A]^T$$

$$= \begin{bmatrix} (s+2)(s-3) & (s-3) & 0 \\ 0 & (s+1)(s-3) & 0 \\ +(s+2) & +1 & (s+1)(s+2) \end{bmatrix}^T$$

$$= \begin{bmatrix} (s+2)(s-3) & 0 & (s+2) \\ (s-3) & (s+1)(s-3) & 1 \\ 0 & 0 & (s+1)\ (s+2) \end{bmatrix}$$

$$|sI-A| = (s+1)(s+2)(s-3)$$

$$\therefore \quad [sI-A]^{-1} = \frac{Adj[sI-A]}{|sI-A|}$$

$$= \begin{bmatrix} \dfrac{1}{s+1} & 0 & \dfrac{1}{(s+1)(s-3)} \\ \dfrac{1}{(s+1)(s+2)} & \dfrac{1}{(s+2)} & \dfrac{1}{(s+1)\ (s+2)(s-3)} \\ 0 & 0 & \dfrac{1}{s-3} \end{bmatrix}$$

$$\therefore \quad T.F. = [1 \quad 1 \quad 0] \begin{bmatrix} \dfrac{1}{s+1} & 0 & \dfrac{1}{(s+1)(s-3)} \\ \dfrac{1}{(s+1)(s+2)} & \dfrac{1}{(s+2)} & \dfrac{1}{(s+1)\ (s+2)(s-3)} \\ 0 & 0 & \dfrac{1}{s-3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= [1 \quad 1 \quad 0] \begin{bmatrix} \dfrac{1}{(s+1)\ (s-3)} \\ \dfrac{1}{(s+1)(s+2)(s-3)} \\ \dfrac{1}{(s-3)} \end{bmatrix}$$

$$= \frac{1}{(s+1)(s-3)} + \frac{1}{(s+1)(s+2)(s-3)}$$

$$= \frac{(s+3)}{(s+1)\ (s+2)(s-3)}$$

Example 3.19 : *Find the transformation matrix P which will convert the following matrix A to diagonal form. The eigen values are* λ_1, λ_2 *and* λ_3.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix}$$

Solution : The matrix $\lambda I - A$ is,

$$[\lambda I - A] = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ a_1 & a_2 & \lambda + a_3 \end{bmatrix}$$

To find P, means to find eigen vectors and modal matrix.

For $\lambda = \lambda_1$, $[\lambda_1 I - A] = \begin{bmatrix} \lambda_1 & -1 & 0 \\ 0 & \lambda_1 & -1 \\ a_1 & a_2 & \lambda_1 + a_3 \end{bmatrix}$

To find eigen vector, find cofactors about 3^{rd} row.

$$M_1 = \begin{bmatrix} C_{31} \\ C_{32} \\ C_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_1 \\ \lambda_1^2 \end{bmatrix}$$

For $\lambda = \lambda_2$, $[\lambda_2 I - A] = \begin{bmatrix} \lambda_2 & -1 & 0 \\ 0 & \lambda_2 & -1 \\ a_1 & a_2 & \lambda_2 + a_3 \end{bmatrix}$

$$\therefore \quad M_2 = \begin{bmatrix} C_{31} \\ C_{32} \\ C_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_2 \\ \lambda_2^2 \end{bmatrix}$$

For $\lambda = \lambda_3$, $[\lambda_3 I - A] = \begin{bmatrix} \lambda_3 & -1 & 0 \\ 0 & \lambda_3 & -1 \\ a_1 & a_2 & \lambda_3 + a_3 \end{bmatrix}$

$$\therefore \quad M_3 = \begin{bmatrix} C_{31} \\ C_{32} \\ C_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_3 \\ \lambda_3^2 \end{bmatrix}$$

Hence transformation matrix P is,

$$P = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}$$

Thus if matrix A is in phase variable form,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 \end{bmatrix}$$

And the eigen values are distinct as $\lambda_1, \lambda_2, \ldots \lambda_n$ then it can be proved as above that the diagonalising matrix is,

$$P = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & & \lambda_n^{n-1} \end{bmatrix}$$

Key Point: *Note that while finding the eigen vectors find the cofactors about last row.*

Example 3.20 : *Determine the transfer matrix for the system.*

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \; : \; \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

(VTU: July/Aug.-2005)

Solution : From the given state model,

$$A = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 4 & 6 \\ -5 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & -1 \\ 8 & 1 \end{bmatrix}, \; D = [\,0\,]$$

$$[sI - A]^{-1} = \frac{\text{Adj}\,[sI - A]}{|sI - A|}, \; [sI - A] = \begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix}$$

$$\text{Adj}\,[sI - A] = \begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix}^T = \begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}$$

$$\therefore \quad [sI-A]^{-1} = \frac{\begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}}{(s^2+3s+2)} = \frac{\begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}}{(s+1)(s+2)}$$

$$\text{T.F.} = C[sI-A]^{-1}B = \frac{\begin{bmatrix} 1 & -1 \\ 8 & 1 \end{bmatrix}\begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ -5 & 0 \end{bmatrix}}{(s+1)\ (s+2)}$$

$$= \frac{\begin{bmatrix} 1 & -1 \\ 8 & 1 \end{bmatrix}\begin{bmatrix} 4s-5 & 6s \\ -5s-23 & -12 \end{bmatrix}}{(s+1)(s+2)}$$

$$\therefore \text{T.F.} = \begin{bmatrix} \frac{9s+18}{(s+1)(s+2)} & \frac{6s+12}{(s+1)(s+2)} \\ \frac{27s-63}{(s+1)(s+2)} & \frac{48s-12}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{9}{s+1} & \frac{6}{s+1} \\ \frac{27s-63}{(s+1)(s+2)} & \frac{48s-12}{(s+1)(s+2)} \end{bmatrix}$$

This is the necessary transfer matrix.

➡ **Example 3.21 :** *Convert the following state model into canonical form.*

(VTU: July/Aug.-2005)

$$\dot{X} = \begin{bmatrix} 1 & -4 \\ 3 & -6 \end{bmatrix} X + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u, \ Y = [1 \quad 0]\ X$$

Solution : From the given model,

$$A = \begin{bmatrix} 1 & -4 \\ 3 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, C = [1 \quad 0]$$

Let us find the eigen values, eigen vectors and model matrix of A.

$$|\lambda I - A| = \left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -4 \\ 3 & -6 \end{bmatrix} \right| = \begin{vmatrix} \lambda-1 & 4 \\ -3 & \lambda+6 \end{vmatrix} = 0$$

$$\therefore \quad (\lambda-1)(\lambda+6)+12 = 0 \text{ i.e. } \lambda^2+5\lambda+6 = 0$$

$$\therefore \quad (\lambda+2)(\lambda+3) = 0 \text{ i.e. } \lambda_1 = -2, \lambda_2 = -3$$

To find eigen vectors,

$$\text{For } \lambda_1 = -2, [\lambda_1 I - A] = \begin{bmatrix} -3 & 4 \\ -3 & 4 \end{bmatrix} \text{ i.e. } M_1 = \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

For $\lambda_2 = -3$, $[\lambda_2 I - A] = \begin{bmatrix} -4 & 4 \\ -3 & 3 \end{bmatrix}$ i.e. $M_2 = \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\therefore \quad M = [M_1 : M_2] = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} = \text{Modal matrix}$$

$$\therefore \quad M^{-1}AM = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} = \Lambda$$

While $\quad \tilde{B} = M^{-1}B$

$$M^{-1} = \frac{\text{Adj } M}{|M|} = \frac{\begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}}{\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix}} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$$

$$\therefore \quad \tilde{B} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$\tilde{C} = CM = [1 \ 0] \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} = [4 \ 1]$$

The canonical state model is,

$$\dot{Z}(t) = \Lambda Z(t) + \tilde{B}u$$

$$Y(t) = \tilde{C}\, Z(t) \quad \text{where } X(t) = M\, Z(t)$$

Example 3.22 : *Convert the following square matrix A into Jordan canonical form using a suitable non-singular transformation matrix P.* **(VTU: July/Aug.- 2005)**

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -9 & -6 \end{bmatrix}$$

Solution : Find the eigen values of A.

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 4 & 9 & \lambda+6 \end{vmatrix} = 0$$

$$\therefore \quad \lambda^2(\lambda+6)+4+9\lambda = 0 \qquad \text{i.e. } \lambda^3 + 6\lambda^2 + 9\lambda + 4 = 0$$

$$\therefore \quad (\lambda+4)(\lambda+1)(\lambda+1) = 0 \qquad \text{i.e. } \lambda_1 = -4,\ \lambda_2 = -1,\ \lambda_3 = -1$$

For $\lambda_1 = -4$

$$[\lambda_1 I - A] = \begin{bmatrix} -4 & -1 & 0 \\ 0 & -4 & -1 \\ 0 & 9 & 2 \end{bmatrix} \quad \therefore M_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 16 \end{bmatrix}$$

For $\lambda_2 = -1$,

$$[\lambda_2 I - A] = \begin{bmatrix} \lambda_2 & -1 & 0 \\ 0 & \lambda_2 & -1 \\ 4 & 9 & \lambda_2 + 6 \end{bmatrix}$$

$$\therefore \quad M_2 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix}_{\lambda_2 = -1} = \begin{bmatrix} \lambda_2^2 + 6\lambda_2 + 9 \\ -4 \\ -4\lambda_2 \end{bmatrix}_{\lambda_2 = -1} = \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

As $\lambda_2 = -1$ is repeated twice,

$$M_3 = \begin{bmatrix} \frac{1}{1!} & \frac{dC_{11}}{d\lambda_2} \\ \frac{1}{1!} & \frac{dC_{12}}{d\lambda_2} \\ \frac{1}{1!} & \frac{dC_{13}}{d\lambda_2} \end{bmatrix} = \begin{bmatrix} 2\lambda_2 + 6 \\ 0 \\ -4 \end{bmatrix}_{\lambda_2 = -1} = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\therefore \quad M = [M_1 : M_2 : M_3] = \begin{bmatrix} 1 & 1 & 1 \\ -4 & -1 & 0 \\ 16 & 1 & -1 \end{bmatrix}$$

$$\text{Adj } [M] = \begin{bmatrix} 1 & -4 & 12 \\ 2 & -17 & 15 \\ 1 & -4 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 1 \\ -4 & -17 & -4 \\ 12 & 15 & 3 \end{bmatrix}$$

$$\therefore \quad M^{-1} = \frac{\text{Adj } M}{|M|} = \frac{\begin{bmatrix} 1 & 2 & 1 \\ -4 & -17 & -4 \\ 12 & 15 & 3 \end{bmatrix}}{9}$$

$$\therefore \quad M^{-1}AM = \frac{1}{9}\begin{bmatrix} 1 & 2 & 1 \\ -4 & -17 & -4 \\ 12 & 15 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -9 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -4 & -1 & 0 \\ 16 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{.....Diagonal matrix with Jordan block}$$

Example 3.23 : *Consider the matrix* **(VTU: Jan./Feb.-2006)**

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

i) Find the eigen values and eigen vectors of A.

ii) Write the modal matrix.

iii) Show that the modal matrix indeed diagonalizes A.

Solution : For eigen values $|\lambda I - A| = 0$

$$\therefore \begin{vmatrix} \lambda-2 & 2 & -3 \\ -1 & \lambda-1 & -1 \\ -1 & -3 & \lambda+1 \end{vmatrix} = 0$$

i.e. $(\lambda-1)(\lambda+1)(\lambda-2)+2-9-3(\lambda-1)+2(\lambda+1)-3(\lambda-2)=0$

i.e. $\lambda^3-2\lambda^2-5\lambda+6=0$ i.e. $(\lambda-1)(\lambda+2)(\lambda-3)=0$

$\therefore \quad \lambda_1=+1, \quad \lambda_2=-2, \quad \lambda_3=3$

For $\lambda_1=1$,

$$[\lambda I - A] = \begin{bmatrix} -1 & 2 & -3 \\ -1 & 0 & -1 \\ -1 & -3 & 2 \end{bmatrix} \quad \therefore M_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda_2=-2$,

$$[\lambda I - A] = \begin{bmatrix} -4 & 2 & -3 \\ -1 & -3 & -1 \\ -1 & -3 & -1 \end{bmatrix} \quad \therefore M_2 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As $C_{11} = C_{12} = C_{13} = 0$, calculate cofactors about other row.

$$M_2 = \begin{bmatrix} C_{21} \\ C_{22} \\ C_{23} \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \\ -14 \end{bmatrix}$$

For $\lambda_3 = 3$,

$$[\lambda I - A] = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 2 & -1 \\ -1 & -3 & 4 \end{bmatrix} \quad M_3 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad M = [M_1 : M_2 : M_3] = \begin{bmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -14 & 1 \end{bmatrix}$$

$$M^{-1} = \frac{\text{Adj } M}{|M|} = \frac{[\text{Cofactor matrix of } M]^T}{|M|}$$

$$= \frac{1}{-30} \begin{bmatrix} 15 & -25 & 10 \\ 0 & -2 & 2 \\ -15 & -3 & -12 \end{bmatrix}$$

$$\therefore \quad M^{-1}AM = -\frac{1}{30} \begin{bmatrix} 15 & -25 & 10 \\ 0 & -2 & 2 \\ -15 & -3 & -12 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -14 & 1 \end{bmatrix}$$

$$= -\frac{1}{30} \begin{bmatrix} 15 & -25 & 10 \\ 0 & 4 & -4 \\ -45 & -9 & -36 \end{bmatrix} \begin{bmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -14 & 1 \end{bmatrix}$$

$$= -\frac{1}{30} \begin{bmatrix} -30 & 0 & 0 \\ 0 & 60 & 0 \\ 0 & 0 & -90 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \Lambda$$

Thus modal matrix indeed diagonalizes A.

Example 3.24 : *Given the state model* $\dot{X} = AX - Bu,\ y = CX$ **(VTU: Jan./Feb.-2006)**

where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ *and* $C = [1 \ \ 0 \ \ 0]$

i) Simulate and find the transfer function $\frac{Y(s)}{U(s)}$ *using Mason's gain formula.*

ii) Determine the transfer function from the state model formulation.

Solution : The state equations are,

$$\dot{X}_1 = X_2, \quad \dot{X}_2 = X_3, \quad \dot{X}_3 = -X_1 - 2X_2 - 3X_3 + u, \; y = X_1$$

Hence the signal flow graph is,

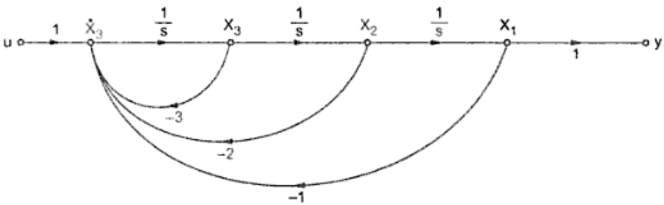


Fig. 3.2

From **Mason's gain formula,**

$$\frac{Y(s)}{U(s)} = \frac{T_1\Delta_1 + T_2\Delta_2 + \ldots\ldots + T_K\Delta_K}{\Delta} \text{ where K = Number of forward paths}$$

Here K = 1, $\quad T_1$ = Forward path gain $= \frac{1}{s} \times \frac{1}{s} \times \frac{1}{s} = \frac{1}{s^3}$

The various loop gains are,

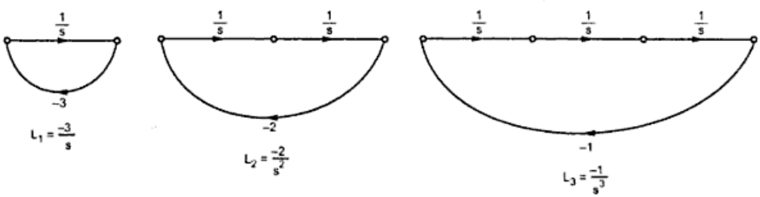


Fig. 3.3

All loops are touching to each other.

$$\therefore \qquad \Delta = 1 - [L_1 + L_2 + L_3] = 1 + \frac{3}{s} + \frac{2}{s^2} + \frac{1}{s^3}$$

Δ_K = Eliminating all loop gains from Δ which are touching to K^{th} forward path

$\therefore \; \Delta_1 = 1$

$$\therefore \quad \frac{Y(s)}{U(s)} = \frac{\frac{1}{s^3}\times 1}{1+\frac{3}{s}+\frac{2}{s^2}+\frac{1}{s^3}} = \boxed{\frac{1}{s^3+3s^2+2s+1}}$$

Now let us use the **state model method** of finding the transfer function.

$$\therefore \quad \frac{Y(s)}{U(s)} = C[sI-A]^{-1}B+D \qquad \text{...D = 0}$$

$$[sI-A] = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$\text{Adj }[sI-A] = \begin{bmatrix} s^2+3s+2 & -1 & -s \\ s+3 & s(s+3) & -2s-1 \\ 1 & s & s^2 \end{bmatrix}^T = \begin{bmatrix} s^2+3s+2 & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -2s-1 & s^2 \end{bmatrix}$$

$$\therefore \quad [sI-A]^{-1} = \frac{\text{Adj }[sI-A]}{|sI-A|} = \frac{\begin{bmatrix} s^2+3s+2 & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -2s-1 & s^2 \end{bmatrix}}{s^3+3s^2+2s+1}$$

$$\therefore \quad \frac{Y(s)}{U(s)} = \frac{[1 \;\; 0 \;\; 0]\begin{bmatrix} s^2+3s+2 & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -2s-1 & s^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{s^3+3s^2+2s+1}$$

$$\therefore \quad \boxed{\frac{Y(s)}{U(s)} = \frac{1}{s^3+3s^2+2s+1}} \qquad \text{....Same as obtained above}$$

Example 3.25 : *The vector* $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ *is an eigen vector of* $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ *Find the eigen value of A corresponding to the vector given.* **(VTU: July/Aug.-2006)**

Solution : The eigen vector is that vector which is non-zero such that

$$AX_i = \lambda_i X_i$$

where $\quad X_i = \text{Eigen vector} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$$\lambda_i = \text{Eigen value} = \text{To be obtained}$$

$$\therefore \quad \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \lambda_i \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix} = \begin{bmatrix} \lambda_i \\ 2\lambda_i \\ -\lambda_i \end{bmatrix}$$

$$\therefore \quad \lambda_i = 5 \qquad \text{...Corresponding eigen value.}$$

Review Questions

1. *Derive the transfer function from state model.*
2. *What is characteristic equation of a system matrix A ?*
3. *Define eigen values and eigen vectors of a matrix.*
4. *When the generalised eigen vectors are used ? How to use them ?*
5. *What is modal matrix ? State its importance.*
6. *State the advantages of diagonalisation of a matrix.*
7. *How to achieve diagonalisation of matrix A ?*
8. *Reduce the given state model in diagonal form,*

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ 8 & -6 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \end{bmatrix} U(t) \text{ and } Y(t) = [1 \;\; 0] \; X(t)$$

9. *What is Vander Monde matrix ? When it exists ? What is its importance ?*
10. *Find the T.F. of the systems with following state models,*

a) $$\dot{X} = \begin{bmatrix} 0 & 1 \\ -1 & -4 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U, \; Y = [1 \;\; 0] \; X$$

b) $$\dot{X} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U, \; Y = [0 \;\; 1] \; X$$

c) $$\dot{X} = \begin{bmatrix} 0 & 0 & -2 \\ -1 & 0 & 0 \\ -4 & -2 & -5 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} U, \; Y = [1 \;\; 0 \;\; 0] \; X$$

11. *Obtain the T.F. of the system having state model,*

$$\dot{X}(t) = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} X(t) + \begin{bmatrix} 2 \\ 5 \end{bmatrix} U(t)$$

$$Y(t) = [1 \;\; 2] \; X(t)$$

$$\left(\textbf{Ans.} \; \frac{12s + 59}{(s+2)(s+4)} \right)$$

4 Solution of State Equations

4.1 Background

Uptill now the methods of obtaining state model in various forms and obtaining transfer function from the state model are discussed. It is seen that the output response depends on the state variables and their initial values. Hence it is necessary to obtain the state vector X(t) which satisfies the state equation $\dot{X}(t) = A\ X(t) + B\ U(t)$ at any time t. This is called solution of state equations, which helps to obtain the output response of a system.

Consider the state equatio n of linear time invariant system as,

$$\dot{X}(t) = A\ X(t) + B\ U(t)$$

The matrices A and B are constant matrices. This state equation can be of two types,

1. Homogeneous and 2. Nonhomogeneous

4.1.1 Homogeneous Equation

If A is a constant matrix and input control forces are zero then the equation takes the form,

$$\dot{X}(t) = A\ X(t) \qquad \text{...(1)}$$

Such an equation is called **homogeneous equation**. The obvious equation is if input is zero,how output can exist ? In such systems, the driving force is provided by the initial conditions of the system to produce the output. For example, consider a series RC circuit in which capacitor is initially charged to V volts. The current is the output. Now there is no input control force i.e.. external voltage applied to the system. But the initial voltage on the capacitor drives the current through the system and capacitor starts discharging through the resistance R. Such a system which works on the initial conditions without any input applied to it is called homogeneous system.

4.1.2 Nonhomogeneous Equation

If A is a constant matrix and matrix U(t) is non zero vector i.e. the input control forces are applied to the system then the equation takes normal form as,

$$\dot{X}(t) = A\,X(t) + B\,U(t) \qquad \text{... (2)}$$

Such an equation is called **nonhomogeneous equation**. Most of the practical systems require inputs to drive them. Such systems are nonhomogeneous linear systems.

The solution of the state equation is obtained by considering basic method of finding the solution of homogeneous equation.

4.2 Review of Classical Method of Solution

Consider a scalar differential equation as,

$$\frac{dx}{dt} = ax \quad \text{where} \quad x(0) = x_0 \qquad \text{... (1)}$$

This is a homogeneous equation without the input vector.

Assume the solution of this equation as,

$$x(t) = b_0 + b_1 t + b_2 t^2 + \ldots + b_k t^k \qquad \text{... (2)}$$

At $t = 0$, $$x(0) = x_0 = b_0 \qquad \text{... (3)}$$

The solution has to satisfy the original differential equation hence using (2) in (1),

$$\frac{d}{dt}[b_0 + b_1 t + \ldots + b_k t^k] = a\,[b_0 + b_1 t + \ldots + b_k t^k]$$

$$\therefore \quad b_1 + 2b_2 t + \ldots + k b_k t^{k-1} = ab_0 + ab_1 t + \ldots + ab_k t^k$$

For validity of this equation, the coefficients of various powers of 't' on both sides, must be equal.

$$\therefore \quad b_1 = ab_0,\ 2b_2 = ab_1,\ \ldots\ kb_k = ab_{k-1}$$

$$\therefore \quad b_1 = a\,b_0$$

$$b_2 = \frac{1}{2}\,a\,b_1 = \frac{1}{2}\,a^2 b_0 = \frac{1}{2!}\,a^2\,b_0$$

$$b_3 = \frac{1}{3}\,a\,b_2 = \frac{1}{3\times 2}\,a^3 b_0 = \frac{1}{3!}\,a^3\,b_0$$

$$\vdots$$

$$b_k = \frac{1}{k!}\,a^k\,b_0 \quad \text{and} \quad x(0) = b_0$$

1

State Variable Analysis and Design

1.1 Background

The conventional approach used to study the behaviour of linear time invariant control systems, uses the time domain or frequency domain methods. In all these methods, the systems are modelled using transfer function approach, which is the ratio of Laplace transform of output to input, neglecting all the initial conditions. Thus this conventional analysis faces all the limitations associated with the transfer function approach.

1.1.1 Limitations of Conventional Approach

Some of its limitations can be stated as :

1) Naturally, significant initial conditions in obtaining precise solution of any system, loose their importance in conventional approach.
2) The method is insufficient and troublesome to give complete time domain solution of higher order systems.
3) It is not very much convenient for the analysis of Multiple Input Multiple Output systems.
4) It gives analysis of system for some specific types of inputs like Step, Ramp etc.
5) It is only applicable to Linear Time Invariant Systems.
6) The classical methods like Root locus, Bode plot etc. are basically trial and error procedures which fail to give the optimal solution required.

Hence it is absolutely necessary to use a method of analysing systems which overcomes most of the above said difficulties. The modern method discussed in this chapter uses the concept of total internal state of the system considering all initial conditions. This technique which uses the concept of state is called **State Variable Analysis** or **State Space Analysis**.

Key Point: *State variable analysis is essentially a time domain approach but it has number of advantages compared to conventional methods of analysis.*

1.1.2 Advantages of State Variable Analysis

The various advantages of state variable analysis are,

1) The method takes into account the effect of all initial conditions.
2) It can be applied to nonlinear as well as time varying systems.
3) It can be conveniently applied to Multiple Input Multiple Output systems.
4) The system can be designed for the optimal conditions precisely by using this modern method.
5) Any type of the input can be considered for designing the system.
6) As the method involves matrix algebra, can be conveniently adopted for the digital computers.
7) The state variables selected need not necessarily be the physical quantities of the system.
8) The vector matrix notation greatly simplifies the mathematical representation of the system.

1.2 Concept of State

Consider a football match. The score in the football match must be updated at every instant from the knowledge and information of the total score before that instant. This procedure of updating the score continues till the end of the match when we get exact and precise score of the entire match. This updating procedure has main importance in the understating of the concept of state.

Consider the network as shown in the Fig. 1.1

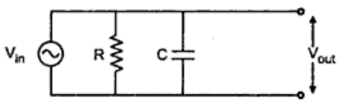


Fig. 1.1

To find V_{out}, the knowledge of the initial capacitor voltage must be known. Only information about V_{in} will not be sufficient to obtain precisely the V_{out} at any time $t \geq 0$. Such systems in which the output is not only dependent on the input but also on the initial conditions are called the *systems with memory* or *dynamic systems.*

While if in the above network capacitor `C' is replaced by another resistance `R_1' then output will be dependent only on the input applied V_{in}.

Such systems in which the output of the system depends only on the input applied at t = 0, are called *systems with zero memory or static systems.*

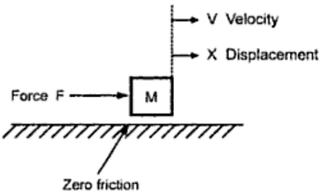


Fig. 1.2

Consider another example of simple mechanical system as shown in the Fig. 1.2

Now according to Newton's law of motion,

$$F = Ma$$

$$a = \text{Acceleration of mass M}$$

$$\therefore \quad F = M\frac{d}{dt}(v(t))$$

i.e.
$$v(t) = \frac{1}{M}\int f(t)\,dt$$

Now velocity at any time 't' is the result of the force F applied to the particle in the entire past,

$$v(t) = \frac{1}{M}\int_{-\infty}^{t} f(t)\,dt = \frac{1}{M}\int_{-\infty}^{t_0} f(t)\,dt + \frac{1}{M}\int_{t_0}^{t} f(t)\,dt$$

where t_0 = Initial time.

Now $\frac{1}{M}\int_{-\infty}^{t_0} f(t)\,dt = v(t_0)$

$$\therefore \quad \boxed{v(t) = v(t_0) + \int_{t_0}^{t} f(t)\,dt}$$

From the above equation it is clear that for the same input f(t), we get the different values of the velocities v(t) depending upon our choice of parameter t_0 and the value of $v(t_0)$.

If $v(t_0)$ is known and the input vector from 't_0' to 't' is known then we can obtain a unique value of the output v(t) at any time $t > t_0$.

Key Point : *Thus initial conditions i.e. memory affects the system characterisation and subsequent behaviour.*

Thus initial conditions describe the status or state of the system at $t = t_0$. **The state can be regarded as a compact and concise representation of the past history of the system.** So the state of the system in brief separates the future from the past so that the state contains all the information concerning the past history of the system necessarily required to determine the response of the system for any given type of input.

The state of the system at any time 't' is actually the combined effect of the values of all the different elements of the system which are associated with the initial conditions of the system. **Thus the complete state of the system can be considered to be a vector having components which are the variables of system which are closely associated with**

initial conditions. So state can be defined as vector X(t) called state vector. This X(t) i.e. state at any time `t' is `n' dimensional vector i.e. column matrix $n \times 1$ as indicated below.

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ : \\ : \\ X_n(t) \end{bmatrix}$$

Now these variables $X_1(t)$, $X_2(t)$, ... $X_n(t)$ which constitute the state vector X(t) are called the *state variables of the system.*

If state at $t = t_1$ is to be decided then we must know $X(t_0)$ and knowledge of the input applied between $t_0 \rightarrow t_1$. This new state will be $X(t_1)$ which will act as initial state to find out the state at any time $t > t_1$. This is called the updating of the state. The output of the system at $t = t_1$ will be the function of $X(t_1)$ and the instantaneous value of the input at $t = t_1$, if any. The number of the state variables for a system is generally equal to the order of the system. The number of independent state variables is normally equal to the number of energy storing elements (e.g.: capacitor voltage, current in inductor) contained in the system.

1.2.1 Important Definitions

1) **State** : The state of a dynamic system is defined as a minimal set of variables such that the knowledge of these variables at $t = t_0$ together with the knowledge of the inputs for $t \geq t_0$, completely determines the behaviour of the system for $t > t_0$.

2) **State Variables** : The variables involved in determining the state of a dynamic system X(t), are called the state variables. $X_1(t)$, $X_2(t)$ ......$X_n(t)$ are nothing but the state variables. These are normally the energy storing elements contained in the system.

3) **State Vector** : The `n' state variables necessary to describe the complete behaviour of the system can be considered as `n' components of a vector X(t) called the state vector at time `t'. The state vector X(t) is the vector sum of all the state variables.

4) **State Space** : The space whose co-ordinate axes are nothing but the `n' state variables with time as the implicit variable is called the state space.

5) **State Trajectory** : It is the locus of the tips of the state vectors, with time as the implicit variable.

Definitions can be explained by considering second order system :

Order is 2 so number of state variables required is 2 say $X_1(t)$ and $X_2(t)$.

State vector will be the matrix of order 2×1.

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}$$

i.e. $$\overrightarrow{X(t)} = \overrightarrow{X_1(t)} + \overrightarrow{X_2(t)} \quad \text{(vector addition)}$$

The state space will be a plane in this case as the number of variables are two. Thus state space can be shown as in the Fig. 1.3.

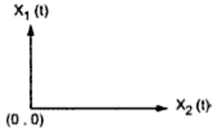


Fig. 1.3

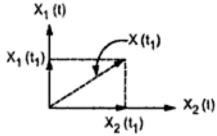


Fig. 1.4

Now consider $t = t_1$

$\therefore$ $$\overrightarrow{X(t_1)} = \overrightarrow{X_1(t_1)} + \overrightarrow{X_2(t_1)}$$

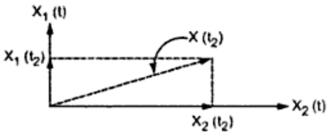


Fig. 1.5

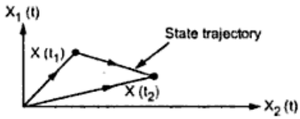


Fig. 1.6

Now consider $t = t_2$

$\therefore$ $$\overrightarrow{X(t_2)} = \overrightarrow{X_1(t_2)} + \overrightarrow{X_2(t_2)}$$

The state trajectory can be shown by joining the tips of the two state vectors i.e. $X(t_1)$ and $X(t_2)$.

1.3 State Model of Linear Systems

Consider Multiple Input Multiple Output, nth order system as shown in the Fig. 1.7.

Number of inputs = m

Number of outputs = p

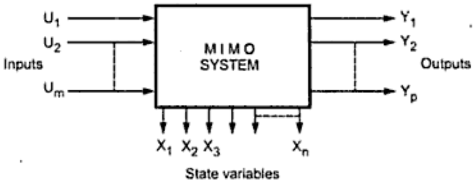


Fig. 1.7

$$U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ : \\ : \\ U_m(t) \end{bmatrix}, X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ : \\ : \\ X_n(t) \end{bmatrix}, Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \\ : \\ : \\ Y_p(t) \end{bmatrix}$$

All are column vectors having orders $m \times 1$, $n \times 1$ and $p \times 1$ respectively.

For such a system, the state variable representation can be arranged in the form of 'n' first order differential equations.

$$\frac{dX_1(t)}{dt} = \dot{X}_1(t) = f_1(X_1, X_2, \ldots\ldots X_n, U_1, U_2 \ldots\ldots\ldots U_m)$$

$$\frac{dX_2(t)}{dt} = \dot{X}_2(t) = f_2(X_1, X_2, \ldots\ldots X_n, U_1, U_2 \ldots\ldots\ldots U_m)$$

$$:$$

$$:$$

$$\frac{dX_n(t)}{dt} = \dot{X}_n(t) = f_n(X_1, X_2, \ldots\ldots X_n, U_1, U_2 \ldots\ldots\ldots U_m)$$

Where $f = \begin{bmatrix} f_1 \\ f_2 \\ : \\ : \\ f_n \end{bmatrix}$ is the functional operator.

Integrating the above equation,

$$X_i(t) = X_i(t_0) + \int_{t_0}^{t} f_i (X_1, X_2 ......X_n \; U_1, U_2 .....U_m) \, dt$$

where i = 1, 2, ...n.

Thus `n' state variables and hence state vector at any time `t' can be determined uniquely.

Any `n' dimensional time invariant system has state equations in the functional form as,

$$\dot{X}(t) = f(X, U)$$

While outputs of such system are dependent on the state of system and instantaneous inputs.

∴ Functional output equation can be written as,

$$Y(t) = g(X, U) \quad \text{where `g' is the functional operator.}$$

For time variant system, the same equations can be written as,

$$\boxed{\begin{aligned} \dot{X}(t) &= f(X, U, t) \text{ .... State equation} \\ Y(t) &= g(X, U, t) \text{ .... Output equation} \end{aligned}}$$

Diagramatically this can be represented as in the Fig. 1.8.

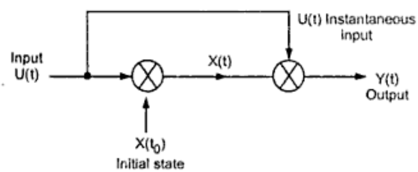


Fig. 1.8 Input-output state description of a system

The functional equations can be expressed interms of linear combination of system states and the input as,

$$\dot{X}_1 = a_{11} X_1 + a_{12}X_2 + \ldots + a_{1n}X_n + b_{11}U_1 + b_{12}U_2 + \ldots + b_{1m} U_m$$

$$\dot{X}_2 = a_{21} X_1 + a_{22}X_2 \ldots + a_{2n}X_n + b_{21}U_1 + b_{22}U_2 + \ldots + b_{2m} U_m$$

$$\vdots$$

$$\dot{X}_n = a_{n1} X_1 + a_{n2}X_2 + \ldots + a_{nn}X_n + b_{n1}U_1 + b_{n2}U_2 + \ldots + b_{nm} U_m$$

For the linear time invariant systems, the coefficients a_{ij} and b_{ij} are constants. Thus all the equations can be written in vector matrix form as,

$$\boxed{\dot{X}(t) = A\, X(t) + B\, U(t)}$$

where $X(t)$ = State vector matrix of order $n \times 1$

$U(t)$ = Input vector matrix of order $m \times 1$

A = **System matrix or Evolution matrix** of order **$n \times n$**

B = **Input matrix or control matrix** of order **$n \times m$**

Similarly the output variables at time t can be expressed as the linear combinations of the input variables and state variables at time t as,

$$Y_1(t) = c_{11} X_1(t) + \ldots + c_{1n} X_n(t) + d_{11} U_1(t) + \ldots + d_{1m} U_m(t)$$

$$\vdots$$

$$Y_p(t) = c_{p1} X_1(t) + \ldots + c_{p1} X_n(t) + d_{p1} U_1(t) + \ldots + d_{pm} U_m(t)$$

For the linear time invariant systems, the coefficients c_{ij} and d_{ij} are constants. Thus all the output equations can be written in vector matrix form as,

$$\boxed{Y(t) = C\, X(t) + D\, U(t)}$$

where $Y(t)$ = Output vector matrix of order $p \times 1$

C = **Output matrix or observation matrix** of order $p \times n$

D = **Direct transmission matrix** of order $p \times m$

The two vector equations together is called the **state model** of the linear system.

$\therefore$

$$\dot{X}(t) = A\, X(t) + B\, U(t) \quad \text{...State equation}$$

$$Y(t) = C\, X(t) + D\, U(t) \quad \text{...Output equation}$$

This is state model of a system.

For linear time-variant systems, the matrices A, B, C and D are also time dependent. Thus,

$$\left.\begin{aligned}\dot{X}(t) &= A(t)\,X(t) + B(t)\,U(t)\\ Y(t) &= C(t)\,X(t) + D(t)\,U(t)\end{aligned}\right\}\ \text{For linear time variant system}$$

1.3.1 State Model of Single Input Single Output System

Consider a single input single output system i.e. m = 1 and p = 1. But its order is 'n' hence n state variables are required to define state of the system. In such a case, the state model is

$$\dot{X}(t) = A\,X(t) + B\,U(t)$$

$$Y(t) = C\,X(t) + d\,U(t)$$

where $A = n \times n$ matrix, $B = n \times 1$ matrix

$C = 1 \times n$ matrix, d = constant

and $U(t)$ = single scalar input variable

In general remember the orders of the various matrices.

$\therefore$

$$\boxed{\begin{aligned}&\mathbf{A} = \text{Evolution matrix} \Rightarrow \mathbf{n \times n}\\ &\mathbf{B} = \text{Control matrix} \Rightarrow \mathbf{n \times m}\\ &\mathbf{C} = \text{Observation matrix} \Rightarrow \mathbf{p \times n}\\ &\mathbf{D} = \text{Transmission matrix} \Rightarrow \mathbf{p \times m}\end{aligned}}$$

1.4 State Diagram Representation

It is the pictorial representation of the state model derived for the given system. It forms a close relationship amongst the state model, differential equations of the system and its solution. It is basically a block diagram type approach which is designed from the view of programming of a computer. The basic advantage of state diagram is when it is impossible to select the state variables as physical variables. When transfer function of system is given then state diagram may be obtained first. And then by assigning mathematical state variables there in, standard state model can be obtained.

State diagram of a linear time invariant continuous system is discussed here for the sake of simplicity. It is a proper interconnection of three basic units.

i) Scalars ii) Adders iii) Integrators

Scalars are nothing but like amplifiers having required gain.

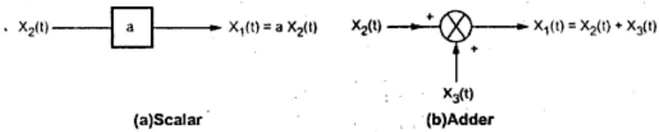


Fig. 1.9

Adders are nothing but summing points.

Integrators are the elements which actually integrate the differentiation of state variable to obtain required state variable.

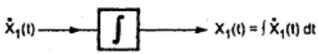


Fig. 1.9 (c)

Now integrators are denoted as '1/s' in Laplace transform. So **transfer function of any integrator is always 1/s.** With these three basic units we can draw the state diagrams of any order system.

Key Point : *The output of each integrator is the state variable.*

1.4.1 State Diagram of Standard State Model

Consider standard state model

$$\dot{X}(t) = A\,X(t) + B\,U(t) \quad \text{and}$$

$$Y(t) = C\,X(t) + D\,U(t)$$

So its state diagram will be as in the Fig. 1.10.

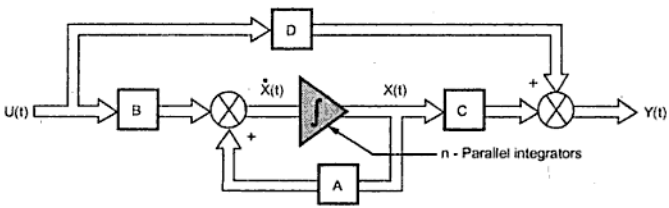


Fig. 1.10 State diagram of MIMO system

The thick arrows indicate that there are multiple number of input, output and state variables. There must be n parallel integrators for n state variables. The output of each integrator is a seperate state variable. If such a state diagram for the system is obtained then the state model from the diagram can be easily obtained.

Remarks

1. To obtain the state model from state diagram, always choose **output of each integrator as a state variable**. Number of integrators always equals the order of the system i.e. 'n'
2. Differentiators are not used in the state diagram as they amplify the inevitable noise.

Example 1.1 : *Obtain the state diagram of SISO system represented by equations,*

$$\dot{X}_1(t) = a_1 X_1(t) + b_1 U(t), \quad \dot{X}_2(t) = a_2 X_1(t) + a_3 X_2(t) + b_2 U(t)$$

and $$Y(t) = c_1 X_1(t) + c_2 X_2(t)$$

Solution : As there are 2 state variables $X_1(t)$ and $X_2(t)$, the two integrators are required.

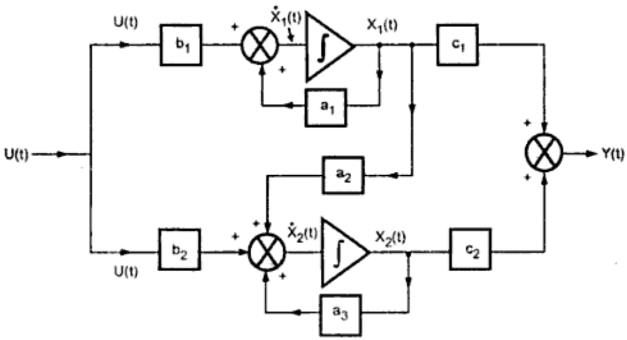


Fig. 1.11

1.5 Non Uniqueness of State Model

Consider a standard state model for a system

$$\dot{X}(t) = A\, X(t) + B\, U(t) \qquad \text{... (1)}$$

$$y(t) = C\, X(t) + D\, U(t) \qquad \text{... (2)}$$

Let X = M Z where M is a *non singular* constant matrix.

Key Point: *A matrix whose determinant is nonzero is called nonsingular matrix.*

This means Z is new set of state variables which is obtained by linear combinations of the original state variables.

$$X = MZ \quad \text{hence} \quad \dot{X} = M\dot{Z}$$

Substituting in a state model

$$M\,\dot{Z}(t) = AM\,Z(t) + B\,U(t) \qquad \text{... (3)}$$

$$Y(t) = CM\,Z(t) + D\,U(t) \qquad \text{... (4)}$$

$\therefore$ Premultiplying equation (3) by M^{-1}

$$\therefore \quad \dot{Z}(t) = M^{-1}AM\,Z(t) + M^{-1}\,B\,U(t)$$

where $\quad M^{-1}M = I$ i.e. Identity matrix

$$\therefore \quad \dot{Z}(t) = \hat{A}\,Z(t) + \hat{B}\,U(t) \qquad \text{... (5)}$$

where
$$\hat{A} = M^{-1}\,AM$$
$$\hat{B} = M^{-1}\,B$$

and
$$Y(t) = \hat{C}\,Z(t) + D\,U(t) \qquad \text{... (6)}$$

Where
$$\hat{C} = CM$$

Equations (5) and (6) forms a new state model of the system.

This shows that state model is not a unique property.

Key Point: *Any linear combinations of the original set of state variables results into a valid new set of state variables.*

1.6 Linearization of State Equation

Any general time invariant system is said to be in equilibrium, at a point (X_0, U_0) when,

$$\boxed{\dot{X} = f(X_0, U_0) = 0}$$

The derivatives of all the state variables are zero at a point of equilibrium. The system has a tendency to lie at the equilibrium point unless and untill it is forcefully disturbed.

The state equation $\dot{X}(t) = f(X, U)$ can be linearized for small deviations about an equilibrium point (X_0, U_0). This is possible by expanding the state equation using Taylor series and considering only first order terms, neglecting second and higher order terms.

So expanding k^{th} state equation,

$$\dot{X}_k = f_k(X_0, U_0) + \sum_{l=1}^{n} \left.\frac{\partial f_l(X,U)}{\partial X_l}\right|_{\substack{X=X_0\\U=U_0}} (X_l - X_{l0}) + \sum_{j=1}^{m} \left.\frac{\partial f_k(X,U)}{\partial U_j}\right|_{\substack{X=X_0\\U=U_0}} (U_j - U_{j0})$$

Now $\quad f_k(X_0, U_0) = 0$ at the equilibrium point.

And let the variation about the operating point is,

$$\tilde{X}_l = X_l - X_{l0} \quad \text{hence } \dot{\tilde{X}} = \dot{X}_l$$

and $$\tilde{U}_j = U_j - U_{j0}$$

Hence the k^{th} state equation can be linearized as,

$$\dot{\tilde{X}}_k = \sum_{l=1}^{n} \left.\frac{\partial f_k(X,U)}{\partial X_l}\right|_{\substack{X=X_0\\U=U0}} \tilde{X}_l + \sum_{j=1}^{m} \left.\frac{\partial f_k(X,U)}{\partial U_j}\right|_{\substack{X=X_0\\U=U_0}} \tilde{U}_j$$

In this equation $\dot{\tilde{X}}_k$, $\tilde{X}_l$ and $\tilde{U}_j$ are the vector matrices and the remaining terms are the matrices of order n × n and n × m respectively. Hence the linearized equation can be expressed in the vector matrix form as,

$$\boxed{\dot{\tilde{X}} = A\tilde{X} + B\tilde{U}}$$

where
$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \text{ is } n \times n \text{ matrix}$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial U_1} & \frac{\partial f_1}{\partial U_2} & \cdots & \frac{\partial f_1}{\partial U_m} \\ \frac{\partial f_2}{\partial U_1} & \frac{\partial f_2}{\partial U_2} & \cdots & \frac{\partial f_2}{\partial U_m} \\ \vdots & & \cdots & \vdots \\ \frac{\partial f_n}{\partial U_1} & \frac{\partial f_n}{\partial U_2} & \cdots & \frac{\partial f_n}{\partial U_m} \end{bmatrix} \text{ is } n \times m \text{ matrix}$$

All the partial derivatives of the matrices A and B are to be obtained at an equilibrium state (X_0, U_0). Such matrices A and B defined above interms of partial derivatives are called the **Jacobian matrices**.

Review Questions

1. *Explain the concept of state.*
2. *Define and explain the following terms,*
   - *a. State variables*
   - *b. State vector*
   - *c. State trajectory*
   - *d. State*
   - *e. State space*
3. *Explain advantages of state variable method over conventional one.*
4. *Write a short note on advantages and limitations of state variable approach.*
5. *Write a note on linearization of state equations.*
6. *Prove the nonuniqueness of the state model.*

❑❑❑

2

State Space Representation

2.1 State Variable Representation using Physical Variables

The state variables are minimum number of variables which are associated with all the initial conditions of the system. As their sequence is not important, the state model of the system is not unique. But for all the state models it is necessary that the number of state variables is equal and minimal. This number 'n' indicates the **order** of the system. For second order system minimum two state variables are necessary and so on.

To obtain the state model for a given system, it is necessary to select the state variables. Many a times, the various physical quantities of system itself are selected as the state variables.

For the **electrical systems,** the **currents through various inductors** and the **voltage across the various capacitors** are selected to be the state variables. Then by any method of network analysis, the equations must be written interms of the selected state variables, their derivatives and the inputs. The equations must be rearranged in the standard form so as to obtain the required state model.

Key Point: *It is important that the equation for differentiation of one state variable should not involve the differentiation of any other state variable.*

In the mechanical systems the displacements and velocities of energy storing elements such as spring and friction are selected as the state variables.

In general, **the physical variables associated with energy storing elements, which are responsible for initial conditions, are selected as the state variables of the given system.**

Example 2.1 : *Obtain the state model of the given electrical system.*

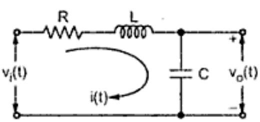


Fig. 2.1

(VTU : July/Aug.-2006)

Solution : There are two energy storing elements L and C. So the two state variables are current through inductor i(t) and voltage across capacitor i.e. $v_o(t)$.

$\therefore \quad X_1(t) = i(t) \quad \text{and} \quad X_2(t) = v_o(t)$

And $\quad U(t) = v_i(t) = \text{Input variable}$

Applying KVL to the loop,

$$v_i(t) = i(t)\, R + L\,\frac{di(t)}{dt} + v_o(t)$$

Arrange it for di(t)/dt,

$$\therefore \quad \frac{di(t)}{dt} = \frac{1}{L}v_i(t) - \frac{R}{L}i(t) - \frac{1}{L}v_o(t) \quad \text{but} \quad \frac{di(t)}{dt} = \dot{X}_1(t)$$

$$\text{i.e.} \quad \dot{X}_1(t) = -\frac{R}{L}X_1(t) - \frac{1}{L}X_2(t) + \frac{1}{L}U(t) \qquad \dots (1)$$

$$\text{While} \quad v_o(t) = \text{Voltage across capacitor} = \frac{1}{C}\int i(t)\,dt$$

$$\therefore \quad \frac{dv_o(t)}{dt} = \frac{1}{C}i(t) \quad \text{but} \quad \frac{dv_o(t)}{dt} = \dot{X}_2(t)$$

$$\text{i.e.} \quad \dot{X}_2(t) = \frac{1}{C}X_1(t) \qquad \dots (2)$$

The equations (1) and (2) give required state equation.

$$\begin{bmatrix} \dot{X}_1(t) \\ \dot{X}_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} U(t)$$

$$\text{i.e.} \quad \dot{X}(t) = A\,X(t) + B\,U(t)$$

While the output variable $Y(t) = v_o(t) = X_2(t)$

$$\therefore \quad Y(t) = [0 \quad 1] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + [0]\,U(t)$$

$$\text{i.e.} \quad Y(t) = C\,X(t) \quad \text{and} \quad D = [0]$$

This is the required state model. As n = 2, it is second order system.

> **Note :** The order of the state variables is not important. $X_1(t)$ can be $v_o(t)$ and $X_2(t)$ can be i(t) due to which state model matrices get changed. Hence **state model is not the unique property of the system.**

Example 2.2 : *Obtain the state model of the given electrical network in the standard form.*

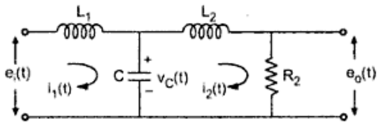


Fig. 2.2

Solution :

$$U(t) = \text{input} = e_i(t)$$

$$Y(t) = \text{output} = e_o(t)$$

State variables : $X_1(t) = i_1(t),\ X_2(t) = i_2(t),\ X_3(t) = v_C(t)$

Writing the equations :

$$e_i(t) = L_1 \frac{di_1(t)}{dt} + v_C(t)$$

i.e.
$$\frac{di_1(t)}{dt} = \frac{1}{L_1} e_i(t) - \frac{1}{L_1} v_C(t)$$

$\therefore$
$$\dot{X}_1(t) = \frac{1}{L_1} U(t) - \frac{1}{L_1} X_3(t) \qquad \text{... (1)}$$

Then,
$$v_C(t) = L_2 \frac{di_2(t)}{dt} + i_2(t) R_2$$

$\therefore$
$$\frac{di_2(t)}{dt} = \frac{1}{L_2} v_C(t) - \frac{R_2}{L_2} i_2(t)$$

$\therefore$
$$\dot{X}_2(t) = \frac{1}{L_2} X_3(t) - \frac{R_2}{L_2} X_2(t) \qquad \text{... (2)}$$

and
$$C\frac{dv_C(t)}{dt} = i_1(t) - i_2(t) = \text{Current through capacitor}$$

$\therefore$
$$\frac{dv_C(t)}{dt} = \frac{1}{C} i_1(t) - \frac{1}{C} i_2(t)$$

$\therefore$
$$\dot{X}_3(t) = \frac{1}{C} X_1(t) - \frac{1}{C} X_2(t) \qquad \text{... (3)}$$

$$\therefore \quad \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} U(t)$$

i.e. $\dot{X}(t) = AX(t) + BU(t)$

and $e_o(t) = i_2(t) R_2$

$\therefore$ $Y(t) = X_2(t) \; R_2$

$$\therefore \quad Y(t) = \begin{bmatrix} 0 & R_2 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

i.e. $Y(t) = CX(t) + DU(t)$

where D = 0. This is the required state model.

2.1.1 Advantages

The advantages of using available physical variables as the state variables are,

1. The physical variables which are selected as the state variables are the physical quantities and can be measured.
2. As state variables can be physically measured, the feedback may consists the information about state variables in addition to the output variables. Thus design with state feedback is possible.
3. Once the state equations are solved and solution is obtained, directly the behaviour of various physical variables with time is available.

But the important limitation of this method is that obtaining solution of such state equation with state variables as physical variables is very difficult and time consuming.

2.2 State Space Representation using Phase Variables

Let us study how to obtain state space model using phase variables. The **phase variables** are those state variables which are obtained by assuming one of the system variable as a state variable and other state variables are the derivatives of the selected system variable. Most of the time, the system variable used is the output variable which is used to select the state variable.

Such set of phase variables is easily obtained if the differential equation of the system is known or the system transfer function is available.

2.2.1 State Model from Differential Equation

Consider a linear continuous time system represented by n^{th} order differential equation as,

$$Y^n + a_{n-1} Y^{n-1} + a_{n-2} Y^{n-2} + \ldots + a_1 \dot{Y} + a_0 Y(t) = b_0 U + b_1 \dot{U} + \ldots + b_{m-1} U^{m-1} + b_m U^m \quad \ldots (1)$$

In the equation, $Y^n(t) = \dfrac{dY^n(t)}{dt^n} = n^{th}$ derivative of Y(t).

For time invariant system, the coefficients $a_{n-1}, a_{n-2}, \ldots a_0, b_0, b_1, \ldots b_m$ are constants.

For the system,

$$Y(t) = \text{Output variable}$$

$$U(t) = \text{Input variable}$$

$Y(0), \dot{Y}(0), \ldots Y(0)^{n-1}$ represent the initial conditions of the system.

Consider the **simple case** of the system in which derivatives of the control force U(t) are absent.

Thus $\quad \dot{U}(t) = \ddot{U}(t) = \ldots = U^m(t) = 0$

$$\therefore Y^n + a_{n-1} Y^{n-1} + \ldots + a_1 \dot{Y} + a_0 Y(t) = b_0 U(t) \quad \ldots (2)$$

Choice of state variable is generally output variable Y(t) itself. And other state variables are derivatives of the selected state variable Y(t).

$$\therefore \quad X_1(t) = Y(t)$$

$$\therefore \quad X_2(t) = \dot{Y}(t) = \dot{X}_1(t)$$

$$\therefore \quad X_3(t) = \ddot{Y}(t) = \ddot{X}_1(t) = \dot{X}_2(t)$$

$$\vdots$$

Thus the various state equations are,

$$\dot{X}_1(t) = X_2(t)$$

$$\dot{X}(t) = X_3(t)$$

$$\vdots$$

$$\dot{X}_{n-1}(t) = X_n(t)$$

$$\dot{X}_n(t) = ?$$

Note that only n variables are to be defined to keep their number minimum. Thus $\dot{X}_{n-1}(t)$ gives n^{th} state variable $X_n(t)$. But to complete state model $\dot{X}_n(t)$ is necessary.

Important : $\dot{X}_n(t)$ is to be obtained by substituting the selected state variables in the original differential equation (2). We have $Y(t) = X_1$, $\dot{Y}(t) = X_2$, $\ddot{Y}(t) = X_3$, ...

$$Y^{n-1}(t) = X_n(t),\ Y^n(t) = \dot{X}_n(t)$$

$$\therefore \dot{X}_n(t) + a_{n-1} X_n(t) + a_{n-2} X_{n-1}(t) + \ldots + a_1 X_2 t + a_0 X_1(t) = b_0 U(t)$$

$$\therefore \dot{X}_n(t) = -a_0 X_1 - a_1 X_2 \ldots - a_{n-2} X_{n-1} - a_{n-1} X_n + b_0 U(t) \qquad \ldots (3)$$

Hence all the equations now can be expressed in vector matrix form as,

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \vdots \\ \dot{X}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ -a_0 & -a_1 & \cdots & \ldots & -a_{n-1} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_0 \end{bmatrix} U(t)$$

i.e.

$$\dot{X} = AX(t) + BU(t)$$

Such set of state variables is called set of **phase variables.**

The matrix A is called matrix in **phase variable form** and it has following features,

1) Upper off diagonal i.e. upper parallel row to the main principle diagonal contains all elements as 1.
2) All other elements except last row are zeros.
3) Last row consists of the negatives of the coefficients contained by the original differential equation.

Such a form of matrix A is also called **Bush form** or **Companion form.** Hence the method is also called **companion form realization.**

The output equation is,

$$Y(t) = X_1(t)$$

$$\therefore \quad Y(t) = [\,1 \;\; 0 \ldots. 0\,] \begin{bmatrix} X_1(t) \\ X_2(t) \\ : \\ : \\ X_n(t) \end{bmatrix}$$

i.e. $\quad Y(t) = CX(t) \quad$ where $D = 0$

This model in the Bush form can be shown in the state diagram as in the Fig. 2.3.

Output of each integrator is a state variable.

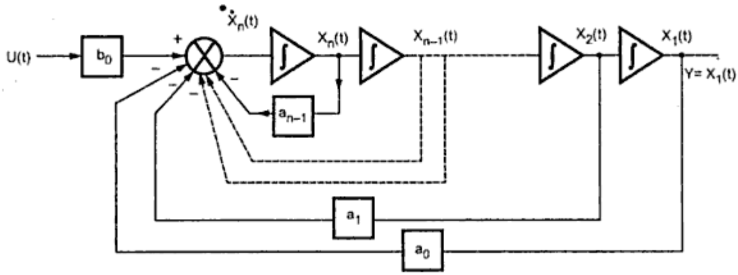


Fig. 2.3 State diagram for phase variable form

> Observe that the transfer function of the blocks in the various feedback paths are the coefficients existing in the original differential equation.

If the differential equation consists of the derivatives of the input control force U(t) then this method is not useful. In such a case, the state model is to be obtained from the transfer function.

Example 2.3 : *Construct the state model using phase variables if the system is described by the differential equation,*

$$\frac{d^2Y(t)}{dt^3} + 4\frac{d^2Y(t)}{dt^2} + 7\frac{dY(t)}{dt} + 2Y(t) = 5U(t)$$

Draw the state diagram.

Solution : Choose output Y(t) as the state variable $X_1(t)$ and successive derivatives of it give us remaining state variables. As order of the equation is 3, **only 3 state variables are allowed.**

$$X_1(t) = Y(t)$$

$\therefore$ $$X_2(t) = \dot{X}_1(t) = \dot{Y}(t) = \frac{dY(t)}{dt}$$

and $$X_3(t) = \dot{X}_2(t) = \ddot{Y}(t) = \frac{d^2Y(t)}{dt^2}$$

Thus $$\dot{X}_1(t) = X_2(t) \qquad \text{... (1)}$$

$$\dot{X}_2(t) = X_3(t) \qquad \dots (2)$$

To obtain $\dot{X}_3(t)$, substitute state variables obtained in the differential equation.

$$\frac{d^3Y(t)}{dt^3} = \dddot{Y}(t) = \frac{d}{dt}[\ddot{Y}(t)] = \frac{dX_3}{dt} = \dot{X}_3(t)$$

$$\therefore \dot{X}_3(t) + 4X_3(t) + 7X_2(t) + 2X_1(t) = 5U(t)$$

$$\therefore \quad \dot{X}_3(t) = -2X_1(t) - 7X_2(t) - 4X_3(t) + 5U(t) \qquad \dots (3)$$

The equations (1), (2) and (3) give us required state equation.

$$\therefore \quad \dot{X}(t) = AX(t) + BU(t)$$

where
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -7 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

The output is, $Y(t) = X_1(t)$

$$\therefore \quad Y(t) = CX(t) + DU(t)$$

where $C = [1\ 0\ 0]$, $D = 0$

This is the required state model using phase variables.

The state diagram is shown in the Fig. 2.4.

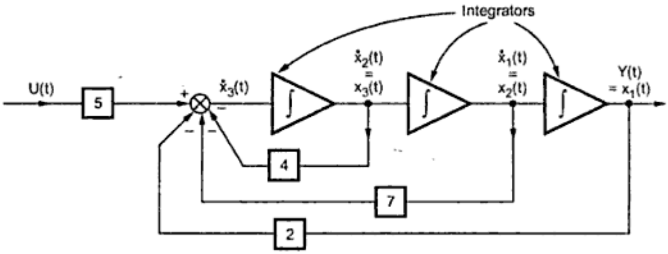


Fig. 2.4

2.2.2 State Model from Transfer Function

Consider a system characterized by the differential equation containing derivatives of the input variable U(t) as,

$$Y^n + a_{n-1} Y^{n-1} + \dots + a_1 \dot{Y} + a_0 Y(t) = b_0 U + b_1 \dot{U} + \dots + b_{m-1} U^{m-1} + b_m U^m \qquad \dots (1)$$

In such a case, it is advantageous to obtain the transfer function, assuming zero initial conditions. Taking Laplace transform of both sides of equation (1) and neglecting initial conditions we get,

$$Y(s)\,[s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0] = [b_0 + sb_1 + \ldots + b_{m-1} s^{m-1} + b_m s^m]\, U(s)$$

$$\therefore \qquad \frac{Y(s)}{U(s)} = T(s) = \frac{b_0 + sb_1 + \ldots + b_{m-1} s^{m-1} + b_m s^m}{a_0 + sa_1 + \ldots + a_{n-1} s^{n-1} + s^n} \qquad \ldots (2)$$

Practically in most of the control systems m < n but for general case, let us assume m = n.

$$\therefore \qquad T(s) = \frac{b_0 + sb_1 + \ldots + b_{n-1} s^{n-1} + b_n s^n}{a_0 + sa_1 + \ldots + a_{n-1} s^{n-1} + s^n}$$

From such a transfer function, state model can be obtained and then zero initial conditions can be replaced by the practical initial conditions to get required result.

There are two methods of obtaining state model from the transfer function,

1) Using signal flow graph approach

2) Using direct decomposition of transfer function

1) Using Signal Flow Graph Approach

The Mason's gain formula for signal flow graph states that,

$$T(s) = \frac{\Sigma T_k \Delta_k}{\Delta}$$

where

$$T_k = \text{Gain of } k^{th} \text{ forward path}$$

$$\Delta = \text{System determinant}$$

$$= 1 - \{\Sigma \text{ all loop gains}\} + \left\{\begin{matrix}\Sigma \text{ Gain} \times \text{Gain product of} \\ \text{all combinations of two} \\ \text{non - touching loops}\end{matrix}\right\} - \ldots$$

$$\Delta_k = \text{Value of } \Delta \text{ elliminating those loop gains and products which are touching to } k^{th} \text{ forward path}$$

According to this formula, construct the signal flow graph from the transfer function. From the signal flow graph, state model can be obtained. While obtaining signal flow graph, try to get the gains of branches as '1/s' representing the integrators. This helps to obtain the required state model. To clear this idea, let us consider the system having transfer function,

$$T(s) = \frac{b_0 + sb_1 + s^2 b_2 + s^3 b_3}{a_0 + sa_1 + s^2 a_2 + s^3} = \frac{Y(s)}{U(s)}$$

Divide both numerator and denominator by highest power of s,

$$\therefore \quad T(s) = \frac{\frac{b_0}{s^3} + \frac{b_1}{s^2} + \frac{b_2}{s} + b_3}{1 + \frac{a_2}{s} + \frac{a_1}{s^2} + \frac{a_0}{s^3}} = \frac{b_3 + \frac{b_2}{s} + \frac{b_1}{s^2} + \frac{b_0}{s^3}}{1 - \left[-\frac{a_2}{s} - \frac{a_1}{s^2} - \frac{a_0}{s^3} \right]}$$

$$= \frac{T_1 \Delta_1 + T_2 \Delta_2 + T_3 \Delta_3 + T_4 \Delta_4}{1 - [\Sigma \text{ All loop gains}] + [\Sigma \text{ Gain products of 2 non-touching loops}] - \cdots}$$

Assuming that there are no combinations of 2 and more non-touching loops.

$\therefore$ Loop gains are, $L_1 = -\frac{a_2}{s}$, $L_2 = -\frac{a_1}{s^2}$, $L_3 = -\frac{a_0}{s^3}$

And let $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 1$ i.e. all loops are touching to all the forward paths. Hence forward path gains are,

$$T_1 = b_3, \ T_2 = \frac{b_2}{s}, \ T_3 = \frac{b_1}{s^2}, \quad T_4 = \frac{b_0}{s^3}$$

Involving various branches having gains $\frac{1}{s}$, the signal flow graph can be obtained as,

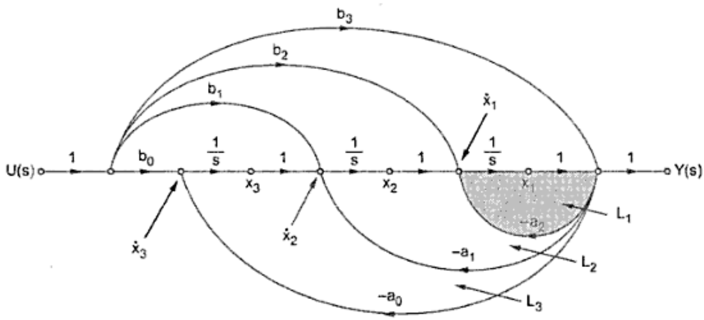


Fig. 2.5

Each branch with gain $\frac{1}{s}$ represents an integrator. **Output of each integrator is a state variable.** According to signal flow graph, value of the variable at the node is an algebraic sum of all the signals entering at that node. Outgoing branches does not affect the value of variable. Hence from signal flow graph,

$$\dot{X}_1 = b_2U + X_2 - a_2 Y$$

$$\dot{X}_2 = b_1U + X_3 - a_1 Y$$

and $$\dot{X}_3 = b_0U - a_0 Y$$

$$Y = b_3 U + X_1$$

Substituting Y in all the equations,

$$\dot{X}_1 = -a_2 X_1 + X_2 + [b_2 - a_2b_3] U(t) \quad \text{... (1a)}$$

$$\dot{X}_2 = -a_1X_1 + X_3 + [b_1 - a_1b_3] U(t) \quad \text{... (1b)}$$

$$\dot{X}_3 = -a_0 X_1 + [b_0 - a_0 b_3] U(t) \quad \text{... (1c)}$$

These equations give the required state model.

$\therefore$ $$\dot{X}(t) = A X(t) + B U(t)$$

where $$A = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} b_2 - a_2b_3 \\ b_1 - a_1b_3 \\ b_0 - a_0b_3 \end{bmatrix}$$

and $$Y(t) = CX(t) + DU(t)$$

where $$C = [1 \ \ 0 \ \ 0], \quad D = b_3$$

Now as $Y(0)$, $\dot{Y}(0)$, $\ddot{Y}(0)$, $U(0)$, $\dot{U}(0)$ and $\ddot{U}(0)$ are the known initial conditions, using the derived state model, the initial conditions $X_1(0)$, $X_2(0)$ and $X_3(0)$ can be obtained.

Example 2.4 : *A feedback system is characterized by the closed loop transfer function,*

$$T(s) = \frac{s^2+3s+3}{s^3+2s^2+3s+1}$$

Draw a suitable signal flow graph and obtain the state model.

Solution : Divide N and D by s^3.

$$\therefore \quad T(s) = \frac{\frac{1}{s}+\frac{3}{s^2}+\frac{3}{s^3}}{1+\frac{2}{s}+\frac{3}{s^2}+\frac{1}{s^3}} = \frac{\frac{1}{s}+\frac{3}{s^2}+\frac{3}{s^3}}{1-\left[-\frac{2}{s}-\frac{3}{s^2}-\frac{1}{s^3}\right]}$$

$$\therefore \qquad L_1 = -\frac{2}{s}, \quad L_2 = -\frac{3}{s^2}, \quad L_3 = -\frac{1}{s^3}$$

$$T_1 = \frac{1}{s}, \quad T_2 = \frac{3}{s^2}, \quad T_3 = \frac{3}{s^3}$$

And $\Delta_1 = \Delta_2 = \Delta_3 = 1$ with no combinations of non-touching loops.

There are many signal flow graphs which can be obtained to satisfy above transfer function.

Method 1 : The signal flow graph is as shown in the Fig. 2.6.

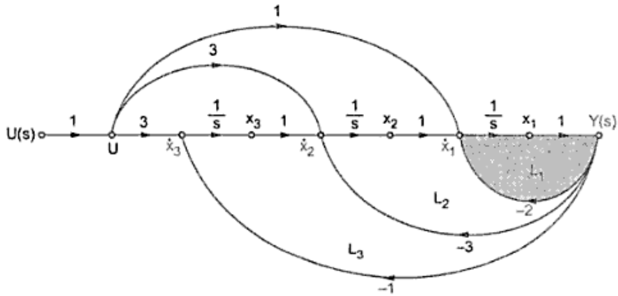


Fig. 2.6

From signal flow graph,

$$Y = X_1$$

$$\dot{X}_1 = X_2 + U - 2Y = -2X_1 + X_2 + U$$

$$\dot{X}_2 = X_3 + 3U - 3Y = -3X_1 + X_3 + 3U$$

$$\dot{X}_3 = 3U - Y = -X_1 + 3U$$

Hence the state model is,

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} U(t)$$

i.e. $$\dot{X} = AX(t) + BU(t)$$

and $$Y(t) = [1 \quad 0 \quad 0] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

i.e. $$Y(t) = X(t) \quad \text{with } D = 0$$

Method 2 : When m ≠ n and m < n then signal flow graph can be constructed so as to obtain matrix A in phase variable form. This is shown in the Fig. 2.7.

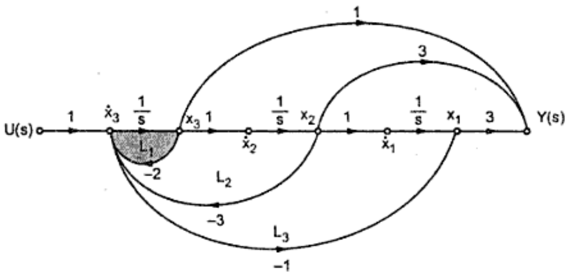


Fig. 2.7

From the signal flow graph,

$$\dot{X}_1 = X_2$$

$$\dot{X}_2 = X_3$$

$$\dot{X}_3 = -X_1 - 3X_2 - 2X_3 + U$$

and $$Y = 3X_1 + 3X_2 + X_3$$

Hence state model has,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = [3 \quad 3 \quad 1] \text{ and}$$

$$D = 0$$

Thus the matrix A is in phase variable form.

Note : When m < n, then transmission matrix D = 0.

2) Using Direct Decomposition of Transfer Function

This is also called direct programming. In this method, **denominator of transfer function is rearranged in a** specific form. To understand the rearrangement, consider an element with transfer function $\frac{1}{s+a}$. From block diagram algebra, the transfer function of minor feedback loop is $\frac{G}{1+GH}$ for negative feedback.

Let $$\frac{1}{s+a} = \frac{\frac{1}{s}}{1+\frac{a}{s}} = \frac{G}{1+GH}$$

where $$G = \frac{1}{s} = \text{integrator and } H = a$$

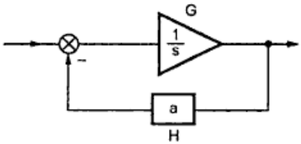


Fig. 2.8

The feedback is negative and the transfer function can be simulated as shown in the Fig. 2.8. with a minor feedback loop.

Now if such a loop is added in the forward path of another such loop then we get the block diagram as shown in the Fig. 2.9.

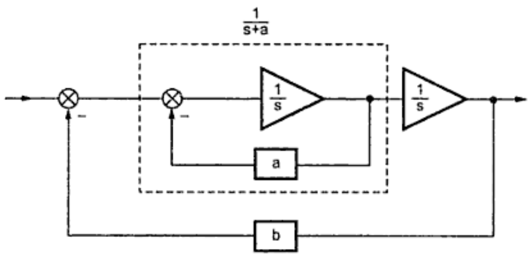


Fig. 2.9

The transfer function now becomes,

$$= \frac{\frac{1}{s(s+a)}}{1+\frac{b}{s(s+a)}} = \frac{1}{s(s+a)+b} = \frac{1}{sX+b}$$

where $\quad X = (s + a)$

If now the entire block shown in the Fig. 2.9 is added in the forward path of another minor loop with an integrator and feedback gain 'c', we get the transfer function as,

$$= \frac{1}{sY+c} \text{ where } Y = sX + b$$

Thus the denominator of transfer function becomes

$s(sX + b) = [s(s\{s + a\} + b)] = s^3 + as^2 + bs$

Thus denominator of any order can be directly programmed as discussed above.

$$s^2 + as + b \Rightarrow \{s(s + a) + b\}$$

$$s^3 + as^2 + bs + c \Rightarrow \{[(s + a)s + b]s + c\}$$

$$s^4 + as^2 + bs^2 + cs + d \Rightarrow \{([(s + a)s + b]s + c)s + d\} \text{ and so on.}$$

Now if numerator is $b_1 s + b_0$ and denominator. Simulation is obtained directly then the block diagram is as shown in the Fig. 2.10. But $s = \frac{d}{dt}$, which is differentiator and is not used to obtain state model. In such a case, take off point 't' is shifted before the last integrator block.

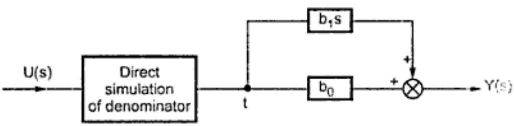


Fig. 2.10 (a)

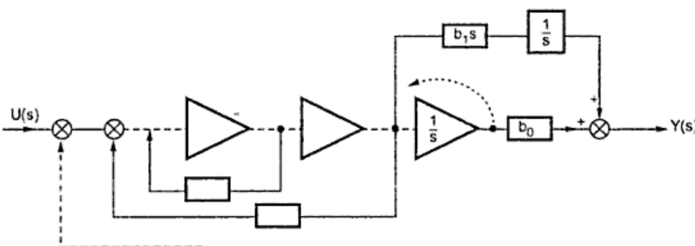


Fig. 2.10 (b)

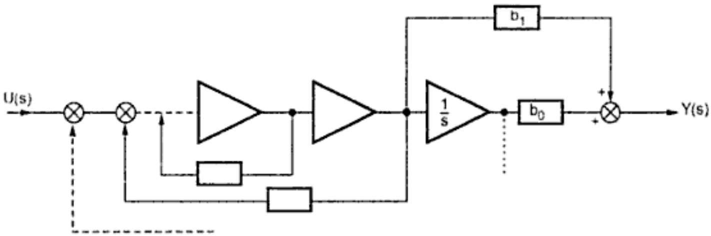


Fig. 2.10 (c)

According to block diagram reduction rule, while shifting take off point before the block, the take off signal must be multiplied by transfer function of block before which it is to be shifted. Thus we get block of b_1 with take off from input of last integrator.

Similarly if there is a term b_2s^2 in the numerator then shift take off point before one more integrator as shown in the Fig. 2.10 (d).

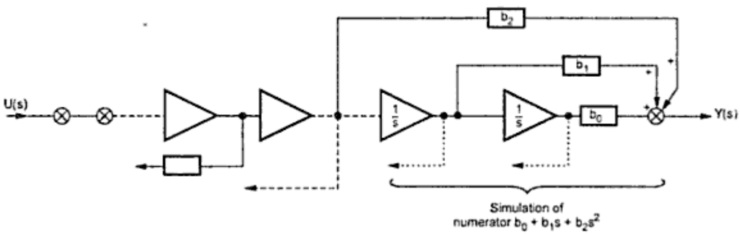


Fig. 2.10 (d)

Thus for any order of numerator, complete simulation of the transfer function can be achieved.

Then **assigning output of each integrator as the state variable, state model in the phase variable form can be obtained.**

Example 2.5 : *Obtain state model by direct decomposition method of a system whose transfer function is*

$$\frac{Y(s)}{U(s)} = \frac{5s^2+6s+8}{s^3+3s^2+7s+9}$$

Solution : Decompose denominator as below,

$$s^3+3s^2+7s+9 = \{([s+3]s+7)s+9\}$$

Its simulation starts from (s + 3) in denominator

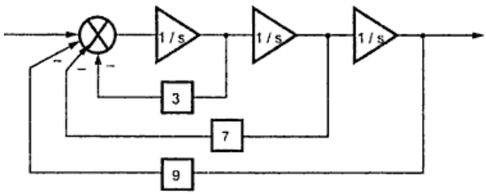


Fig. 2.11

To simulate numerator, shift take-off point once for 6s and shift twice for $5s^2$.

Therefore complete state diagram can be obtained as follows.

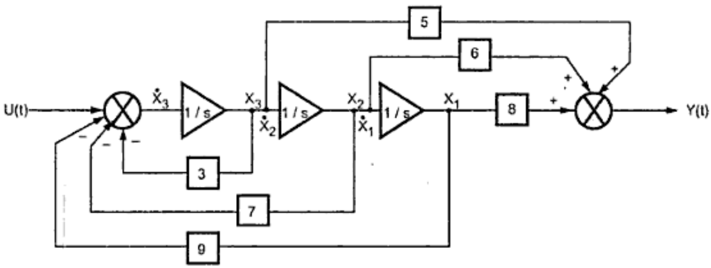


Fig. 2.12

Assign output of each integrator as the state variable.

$$\dot{X}_1 = X_2$$

$$\dot{X}_2 = X_3$$

$$\dot{X}_3 = U(t) - 9X_1(t) - 7X_2(t) - 3X_3(t)$$

While output, $Y(t) = 8X_1(t) + 6X_2(t) + 5X_3(t)$

$\therefore$ State model is,

$$\dot{X}(t) = AX(t) + BU(t)$$

and $Y(t) = CX(t) + DU(t)$

where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & -7 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$C = [8 \quad 6 \quad 5], \qquad D = 0$$

The matrix 'A' obtained is in the Bush form or Phase Variable form.

2.2.3 Advantages

The various advantages of phase variables i.e. direct programming method are,

1. Easy to implement.
2. The phase variables need not be physical variables hence mathematically powerful to obtain state model.
3. It is easy to establish the link between the transfer function design and time domain design using phase variables.
4. In many simple cases, just by inspection, the matrices A, B, C and D can be obtained.

2.2.4 Limitations

The various limitations of phase variables are,

1. The phase variables are not the physical variables hence they loose the practical significance. They have mathematical importance.
2. The phase variables are mathematical variables hence not available for the measurement point of view.
3. Also these variables are not available from control point of view.
4. The phase variables are the output and its derivatives, if derivatives of input are absent. But it is difficult to obtain second and higher derivatives of output.
5. The phase variable form, though special, does not offer any advantage from the mathematical analysis point of view.

Due to all these disadvantages, canonical variables are very popularly used to obtain the state model.

2.3 State Space Representation using Canonical Variables

This method of obtaining the state model using the canonical variables is also called **parallel programming** method and matrix A obtained using this method is said to have **canonical form, normal form** or **Foster's form**. The matrix A in such a case is a diagonal matrix and plays an important role in the state space analysis.

The method is basically based on Partial Fraction Expansion of the given transfer function T(s).

Consider the transfer function T(s) as,

$$T(s) = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \ldots + b_m}{(s+a_1)(s+a_2)\ldots(s+a_n)}$$

Case 1 : If the degree 'm' is **less than** 'n' (m < n), then T(s) can be expressed using partial fraction expansion as,

$$T(s) = \frac{c_1}{s+a_1} + \frac{c_2}{s+a_2} + \ldots + \frac{c_n}{s+a_n} = \sum_{i=1}^{n} \frac{c_i}{s+a_i}$$

Now each group $\frac{c_1}{s+a_1}$ can be simulated using the minor loop in state diagram as shown in the Fig. 2.13.

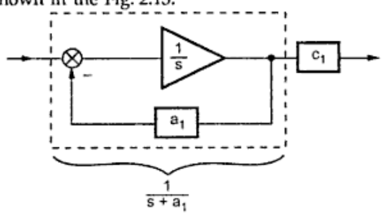


Fig. 2.13

The outputs of all such groups are to be added to obtain the resultant output.

To add the outputs, all the groups must be connected in parallel with each other. The input U(s) to all of them is same. Hence the method is called parallel programming. The overall state diagram is shown in the Fig. 2.14.

Then **assign output of each integrator as a state variable** and write the state equations as,

$$\dot{X}_1 = -a_1 X_1 + U$$

$$\dot{X}_2 = -a_2 X_2 + U$$

$$\vdots$$

$$\dot{X}_n = -a_n X_n + U$$

While $\quad Y = c_1 X_1 + c_2 X_2 + \ldots + c_n X_n$

$\therefore \quad \dot{X} = AX + BU$ and $Y = CX + DU$

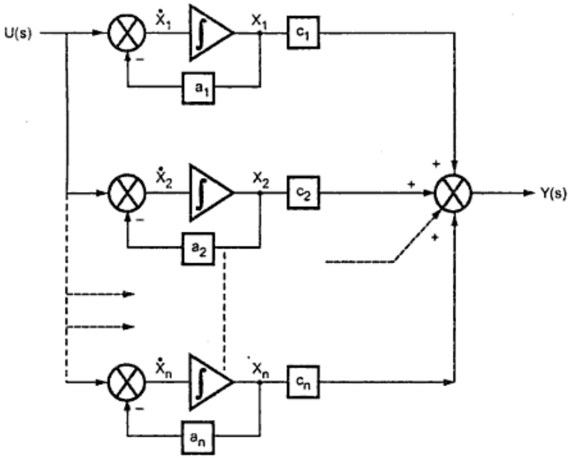


Fig. 2.14 Foster's form simulation

where $$A = \begin{bmatrix} -a & 0 & 0 & \cdots & 0 \\ 0 & -a_2 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & -a_n \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [c_1 \;\; c_2 \;\; \ldots\ldots \;\; c_n] \quad \text{and} \quad D = 0$$

Case 2 : If the degree m = n i.e. numerator and the denominator have same degree then first divide the numerator by denominator and then obtain partial fractions of remaining factor.

$$\therefore \qquad T(s) = \frac{N(s)}{D(s)} = c_0 + \sum_{i=1}^{n} \frac{c_i}{s+a_i}$$

where c_0 = Constant obtained by dividing N(s) by D(s).

In such a case, the state diagram for partial fraction terms remains same as before and in addition to all the outputs, c_0 U(t) gets added to obtain the resultant output as shown in the Fig. 2.15.

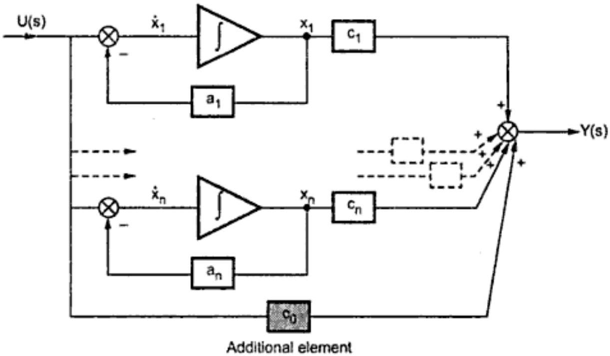


Fig. 2.15 State diagram for T(s) with m = n

Thus the state model consists of the matrices as,

$$A = \begin{bmatrix} -a_1 & 0 & \dots & 0 \\ 0 & -a_2 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & -a_n \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [c_1 \quad c_2 \quad \dots \quad c_n], \quad D = c_0$$

Key Point: *Thus for m = n, direct transmission matrix D exists in the state model.*

When the denominator D(s) of the transfer function T(s) has **non-repeated roots** then the matrix A obtained by parallel programming has following features,

1. Matrix A is diagonal i.e. in canonical form or normal form.
2. The diagonal consists of the elements which are the gains of all the feedback paths associated with the integrators.
3. The **diagonal elements are the poles of the transfer function T(s).**

Example 2.6 : *Obtain the state model by Foster's form of a system whose T.F. is,*

$$\frac{s^2+4}{(s+1)(s+2)(s+3)}$$

Solution : Find out partial fraction expansion of it

$$\frac{s^2+4}{(s+1)(s+2)(s+3)} = \frac{2.5}{s+1} - \frac{8}{s+2} + \frac{6.5}{s+3}$$

∴ Total state diagram is as shown in the Fig. 2.16.

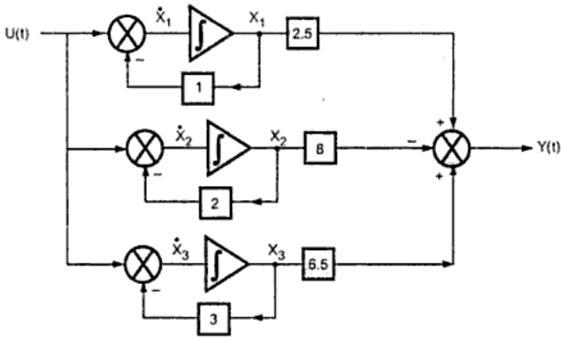


Fig. 2.16

$$\dot{X}_1 = U(t) - X_1(t)\ ,\ \dot{X}_2 = U(t) - 2X_2(t)\ ,$$

$$\dot{X}_3 = U(t) - 3X_3(t)$$

$$Y(t) = 2.5X_1(t) - 8X_2(t) + 6.5X_3(t)$$

∴ State model is, $\dot{X} = AX + BU$

and $Y = CX + DU$

where $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$C = [2.5 \ \ -8 \ \ 6.5] \quad D = 0$

2.3.1 Jordan's Canonical Form

In the Foster's form it is assumed that the roots of denominator of T(s) are non-repeated, simple and distinct. But if the roots are repeated then the parallel programming results matrix A in a form called **Jordan's Canonical form.**

Let T(s) has pole at $s = -a_1$ which is repeated for r times as,

$$T(s) = \frac{N(s)}{(s+a_1)^r(s+a_2)\dots(s+a_n)}$$

The method of obtaining partial fraction for such a case is,

$$T(s) = \frac{c_1}{(s+a_1)^r}+\frac{c_2}{(s+a_1)^{r-1}}+\dots+\frac{c_r}{(s+a_1)}+\frac{c_{r+1}}{(s+a_2)}+\dots+\frac{c_n}{(s+a_n)} \qquad \dots m < n$$

If the degree of N(s) and D(s) is same i.e. m = n we get additional constant c_0 as,

$$T(s) = c_0+\frac{c_1}{(s+a_1)^r}+\dots+\frac{c_r}{(s+a_1)}+\frac{c_{r+1}}{(s+a_2)}+\dots+\frac{c_n}{(s+a_n)} \qquad \dots m = n$$

This can be mathematically expressed as,

$$T(s) = \sum_{i=1}^{r}\frac{c_i}{(s+a_1)^{r-i+1}}+\sum_{i=r+1}^{n}\frac{c_i}{(s+a_i)} \qquad \dots m < n$$

and

$$T(s) = c_0+\sum_{i=1}^{r}\frac{c_i}{(s+a_1)^{r-i+1}}+\sum_{i=r+1}^{n}\frac{c_i}{(s+a_i)} \qquad \dots m = n$$

> **Key Point**: ***Note that in partial fraction expansion, a seperate coefficient is assumed for each power of repeated factor.***

In simulating such an equation by parallel programming, $\frac{1}{(s+a_1)^r}$ is simulated by connecting $\frac{1}{(s+a_1)}$ groups, r times in series first. While all other distinct factors are simulated by parallel programming as before. The components of each power of $\frac{1}{(s+a_1)}$ to be added to get output is to be taken from output of each integrator which are connected in series. This is shown in the Fig. 2.17.

Now assign state variables at the output of each integrator. **For series integrators, assign the state variables from right to left, as shown in the Fig. 2.17.**

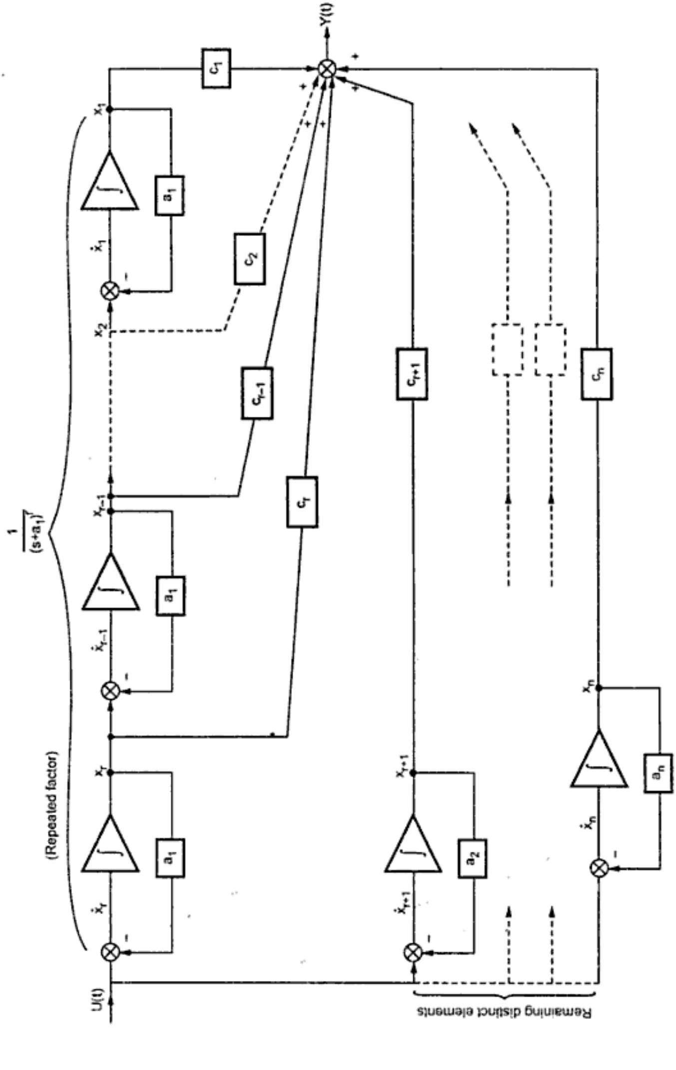


Fig. 2.17 State diagram for Jordan's form with m < n

For series integrators, the state equations are,

$$\dot{X}_1 = -a_1X_1 + X_2$$

$$\dot{X}_2 = -a_1X_2 + X_3$$

$$\vdots$$

$$\dot{X}_{r-1} = -a_1X_{r-1} + X_r$$

$$\dot{X}_r = -a_1X_r + U(t)$$

While for parallel integrators, the state equations are,

$$\dot{X}_{r+1} = -a_2\,X_{r+1} + U(t)$$

$$\vdots$$

$$\dot{X}_n = -a_n\,X_n + U(t)$$

While $$Y(t) = c_1X_1 + c_2X_2 + \ldots + c_r\,X_r + c_{r+1}\,X_{r+1} + \ldots + c_n\,X_n$$

Key Point: *$Y(t)$ has additional c_0 $U(t)$ term if $m = n$.*

Hence the state model has matrices in the form,

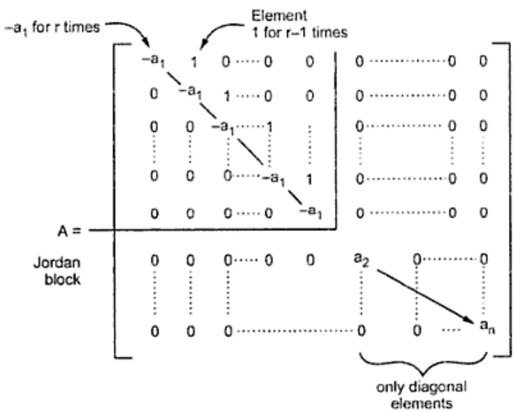


Fig. 2.18

$$B = \begin{bmatrix} \left.\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right\} \begin{matrix} r-1 \\ \text{times} \end{matrix} \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad C = [c_1 \; c_2 \; \dots \; c_r \; c_{r+1} \; \dots \; c_n]$$

The matrix D is zero if $m < n$ and is 'c_0' if $m = n$. The matrix A in such a case has a Jordan block for the repeated factor and many times A is denoted as J.

$$\therefore \qquad \dot{X}(t) = JX(t) + BU(t) \qquad \text{... For repeated roots}$$

The matrix A = J has the following features,

1. The principle diagonal consists of all the poles of transfer function with repeated pole 'r' times and other nonrepeated poles.
2. Upper off diagonal consists of $(r - 1)$ times unity element, as indicated in the Jordan block.
3. All the remaining elements are zero.

Note that the matrix B has '$r - 1$' zeros and all other elements as unity. The matrix C has all partial fraction coefficients $c_1, c_2 \dots c_n$.

2.3.2 Advantages of Canonical Variables

The main advantages of canonical variables are

1. The matrix A is diagonal
2. The diagonal element is very important in the mathematical analysis.
3. Due to diagonal feature, the decoupling between the state variable is possible. This means all 'n' differential equations are independent of each other. Thus $\dot{X}_1$ depends on X_1 alone, $\dot{X}_2$ depends on X_2 alone and so on. Such decoupling is important in system design from controlling point of view.

2.3.3 Disadvantages of Canonical Variables

The main disadvantages of canonical variables is similar to the phase variables. These are not the physical variables hence practically difficult to measure and control. Hence such variables are not practically advantageous, though mathematically are very important.

Example 2.7 : *Obtain the state model in Jordan's canonical form of a system whose T.F. is* $\dfrac{1}{(s+2)^2 (s+1)}$

Solution : Finding partial fraction expansion,

$$T(s) = \frac{A}{(s+2)^2} + \frac{B}{(s+2)} + \frac{C}{(s+1)}$$

Take LCM on right hand side and equate numerator with numerator of T(s),

$\therefore A(s+1) + B(s+2)(s+1) + C(s+2)^2 = 1$

Equate coefficients of all powers of s on both sides,

$$\therefore \quad B + C = 0, \quad \text{... from power of } s^2$$

$$A + 3B + 4C = 0, \quad \text{... from power of } s$$

$$A + 2B + 4C = 1 \quad \text{... from constant term}$$

Solving we get, $A = -1,\ B = -1,\ C = 1$

$$T(s) = \frac{-1}{(s+2)^2} - \frac{1}{(s+2)} + \frac{1}{(s+1)}$$

Simulate first term by series integrators while other nonrepeated terms by parallel integrators.

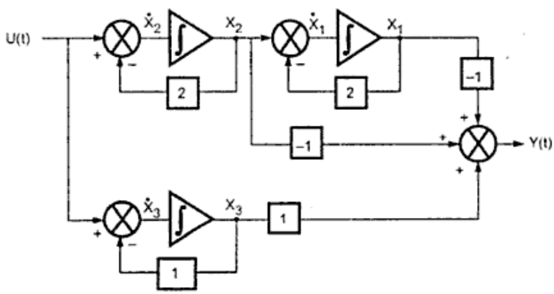


Fig. 2.19

Total simulation is,

$$\dot{X}_1 = -2X_1 + X_2 \qquad \dot{X}_2 = U(t) - 2X_2 \qquad \dot{X}_3 = U(t) - X_3(t)$$

$$Y(t) = -X_1(t) - X_2(t) + X_3(t)$$

$\therefore$ State model is , $\dot{X} = AX + BU$ and $Y = CX + DU$

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$C = [-1 \quad -1 \quad 1], \quad D = 0$$

The matrix A consists of Jordan block.

Important Note : If some of the poles of T(s) are complex in nature, then a mixed approach can be used. The quadratic or higher order polynomial having complex roots can be simulated by direct decomposition while real distinct roots can be simulated by the parallel programming using canonical variable, the example 2.8 below explains this procedure.

Example 2.8 : *Combination of Direct Decomposition and Foster's Form. Obtain the state model for the given T.F.*

$$T(s) = \frac{2}{(s+3)(s^2+2s+2)}$$

Solution : Obtain partial fractions as,

$$T(s) = \frac{2}{(s+3)(s^2+2s+2)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+2s+2}$$

Find LCM on right hand side and equate numerators of both sides,

$$\therefore \quad A(s^2+2s+2)+(Bs+C)(s+3) = 2$$

$$\therefore \quad As^2+2As+2A+Bs^2+Cs+3Bs+3C = 2$$

Equating coefficients of all powers of s,

$$A+B = 0 \quad 2A+C+3B = 0 \qquad 2A+3C = 2$$

$$2A+C-3A = 0 \quad \therefore \quad 2A+3A = 2$$

$$C-A = 0$$

$$C = A$$

Solving, $A = \frac{2}{5}, \quad B = -\frac{2}{5}, C = \frac{2}{5}$

$$T(s) = \frac{\frac{2}{5}}{(s+3)} + \frac{-\frac{2}{5}s+\frac{2}{s}}{s^2+2s+2} = \frac{\frac{2}{5}}{s+3} + \frac{-\frac{2}{5}s+\frac{2}{5}}{\{(s+2)s+2\}}$$

The quadratic $s^2 + 2s + 2$ having complex roots is decomposed directly.

$\therefore$ Complete state diagram is as shown in the Fig. 2.20

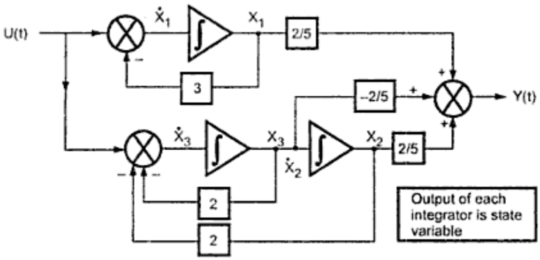


Fig. 2.20

$$\dot{X}_1 = U(t) - 3X_1$$

$$\dot{X}_2 = X_3$$

$$\dot{X}_3 = U(t) - 2X_2 - 2X_3$$

and $$Y(t) = \frac{2}{5}X_1(t) + \frac{2}{5}X_2(t) - \frac{2}{5}X_3(t)$$

$\therefore$ State model is, $\dot{X} = AX + BU$ and $Y = CX$

where $$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} \frac{2}{5} & -\frac{2}{5} & \frac{2}{5} \end{bmatrix}, D = 0$$

Note that in such a case matrix A does not have any specific form.

2.4 State Model by Cascade Programming

This is also called **Pole-Zero Form or Gullemin's Form**. This can be effectively used when both numerator and denominator can be factorised.

In such form, group a pole and zero together and arrange given transfer function as the product of all such groups. Then simulate each group separately and connect all such simulations in cascade to get complete simulation. Then assigning output of each integrator as a state variable obtain a state model in standard form.

Simulation of group : Consider $\frac{s+a}{s+b}$

First simulate denominator $\frac{1}{s+b}$ as in the Fig. 2.21 (a) and then as discussed earlier simulate the numerator (s + a) as shown in the Fig. 2.21 (b).

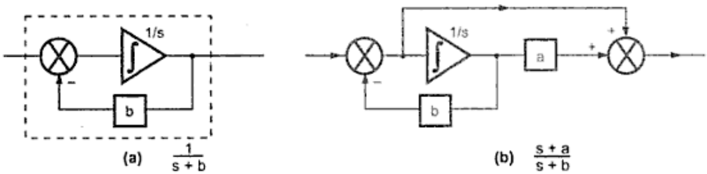


Fig. 2.21 Simulation of a group of pole-zero

Example 2.9 : *Obtain the state model of a system by cascade programming whose transfer function is*

$$T(s) = \frac{Y(s)}{U(s)} = \frac{(s+2)(s+4)}{s(s+1)(s+3)}$$

Solution : Arrange the given T.F. as below

$$\frac{Y(s)}{U(s)} = \underbrace{\frac{(s+2)}{(s+1)}}_{\text{Group 1}} \cdot \underbrace{\frac{(s+4)}{(s+3)}}_{\text{Group 2}} \cdot \underbrace{\frac{1}{s}}_{\text{Group 3}}$$

Now simulate each group as discussed and connect all of them in series to obtain T(s). The complete simulation is shown in the Fig. 2.22.

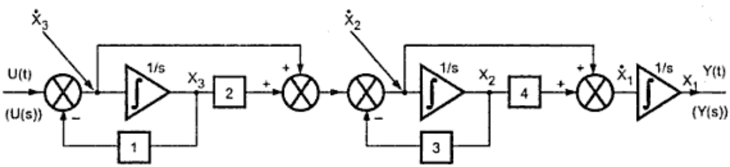


Fig. 2.22

Now $\quad \dot{X}_1 = \dot{X}_2 + 4X_2, \quad \dot{X}_2 = -3X_2 + 2X_3 + \dot{X}_3$

and $\dot{X}_3 = U(t) - X_3$

Substituting $\dot{X}_3$ into $\dot{X}_2$ equation,

$$\dot{X}_2 = -3X_2 + 2X_3 + U(t) - X_3 = U(t) - 3X_2 + X_3$$

Substituting $\dot{X}_2$ into $\dot{X}_1$ equation,

$$\dot{X}_1 = U(t) - 3X_2 + X_3 + 4X_2 = U(t) + X_2 + X_3$$

Model becomes,

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} U(t)$$

and $Y(t) = X_1(t)$

i.e. $$Y(t) = [1 \quad 0 \quad 0] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

So $$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -1 \end{bmatrix} \; B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$C = [1 \; 0 \; 0] \qquad D = 0$$

Feature of A : Feature of Matrix A is its principle diagonal contains gains of all feedback paths associated with all integrators i.e. 0, –3, –1 in above problem. All terms below principle diagonal are zero. Thus the features of A having all poles of T(s) in its principle diagonal still continues in this method of programming.

This method is also known as **Iterative programming.**

Examples with Solutions

➡ **Example 2.10 :** *Obtain the state model of the given network in the standard form.*

Assume $R_1 = 1\ \Omega \qquad C_1 = 1\ F$

$R_2 = 2\ \Omega \qquad C_2 = 1\ F$

$R_3 = 3\ \Omega$

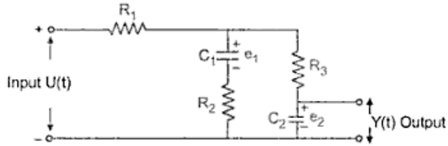


Fig. 2.23

Solution : Selecting state variables as voltages across capacitors C_1 and C_2 i.e. e_1 and e_2,

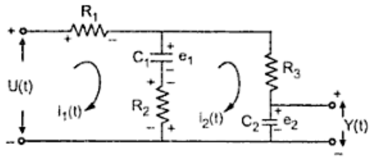


Fig. 2.23 (a)

$$e_1 = X_1(t)$$

$$e_2 = X_2(t)$$

Applying Kirchhoff's laws,

$$U(t) - i_1 R_1 - e_1 - R_2 (i_1 - i_2) = 0$$

$$U(t) = i_1 R_1 + e_1 + R_2 (i_1 - i_2) \quad \text{... (1)}$$

for second loop,

$$e_1 - i_2 R_3 - e_2 - (i_2 - i_1) R_2 = 0$$

$$\therefore \quad e_1 = i_2 R_3 + e_2 + (i_2 - i_1) R_2 \quad \text{... (2)}$$

Solving simultaneously equations (1) and (2)

$$\therefore \quad U(t) = i_1 (R_1 + R_2) - i_2 R_3 + e_1$$

$$e_1 = -i_1 R_2 + i_2 (R_2 + R_3) + e_2$$

Substituting the values,

$$U(t) = i_1 (1 + 2) - 2i_2 + e_1$$

$$\text{i.e.} \quad U(t) = 3 i_1 - 2i_2 + e_1 \quad \text{... (3)}$$

$$e_1 = -2i_1 + 5i_2 + e_2 \quad \text{... (4)}$$

Multiply equation (3) by 2 and equation (4) by 3

$$2\,U(t) = 6i_1 - 4i_2 + 2e_1$$

$$3e_1 = -6i_1 + 15i_2 + 3e_2 \quad \text{and adding}$$

$$3\,U(t) + 3e_1 = 11i_2 + 3e_2 + 2e_1$$

$$i_2 = \frac{3}{11} U(t) + \frac{1}{11} e_1 - \frac{3}{11} e_2 \quad \text{... (5)}$$

Now from equation (3)

$$U(t) = 3i_1 - 2i_2 + e_1$$

Substituting i_2 from equation (5)

$$3i_1 = U(t) + \frac{6}{11} U(t) + \frac{2}{11} e_1 - e_1 - \frac{6}{11} e_2$$

$$U(t) = 3i_1 - \frac{6}{11} U(t) - \frac{2}{11} e_1 + \frac{6}{11} e_2 + e_1$$

$$\therefore \quad i_1 = \frac{17}{33} U(t) - \frac{9}{33} e_1 - \frac{6}{33} e_2 \quad \text{... (6)}$$

Now $\quad C_1 \frac{de_1}{dt} = i_1 - i_2 \quad$ ...Capacitor current is C $[dv_C/dt]$

as $C_1 = 1$,

$$\therefore \quad \frac{de_1}{dt} = \frac{17}{33} U(t) - \frac{9}{33} e_1 - \frac{6}{33} e_2 - \frac{3}{11} U(t) - \frac{1}{11} e_1 + \frac{3}{11} e_2$$

$$\frac{de_1}{dt} = \frac{8}{33} U(t) - \frac{12}{33} e_1 + \frac{3}{33} e_2$$

$$\therefore \quad \dot{X}_1 = \frac{8}{33} U(t) - \frac{12}{33} X_1 + \frac{1}{11} X_2 \quad \text{... (7)}$$

and $\quad C_2 \frac{de_2}{dt} = i_2 \quad$ ...Capacitor current is C $[dv_C/dt]$

as $\quad C_2 = 1$

$$\therefore \quad \frac{de_2}{dt} = \frac{3}{11} U(t) + \frac{1}{11} e_1 - \frac{3}{11} e_2$$

$$\therefore \quad \dot{X}_2 = \frac{3}{11} U(t) + \frac{1}{11} X_1 - \frac{3}{11} X_2 \quad \text{... (8)}$$

and $\quad Y(t) = e_2 = X_2$

$\therefore$ State model is, $\dot{X} = AX + BU$ and $Y = CX + DU$

where
$$A = \begin{bmatrix} -\frac{12}{33} & \frac{1}{11} \\ \frac{1}{11} & -\frac{3}{11} \end{bmatrix}, \; B = \begin{bmatrix} \frac{8}{33} \\ \frac{3}{11} \end{bmatrix} \quad C = [0\ ,1] \quad D = [\ 0\ ]$$

Example 2.11 : *Consider the mechanical system shown in figure. For shown displacements and velocities obtain the state model in standard form.*

Assume velocity of M_2 as output.

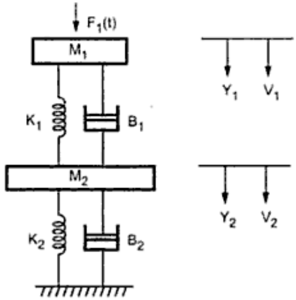


Fig. 2.24

Solution : Select the state variables as energy storing elements i.e. displacements and velocities related to spring and friction.

$$\therefore \quad X_1(t) = Y_1(t), \quad X_3 = V_1(t)$$

$$X_2(t) = Y_2(t), \quad X_4 = V_2(t)$$

$$U(t) = F_1(t), \quad Y(t) = V_2(t)$$

Draw the equivalent mechanical system. Due to $F_1(t)$, M_1 will displace by Y_1. Due to spring K_1 and friction B_1 which are between the two masses, the displacement change from Y_1 to Y_2. While mass M_2, spring K_2 and friction B_2 are under the influence of Y_2 alone as K_2 and B_2 are with reference to fixed support and not between two moving points. Represent each displacement by separate node. Connect the elements in parallel

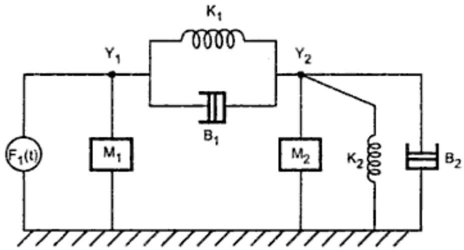


Fig. 2.25

which are under the influence of same displacement, thus K_1, B_1 parallel between Y and Y_2, m_2, K_2 and B_2 parallel between Y_2 and reference i.e. fixed support and so on.

The spring force is proportional to net displacement in spring while frictional force is proportional to the velocities.

At node Y_1,

$$F_1 = M_1 \frac{d^2 Y_1}{dt^2} + K_1 (Y_1 - Y_2) + \frac{B_1 d(Y_1 - Y_2)}{dt} \quad \text{... (1)}$$

At node Y_2,

$$0 = M_2 \frac{d^2 Y_2}{dt^2} + K_2 Y_2 + B_2 \frac{dY_2}{dt} + K_1 (Y_2 - Y_1) + \frac{B_1 d(Y_2 - Y_1)}{dt} \quad \text{... (2)}$$

Substituting all values in terms of state variables we get,

$$Y_1 = X_1, \qquad \frac{dY_1}{dt} = \dot{X}_1 = V_1 = X_3, \qquad \frac{d^2Y_1}{dt^2} = \dot{X}_3$$

$$Y_2 = X_2, \qquad \frac{dY_2}{dt} = \dot{X}_2 = V_2 = X_4, \qquad \frac{d^2Y_2}{dt^2} = \dot{X}_4$$

∴ Substituting in equation (1) and equation (2)

$$U(t) = M_1 \dot{X}_3 + K_1 [X_1 - X_2] + B_1 [X_3 - X_4] \quad \text{... (3)}$$

$$0 = M_2 \dot{X}_4 + K_2 X_2 + B_2 X_4 + K_1 [X_2 - X_1] + B_1 [X_4 - X_3] \quad \text{... (4)}$$

From equation (3) and equation (4) we can write

$$\dot{X}_3 = \frac{1}{M_1} [U(t) - K_1 (X_1 - X_2) - B_1(X_3 - X_4)] \quad \text{... (5)}$$

and

$$\dot{X}_4 = \frac{1}{M_2} [-K_2 X_2 - B_2 X_4 - K_1 (X_2 - X_1) - B_1 (X_4 - X_3)] \quad \text{... (6)}$$

and

$$\dot{X}_1 = X_3, \quad \dot{X}_2 = X_4, \quad Y(t) = V_2(t) = X_4(t)$$

∴ State model can be constructed in the standard form

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

Where
$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K_1}{M_1} & +\frac{K_1}{M_1} & -\frac{B_1}{M_1} & +\frac{B_1}{M_1} \\ \frac{K_1}{M_2} & \frac{-(K_1+K_2)}{M_2} & \frac{B_1}{M_2} & \frac{-(B_1+B_2)}{M_2} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1/M_1 \\ 0 \end{bmatrix}, \quad C = [0 \ 0 \ 0 \ 1]$$

Example 2.12 : *For the given T.F. of a system obtain the state model by*

i) Direct decomposition ii) Gullemin's Form iii) Foster's Form

$$T(s) = \frac{(s+2)(s+3)}{s(s+1)(s^2+9s+20)}$$

Solution : i) Direct decomposition

$$T(s) = \frac{s^2+5s+6}{s(s^3+10s^2+29s+20)} = \frac{s^2+5s+6}{\{([s+10]s+29)s+20\}s}$$

State diagram is as follows

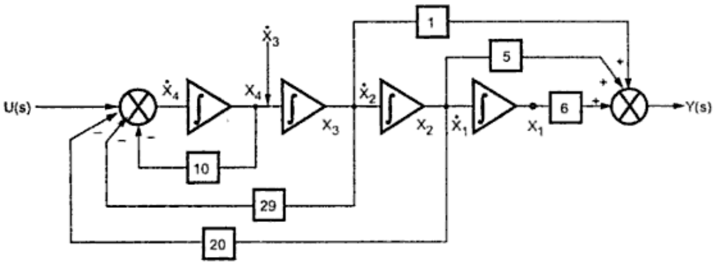


Fig. 2.26

$$\dot{X}_1 = X_2, \quad \dot{X}_2 = X_3, \quad \dot{X}_3 = X_4,$$

$$\dot{X}_4 = U - 20X_2 - 29X_3 - 10X_4,$$

$$Y = 6X_1 + 5X_2 + X_3$$

$\therefore$ State model is

$$\dot{X} = AX + BU \text{ and } Y = CX + DU$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -20 & -29 & -10 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = [\,6 \;\; 5 \;\; 1 \;\; 0\,] \qquad D = [\,0\,]$$

ii) Gullemin's Form

$$T(s) = \frac{(s+2)(s+3)}{s(s+1)(s+4)(s+5)} = \frac{(s+2)}{(s+1)} \cdot \frac{(s+3)}{(s+4)} \cdot \frac{1}{(s+5)} \cdot \frac{1}{s}$$

State diagram is,

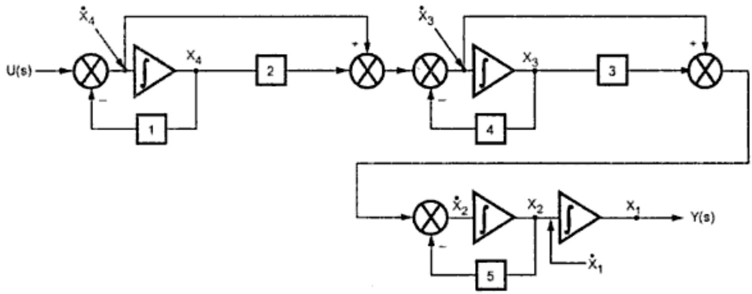


Fig. 2.27

$$\dot{X}_1 = X_2 \qquad \dot{X}_2 = -5X_2 + 3X_3 + \dot{X}_3$$

$$\dot{X}_3 = -4X_3 + 2X_4 + \dot{X}_4\,, \quad \dot{X}_4 = U(s) - X_4$$

Substituting $\dot{X}_4$ back in $\dot{X}_3$

$$\dot{X}_3 = -4X_3 + 2X_4 + U(s) - X_4 = -4X_3 + X_4 + U(s)$$

Substituting $\dot{X}_3$ back in $\dot{X}_2$

$$\dot{X}_2 = -5X_2 + 3X_3 - 4X_3 + X_4 + U(s) = -5X_2 - X_3 + X_4 + U(s)$$

and $\quad Y = X_1$

∴ State model becomes

$$\dot{X} = AX + BU \quad \text{and } Y = CX$$

where
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -5 & -1 & 1 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad C = [\,1\ 0\ 0\ 0\,]$$

iii) For Foster's form :

Find out partial fraction expansion of T(s)

$$T(s) = \frac{A}{s+1} + \frac{B}{s+4} + \frac{C}{s+5} + \frac{D}{s}$$

$$= \frac{-1/6}{s+1} + \frac{1/6}{s+4} - \frac{3/10}{s+5} + \frac{3/10}{s}$$

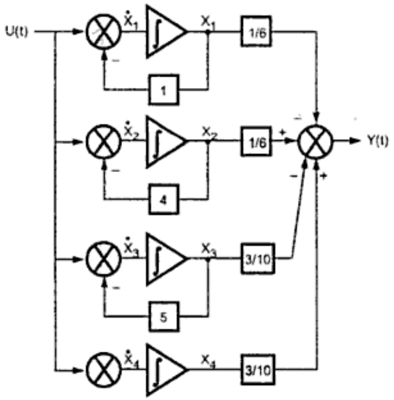


Fig. 2.28

$$\dot{X}_1 = U(t) - X_1, \qquad Y = -\frac{1}{6}X_1 + \frac{1}{6}X_2 - \frac{3}{10}X_3 + \frac{3}{10}X_4$$

$$\dot{X}_2 = U(t) - 4X_2$$

$$\dot{X}_3 = U(t) - 5X_3$$

$$\dot{X}_4 = U(t)$$

State model is,

$$\dot{X} = AX + BU$$

and $$Y = CX$$

where $$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} -\frac{1}{6} & \frac{1}{6} & -\frac{3}{10} & \frac{3}{10} \end{bmatrix}$$

Example 2.13 *Derive the state model in Jordan's canonical form for a system having T.F.*

$$T(s) = \frac{1}{s^3 + 4s^2 + 5s + 2}$$

Solution : $$T(s) = \frac{1}{s^3 + 4s^2 + 5s + 2} = \frac{1}{(s+1)^2 (s+2)} = \frac{A}{(s+1)^2} + \frac{B}{(s+1)} + \frac{C}{(s+2)}$$

i.e. $A(s + 2) + B(s + 1)(s + 2) + C(s+1)^2 = 1$

$$As + 2A + Bs^2 + 3Bs + 2B + Cs^2 + 2Cs + C = 1$$

$$B + C = 0 \text{ , } A + 3B + 2C = 0 \text{ ,}$$

$$2A + 2B + C = 1$$

Now $C = 1$, $B = -1$, $A = 1$

$$T(s) = \frac{1}{(s+1)^2} - \frac{1}{(s+1)} + \frac{1}{(s+2)}$$

State diagram is ,

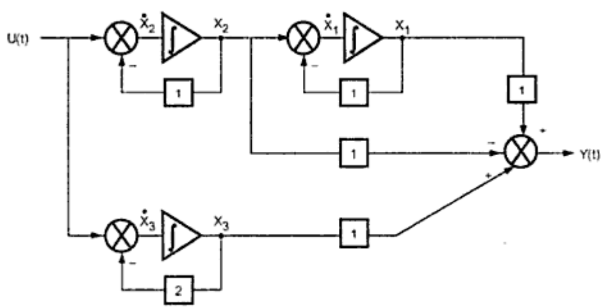


Fig. 2.29

$$\dot{X}_1 = -X_1 + X_2$$

$$\dot{X}_2 = U(t) - X_2$$

$$\dot{X}_3 = U(t) - 2X_3$$

$$Y(s) = X_1 - X_2 + X_3$$

$\therefore$ State model is,

$$\dot{X} = AX + BU \qquad Y = CX$$

where $$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

$$C = [1 \ \ -1 \ \ 1]$$

Example 2.14 : *Obtain the state model of system whose T.F. is*

$$T(s) = \frac{s^3 + 3s^2 + 2s}{s^3 + 12s^2 + 47s + 60} \text{ by Foster's form.}$$

Solution : $$T(s) = \frac{s^3 + 3s^2 + 2s}{s^3 + 12s^2 + 47s + 60}$$

As numerator and denominator are of same order we cannot directly find out partial fractions.

For partial fraction, numerator degree must be less than denominator. So directly divide N(s) by D(s) and find partial fractions of the remainder.

$$s^3 + 12s^2 + 47s + 60 \;)\; s^3 + 3s^2 + 2s \;(\; 1$$

$$\underline{s^3 + 12s^2 + 47s + 60}$$

$$-9s^2 - 45s - 60$$

$$\therefore \quad T(s) = \frac{Y(s)}{U(s)} = 1 - \left[\frac{9s^2 + 45s + 60}{s^3 + 12s^2 + 47s + 60}\right]$$

$$= 1 - \left[\frac{9s^2 + 45s + 60}{(s+3)(s+4)(s+5)}\right] = 1 - \frac{A}{s+3} - \frac{B}{s+4} - \frac{C}{s+5}$$

$$= 1 - \frac{3}{s+3} + \frac{24}{s+4} - \frac{30}{s+5}$$

State diagram :

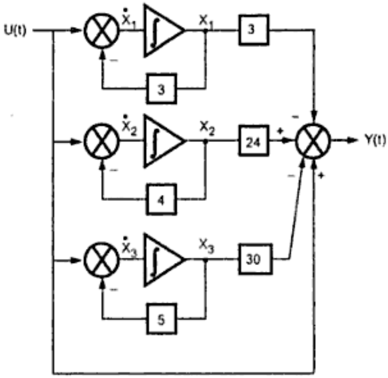


Fig. 2.30

$$\dot{X}_1 = U(t) - 3X_1, \quad \dot{X}_2 = U(t) - 4X_2,$$

$$\dot{X}_3 = U(t) - 5X_3$$

$$Y(s) = -3X_1 + 24X_2 - 30X_3 + U(s)$$

$\therefore$ State model is,

$$\dot{X} = AX + BU \quad \text{and} \quad Y = CX + DU$$

where $A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$C = [-3 \quad 24 \quad -30], \quad D = [1]$$

When N(s) and D(s) are having same order, or degree of N(s) > D(s) then, there is always direct transmission matrix D present in the model.

➠ **Example 2.15 :** *Obtain a state space model of the system with transfer function*

$$\frac{Y(s)}{U(s)} = \frac{6}{s^3 + 6s^2 + 11s + 6}$$ **(VTU:Jan./Feb.-2007,July/Aug.-2007,Dec./Jan-2008)**

Solution :

The T.F. is $\frac{Y(s)}{U(s)} = \frac{6}{s^3+6s^2+11s+6}$

Using factorisation of denominator,

$$\frac{Y(s)}{U(s)} = \frac{6}{(s+1)\ (s+2)\ (s+3)}$$

Taking partial fractions,

$$\frac{Y(s)}{U(s)} = \frac{3}{s+1} - \frac{6}{s+2} + \frac{3}{s+3}$$

Hence the state diagram is as shown in the Fig. 2.31

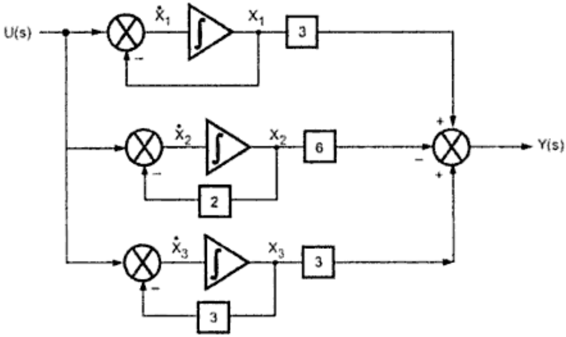


Fig. 2.31

This is Foster's form of representation. From above Fig. 2.31 we get,

$$\dot{X}_1 = U(s) - X_1$$

$$\dot{X}_2 = U(s) - 2X_2$$

$$\dot{X}_3 = U(s) - 3X_3$$

and $\quad Y(s) = 3X_1 - 6X_2 + 3X_3$

Hence the state space model is,

$$\dot{X} = AX + BU$$

and $$Y = CX$$

where $$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$C = [3 \quad -6 \quad 3]$$

Example 2.16 : *Obtain the state model for system represented by*

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 10y = 3U(t)$$

(VTU : May-99, Dec.2007/Jan.-2008)

Solution : System is 3rd order, n = 3

3 integrators and variables are required. Select $y = X_1$ and then successive differentiation of y as next variable.

$$\therefore \quad \dot{X}_1 = X_2 = dy/dt \quad \text{... (1)}$$

$$\dot{X}_2 = X_3 = \frac{d^2y}{dt^2} \quad \text{... (2)}$$

Now as 3 variables are defined, $\dot{X}_3 \neq X_4$ but $\dot{X}_3$ must be obtained by substituting all selected variables in original differential equation.

$$\therefore \quad \dot{X}_3 + 6X_3 + 11X_2 + 10X_1 = 3U \quad \text{as} \quad \dot{X}_3 = \frac{d^3y}{dt^3}$$

$$\therefore \quad \dot{X}_3 = 3U - 10X_1 - 11X_2 - 6X_3 \quad \text{... (3)}$$

$$y = X_1$$ which is output equation.

$\therefore$ State model can be written as,

$$\dot{X} = AX + BU$$

and $$y = CX + DU$$

Where $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -11 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, C = [\,1 \quad 0 \quad 0\,], \quad D = [0]$$

State diagram :

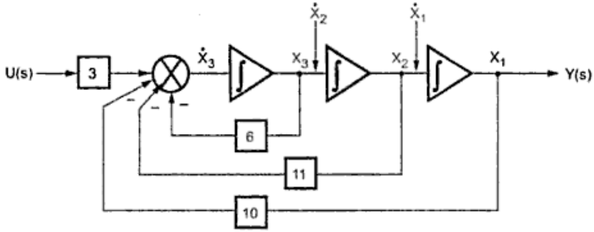


Fig. 2.32

Example 2.17 : *Obtain the state model of the differential equation*

$$4\frac{d^3c(t)}{dt^3}+3\frac{d^2c(t)}{dt^2}+\frac{dc(t)}{dt}+2c(t)=5r(t)$$ **(VTU : Aug-95)**

Solution : The equation can be written as,

$$4\dddot{c}(t)+3\ddot{c}(t)+\dot{c}(t)+2c(t) = 5r(t)$$

The order of the equation is 3. Selecting state variables as,

$$c(t) = X_1(t)$$

$$X_2(t) = \dot{X}_1(t) = \dot{c}(t) \quad ...(1)$$

$$X_3(t) = \dot{X}_2(t) = \ddot{c}(t) \quad ...(2)$$

$$\dot{X}_3(t) = \dddot{c}(t)$$

Substituting in the original equation,

$$4\dot{X}_3(t)+3X_3(t)+X_2(t)+2X_1(t) = 5r(t)$$

$$\therefore \quad \dot{X}_3(t) = -\frac{2}{4}X_1(t)-\frac{1}{4}X_2(t)-\frac{3}{4}X_3(t)+\frac{5}{4}r(t) \quad ...(3)$$

Hence the state model is,

$$\begin{bmatrix}\dot{X}_1\\ \dot{X}_2\\ \dot{X}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\frac{2}{4} & -\frac{1}{4} & -\frac{3}{4}\end{bmatrix}\begin{bmatrix}X_1(t)\\ X_2(t)\\ X_3(t)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \frac{5}{4}\end{bmatrix}r(t)$$

While the output equation is,

$$c(t) = X_1(t)$$

i.e.
$$y(t) = [\,1\ 0\ 0\,]\begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix}$$

Example 2.18 : *Obtain the state model of the system whose closed loop transfer function is*

$$\frac{C(s)}{R(s)} = \frac{2(s+3)}{(s+1)(s+2)}$$

(VTU : Feb-97, Oct-98)

Solution : Let us use the parallel programming i.e. Foster's form. Finding partial fractions of C(s)/R(s).

$$\frac{C(s)}{R(s)} = \frac{4}{s+1} - \frac{2}{s+2}$$

Hence the state diagram is as shown in the Fig. 2.33

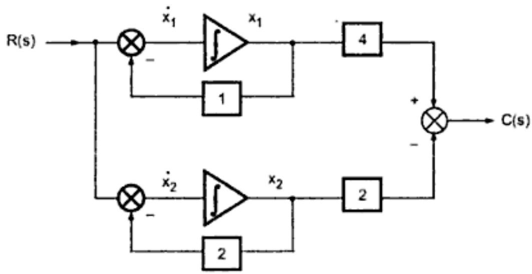


Fig. 2.33

Hence the state equations from the Fig. 2.33 are

$$\dot{X}_1 = R(s) - X_1 \quad ...(1)$$

$$\dot{X}_2 = R(s) - 2X_2 \quad ...(2)$$

and
$$C(s) = 4X_1 - 2X_2$$

Hence the state model is,

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} r(t)$$

and $$c(t) = \begin{bmatrix} 4 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Example 2.19 : *Write a set of state equations for the network shown in the Fig. 2.34 Choose i_1, i_2 and v_C as state variables.* **(VTU : April-98,July/Aug.-2007)**

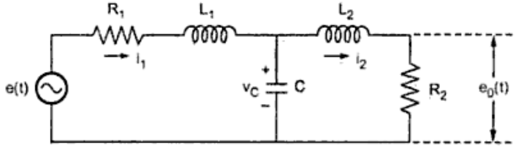


Fig. 2.34

Solution : Let $e(t)$ = Input

$$\text{Voltage across } R_2 = e_0(t) = \text{output}$$

$$i_1(t) = X_1(t)$$

$$i_2(t) = X_2(t)$$

$$v_C(t) = X_3(t)$$

Let $i_1(t)$ and $i_2(t)$ are the loop currents. Applying KVL to the loops.

$$e(t) = R_1 i_1 + L_1 \frac{d i_1}{d t} + v_C$$

$$\therefore \quad \frac{d i_1}{d t} = \frac{1}{L_1}[-i_1 R_1 - v_C + e(t)]$$

$$\therefore \quad \dot{X}_1(t) = \frac{1}{L_1}[-R_1 X_1 - X_3 + e(t)] \qquad \text{... (1)}$$

Then, $$v_C(t) = L_2 \frac{d i_2}{d t} + i_2 R_2$$

$$\therefore \quad \frac{d i_2}{d t} = \frac{1}{L_2}[-i_2 R_2 + v_C]$$

$$\therefore \quad \dot{X}_2(t) = \frac{1}{L_2}[-R_2 X_2 + X_3] \qquad \text{... (2)}$$

and $$C \frac{d v_C}{d t} = i_1 - i_2 \qquad \text{...Capacitor current}$$

$$\therefore \quad \frac{d v_C}{d t} = \frac{1}{C}(i_1 - i_2)$$

$$\therefore \qquad \dot{X}_3 = \frac{1}{C}(X_1 - X_2) \qquad \ldots (3)$$

While $$e_0(t) = i_2 R_2 = R_2 X_2 \qquad \ldots (4)$$

Hence the state model is,

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & +\frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} U(t) \quad \text{where } U(t) = e(t)$$

and $$Y(t) = e_0(t) = \begin{bmatrix} 0 & R_2 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

Example 2.20 : *Considering v_C and I_i as state variables and I_x as the output variables in the circuit shown below, obtain the state model.*

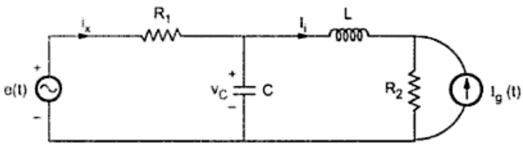


Fig. 2.35

(VTU : Aug. - 97)

Solution : Convert the current source to voltage source as shown.

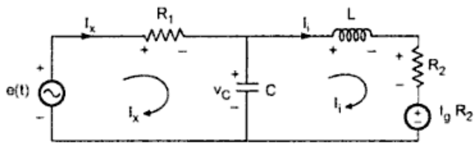


Fig. 2.35 (a)

Two inputs, $e(t)$ and $I_g(t)$ i.e. $U_1 = e(t)$ and $U_2 = I_g(t)$

One output, $I_x(t)$ i.e. $Y(t) = I_x(t)$

Variables, $v_C = X_1(t)$ and $I_i = X_2(t)$

Let $I_x(t)$ and $I_i(t)$ be the loop currents.

Applying KVL to the two loops,

Loop 1, $-I_x R_1 - v_C + e(t) = 0$

$$\therefore \quad I_x R_1 = e(t) - v_C$$

$$\therefore \quad I_x = \frac{1}{R_1} e(t) - \frac{1}{R_1} v_C(t)$$

$$\therefore \quad Y(t) = \frac{1}{R_1} U_1 - \frac{1}{R_1} X_1(t) \quad \text{... Output equation}$$

Loop 2, $-L\frac{dI_i}{dt} - I_i R_2 - I_g R_2 + v_C = 0$

$$\therefore \quad \frac{dI_i}{dt} = \frac{1}{L} v_C(t) - \frac{R_2}{L} I_i - \frac{R_2}{L} I_g$$

$$\therefore \quad \dot{X}_2 = \frac{1}{L} X_1(t) - \frac{R_2}{L} X_2(t) - \frac{R_2}{L} U_2 \quad \text{... State equation}$$

and current through capacitor,

$$I_x - I_i = C\frac{dv_C}{dt}$$

$$\therefore \quad \frac{dv_C}{dt} = \frac{1}{C}[I_x - I_i]$$

Substituting I_x from output equation,

$$\dot{X}_1 = \frac{1}{CR_1} U_1 - \frac{1}{CR_1} X_1(t) - \frac{1}{C} X_2(t) \quad \text{... State equation}$$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 C} & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R_2}{L} \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{CR_1} & 0 \\ 0 & -\frac{R_2}{L} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$Y = \begin{bmatrix} -\frac{1}{R_1} & 0 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1} & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

Example 2.21 : *Derive two state models for the system described by the differential equation*

$$D^3 y + 4D^2 y + 5Dy + 2y = 2D^2 u + 6Du + 5u \quad \text{where } D = d/dt$$

i) One in phase variable form.

ii) Other in Jordan-Canonical form. **(VTU: March-2001)**

Solution : Take the Laplace transform of both sides of the equation and neglect the initial conditions to obtain transfer function of the system as,

$$s^3 Y(s) + 4s^2 Y(s) + 5sY(s) + 2Y(s) = 2s^2 U(s) + 6s\, U(s) + 5U(s)$$

$$\therefore \quad \frac{Y(s)}{U(s)} = \frac{2s^2 + 6s + 5}{s^3 + 4s^2 + 5s + 2}$$

i) Phase variable form

Use the direct decomposition.

$$\frac{Y(s)}{U(s)} = \frac{2s^2 + 6s + 5}{\{[(s+4)s+5]s+2\}}$$

The state diagram can be shown as in the Fig. 2.36

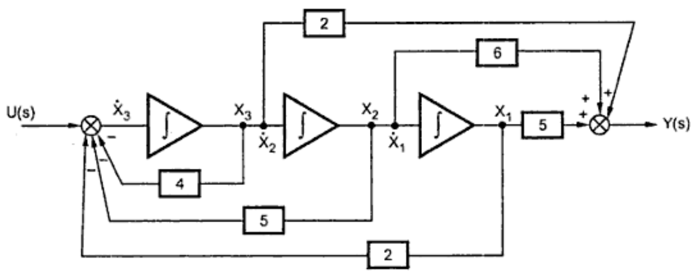


Fig. 2.36

So $\quad \dot{X}_1 = X_2 \qquad \dot{X}_2 = X_3$

$$\dot{X}_3 = U(s) - 2X_1 - 5X_2 - 4X_3$$

and $\quad Y(s) = 5X_1 - 6X_2 + 2X_3$

So state model is having matrices,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } C = [5\ 6\ 2]$$

ii) Jordan canonical form :

Factorise the denominator as,

$$\frac{Y(s)}{U(s)} = \frac{2s^2+6s+5}{(s+1)^2\ (s+2)} = \frac{A}{(s+1)^2}+\frac{B}{(s+1)}+\frac{C}{(s+2)}$$

$$\therefore\ A(s+2)+B(s+1)\ (s+2)+C(s+1)^2 = 2s^2+6s+5$$

$$\therefore\ As+2A+Bs^2+3Bs+2B+Cs^2+2Cs+C = 2s^2+6s+5$$

$$\therefore\ B+C=2, \quad A+3B+2C=6, \quad 2A+2B+C=5$$

$$\therefore\ A=1, \quad B=1, \quad C=1$$

$$\therefore\ \frac{Y(s)}{U(s)} = \frac{1}{(s+1)^2}+\frac{1}{(s+1)}+\frac{1}{(s+2)}$$

The state model is shown in the Fig.2.37 (a)

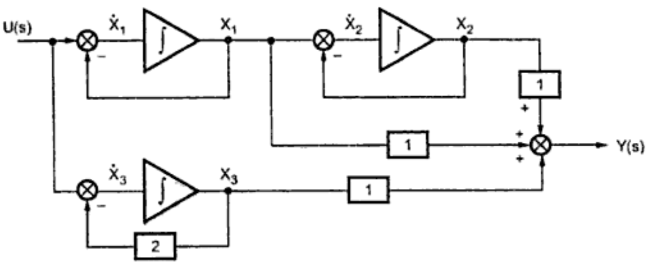


Fig. 2.37

So $\qquad \dot{X}_1 = U(s)-X_1 \qquad \dot{X}_2 = X_1-X_2 \qquad \dot{X}_3 = U(s)-2X_3$

and $\qquad Y(s) = X_1+X_2+X_3$

So state model is having matrices,

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } C = [1\ 1\ 1]$$

Example 2.22 : *For the system shown in the Fig. 2.38, obtain the state model choosing $v_1(t)$ and $v_2(t)$ as the state variables,* **(VTU: July/Aug.-2005, Jan./Feb.-2007)**

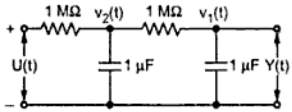


Fig. 2.38

Solution : Select the two currents as shown in the Fig. 2.38(a). And write the equations using KVL and KCL.

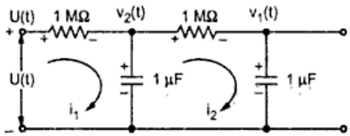


Fig. 2.38 (a)

$$i_1(t) = \frac{U(t)-v_2(t)}{1\times 10^6} \quad \text{... (1)}$$

$$v_2(t) = \frac{1}{C}\int (I_1 - I_2)\, dt$$

i.e. $$i_1 - i_2 = C\frac{dv_2(t)}{dt} \quad \text{... (2)}$$

$$i_2(t) = \frac{v_2 - v_1}{1\times 10^6} \quad \text{... (3)}$$

And $$v_1(t) = \frac{1}{C}\int i_2\, dt$$

$\therefore$ $$i_2(t) = C\frac{dv_1(t)}{dt} \quad \text{... (4)}$$

Elliminate i_1 and i_2 from above equations and $C = 1 \times 10^{-6}$ F.

Substituting (1) and (3) in (2) we get,

$$C\frac{dv_2}{dt} = \left[\frac{U(t)-v_2(t)}{1\times10^6}\right]-\left[\frac{v_2(t)-v_1(t)}{1\times10^6}\right]$$

$$\therefore \quad \frac{dv_2}{dt} = \frac{1}{C}\left[\frac{U(t)}{1\times10^6}\right]+\frac{1}{C}\,\frac{v_1(t)}{1\times10^6}-\frac{1}{C}\left[\frac{2v_2(t)}{1\times10^6}\right]$$

$$\therefore \quad \frac{dv_2}{dt} = v_1(t) - 2v_2(t) + U(t) \qquad \text{... (5)}$$

Using (3) in (4),

$$C\frac{dv_1}{dt} = \frac{v_2 - v_1}{1\times10^6}$$

$$\therefore \quad \frac{dv_1}{dt} = -v_1(t) + v_2(t) \qquad \text{... (6)}$$

Select $X_1(t) = v_1(t)$ and $X_2(t) = v_2(t)$

Using selected state variables,

$$\dot{X}_1 = -X_1 + X_2$$

and $\dot{X}_2 = X_1 - 2X_2 + U(t)$

and $Y(t) = v_1(t) = X_1(t)$

Hence the state model is,

$$\dot{X}(t) = AX(t) + BU(t)$$

and $Y(t) = CX(t) + DU(t)$

where $A = \begin{bmatrix} -1 & +1 \\ 1 & -2 \end{bmatrix},\; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\; C = [1 \;\; 0],\; D = 0$

Example 2.23 : *Derive two state models for the system with transfer function,*

$$\frac{Y(s)}{U(s)} = \frac{50\left(1+\frac{s}{5}\right)}{s\left(1+\frac{s}{2}\right)\left(1+\frac{s}{50}\right)}$$

i) For first model, matrix A must be in companion form.

ii) For second model, matrix A must be in diagonal form.

Solution : Arrange the transfer function as,

$$T(s) = \frac{50\times\frac{1}{5}\times(s+5)}{s\times\frac{1}{2}\times(s+2)\times\frac{1}{50}\times(s+50)} = \frac{1000(s+5)}{s(s+2)(s+50)}$$

i) The companion form means phase variable form for which use direct decomposition of denominator,

$$T(s) = \frac{1000(s+5)}{[s^2+52s+100]s} = \frac{1000s+5000}{\{[(s+52)s+100]s\}}$$

The state diagram is as shown in the Fig. 2.39.

From state diagram,

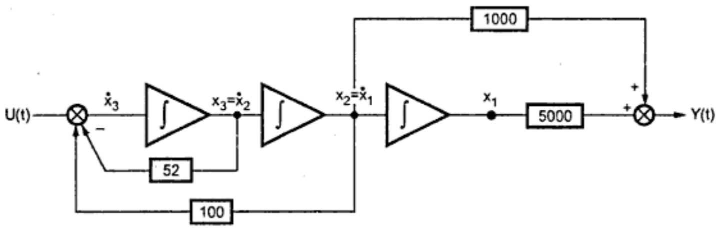


Fig. 2.39

$$\dot{X}_1 = X_2, \quad \dot{X}_2 = X_3$$

and $$\dot{X}_3 = -100X_2 - 52X_3 + U(t)$$

While $$Y(t) = 5000X_1 + 1000X_2$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -100 & -52 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C = [5000 \quad 1000 \quad 0], \quad D = 0$$

ii) For diagonal form use partial fractions,

$$T(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+50}$$

$$A = T(s)\,s\big|_{s=0} = \frac{1000\times5}{2\times50} = 50$$

$$B = T(s)(s+2)|_{s=-2} = \frac{1000(3)}{(-2)(48)} = -31.25$$

$$C = T(s)(s+50)|_{s=-50} = \frac{1000(-45)}{(-50)(-48)} = -18.75$$

$$\therefore \quad T(s) = \frac{50}{s} - \frac{31.25}{s+2} - \frac{18.75}{s+50}$$

The state diagram is,

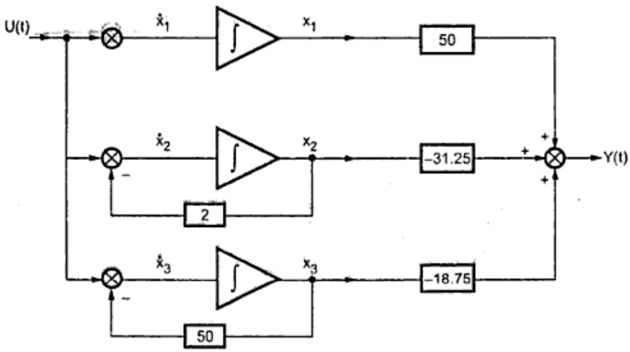


Fig. 2.39 (a)

Thus the state equations are,

$$\dot{X}_1 = U(t),\ \dot{X}_2 = -2X_2 + U(t),\ \dot{X}_3 = -50X_3 + U(t)$$

and $\quad Y(t) = 50X_1 - 31.25X_2 - 18.75X_3$

Thus, $\quad A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -50 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$

$$C = [50 \quad -31.25 \quad -18.75],\ D = 0$$

Example 2.24 : *Derive the state model for two input two output system shown in the Fig. 2.40.*

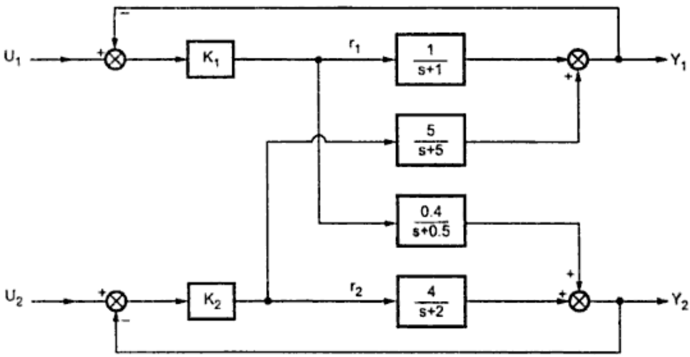


Fig. 2.40

Select output of simple lags as state variables.

Solution : As suggested, the output of $\frac{1}{s+1}$ is X_1,

$\frac{1}{s+5}$ is X_2, $\frac{1}{s+0.5}$ is X_3 and $\frac{1}{s+2}$ is X_4.

The state diagrams for simple lags are,

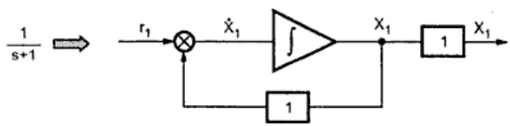


Fig. 2.40 (a)

$$\dot{X}_1 = r_1 - X_1 \qquad \ldots(1)$$

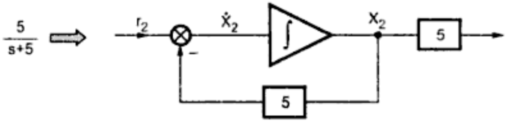


Fig. 2.40 (b)

$$\therefore \quad \dot{X}_2 = r_2 - 5X_2 \qquad \ldots(2)$$

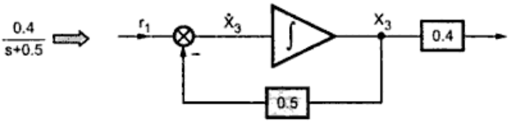


Fig. 2.40 (c)

$$\therefore \quad \dot{X}_3 = r_1 - 0.5X_3 \qquad \ldots(3)$$

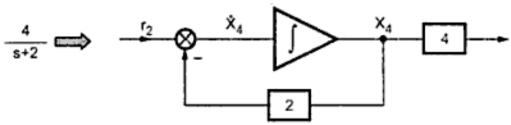


Fig. 2.40 (d)

$$\therefore \quad \dot{X}_4 = r_2 - 2X_4 \qquad \ldots(4)$$

From the given block diagram,

$$r_1 = K_1 [U_1 - Y_1] \quad \text{and} \quad r_2 = K_2 [U_2 - Y_2] \qquad \ldots(5)$$

and $$Y_1 = X_1 + 5X_2 \quad ...(6\ a)$$

$$Y_2 = 0.4X_3 + 4X_4 \quad ...(6\ b)$$

Substituting in (5) we get,

$$r_1 = -K_1 X_1 - 5K_1 X_2 + K_1 U_1 \quad ...(7)$$

$$r_2 = -0.4K_2 X_3 - 4K_2 X_4 + K_2 U_2 \quad ...(8)$$

Using (7) and (8) in the equations (1) to (4),

$$\dot{X}_1 = (-1 - K_1) X_1 - 5K_1 X_2 + K_1 U_1 \quad ...(9\ a)$$

$$\dot{X}_2 = -5X_2 - 0.4K_2 X_3 - 4K_2 X_4 + K_2 U_2 \quad ...(9\ b)$$

$$\dot{X}_3 = -K_1 X_1 - 5K_1 X_2 - 0.5X_3 + K_1 U_1 \quad ...(9\ c)$$

$$\dot{X}_4 = -0.4K_2 X_3 + (-4K_2 - 2) X_4 + K_2 U_2 \quad ...(9d)$$

The equations (9a) to (9d) and (6a) to (6b) give the required state model as,

$$\dot{X} = AX + BU \quad \text{and } Y = CX + DU \text{ where}$$

$$A = \begin{bmatrix} (-1-K_1) & -5K_1 & 0 & 0 \\ 0 & -5 & -0.4K_2 & -4K_2 \\ -K_1 & -5K_1 & -0.5 & 0 \\ 0 & 0 & -0.4K_2 & (-4K_2-2) \end{bmatrix}, \quad B = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \\ K_1 & 0 \\ 0 & K_2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 0 & 0.4 & 4 \end{bmatrix}, \quad D = [0]$$

Example 2.25 : *A series RLC circuit with R = 1 Ω, L = 1 H and C = 1 F is excited by v = 10 V. Write the state equations in the matrix form.* **[Bangalore Univ., Dec.-95]**

Solution : The circuit is shown in the Fig. 2.41.

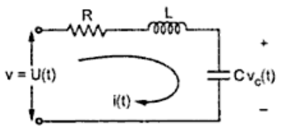


Fig. 2.41

Applying KVL to the circuit,

$$v = i(t) R + L \frac{dI(t)}{dt} + v_C(t) \quad ...(1)$$

and $$v_C(t) = \frac{1}{C}\int i(t)\,dt$$

$$\therefore \quad \frac{dv_C(t)}{dt} = \frac{1}{C}\,i(t) \qquad \text{... (2)}$$

Let $i(t) = X_1$ = Current through inductor

$v_C(t) = X_2$ = Voltage across capacitor

$V = U(t) = 10\ V$

Substituting all the values,

$$10 = X_1 + \dot{X}_1 + X_2$$

$$\therefore \quad \dot{X}_1 = -X_1 - X_2 + 10 \qquad \text{... (3)}$$

and $$\dot{X}_2 = X_1 \qquad \text{... (4)}$$

Hence state equations in matrix form are,

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \end{bmatrix} U(t)$$

Example 2.26 : *Obtain the state space representation in phase variable form for the system represented by,* $D^4y + 20D^3y + 45D^2y + 18\,Dy + 100\,y = 10D^2u + 5\,Du + 100\,u$ *with y as output and u as input.* **(VTU: Jan./Feb.-2005)**

Solution : Phase varaible form

Taking Laplace transform of both sides and neglecting initial conditions,

$$s^4Y(s) + 20s^3Y(s) + 45s^2Y(s) + 18s\,Y(s) + 100\,Y(s) = 10s^2U(s) + 5s\,U(s) + 100\,U(s)$$

$$\therefore \quad \frac{Y(s)}{U(s)} = \frac{10s^2 + 5s + 100}{s^4 + 20s^3 + 45s^2 + 18s + 100}$$

Using direct decomposition,

$$\frac{Y(s)}{U(s)} = \frac{10s^2 + 5s + 100}{\{[([s+20]s + 45)\,s + 18]s + 100\}}$$

The state diagram is as shown in Fig. 2.42

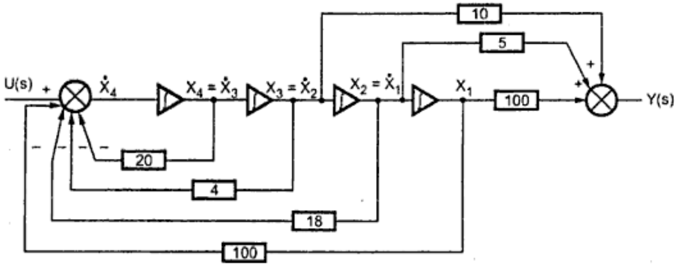


Fig. 2.42

The state equations are,

$$\dot{X}_1 = X_2,\ \dot{X}_2 = X_3,\ \dot{X}_3 = X_4,\ \dot{X}_4 = -100X_1 - 18X_2 - 4X_3 - 20X_4 + U$$

and $$y = 100\,X_1 + 5X_2 + 10X_3$$

$$\therefore \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -100 & -18 & -4 & -20 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},\ C = [100\ \ 5\ \ 10\ \ 0]$$

Example 2.27 : *Obtain the state model of the system responsed by the following differential equation :* $\dddot{y} + 6\ddot{y} + 5\dot{y} + y = u.$ **(VTU: July/Aug.- 2005)**

Solution : Take Laplace transform of both sides neglecting initial conditions,

$$s^3Y(s) + 6s^2Y(s) + 5sY(s) + Y(s) = U(s)$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^3 + 6s^2 + 5s + 1}$$

$$\therefore \quad \frac{Y(s)}{U(s)} = \frac{1}{\{[(s+6)s+5]s+1\}}$$

The state diagram is shown in the Fig. 2.43

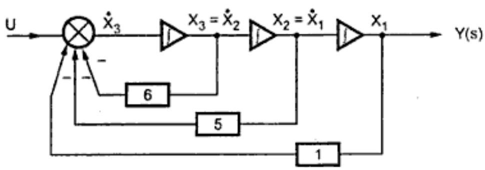


Fig. 2.43

The state equations are,

$$\dot{X}_1 = X_2,\ \dot{X}_2 = X_3,\ \dot{X}_3 = -X_1 - 5X_2 - 6X_3 + U$$

and $$Y = X_1$$

$$\therefore \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C = [1\ 0\ 0]$$

Example 2.28 : *Obtain two different state models for a system represented by the following transfer function. Write suitable state diagram in each case.*

(VTU: July/Aug.-2005)

$$\frac{Y(s)}{U(s)} = \frac{8s^2 + 17s + 8}{(s+1)\left(s^2 + 8s + 15\right)}$$

Solution : 1) Direct decomposition :

$$\frac{Y(s)}{U(s)} = \frac{8s^2 + 17s + 8}{s^3 + 9s^2 + 23s + 15} = \frac{8s^2 + 17s + 8}{\{[(s+9)s + 23]s + 15\}}$$

The state diagram is shown in the Fig.2.44

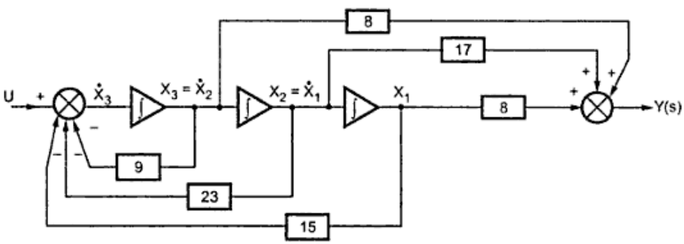


Fig. 2.44

$$\therefore \quad \dot{X}_1 = X_2,\ \dot{X}_2 = X_3,\ \dot{X}_3 = -15X_1 - 23X_2 - 9X_3 + U$$

$$Y = 8X_1 + 17X_2 + 8X_3$$

$$\therefore \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -23 & -9 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C = [8\ 17\ 8]$$

2) Foster's term :

$$\frac{Y(s)}{U(s)} = \frac{8s^2+17s+8}{(s+1)(s+3)(s+5)} = \frac{A}{s+1}+\frac{B}{s+3}+\frac{C}{s+5}$$

A = – 0.125, B = –7.25, C = 15.375

$$\therefore \qquad \frac{Y(s)}{U(s)} = \frac{-0.125}{s+1}-\frac{7.25}{s+3}+\frac{15.375}{s+5}$$

The state diagram is shown in the Fig. 2.45

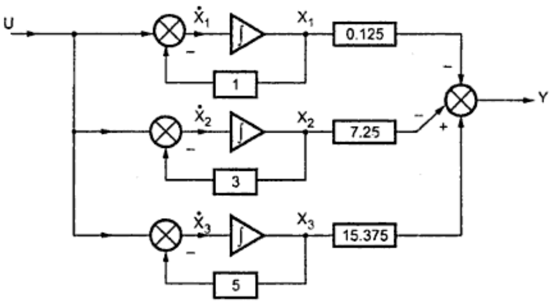


Fig. 2.45

The state equations are,

$$\dot{X}_1 = -X_1 + U, \qquad \dot{X}_2 = -3X_2 + U, \qquad \dot{X}_3 = -5X_3 + U$$

$$Y = -0.125X_1 - 7.25X_2 + 15.375X_3$$

$$\therefore \qquad A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; C = [-0.125 \; -7.25 \; 15.375]$$

Example 2.29 : *Choosing appropriate physical variables as state variables, obtain the state model for the electric circuit shown in Fig. 2.46* **(VTU: Jan./Feb.-2006)**

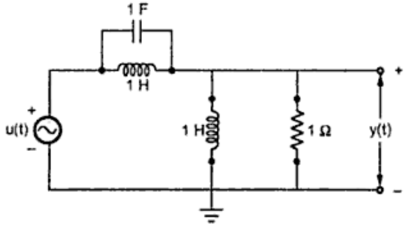


Fig. 2.46

Solution : The various currents are shown in the Fig. 2.46(a).

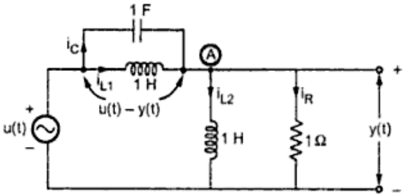


Fig. 2.46 (a)

The volatage across resistance and second inductor is y(t).

$$\therefore \qquad i_R = \frac{y(t)}{R} = y(t) \qquad \text{....(1)}$$

$$\text{and} \qquad 1\times \frac{di_{L2}}{dt} = y(t) \qquad \text{....(2)}$$

The voltage across capacitor is, $\quad v_C(t) = u(t) - y(t)$

$$\therefore \qquad i_C(t) = C\frac{dv_C(t)}{dt} = \frac{dv_C(t)}{dt} \qquad \text{....(3)}$$

$$\text{and} \qquad 1\times \frac{di_{L1}}{dt} = u(t) - y(t) = v_C\,(t) \qquad \text{.....(4)}$$

From (4), $\quad y(t) = u(t) - \dfrac{di_{L1}}{dt} = u(t) - v_C(t)$

Using in (2), $\quad \dfrac{di_{L2}}{dt} = u(t) - v_C(t) \qquad \text{....(5)}$

Applying KCL at node A,

$$i_C + i_{L1} = i_{L2} + i_R$$

$$\therefore \quad \frac{dv_C(t)}{dt} + i_{L1} = i_{L2} + y(t)$$

$$\frac{dv_C(t)}{dt} = u(t) - v_C(t) - i_{L1} + i_{L2} \qquad ...(6)$$

Select $X_1 = i_{L1}$, $X_2 = i_{L2}$, $X_3 = v_C$

$$\therefore \quad \dot{X}_1 = \frac{di_{L1}}{dt} = X_3 \qquad ...\text{from (4)}$$

$$\dot{X}_2 = \frac{di_{L2}}{dt} = -X_3 + U \qquad ...\text{from (5)}$$

$$\dot{X}_3 = \frac{dv_C(t)}{dt} = -X_1 + X_2 - X_3 + U \qquad ...\text{from (6)}$$

and

$$Y = \frac{di_{L2}}{dt} = \dot{X}_2 = -X_3 + U \qquad ...\text{from (2)}$$

$$\therefore \quad A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}, D = [1]$$

Example 2.30 : *For the transfer function* $\frac{Y(s)}{R(s)} = \frac{s(s+2)(s+3)}{(s+1)^2(s+4)}$ **(VTU: Jan./Feb.-2006)**

Obtain the state model in

i) Phase variable canonical form

ii) Jordan Canonical form

Solution : i) Phase variable form

$$\frac{Y(s)}{R(s)} = \frac{s(s+2)(s+3)}{(s+1)^2(s+4)} = \frac{s^3+5s^2+6s}{s^3+6s^2+9s+4}$$

Using direct decomposition,

$$\frac{Y(s)}{R(s)} = \frac{s^3+5s^2+6s}{\{[(s+6)s+9]s+4\}}$$

The state diagram is shown in the Fig. 2.47.

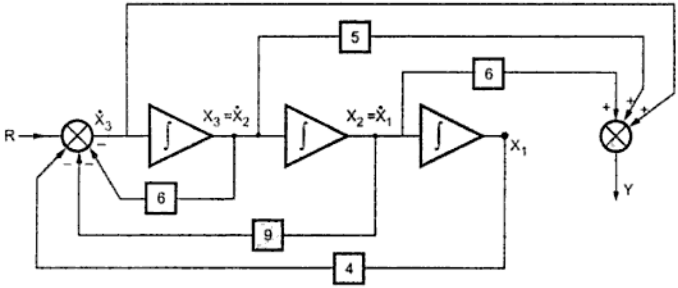


Fig. 2.47

$$\dot{X}_1 = X_2,\ \dot{X}_2 = X_3,\quad \dot{X}_3 = -4X_1 - 9X_2 - 6X_3 + R$$

$$Y = 6X_2 + 5X_3 + \dot{X}_3 = 6X_2 + 5X_3 - 4X_1 - 9X_2 - 6X_3 + R$$

$$\therefore \quad Y = -4X_1 - 3X_2 - X_3 + R$$

$$\therefore \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -9 & -6 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C = [-4\ \ -3\ \ -1],\ D = [1]$$

ii) Jordan Canonical form :

As the degree of denominator and numerater is same, first divide and then obtain partial fractions

$$\begin{array}{r} s^3+6s^2+9s+4)\ s^3+5s^2+6s\ (1 \\ s^3+6s^2+9s+4 \\ \underline{-\quad -\quad -\quad -} \\ -s^2-3s\ -4 \end{array}$$

$$\therefore \quad \frac{Y(s)}{R(s)} = 1 - \frac{(s^2+3s+4)}{(s+1)^2(s+4)} = 1 - \left\{ \frac{A}{(s+1)^2} + \frac{B}{s+1} + \frac{C}{s+4} \right\}$$

$$\therefore \quad A(s+4) + B(s+1)(s+4) + C(s+1)^2 = s^2 + 3s + 4$$

$$\therefore \quad As+4A+Bs^2+5sB+4B+Cs^2+2Cs+C = s^2+3s+4$$

$$\therefore B+C=1, \quad A+5B+2C=3, \quad 4A+4B+C=4$$

$$A=0.666, \quad B=0.111, \quad C=+0.888$$

$$\therefore \quad \frac{Y(s)}{R(s)} = 1-\left\{\frac{0.666}{(s+1)^2}+\frac{0.111}{(s+1)}+\frac{0.888}{(s+4)}\right\}$$

$$= 1-\frac{0.666}{(s+1)^2}-\frac{0.111}{s+1}-\frac{0.888}{s+4}$$

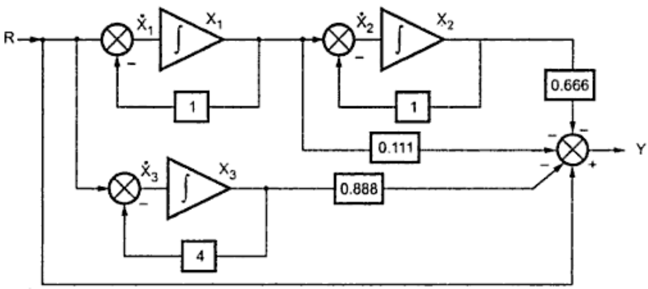


Fig. 2.48

$$\therefore \dot{X}_1=-X_1+R, \quad \dot{X}_2=-X_2+X_1, \quad \dot{X}_3=-4X_3+R$$

$$Y = -0.111X_1-0.666X_2-0.888X_3+R$$

$$\therefore \quad A=\begin{bmatrix}-1 & 0 & 0\\ 1 & -1 & 0\\ 0 & 0 & -4\end{bmatrix}, B=\begin{bmatrix}1\\0\\1\end{bmatrix}, C=\begin{bmatrix}-0.111 & -0.666 & -0.888\end{bmatrix}, D=[1]$$

Example 2.31 : *Fig. 2.49 shows the block diagram of a speed control system with state variable feedback. The drive motor is an armature controlled d.c. motor with armature resistance R_a, armature inductance L_a, motor torque constant K_T, inertia referred to constant K_b and tachometer K_t. The applied armature voltage is controlled by a three phase full-converter. e_c is control voltage, e_a is armature voltage, e_r is the reference voltage corresponding to the desired speed. Taking $X_1 = \omega$ (speed) and $X_2 = i_a$ (armature current) as the state variables, $u = e_r$ as the input, and $y = \omega$ as the output, derive a state variable mode for the feedback system.* **(VTU: July/Aug.-2006)**

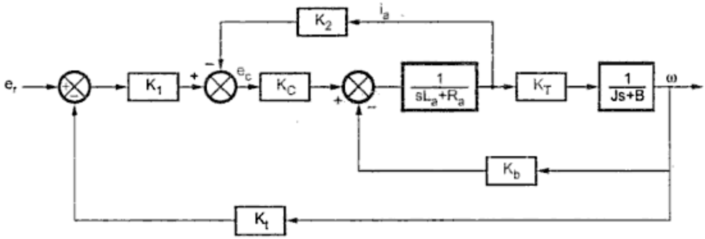


Fig. 2.49

Solution : For armature controlled d.c. motor,

$$v_a(t) = L_a\frac{di_a}{dt} + i_a R_a + e_b(t) \qquad \text{......(1)}$$

and

$$e_b(t) = K_b\omega(t) \qquad \text{......(2)}$$

$$T_m = K_T i_a \qquad \text{......(3)}$$

The torque produced is used to drive the load against inertia J and friction B.

$$\therefore \quad T_m = J\frac{d\omega}{dt} + B\omega \qquad \text{......(4)}$$

$$\therefore \quad K_T i_a = J\frac{d\omega}{dt} + B\omega$$

$$\therefore \quad \frac{d\omega}{dt} = \frac{K_T}{J}i_a - \frac{B}{J}\omega \qquad \text{......(5)}$$

The state variables are $X_1 = \omega$ and $X_2 = i_a$

$$\therefore \quad \dot{X}_1 = \frac{K_T}{J}X_2 - \frac{B}{J}X_1 \qquad \text{......(6)}$$

From the equation (1),

$$v_a(t) = L_a\frac{di_a}{dt} + i_a R_a + K_b\omega(t) \qquad \text{......(7)}$$

But the armature voltage v_a is controlled by three phase full converter.

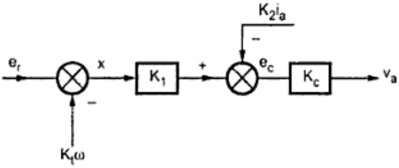


Fig. 2.50

$$\therefore v_a = K_c e_c$$
$$= K_c[K_1 x - K_2 i_a]$$
$$= K_c K_1 x - K_c K_2 i_a$$
$$= K_c K_1 [e_r - K_t\omega] - K_c K_2 i_a$$
$$\therefore v_a = K_c K_1 e_r - K_c K_1 K_t \omega - K_c K_2 i_a$$

Using this in the equation (7) above,

$$K_c K_1 e_r - K_c K_1 K_t \omega - K_c K_2 i_a = La\frac{di_a}{dt} + i_a R_a + K_b\omega$$

$$\therefore \quad \frac{di_a}{dt} = -\frac{(K_c K_1 K_t + K_b)}{L_a}\omega - \frac{(K_c K_2 + R_a)}{L_a} i_a + \frac{K_c K_1}{L_a} e_r \quad \text{....(8)}$$

$$\therefore \quad \dot{X}_2 = -\frac{(K_c K_1 K_t + K_b)}{L_a} X_1 - \frac{(K_c K_2 + R_a)}{L_a} X_2 + \frac{K_c K_1}{L_a} u \quad \text{....(9)}$$

And $\quad y = \omega = X_1$

The equations (6), (9) and (10) give us the required state model with,

$$A = \begin{bmatrix} -\frac{B}{J} & \frac{K_T}{J} \\ \frac{-(K_c K_1 K_t + K_b)}{L_a} & \frac{-(K_c K_2 + R_a)}{L_a} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{K_c K_1}{L_a} \end{bmatrix}, C = [1 \quad 0]$$

Example 2.32 : *For a transfer function given by* $G(s) = \frac{2}{s^2 + 3s + 2}$, *write the state model in the :*

i) Phase variable form ii) Diagonal form

Solution : i) Phase variable form using direct decomposition

$$G(s) = \frac{2}{s^2 + 3s + 2} = \frac{2}{[(s+3)s+2]}$$

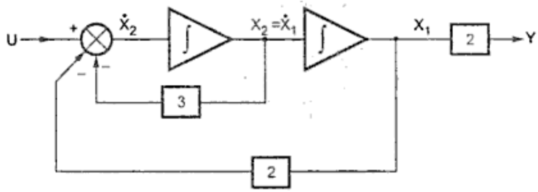


Fig. 2.51

$\dot{X}_1 = X_2, \quad \dot{X}_2 = -2X_1 - 3X_2 + U, \qquad Y = 2X_1$

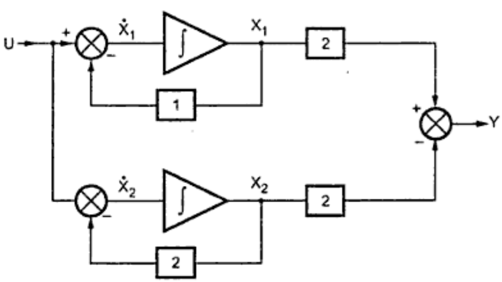
$$\therefore \qquad A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 2 & 0 \end{bmatrix}$$

ii) Diagonal form using Foster's form

$$G(s) = \frac{2}{s^2 + 3s + 2} = \frac{2}{(s+1)\,(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$\therefore \qquad A = \frac{2}{(2-1)} = 2, \; B = \frac{2}{(-2+1)} = -2$$

$$\therefore \qquad G(s) = \frac{2}{s+1} - \frac{2}{s+2}$$



$\dot{X}_1 = -X_1 + U$

$\dot{X}_1 = -2X_2 + U$

$Y = 2X_1 + 2X_2$

$\therefore A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore C = \begin{bmatrix} 2 & 2 \end{bmatrix}$

Fig. 2.52

Example 2.33 : *For the network shown in Fig. 2.53, choosing $i_1(t) = X_1(t)$ and $i_2(t) = X_2(t)$ as state variables, obtain the state equation and output equation in vector matrix form.*

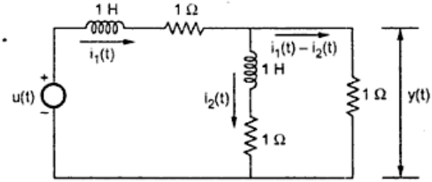


Fig. 2.53

(VTU: Dec.-2007/Jan.-2008)

Solution : Applying KVL to the two loops,

$$-\frac{di_1}{dt} - i_1 - \frac{di_2}{dt} - i_2 + u(t) = 0 \quad \text{...Loop 1}$$

$$-(i_1 - i_2) + i_2 + \frac{di2}{dt} = 0 \quad \text{...Loop 2}$$

$$\therefore \quad \frac{di2}{dt} = i_1 - 2i_2 \quad \text{...(1)}$$

Using (1) in equation for loop 1,

$$\frac{di_1}{dt} = -i_1 - (i_1 - 2i_2) - i_2 + u(t) = -2i_1 + i_2 + u(t) \quad \text{...(2)}$$

using $i_1(t) = X_1$ and $i_2(t) = X_2$

$$\therefore \quad \dot{X}_1 = -2X_1 + X_2 + u(t) \quad \text{...(3)}$$

$$\dot{X}_2 = X_1 - 2X_2 \quad \text{...(4)}$$

and $$y(t) = i_1 - i_2 = X_1 - X_2 \quad \text{...(5)}$$

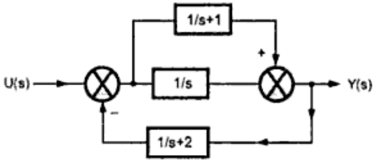
$$\therefore \quad \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

and $$y(t) = [1 \quad -1] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

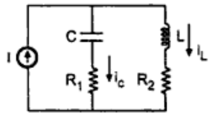
$$\therefore \quad A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = [1 \quad -1], D = [0]$$

Review Questions

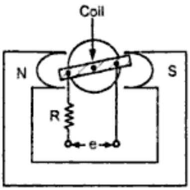
1. *Discuss the derivation of the state model using following methods of programming.*
   i) Bush form *ii) Gullemin's form*
   iii) Foster's form *iv) Jordan's form*
2. *Elaborate upon the basis of selecting suitable state variables for a system.*
3. *Obtain the state model for the block diagram shown.*



4. *Voltage across capacitor is* v_C. *With* v_C and i_L *as a set of state variables derive the state model.*



5. *Consider the permanent magnet moving coil instrument as shown,*
   Here, e = Voltage to be measured,
   R = Armature Resistance,
   L = Armature Inductance , K_b= *Back e.m.f. constant relating*
   i = Current through armature coil ,
   J = M.I. of rotating part , K_s = *Spring constant ,*
   K_T = *Torque constant relating i and T ,*
   Derive its state model.

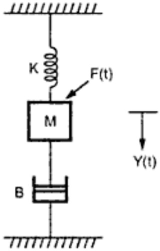


Ans. : $\dot{X} = AX + BU \quad Y = CX + DU$

$$\text{where } A = \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{K_b}{L} \\ 0 & 0 & 1 \\ \frac{K_T}{J} & -\frac{K_s}{J} & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix} \quad C = [0 \ 1 \ 0]$$

6. *For a mechanical system shown below obtain its state model in standard form. Assume output as Y(t).*

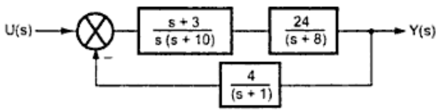
Ans. : $\dot{X} = AX + BD, \; Y = CX$



$$\text{where } A = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix} B = \begin{bmatrix} 0 \\ 1/M \end{bmatrix}, \; C = [0 \;\; 1]$$

7. *Obtain the state model in standard Bush form of a system shown in figure.*

Ans. : $\dot{X} = AX + BU, \quad Y = CX$



$$\text{where } A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -288 & -176 & -98 & -19 \end{bmatrix} \; B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \; C = [72 \;\; 96 \;\; 24 \;\; 0]$$

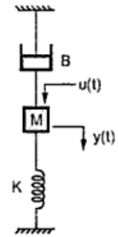
8. *State whether the following statement is True or False with reasons :*

   The state space approach of the system analysis is a frequency domain analysis.

9. *State whether the following statement is True or False with reason.*

   The state space approach of the system analysis is a time domain analysis.

10. *For the mechanical system shown in figure input is u(t) and output is y(t). Obtain state equation and output equation. Represent the system by block diagram.*



❑❑❑

3 Matrix Algebra and Derivation of Transfer Function

3.1 Background

The methods of obtaining the state model in various forms, from the transfer function of a system are discussed in the last chapter. It is also seen that the state models can be different and not a unique property for the given system.

But most important point about the system is its **transfer function is always unique.** Though number of state models are derived for the system, the transfer function of the system, obtained from all of them is always unique for single input single output system. In this chapter the method of obtaining transfer function of the system from its state model is discussed. The concepts of diagonalisation of system matrix A, eigen values and eigen vectors are also included in this chapter. Number of methods from matrix algebra are used to understand the relation between state model and transfer function hence revision of matrix algebra is included in the beginning of this chapter.

3.2 Definition of Matrix

A matrix is an ordered rectangular array of elements which may be real numbers, complex numbers, functions or mathematical operators. The order of matrix is always defined as m × n.

where,

$$\boxed{\begin{aligned} m &= \text{Number of rows} \\ n &= \text{Number of columns} \end{aligned}}$$

For example, consider matrix A of order 2 × 4 of real numbers.

$$A = \begin{array}{c} \begin{bmatrix} 4 & 3 & -2 & 1 \\ 6 & 1 & 0 & 8 \end{bmatrix} \begin{array}{l} \rightarrow \\ \rightarrow \end{array} \text{2 Rows} \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \text{4 Columns} \end{array}$$

Any element of a matrix is denoted as a_{ij} where i indicates position of row and j indicates position of column. Thus for matrix A above $a_{11} = 4$, $a_{13} = -2$, $a_{22} = 1$ and so on.

Key Point: *Note that matrix does not have a value but its determinant has a value.*

A matrix with number of columns as 1 i.e. order m × 1 is called **column matrix** while a matrix with number of rows as 1 i.e. order 1 × n is called **row matrix**. It is seen that vector matrices $\dot{X}$, X, Y are the column matrices.

3.2.1 Types of Matrices

The various types of matrices are,

1. **Square Matrix** : If number of rows (m) is equal to number of columns (n) of a matrix, it is called a square matrix. The system matrix A in a state model is always a square matrix of order n × n.

2. **Diagonal Matrix** : It is a square matrix with all $a_{ij} = 0$ for all $i \neq j$. Only main diagonal elements are present which are nonzero while all other elements are zero. For example,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

3. **Unit Matrix** : This is a diagonal matrix, with all the main diagonal elements equal to unity. It is denoted as I and also called **identity matrix**.

   The unit matrix of order 3 × 3 is,

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. **Null Matrix** : This is a matrix having all its elements as zero. It need not be a square matrix.

5. **Transpose of a Matrix** : If the rows of columns of m × n order matrix are interchanged then resulting n × m order matrix is called transpose of the original matrix. If given matrix is A then its transpose is denoted as A^T.

$$\underset{2\times 3}{A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 5 & 4 \end{bmatrix}}, \qquad \underset{3\times 2}{A^T = \begin{bmatrix} 0 & 2 \\ 1 & 5 \\ 2 & 4 \end{bmatrix}}$$

The **properties** of transpose are,

1. $(A + B)^T = A^T + B^T$
2. $(AB)^T = B^T A^T$

6. Symmetric Matrix : If transpose of a matrix is matrix itself, it is called symmetric matrix. It is a square matrix.

$$A^T = A$$

$$A = \begin{bmatrix} 1 & -4 \\ -4 & 1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & -4 \\ -4 & 1 \end{bmatrix}$$

7. Conjugate Matrix : The conjugate of a matrix is the matrix obtained in which each element is the complex conjugate of the corresponding element of the original matrix. It is denoted as A^*.

$$A = \begin{bmatrix} 1 & +j2 & 1+j1 \\ 2 & -j4 & 3+j2 \end{bmatrix}, \quad A^* = \begin{bmatrix} 1 & -j2 & 1-j1 \\ 2 & +j4 & 3-j2 \end{bmatrix}$$

8. Singular Matrix : A square matrix having its determinant value zero is called singular matrix.

$$A = \begin{bmatrix} 1 & 4 \\ 0.5 & 2 \end{bmatrix} \text{ and } |A| = \begin{vmatrix} 1 & 4 \\ 0.5 & 2 \end{vmatrix} = 2 - 2 = 0$$

Thus A is singular matrix.

9. Nonsingular Matrix : A square matrix whose determinant is nonzero is called nonsingular matrix.

$$A = \begin{bmatrix} 4 & 2 \\ 5 & 6 \end{bmatrix} \text{ and } |A| = \begin{vmatrix} 4 & 2 \\ 5 & 6 \end{vmatrix} = 24 - 10 = 14 \neq 0$$

10. Skew Symmetric Matrix : A square matrix which is equal to its negative transpose is called a skew symmetric matrix.

$$A^T = -A$$

3.2.2 Important Terminologies

Let us study the important terms and their meanings related to the matrix.

1. Minor of an element : Consider a square $(n \times n)$ matrix A and the element a_{ij}. If now i^{th} row and j^{th} column are deleted then the remaining $(n - 1)$ rows and columns form a determinant M_{ij}. The value of this determinant is called the minor of an element a_{ij}.

Consider a square matrix A as,

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 3 \\ 3 & 2 & 5 \end{bmatrix}$$

Let us obtain minor of element $a_{21} = 1$. Delete second row and first column to get determinant M_{21}.

$$\therefore \qquad M_{21} = \begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} = 15 - 8 = 7$$

Thus minor of 'a_{21}' is 7.

2. **Cofactor of an element :** If M_{ij} is the minor of an element a_{ij} then cofactor C_{ij} is defined as,

$$\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$$

Thus for the matrix A considered above, the cofactor of a_{21} is,

$$C_{21} = (-1)^{2+1} M_{21} = (-1)^3 \times 7 = -7$$

3. **Adjoint of a matrix :** The adjoint matrix is the transpose of a cofactor matrix. The cofactor matrix is the matrix obtained by replacing each element of a matrix by the respective cofactor.

$$\therefore \qquad \boxed{\text{Adj } A = [\text{Cofactor matrix of } A]^T}$$

Example 3.1 : *Find the adjoint matrix of the matrix A,*

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 2 \end{bmatrix}$$

Solution : The cofactors of all the elements are,

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 1 \\ 2 & 2 \end{vmatrix} = 6, \qquad C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = -1$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} = -8, \qquad C_{21} = (-1)^{2+1} \begin{vmatrix} 0 & 3 \\ 2 & 2 \end{vmatrix} = +6$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = -7, \qquad C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = -2$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 3 \\ 4 & 1 \end{vmatrix} = -12, \quad C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = +5$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 2 & 4 \end{vmatrix} = 4$$

$$\therefore \quad \text{Cofactor matrix} = \begin{bmatrix} 6 & -1 & -8 \\ 6 & -7 & -2 \\ -12 & +5 & 4 \end{bmatrix}$$

$$\therefore \quad \text{Adj } A = \begin{bmatrix} 6 & -1 & -8 \\ 6 & -7 & -2 \\ -12 & +5 & 4 \end{bmatrix}^T = \begin{bmatrix} 6 & 6 & -12 \\ -1 & -7 & 5 \\ -8 & -2 & 4 \end{bmatrix}$$

4. **Rank of a Matrix :** To find the rank of a matrix means to search for a highest order determinant from a given matrix which is having **nonzero value.** Thus if $r \times r$ determinant has a nonzero value in a given matrix then the rank of a matrix is 'r' and any determinant having order $r + 1$ or more than that has a zero value.

For example, consider matrix A as,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\text{Now} \quad \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -5 \neq 0$$

Thus 3×3 determinant is nonzero hence rank of matrix A is 3.

Example 3.2 : *Find rank of* $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

Solution : For the matrix A,

$$\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0 \quad \text{hence rank is not 2.}$$

Any other determinant of order 1×1 is nonzero. Hence rank of matrix A is 1.

5. **Equality of Matrices :** The two matrices A and B are said to be equal if they have same number of rows and columns and the elements of the corresponding orientations are equal.

3.3 Elementary Matrix Operations

The various elementary matrix operations are,

1. **Addition :** The two matrices A and B can be added to get C = A + B if both A and B are of **same order** and,

$$\boxed{C_{ij} = a_{ij} + b_{ij} \text{ for all i and j}}$$

2. **Subtraction :** The two matrices A and B can be subtracted to get C = A − B if both A and B are of **same order** and,

$$\boxed{C_{ij} = a_{ij} - b_{ij} \text{ for all i and j}}$$

Associative law holds good for both addition and subtraction of the matrices.

$$\therefore \quad [A \pm B] \pm C = A \pm [B \pm C]$$

where all A, B and C matrices have same order.

3. **Multiplication by scalar 'α' :** A matrix is multiplied by a scalar 'α' if all the elements of a matrix are multiplied by that scalar 'α'.

$$\therefore \quad \alpha \times A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha a_{n1} & \alpha a_{n2} & \cdots & \alpha a_{nn} \end{bmatrix}$$

4. **Multiplication of Matrices :** The multiplication of two matrices is possible if number of columns of the first matrix is equal to the number of rows of the second matrix. Such matrices are called **conformal matrices.**

Thus if A is of order m × n and B is of order n × p. Then the multiplication C = A × B is of order m × p. The elements of matrix C are obtained as,

$$\boxed{C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \text{ for i = 1, 2 ... n and j = 1, 2, ... p}}$$

Thus if A is of order 2 × 2 and B is of order 2 × 3 then C = A × B has a order 2 × 3.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$\therefore \quad C = A \times B$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \end{bmatrix}$$

Using all these values in the assumed solution,

$$x(t) = b_0 + ab_0(t) + \frac{1}{2!}a^2 b_0 t^2 + \ldots + \frac{1}{k!}a^k b_0 t^k$$

$$= b_0\left[1 + at + \frac{1}{2!}a^2t^2 + \ldots + \frac{1}{k!}a^k t^k\right]$$

$$= \left[1 + at + \frac{1}{2!}a^2t^2 + \ldots + \frac{1}{k!}a^k t^k\right] x(0)$$

But $1 + at + \frac{1}{2!}a^2 t^2 + \ldots + \frac{1}{k!}a^k t^k = e^{at}$

$$\therefore \quad \dot{X}(t) = e^{at}\, x(0) \quad \ldots (4)$$

This is the required solution of homogeneous equation in **scalar form.**

Thus if the homogeneous state equation is considered,

$$\dot{X}(t) = A\, X(t)$$

then its solution can be written as,

$$\boxed{X(t) = e^{At}\, X(0)}$$

In this case, e^{At} **is not a scalar but a matrix of order n × n as that of matrix A.**

Observation : It can be observed that without input, initial state X(0) drives the state X(t) at any time t. Thus there is transition of the initial state X(0) from initial time t = 0 to any time t through the matrix e^{At}. As it is an exponential term, the matrix e^{At} is called **matrix exponential.** It is also responsible for the transition of the state X(t) at any time t from initial time hence also called **state transition matrix.** It is denoted as $\phi(t)$.

$$\therefore \quad \phi(t) = e^{At} = \text{State transition matrix}$$

And $$\boxed{e^{At} = I + At + \frac{1}{2!}A^2 t^2 + \ldots + \frac{1}{k!}A^k t^k + \ldots}$$

The each term of the above equation is a matrix of order n × n.

If instead of initial time t = 0, it is selected as $t = t_0$ then the state transition matrix is,

$$\phi(t) = e^{A(t-t_0)}$$

4.2.1 Zero Input Response

The solution of the homogeneous state equation is under the condition of zero input. Such a response is called **zero input response** and for convenience denoted as ZIR.

Thus the behaviour of X(t) under the initial conditions without any input U(t) is called the **zero input response** of the system. It is also called **free, natural** or **unforced response** of the system.

4.3 Solution of Nonhomogeneous Equation

Consider a nonhomogeneous state equation as,

$$\dot{X}(t) = A\,X(t) + B\,U(t)$$

$$\therefore \quad \dot{X}(t) - A\,X(t) = B\,U(t)$$

Premultiplying both sides by e^{-At},

$$e^{-At}[\dot{X}(t) - A\,X(t)] = e^{-At}\,B\,U(t)$$

But $e^{-At}\,\dot{X}(t) - e^{-At}\,A\,X(t) = \frac{d}{dt}\,[e^{-At}\,X(t)]$

Substituting in above equation,

$$\frac{d}{dt}\,[e^{-At}\,X(t)] = e^{-At}\,B\,U(t)$$

Assuming initial time as t = 0 and integrating both sides from t = 0 to t,

$$e^{-At}X(t)\Big|_0^t = \int_0^t e^{-A\tau}\,B\,U(\tau)\,d\tau$$

$$\therefore \quad e^{-At}\,X(t) - X(0) = \int_0^t e^{-A\tau}\,B\,U(\tau)\,d\tau$$

Premultiplying both sides by e^{At},

$$\therefore e^{At}\,e^{-At}\,X(t) - e^{At}\,X(0) = e^{At}\int_0^t e^{-A\tau}\,B\,U(\tau)\,d\tau$$

$$\therefore \quad \boxed{X(t) = e^{At}\,X(0) + \int_0^t e^{A(t-\tau)}\,B\,U(\tau)\,d\tau} \quad \dots (1)$$

This is the complete **solution of the nonhomogeneous equation.**

Observations :

1. The solution is divided into two parts. The first part is $e^{At}\ X(0)$ which is nothing but the homogeneous solution or zero input response (ZIR).
2. The other part $\int_0^t e^{A(t-\tau)}\ B\ U(\tau)\ d\tau$ is the part existing only due to application of input U(t) from time 0 to t. It is called **forced solution** or **zero state response** (ZSR).

Thus the solution is,

$$\boxed{X(t) = \underbrace{e^{At} X(0)}_{\text{ZIR}} + \underbrace{\int_0^t e^{A(t-\tau)}\ BU(\tau)\, d\tau}_{\text{ZSR}}}$$

If the initial time is considered as $t = t_0$ rather than $t = 0$, the solution is,

$$X(t) = e^{A(t-t_0)} X(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}\ B\ U(\tau) d\tau$$

In general, the solution of nonhomogeneous equation consists of both the parts, ZIR (zero input response) and ZSR (zero state response).

4.4 Properties of State Transition Matrix

The various useful properties of the state transition matrix are,

$$\phi(t) = e^{At} = \text{State transition matrix}$$

1. $\phi(0) = e^{A \times 0} = I = \text{Identity matrix}$

2. $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$

i.e. $\phi^{-1}(t) = \phi(-t)$

3. $\phi(t_1 + t_2) = e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2}$

$= \phi(t_1)\ \phi(t_2) = \phi(t_2)\ \phi(t_1)$

4. $e^{A(t+s)} = e^{At}\ e^{As}$

5. $e^{(A+B)t} = e^{At}\ e^{Bt}$ only if $AB = BA$

6. $[\phi(t)]^n = [e^{At}]^n = e^{Ant} = \phi(nt)$

7. $\phi(t_2 - t_1) \cdot \phi(t_1 - t_0) = \phi(t_2 - t_0)$

This property states that the process of transition of state can be divided into number of sequential transition. Thus t_0 to t_2 can be divided as t_0 to t_1 and t_1 to t_2, as stated in the property.

Interms of ϕ (t), the solution is expressed as,

$$X(t) = \phi(t - t_0) X(t_0) + \int_{t_0}^{t} \phi(t-\tau) B U(\tau) d\tau$$

where $\phi(t - t_0) = e^{A(t-t_0)}$

and $\phi(t - \tau) = e^{A(t-\tau)}$

8. ϕ (t) is a **nonsingular matrix** for all finite values of t.

4.5 Solution of State Equation by Laplace Transform Method

The Laplace transform method converts integro differential equations to simple algebraic equations. Due this important property, it is very convenient to use Laplace transform method to obtain the solution of state equation.

Consider the nonhomogeneous state equation as,

$$\dot{X}(t) = A X(t) + B U(t) \quad \text{... (1)}$$

Taking Laplace transform of both sides,

$$s X(s) - X(0) = A X(s) + B U(s)$$

$$\therefore \quad s X(s) - A X(s) = X(0) + B U(s)$$

As s is operator, multiplying it by Identify matrix of order n × n,

$$[sI - A] X(s) = X(0) + B U(s)$$

Premultiplying both sides by $[sI - A]^{-1}$,

$$\therefore [sI - A]^{-1} [sI - A] X(s) = [sI - A]^{-1} \{X(0) + B U(s)\}$$

$$\therefore \quad X(s) = [sI - A]^{-1} X(0) + [sI - A]^{-1} B U(s) \quad \text{... (2)}$$

$$= ZIR + ZSR$$

Comparing zero input response obtained earlier,

$$[sI - A]^{-1} = \phi(s)$$

and $\boxed{L^{-1} [sI - A]^{-1} = \phi(t) = e^{At}}$ ... (3)

The matrix $\phi(s) = [sI - A]^{-1}$ is called **resolvant matrix** of A. All the elements of this matrix are rational functions of s.

Taking inverse Laplace transform of (2),

$$X(t) = L^{-1}\{X(s)\} = L^{-1}\{[sI - A]^{-1}\} X(0) + L^{-1}\{[sI - A]^{-1} BU(s)\}$$

Using $[sI - A]^{-1} = \phi(s)$,

$$X(t) = L^{-1}[\phi(s)] X(0) + L^{-1}[\phi(s) B U(s)] \quad \dots (4)$$

The term $L^{-1}[\phi(s) B U(s)]$ is called zero state response.

From the convolution theorem in Laplace transform it is known that,

$$L^{-1}\{F_1(s) F_2(s)\} = f_1(t) * f_2(t) = \text{convolution}$$

$$= \int_0^t f_1(t-\tau) f_2(\tau) d\tau$$

Thus if $F_1(s) = \phi(s)$ and $F_2(s) = BU(s)$ then,

$$L^{-1}\{\phi(s) B U(s)\} = \int_0^t \phi(t-\tau) BU(\tau) d\tau \quad \dots (5)$$

Thus equation (5) shows that $L^{-1}\{\phi(s) B U(s)\}$ is the zero state response, as obtained earlier by classical approach.

$\therefore$

$$\boxed{\begin{aligned} X(t) &= L^{-1}[\phi(s)] X(0) + L^{-1}[\phi(s) B U(s)] \\ \phi(s) &= [sI - A]^{-1} = \frac{\text{Adj}[sI-A]}{|sI-A|} \\ L^{-1}[\phi(s)] &= \phi(t) = e^{At} \end{aligned}}$$

Key Point : *The advantage of this method is without carrying out actual integration which is time consuming, the zero state response can be easily obtained.*

4.6 Computation of State Transition Matrix

While obtaining the solution of state equation, the computation of state transition matrix e^{At} plays an important role. There are various methods of obtaining state transition matrix from the state model. These methods are,

1. Laplace transform method
2. Power series method
3. Cayley Hamilton method
4. Similarity transformation method

Let us discuss these methods of obtaining e^{At} in detail.

4.7 Laplace Transform Method

While studying the method of obtaining the solution using Laplace transform method, it is seen that the Laplace transform of e^{At} i.e. $\phi(t)$ and given by,

$$\phi(s) = [sI - A]^{-1} = \frac{\text{Adj}\,[sI-A]}{|sI-A|}$$

Thus knowing A, once [sI − A] is obtained then $\phi(s)$ can be easily obtained. This $\phi(s)$ is again a matrix of order n × n i.e. same as that of A. It is called resolvant matrix of A.

The e^{At} i.e. $\phi(t)$ then can be obtained by finding inverse Laplace transform of $\phi(s)$ i.e. inverse Laplace transform of each element of the resolvant matrix $\phi(s)$.

This is most simple method of obtaining state transition matrix e^{At} and hence popularly used.

$$\boxed{\therefore \quad e^{At} = L^{-1}[\phi(s)] = L^{-1}[sI - A]^{-1} = L^{-1}\left\{\frac{\text{Adj}\,[sI-A]}{|sI-A|}\right\}}$$

Example 4.1 : *Find the state transition matrix for,*

$$A = \begin{bmatrix} 0 & -1 \\ +2 & -3 \end{bmatrix}$$

Solution : Use Laplace transform method,

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & +1 \\ -2 & s+3 \end{bmatrix}$$

$$\therefore \quad \text{Adj}\,[sI - A] = \begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix}^T = \begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix}$$

$$|sI - A| = \begin{bmatrix} s & +1 \\ -2 & s+3 \end{bmatrix} = s^2 + 3s + 2 = (s+1)(s+2)$$

$$\therefore \quad [sI - A]^{-1} = \frac{\text{Adj}\,[sI-A]}{|sI-A|} = \frac{\begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix}}{(s+1)(s+2)}$$

$$\therefore \quad \phi(s) = [sI - A]^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{-1}{(s+1)(s+2)} \\ \frac{2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$\therefore \quad e^{At} = L^{-1}[sI - A]^{-1} = L^{-1}\begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{-1}{(s+1)(s+2)} \\ \dfrac{2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

Using partial fraction expansion for all the elements.

$$e^{At} = L^{-1}\begin{bmatrix} \dfrac{2}{s+1} - \dfrac{1}{s+2} & \dfrac{-1}{s+1} + \dfrac{1}{s+2} \\ \dfrac{2}{s+1} - \dfrac{2}{s+2} & \dfrac{-1}{s+1} + \dfrac{2}{s+2} \end{bmatrix}$$

$$\therefore \quad e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & -e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} = \phi(t)$$

This is the required state transition matrix.

4.8 Power Series Method

The state transition matrix is matrix exponential and hence can be expressed in a power series as,

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \ldots + \frac{1}{k!}A^kt^k + \ldots$$

$$\therefore \quad \boxed{e^{At} = \sum_{i=0}^{\infty} \frac{1}{i!} A^i t^i} \qquad \ldots (1)$$

Thus by calculating various powers of A and then few terms of power series, e^{At} can be evaluated. As the power series is infinite, it is necessary to stop after calculating first few terms of the series.

Calculating high powers of A is practically time consuming and hence this method is not practically used to obtain state transition matrix.

Example 4.2 : *Find e^{At} of $A = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}$ using power series method.*

Solution : Let us obtain various powers of A.

$$A^2 = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -6 & 7 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ -6 & 7 \end{bmatrix} = \begin{bmatrix} 6 & -7 \\ 14 & -14 \end{bmatrix}$$

$$\therefore \quad e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \ldots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix} t + \frac{1}{2}\begin{bmatrix} -2 & 3 \\ -6 & 7 \end{bmatrix} t^2 + \frac{1}{6}\begin{bmatrix} 6 & -7 \\ 14 & -15 \end{bmatrix} t^3 + \ldots$$

$$= \begin{bmatrix} 1-t^2+t^3+\ldots & -t+1.5t^2-\frac{7}{6}t^3+\ldots \\ 2t-3t^2+\frac{14}{6}t^3+\ldots & 1-3t+3.5t^2-\frac{15}{6}t^3+\ldots \end{bmatrix}$$

Consider the first element of e^{At} which is $1 - t^2 + t^3 + \ldots$

Now $$e^{-t} = 1 + [-t] + \frac{1}{2!}[-t]^2 + \frac{1}{3!}[-t]^3 + \ldots$$

$$\therefore \quad 2e^{-t} = 2 - 2t + t^2 - \frac{1}{3}t^3 + \ldots \qquad \ldots (1)$$

and $$e^{-2t} = 1 + (-2t) + \frac{1}{2!}(-2t)^2 + \frac{1}{3!}(-2t)^3 + \ldots$$

$$= 1 - 2t + 2t^2 - \frac{8}{6}t^3 + \ldots \qquad \ldots (2)$$

From (1) and (2) we can write,

$$2e^{-t} - e^{-2t} = 1 + 0 - t^2 + t^3 + \ldots = 1 - t^2 + t^3 + \ldots$$

Hence the series present as the first element is $2e^{-t} - e^{-2t}$.

But practically such a task is difficult to identify the combinations of series. Hence this method is rarely used.

4.9 Cayley Hamilton Method

This method is based on the Cayley Hamilton theorem. The theorem states that, **every square matrix A satisfies its own characteristics equation.**

Let $f(\lambda)$ be the characteristic equation which is given by,

$$f(\lambda) = |\lambda I - A| = 0$$

i.e. $$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0 \qquad \ldots (1)$$

According to Cayley Hamilton theorem, the matrix A has to satisfy the characteristic equation. Hence,

$$f(A) = A^n + a_1A^{n-1} + \ldots + a_{n-1}A + a_n I = 0 \qquad \ldots (2)$$

where $\qquad I = \text{Identity matrix}$

The theorem is used for evaluating the function of a matrix.

Consider a matrix polynomial as,

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \ldots + a_n A^n + a_{n+1} A^{n+1} + \ldots \qquad \ldots(3)$$

This polynomial has degree more than the degree of A which is n. The above polynomial can be calculated considering the scalar polynomial,

$$p(\lambda) = a_0 + a_1 \lambda + a_2 \lambda^2 + \ldots + a_n \lambda^n + a_{n+1} \lambda^{n+1} + \ldots \qquad \ldots (4)$$

Divide this by the characteristic polynomial $f(\lambda)$,

$$\therefore \quad \frac{p(\lambda)}{f(\lambda)} = Q(\lambda) + \frac{R(\lambda)}{f(\lambda)} \qquad \ldots (5)$$

where $\qquad R(\lambda) = \text{Remainder polynomial}$

$$\therefore \quad p(\lambda) = f(\lambda)\, Q(\lambda) + R(\lambda) \qquad \ldots (6)$$

Now the remainder polynomial $R(\lambda)$ has order less than the characteristic polynomial and can be expressed as,

$$R(\lambda) = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 + \ldots + \alpha_{n-1} \lambda^{n-1} \qquad \ldots (7)$$

Let $\lambda_1, \lambda_2, \ldots \lambda_n$ are the n distinct eigen values of matrix A and $f(\lambda_i) = 0$ for $i = 1, 2, \ldots n$ as $\lambda_1, \lambda_2, \ldots \lambda_n$ are the roots of $f(\lambda) = 0$. Evaluate $p(\lambda)$ at all the eigen values,

$$p(\lambda_i) = f(\lambda_i)\, Q(\lambda_i) + R(\lambda_i) \quad \text{for } i = 1, 2, \ldots n$$

But $\qquad f(\lambda_i) = 0$ hence,

$$\therefore \quad p(\lambda_i) = R(\lambda_i) = \alpha_0 + \alpha_1 \lambda_i + \alpha_2 \lambda_i^2 + \ldots + \alpha_{n-1} \lambda_i^{n-1} \qquad \ldots (8)$$

Substituting $\lambda_1, \lambda_2, \ldots \lambda_n$ in equation (8) we get the set of simultaneous equations which is to be solved for the values of $\alpha_0, \alpha_1 \ldots \alpha_{n-1}$.

Now replace λ in scalar polynomial by matrix A,

$$\therefore \quad p(A) = f(A)\, Q(A) + R(A)$$

but $\qquad f(A) = 0 \qquad$ ... By Cayley Hamilton theorem

$$\therefore \quad p(A) = R(A)$$

$$= \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \ldots \alpha_{n-1} A^{n-1} \qquad \ldots (9)$$

Thus knowing $\alpha_0, \alpha_1 \ldots \alpha_{n-1}$, any matrix polynomial $p(A)$ can be evaluated.

4.9.1 Procedure to Calculate e^{At}

The procedure to calculate e^{At} using Cayley Hamilton theorem can be generalised as,

1. Find the eigen values of matrix A from $|\lambda I - A| = 0$.
2. Form the remainder polynomial $R(\lambda)$ of order $(n - 1)$.

$$R(\lambda) = \alpha_0 + \alpha_1 \lambda + \ldots + \alpha_{n-1} \lambda^{n-1} = p(\lambda)$$

3. If all the eigen values are distinct then substitute $\lambda_1, \lambda_2 \ldots \lambda_n$ in remainder polynomial $R(\lambda)$ to obtain simultaneous equations interms of $\alpha_0, \alpha_1 \ldots \alpha_{n-1}$.

Important Note : If the eigen value λ_k is repeated r times, any one independent equation can be obtained by substituting λ_k in $R(\lambda)$. Then remaining $(r - 1)$ equations are to obtained by differentiating both sides of the equation,

$$p(\lambda) = R(\lambda)$$

$$\therefore \quad \left.\frac{d^j p(\lambda)}{d\lambda^j}\right|_{\lambda=\lambda_k} = \left.\frac{d^j R(\lambda)}{d\lambda^j}\right|_{\lambda=\lambda_k}, \; j = 0, 1, \ldots r-1 \qquad \ldots (10)$$

4. Solve the simultaneous equations $\alpha_0, \alpha_1, \ldots \alpha_{n-1}$.
5. Then $f(A) = R(A) = \alpha_0 I + \alpha_1 A + \ldots + \alpha_{n-1} A^{n-1}$.

Example 4.3 : *Find $f(A) = A^{12}$ for $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ using Cayley Hamilton theorem.*

Solution : The function to be evaluated is,

$$f(A) = A^{12} \text{ hence } p(\lambda) = \lambda^{12}$$

Step 1 : Find eigen values.

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{vmatrix} = 0$$

$$\therefore \quad \lambda^2 + 3\lambda + 2 = 0 \quad \text{i.e. } (\lambda + 1)(\lambda + 2) = 0$$

$$\therefore \quad \lambda_1 = -1, \lambda_2 = -2$$

Step 2 : $R(\lambda) = \alpha_0 + \alpha_1 \lambda = p(\lambda) \ldots$ as $n = 2$.

Step 3 : Substitute λ_1 and λ_2 in $R(\lambda)$.

$$\therefore \quad \alpha_0 + \alpha_1 \lambda_1 = (\lambda_1)^{12}$$

$$\text{and} \quad \alpha_0 + \alpha_1 \lambda_2 = (\lambda_2)^{12} \quad \text{as } p(\lambda) = \lambda^{12}$$

$$\text{i.e.} \quad \alpha_0 - \alpha_1 = (-1)^{12} = 1 \qquad \ldots (1)$$

and $\alpha_0 - 2\alpha_1 = (-2)^{12} = 4036$ ... (2)

Step 4 : Subtracting (2) from (1),

$\therefore \quad \alpha_1 = -4035 \quad \text{and} \quad \alpha_0 = -4034$

Step 5 :

$$f(A) = A^{12} = R(A)$$

$$= \alpha_0 I + \alpha_1 A = -4034\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - 4035\begin{bmatrix}0 & 1\\-2 & -3\end{bmatrix}$$

$$= \begin{bmatrix}-4034 & -4035\\8070 & 8071\end{bmatrix}$$

$$\therefore \quad A^{12} = \begin{bmatrix}-4034 & -4035\\8070 & 8071\end{bmatrix}$$

Similarly e^{At} which is a power series interms of A can be evaluated using Cayley Hamilton theorem.

Example 4.4 : *Find the state transition matrix for*

$A = \begin{bmatrix}0 & -1\\2 & -3\end{bmatrix}$ *Using Cayley Hamilton Theorem.*

Solution : The function to be evaluated is,

$$f(A) = e^{At} \text{ hence } p(\lambda) = e^{\lambda t}$$

Step 1 : Find the eigen values,

$$|\lambda I - A| = 0 \text{ i.e. } \begin{vmatrix}\lambda & 1\\-2 & \lambda+3\end{vmatrix} = 0$$

$$\therefore \quad \lambda^2 + 3\lambda + 2 = 0 \quad \text{i.e.} \quad (\lambda+1)(\lambda+2) = 0$$

$$\therefore \quad \lambda_1 = -1, \quad \lambda_2 = -2$$

Step 2 : Construct $R(\lambda)$

$\therefore \quad R(\lambda) = \alpha_0 + \alpha_1\lambda + \ldots + \alpha_{n-1}\lambda^{n-1}$ but $n = 2$

$\therefore \quad R(\lambda) = \alpha_0 + \alpha_1\lambda = p(\lambda)$ ... By Cayley Hamilton method

where $p(\lambda) = e^{\lambda t}$ ... replacing A by λ in f(A).

$\therefore \quad \alpha_0 + \alpha_1\lambda = e^{\lambda t}$

Step 3 : Substituting values of λ_1 and λ_2,

$$\alpha_0 - \alpha_1 = e^{-t} \quad \ldots(1)$$

and
$$\alpha_0 - 2\alpha_1 = e^{-2t} \quad \ldots(2)$$

Step 4 : Solving equations (1) and (2),

$$\alpha_1 = e^{-t} - e^{-2t}$$

and
$$\alpha_0 = 2e^{-t} - e^{-2t}$$

Step 5 :
$$f(A) = R(A) = \alpha_0 I + \alpha_1 A \ \ldots \text{ replace } \lambda \text{ by } A \text{ in } R(\lambda)$$

$$\therefore \quad e^{At} = 2e^{-t} - e^{-2t}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^{-t} - e^{-2t})\begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}$$

$$\therefore \quad e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & -e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

This is the required state transition matrix.

Example 4.5 : *Find the state transition matrix of,*

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} \text{ by Cayley Hamilton theorem.}$$

Solution : The function to be evaluated is,

$$f(A) = e^{At} \text{ hence } p(\lambda) = e^{\lambda t}$$

Step 1 : Find the eigen values.

$$|\lambda I - A| = \begin{vmatrix} \lambda & 0 & 2 \\ 0 & \lambda-1 & 0 \\ -1 & 0 & \lambda-3 \end{vmatrix} = 0$$

$$\therefore \quad \lambda(\lambda-1)(\lambda-3) + 2(\lambda-1) = 0$$

$$\therefore \quad (\lambda-1)[\lambda^2 - 3\lambda + 2] = 0 \text{ i.e. } (\lambda-1)(\lambda-1)(\lambda-2) = 0$$

$$\therefore \quad \lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 2$$

Thus $\lambda_1 = 1$ is repeated 2 times

Step 2 :
$$R(\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 \quad \ldots n = 3$$

$$= p(\lambda) = e^{\lambda t}$$

Step 3 : For $\lambda_1 = 1$,

$$\alpha_0 + \alpha_1 + \alpha_2 = e^t \quad \text{... (1)}$$

For $\lambda_2 = 1$, as it is repeated use,

$$\left.\frac{d}{d\lambda} p(\lambda)\right|_{\lambda=\lambda_2} = \left.\frac{d}{d\lambda} R(\lambda)\right|_{\lambda=\lambda_2}$$

i.e. $$\left.\frac{d}{d\lambda} e^{\lambda t}\right|_{\lambda=\lambda_2} = \left.\frac{d}{d\lambda}[\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2]\right|_{\lambda=\lambda_2}$$

$\therefore$ $$\left. t\, e^{\lambda t}\right|_{\lambda=\lambda_2} = \left.\alpha_1 + 2\alpha_2 \lambda\right|_{\lambda=\lambda_2}$$

$\therefore$ $$\alpha_1 + 2\,\alpha_2 = te^t \quad \text{... (2)}$$

Note that differentiation is with respect to λ and not t.

and for $\lambda_3 = 2$,

$$\alpha_0 + 2\,\alpha_1 + 4\,\alpha_2 = e^{2t} \quad \text{... (3)}$$

Step 4 : Solve equations (1), (2) and (3).

From (2), $\alpha_1 = t\,e^t - 2\,\alpha_2$

From (1), $\alpha_0 = e^t - \alpha_1 - \alpha_2 = e^t - te^t + 2\alpha_2 - \alpha_2$

$= e^t(1 - t) + \alpha_2$

Using in (3),

$$e^t\,(1 - t) + \alpha_2 + 2t\,e^t - 4\,\alpha_2 + 4\,\alpha_2 = e^{2t}$$

$\therefore$ $\alpha_2 = e^{2t} - e^t + t\,e^t - 2te^t = e^{2t} - e^t - t\,e^t$

$\therefore$ $\alpha_1 = te^t - 2e^{2t} + 2e^t + 2te^t = 3te^t + 2e^t - 2e^{2t}$

and $\alpha_0 = e^t - te^t + e^{2t} - e^t - te^t = -\,2te^t + e^{2t}$

Step 5 : $f(A) = R(A) = \alpha_0 I + \alpha_1 A + \alpha_2\, A^2$

$$A^2 = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -6 \\ 0 & 1 & 0 \\ 3 & 0 & 7 \end{bmatrix}$$

$\therefore$ $$f(A) = e^{At} = \alpha_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 & 0 & -6 \\ 0 & 1 & 0 \\ 3 & 0 & 7 \end{bmatrix}$$

Substituting values of α_0, α_1 and α_2 and rearranging,

$$\therefore \qquad e^{At} = \begin{bmatrix} 2e^{t}-e^{2t} & 0 & 2e^{t}-2e^{2t} \\ 0 & e^{t} & 0 \\ -e^{t}+e^{2t} & 0 & 2e^{2t}-e^{t} \end{bmatrix}$$

This is the required state transition matrix.

4.10 Similarity Transformation Method

Consider a matrix A of order n × n having eigen values $\lambda_1, \lambda_2, \ldots \lambda_n$. There exists a diagonal matrix Λ such that its diagonal elements are the eigen values of matrix A.

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \text{Diagonal matrix} \qquad \ldots (1)$$

Both A and Λ have same eigen values and hence are called similar matrices.

The diagonal matrix Λ can be obtained by defining a similarity transformation of state variables as,

$$X(t) = M\, Z(t)$$

In such a case, M is the **modal matrix** of A.

And the diagonal matrix is obtained as,

$$\Lambda = M^{-1} AM \qquad \ldots (2)$$

Now $\quad e^{\Lambda t} = I + \Lambda t + \frac{1}{2!}\Lambda^2 t^2 + \ldots$

$$= \begin{bmatrix} 1 & 0 & \text{—} & 0 \\ 0 & 1 & \text{—} & 0 \\ | & | & & \\ 0 & 0 & \text{—} & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 & 0 & \text{—}\,0 \\ 0 & \lambda_2 & 0 & \text{—}\,0 \\ | & & & \\ 0 & \text{—} & \text{—} & \lambda_n \end{bmatrix} t + \frac{1}{2!}\begin{bmatrix} \lambda_1^2 & 0 & 0 & \text{—}\,0 \\ 0 & \lambda_2^2 & 0 & \text{—}\,0 \\ | & & & \\ 0 & \text{—} & \text{—} & \lambda_n^2 \end{bmatrix} t^2 + \text{—}$$

$$= \begin{bmatrix} 1+\lambda_1 t+\frac{1}{2!}\lambda_1^2 t^2 + \text{—} & 0 & 0 \text{———} & 0 \\ 0 & 1+\lambda_2 t+\frac{1}{2!}\lambda_2^2 t^2 & 0 \text{———} & 0 \\ | & | & & \\ 0 & 0 & \text{— —} & 1+\lambda_n t+\frac{1}{2!}\lambda_n^2 t^2 + \text{—} \end{bmatrix}$$

$$\therefore \qquad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \qquad \ldots (3)$$

The transformation used is,

$$X(t) = M\,Z(t)$$

$$\therefore \qquad X(0) = M\,Z(0)$$

$$\therefore \qquad M^{-1}X(0) = M^{-1}M\,Z(0) = Z(0) \qquad \ldots (4)$$

$$\text{and} \qquad \dot{X}(t) = M\dot{Z}(t)$$

$$\text{Thus} \qquad \dot{X}(t) = AX(t) = AMZ(t)$$

$$\therefore \qquad M\dot{Z}(t) = AMZ(t)$$

$$\therefore \qquad M^{-1}M\dot{Z}(t) = M^{-1}AM\,Z(t)$$

$$\therefore \qquad \dot{Z}(t) = \Lambda\,Z(t) \qquad \ldots (5)$$

This is transformed state model.

The solution of homogeneous state equation in X(t) is,

$$X(t) = e^{At}X(0) \qquad \ldots (6)$$

While solution of equation (5) is,

$$Z(t) = e^{\Lambda t}Z(0) \qquad \ldots (7)$$

Using equation (4) in (7),

$$Z(t) = e^{\Lambda t}M^{-1}X(0) \qquad \ldots (8)$$

Premultiplying by M,

$$\therefore \qquad MZ(t) = M\,e^{\Lambda t}M^{-1}X(0)$$

$$\text{But} \qquad MZ(t) = X(t) \qquad \ldots \text{Transformation used}$$

$$\therefore \qquad X(t) = M\,e^{\Lambda t}M^{-1}X(0) \qquad \ldots (9)$$

Comparing equation (6) and (9) it can be written as,

$$\boxed{e^{At} = Me^{\Lambda t}M^{-1}} \qquad \ldots (10)$$

where $\qquad M = $ Modal matrix

and $e^{\Lambda t}$ is given by equation (3).

➡ Example 4.6 : *Find state transition matrix of,*

$$A = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}$$ *by similarity transformation method.*

Solution : Find the eigen values.

$$|\lambda I - A| = \begin{vmatrix} \lambda & 1 \\ -2 & \lambda+3 \end{vmatrix} = \lambda^2 + 3\lambda + 2 = 0$$

$$\therefore \quad (\lambda+1)(\lambda+2) = 0$$

$$\therefore \quad \lambda_1 = -1, \quad \lambda_2 = -2$$

Calculate eigen vector.

For $\lambda_1 = -1$, $\quad [\lambda_1 I - A] = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$

$$\therefore \quad M_1 = \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = -2$, $\quad [\lambda_2 I - A] = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}$

$$\therefore \quad M_2 = \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\therefore \quad M = [M_1 : M_2] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

$$M^{-1} = \frac{\text{Adj}[M]}{|M|} = \frac{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}{1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\therefore \quad e^{At} = M e^{\Lambda t} M^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-2t} - e^{-2t} & -e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$ ... Required state transition matrix

Example 4.7 : *Obtain the complete time response of system given by,*

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} X(t) \qquad \text{where } X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{and} \quad Y(t) = [1 \;\; -1]\, X(t)$$

(VTU: Jan./Feb.-2008)

Solution : Given system is homogeneous whose solution is $X(t) = e^{At} X(0)$

Now $$e^{At} = L^{-1}\left\{(sI - A)^{-1}\right\}$$

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{\text{Adj}(sI - A)}{|sI - A|}$$

$$\text{Adj}(sI - A) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} s & -2 \\ 1 & s \end{bmatrix}^T = \begin{bmatrix} s & 1 \\ -2 & s \end{bmatrix}$$

$$|sI - A| = s^2 + 2$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s}{s^2+2} & \frac{1}{s^2+2} \\ \frac{-2}{s^2+2} & \frac{s}{s^2+2} \end{bmatrix}$$

$$e^{At} = L^{-1}\begin{bmatrix} \frac{s}{s^2+2} & \frac{1}{s^2+2} \\ \frac{-2}{s^2+2} & \frac{s}{s^2+2} \end{bmatrix}$$

$$L^{-1}\left\{\frac{s}{s^2+2}\right\} = L^{-1}\left\{\frac{s}{s^2+(\sqrt{2})^2}\right\} = \cos\sqrt{2}t$$

$$L^{-1}\left\{\frac{1}{s^2+2}\right\} = L^{-1}\left\{\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right\} = \frac{1}{\sqrt{2}} \sin\sqrt{2}\,t$$

$$L^{-1}\left\{\frac{-2}{s^2+2}\right\} = L^{-1}\left\{-\sqrt{2} \cdot \frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right\} = -\sqrt{2}\sin\sqrt{2}\,t$$

$$\therefore \quad e^{At} = \begin{bmatrix} \cos\sqrt{2}\,t & \frac{1}{\sqrt{2}}\sin\sqrt{2}\,t \\ -\sqrt{2}\sin\sqrt{2}\,t & \cos\sqrt{2}\,t \end{bmatrix}$$

$$\therefore \qquad X(t) = e^{At}\ X(0) = e^{At}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}\cos\sqrt{2}\,t + \dfrac{1}{\sqrt{2}}\sin\sqrt{2}\,t \\ \cos\sqrt{2}\,t - \sqrt{2}\sin\sqrt{2}\,t\end{bmatrix}$$

$\therefore$ Output response $Y(t) = [1 \quad -1]\ X(t)$

$$= [1 \quad -1]\begin{bmatrix}\cos\sqrt{2}\,t + \dfrac{1}{\sqrt{2}}\sin\sqrt{2}\,t \\ \cos\sqrt{2}\,t - \sqrt{2}\sin\sqrt{2}\,t\end{bmatrix}$$

$$= \frac{3}{\sqrt{2}}\sin\sqrt{2}\,t$$

$$Y(t) = \frac{3}{\sqrt{2}}\sin\sqrt{2}\,t \qquad \text{is the required response.}$$

Example 4.8 : *Find out the time response for unit step input of a system given by*

$$\dot{X}(t) = \begin{bmatrix}0 & 1\\-2 & -3\end{bmatrix} X(t) + \begin{bmatrix}0\\5\end{bmatrix} U(t) \quad \textit{and } Y(t) = \begin{bmatrix}0 & 1\\-2 & -3\end{bmatrix} X(t) \quad \textit{and } X(0) = \begin{bmatrix}1\\0\end{bmatrix}.$$

Solution :

$$[sI - A] = \begin{bmatrix}s & 0\\0 & s\end{bmatrix} - \begin{bmatrix}0 & 1\\-2 & 0\end{bmatrix} = \begin{bmatrix}s & -1\\2 & s+3\end{bmatrix}$$

$$\text{Adj}\ [\ sI - A] = \begin{bmatrix}C_{11} & C_{12}\\C_{21} & C_{22}\end{bmatrix}^T = \begin{bmatrix}s+3 & -2\\1 & s\end{bmatrix}^T = \begin{bmatrix}s+3 & 1\\-2 & s\end{bmatrix}$$

$$|sI - A| = s^2 + 3s + 2 = (s+1)\ (s+2)$$

$$\therefore \qquad e^{At} = L^{-1}\ [(sI - A)^{-1}]$$

$$= \begin{bmatrix}\dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)}\end{bmatrix} = \phi(s)$$

Finding partial fractions,

$$e^{At} = L^{-1}\ (\phi(s)) = \begin{bmatrix}\dfrac{2}{s+1} - \dfrac{1}{s+2} & \dfrac{1}{s+1} - \dfrac{1}{s+2} \\ \dfrac{-2}{s+1} + \dfrac{2}{s+2} & \dfrac{-1}{s+1} + \dfrac{2}{s+2}\end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\text{ZIR} = e^{At}\, X(0) = e^{At} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} \end{bmatrix}$$

To find $\text{ZSR} = L^{-1} \{ \phi(s)\, B\, U(s) \}$

$$U(t) = \text{Unit step} \quad \therefore U(s) = 1/s$$

$$\therefore \quad \text{ZSR} = L^{-1} \left\{ \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} [1/s] \right\}$$

$$= L^{-1} \begin{Bmatrix} \frac{5}{s(s+1)(s+2)} \\ \frac{5}{(s+1)(s+2)} \end{Bmatrix} = L^{-1} \begin{Bmatrix} \frac{2.5}{s} - \frac{5}{s+1} + \frac{2.5}{s+2} \\ \frac{5}{s+1} - \frac{5}{s+2} \end{Bmatrix}$$

$$= \begin{bmatrix} 2.5 - 5e^{-t} + 2.5\, e^{-2t} \\ 5\, e^{-t} - 5e^{-2t} \end{bmatrix}$$

$$\therefore \quad X(t) = \text{ZIR} + \text{ZSR} = \begin{bmatrix} 2.5 - 3e^{-t} + 1.5\, e^{-2t} \\ 3\, e^{-t} - 3e^{-2t} \end{bmatrix}$$

$$\therefore \quad Y(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} X(t)$$

$$\begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 2.5 - 3e^{-t} + 1.5\, e^{-2t} \\ 3\, e^{-t} - 3e^{-2t} \end{bmatrix} = \begin{bmatrix} 3e^{-t} - 3e^{-2t} \\ -5 + 6e^{-2t} - 3e^{-t} \end{bmatrix}$$

$$\therefore \quad \mathbf{Y_1(t) = 3\,(e^{-t} - e^{-2t})}$$

$$\mathbf{Y_2(t) = -5 + 3\,(2e^{-2t} - e^{-t})}$$

These are the outputs for unit step input applied.

Example 4.9 : *For a system* $\dot{X} = AX$, $X(t) = \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix}$ *when* $X(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ *and*

$X(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$ *for* $X(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Determine the system matrix A.

Solution : System is homogeneous so ZSR = 0

$$\therefore \quad X(t) = e^{At} X(0)$$

Let $$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

Now $$X(t) = \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix} \quad \text{hence } \dot{X}(t) = \begin{bmatrix} -2e^{-2t} \\ 4e^{-2t} \end{bmatrix}$$

and at t = 0 $$\dot{X}(t) = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

So $$\dot{X}(0) = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \text{ when } X(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Similarly $$X(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}, \quad \dot{X}(t) = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} \quad \text{at } t = 0 \ \dot{X}(t) = \begin{bmatrix} -1 \\ +1 \end{bmatrix}$$

So $$\dot{X}(0) = \begin{bmatrix} -1 \\ +1 \end{bmatrix} \text{ when } X(0) = \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

Substituting in $\dot{X}(t) = A\,X(t)$

$$\therefore \quad \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore \quad A_1 - 2A_2 = -2 \qquad \text{... (1)}$$

$$A_3 - 2A_4 = +4 \qquad \text{... (2)}$$

$$+A_1 - A_2 = -1 \qquad \text{... (3)}$$

$$A_3 - A_4 = 1 \qquad \text{... (4)}$$

Solving these 4 equations simultaneously,

$A_2 = 1, \quad A_1 = 0, \quad A_4 = -3, \quad A_3 = -2$

$$\therefore \quad A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

4.11 Controllability and Observability

In a control system analysis, it is necessary to find an optimal control solution for a given control problem. The existence of such a solution depends on the answers of two basic questions which are,

1. For a given system, is it possible to transfer any initial state to any other desired state in a finite time under the effect of suitable control input force ?

2. If the output is measured for finite time then with the knowledge of the input, is it possible to determine initial state of the system ?

   The answer to the first question gives the concept of the controllability while the answer to the second question gives the concept of observability of the system.

4.12 Controllability

The answer to the first question means the concept of controllability of a system which is related to the transfer of any initial state of the system to any other desired state, in a finite length of time by application of proper inputs. Hence controllability can be defined as,

A system is said to be **completely state controllable** if it is possible to transfer the system state from any initial state $X(t_0)$ to any other desired state $X(t_f)$ in a specified finite time interval (t_f) by a control vector U(t).

The concept of controllability and observability were originally introduced by Kalman hence **Kalman's** tests are used to find out whether the system is controllable and observable or not.

4.12.1 Kalman's Test for Controllability

Consider n^{th} order multiple input linear time invariant system represented by its state equation as,

$$\dot{X} = AX(t) + BU(t) \quad \text{...(1)}$$

where A has order $n \times n$ matrix

and U(t) is $m \times 1$ vector i.e. there are m inputs.

X(t) is $n \times 1$ state vector.

> The **necessary and sufficient condition** for the system to be completely state controllable is that **the rank of the composite matrix Q_c is 'n'**.

The composite matrix Q_c is given by,

$$\boxed{Q_c = [B : AB : A^2B : \dots A^{n-1}B]} \quad \text{... (2)}$$

In this composite matrix Q_c, B, AB, A^2B ... are the various columns.

Proof : Let us see the proof for this condition. Assume that the final state is the origin of the state space while the initial time is zero i.e. $t_0 = 0$.

As final state $t = t_f$, is the origin of the state space then $X(t_f) = 0$.

$$\therefore \quad X(t_f) = 0 \quad \text{... (3)}$$

The solution of the state equation (1) is given by,

$$X(t) = e^{At} X(0) + \int_0^t e^{A(t-\tau)} B U(\tau) d\tau \quad \text{... (4)}$$

Substituting $t = t_f$ for the final state,

$$X(t_f) = e^{At_f} X(0) + \int_0^{t_f} e^{A(t_f-\tau)} BU(\tau) d\tau \quad \text{...(5)}$$

Using (3) in (5),

$$0 = e^{At_f} X(0) + \int_0^{t_f} e^{A(t_f-\tau)} B U(\tau) d\tau$$

$$e^{At_f} X(0) = -\int_0^{t_f} e^{At_f} \cdot e^{-A\tau} BU(\tau) d\tau$$

$$\therefore \quad X(0) = -\int_0^{t_f} e^{-A\tau} B U(\tau) d\tau \quad \text{... (6)}$$

This shows that any initial state X(0) must satisfy the equation (6) for complete state controllability. Thus the controllability depends on matrices A and B and is independent of matrix C.

Now mathematically $e^{-A\tau}$ can be expressed as,

$$e^{-A\tau} = \sum_{k=0}^{n-1} \alpha_k(\tau) A^k \quad \text{... (7)}$$

Using (7) in (6),

$$X(0) = -\sum_{k=0}^{n-1} A^k B \int_0^{t_f} \alpha_k(\tau) U(\tau) d\tau \quad \text{... (8)}$$

$$\text{Let} \quad \int_0^{t_f} \alpha_k(\tau) U(\tau) d\tau = \beta_k$$

$$\therefore \quad X(0) = -\sum_{k=0}^{n-1} A^k B \beta_k$$

$$\therefore \quad X(0) = -[B : AB : \ldots : A^{n-1} B] \begin{bmatrix} \beta_o \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix} \quad \text{... (9)}$$

$$\therefore \qquad X(0) = -Q_c \begin{bmatrix} \beta_o \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix} \qquad \ldots (10)$$

The vector matrix of β_k is of order $n \times 1$ and if system has to be state controllable then any X(0) must satisfy equation (9) for which rank of Q_c must be n which is a $n \times n$ matrix. For any rank of Q_c less than 'n' it indicates that some of the states are not controllable.

This proves that **the rank of composite matrix Q_c must be 'n' for complete state controllability of the system.**

The advantage of Kalman's test is that for any form of matrix A whether A is canonical or otherwise, the test can be applied. But its limitation is that it does not give a physical feel of the problem. If the answer to the Kalmans test is uncontrollable then it does not indicate which mode is uncontrollable. But due to mathematical simplicity of its application, Kalman's test is used to test the controllability.

Key Point: *The matrix has a rank r means the determinant of order $r \times r$ of the matrix has nonzero value and any determinant having order $r + 1$ or more than that has zero value.*

$$\boxed{\text{Thus rank of } Q_c = n \text{ for controllability}}$$

Example 4.10 *Find the controllability of the system,*

$$\dot{X} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(t)$$

$$n = 2$$

Solution :

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore \qquad AB = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore \qquad Q_c = [B \;\; AB] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Now $\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$ which is nonzero

$\therefore$ Rank of $Q_c = 2 = n$

Hence the given system is **completely controllable.**

Example 4.11: *Evaluate the controllability of the system*

with, $\dot{X} = AX + BU$

and $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solution : $n = 2$

$\therefore \quad Q_c = [B \quad AB]$

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\therefore \quad Q_c = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Now $\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$ = Determinant of $2\times 2 = 0$

Hence rank of $Q_c r = 1$ and

Rank of $Q_c \neq n$

$\therefore$ The system is **not state controllable.**

4.12.2 Condition for State Controllability in s-Plane

The system is represented by its transfer function in the s-plane. Such a system is completely state controllable only if the denominator and numerator polynomials of the transfer function do not have a common factor, except the constant. Such polynomials are called coprime. In other words, there should not be a pole-zero cancellation in the s-domain transfer function of the system for complete state controllability. If the pole-zero cancellation exists then all the information about the dynamic behaviour of the system is not carried all along the system. In such a case, the system cannot be controlled in the direction of the cancelled mode.

4.12.3 Gilbert's Test for Controllability

For the Gilbert's test it is necessary that the **matrix A must be in canonical form.** Hence the given state model is required to be transformed to the canonical form first, to apply the Gilbert's test.

Consider single input linear time invariant system represented by,

$$\dot{X}(t) = A\,X(t) + B\,U(t)$$

where A is not in the canonical form. Then it can be transformed to the canonical form by the transformation,

$$X(t) = M\,Z(t)$$

where M = Modal matrix

The transformed state model, as derived earlier, takes the form,

$$\dot{Z}(t) = \Lambda\,Z(t) + \tilde{B}\,U(t)$$

where Λ = Diagonal matrix

$$\tilde{B} = M^{-1}B$$

It is assumed that all the eigen values of A are distinct.

In such a case the **necessary and sufficient condition** for the complete state controllability is that **the vector matrix $\tilde{B}$ should not have any zero elements.** If it has zero elements then the corresponding state variables are not controllable.

If the eigen values are repeated then matrix A cannot be transformed to Jordan canonical form. If A has eigen values $\lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_2, \lambda_3, \lambda_4, \ldots \lambda_n$ then the transformation results Jordan canonical form as,

Jordan block

$$J = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & & & & \cdots & \lambda_n \end{bmatrix}$$

In such a case, the condition for the complete state controllability is that the **elements of any row of $\tilde{B}$ that corresponds to the last row of each Jordan block are not all zero.**

Example 4.12 : *Consider the system with state equation.*

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U(t)$$

Estimate the state controllability by

i) Kalman's test and ii) Gilbert's test

Solution : i) Kalman's test

$$Q_c = [B : AB : A^2B] \dots n = 3$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -6 & -11 & -6 \\ 36 & 60 & 25 \end{bmatrix}$$

$$\therefore \quad A^2B = \begin{bmatrix} 0 & 0 & 1 \\ -6 & -11 & -6 \\ 36 & 60 & 25 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 25 \end{bmatrix}$$

$$\therefore \quad Q_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{bmatrix}$$

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{vmatrix} = 1, \quad \text{Thus} \quad |Q_c| \text{ is nonsingular.}$$

Hence the rank of Q_c is 3 which is 'n'.

Thus the system is **completely state controllable.**

ii) Gilbert's test

For this, it is necessary to express A in the canonical form. Find eigen values of A.

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{vmatrix} = 0$$

$$\therefore \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

$$\therefore (\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$$

$$\therefore \quad \lambda_1 = -1, \quad \lambda_2 = -2, \quad \lambda_3 = -3$$

As matrix A is in **phase variable form,** the **modal matrix M** is **Vander Monde Matrix.**

$$\therefore \quad M = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$M^{-1} = \frac{\text{Adj}[M]}{|M|} = \frac{\begin{bmatrix} -6 & 6 & -2 \\ -5 & 8 & -3 \\ -1 & 2 & -1 \end{bmatrix}^T}{-2} = \frac{\begin{bmatrix} -6 & -5 & -1 \\ 6 & 8 & 2 \\ -2 & -3 & -1 \end{bmatrix}}{-2} = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix}$$

$$\tilde{B} = M^{-1}B = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -1 \\ 0.5 \end{bmatrix}$$

As none of the elements of $\tilde{B}$ are zero, the system is **completely state controllable.**

Key Point: *As Gilbert's test requires to transform matrix A into canonical form, it is time consuming and hence Kalman's test is popularly used to test controllability.*

4.12.4 Output Controllability

In the design of some practical systems, it is necessary to control the output variables rather than the state variables. In such a case complete output controllability is defined.

Consider the state model of a linear time invariant system as,

$$\dot{X}(t) = A\,X(t) + B\,U(t)$$
$$Y(t) = C\,X(t) + D\,U(t)$$

The system is said to be **completely output controllable** if it is possible to construct an unconstrained input vector U(t) which will transfer any given initial output $Y(t_0)$ to any final output $Y(t_f)$ in a finite time interval $t_0 \le t \le t_f$.

In such a case, construct the test matrix Q_c as,

$$Q_c = [CB : CAB : CA^2B : \ldots\ CA^{n-1}\,B : D]$$

Thus if the order of matrix C is p × n i.e. there are p outputs then for the complete output controllability, the **rank of test matrix Q_c must be p.**

4.13 Observability

The observability is related to the problem of determining the system state by measuring the output for finite length of time. Hence observability can be defined as,

A system is said to be **completely observable,** if every state $X(t_0)$ can be completely identified by measurements of the outputs Y(t) over a finite time interval. If the system is not completely observable means that few of its state variables are not practically measurable and are shielded from the observation.

Similar to the controllability, the observability of the system can be obtained by using Kalman's test.

4.13.1 Kalman's Test for Observability

Consider n^{th} order multiple input multiple output linear time invariant system, represented by its state equation as,

$$\dot{X} = A\,X(t) + B\,U(t) \qquad ...(1)$$

and $$Y(t) = C\,X(t) \qquad ...(2)$$

where $Y(t) = p\times1$ output vector

and $C = 1\times n$ matrix

The system is completely observable if and only **if the rank of the composite matrix Q_o is 'n'.**

The composite matrix Q_o is given by,

$$\boxed{Q_o = [C^T : A^T C^T : ....(A^T)^{n-1} C^T]}$$

where C^T = Transpose of matrix C

and A^T = Transpose of matrix A

Thus if, rank of Q_o = n, then system is completely observable.

Proof : For the system described by the equation (1) and (2), the solution is,

$$X(t) = e^{At}\,X(0) + \int_0^t e^{A(t-\tau)}\,B\,U(\tau)\,d\tau \qquad ...(3)$$

$$\therefore \quad Y(t) = C\,e^{At}\,X(0) + C\int_0^t e^{A(t-\tau)}\,B\,U(\tau)\,d\tau \qquad ...(4)$$

Now as the matrices A, B and C are known and the input vector U(t) is known, the integral term in the equation (4) is known. This can be subtracted from observed value of Y(t), to modify the lefthand side. Hence the obserability can be investigated by considering the equation,

$$Y(t) = C\,e^{At}\,X(0) \qquad ...(5)$$

as the left hand side is known due to measured output. Thus for observability, the unforced system can be considered i.e. homogeneous state equation can be considered.

Consider the state model of unforced system as,

$$\dot{X} = A\,X(t) \quad \text{and} \quad Y(t) = C\,X(t)$$

The output is given by,

$$Y(t) = C\,e^{At}\,X(0) \qquad ...(6)$$

But $$e^{At} = \sum_{k=0}^{n-1} \alpha_k(t)\, A^k \qquad \dots (7)$$

$$\therefore \quad Y(t) = \sum_{k=0}^{n-1} C\alpha_k(t)\, A^k X(0)$$

$$\therefore \quad Y(t) = \alpha_0(t)\, C\, X(0) + \alpha_1(t)\, CA\, X(0) + \dots + \alpha_{n-1}(t)\, C\, A^{n-1}\, X(0) \qquad \dots (8)$$

For the system to be completely observable, from the given output Y(t) over the interval $0 \le t \le t_f$, the X(0) must be uniquely determined. This is possible if the rank of the matrix,

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

must be n.

Taking transpose of the matrix, the condition can be stared as the rank of matrix Q_o must be n for complete observability where,

$$Q_o = [C^T : A^T C^T : \dots : (A^T)^{n-1} C^T]$$

Example 4.13 : *Evaluate the observability of the system*

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(t)$$

and $$Y(t) = [1 \quad 0] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Solution : The order of the system is, $n = 2$

$$A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \quad \text{and} \quad C = [1 \quad 0]$$

$$\therefore \quad A^T = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Q_o = [C^T \quad A^T C^T]$$

$$A^T C^T = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad Q_o = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Consider the determinant

$$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 = \text{nonzero}$$

$\therefore$ Rank of $Q_o = 2 = n$

Hence the system is **completely observable.**

Example 4.14 : *Evaluate the observability of the system with*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } C = [3 \;\; 4 \;\; 1]$$

Solution : The order of the system is n = 3

$$\therefore \quad Q_o = [C^T \quad A^T C^T \quad (A^T)^2 C^T]$$

$$A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

$$A^T C^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$(A^T)^2 C^T = A^T [A^T C^T] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}$$

$$\therefore \quad Q_o = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 1 & -2 \\ 1 & 1 & -2 \end{bmatrix}$$

Consider the determinant,

$$\begin{vmatrix} 3 & 0 & 0 \\ 4 & 1 & -2 \\ 1 & 1 & -2 \end{vmatrix} = -6 + 0 + 0 + 0 - 0 + 6 = 0$$

Hence a nonzero determinant existing in Q_0 is having order less than 3.

$\therefore$ Rank of $Q_0 \neq 3 \neq n$

Hence the system is **not completely observable.**

4.13.2 Condition for Complete Observability in s-Plane

The condition for complete observability in s-plane also can be stated from the transfer functions or transfer matrices. The conditions remain same as before that the numerator and denominator polynomials of transfer function must be coprime. There should not be cancellation of pole and zero in the transfer function. If the cancellation occurs then the system is not observable and the cancelled mode cannot be observed in the output.

4.13.3 Gilbert's Test for Observability

It is known that for Gilbert's test, the state model must be expressed in the canonical form. Consider the state model of linear time invariant system as,

$$\dot{X}(t) = A\,X(t) + B\,U(t)$$

and $$Y(t) = C\,X(t)$$

Use the transformation $X(t) = MZ(t)$ where M is the modal matrix.

$\therefore$ $$Y(t) = CM\,Z(t) = \tilde{C}\,Z(t)$$

where $$\tilde{C} = CM$$

For a single input single output system,

$$Y(t) = \tilde{C}\,Z(t) = [\tilde{C}_{11} \quad \tilde{C}_{12} \quad \ldots \quad \tilde{C}_{1n}] \begin{bmatrix} Z_1(t) \\ Z_2(t) \\ \vdots \\ Z_n(t) \end{bmatrix}$$

$$= \tilde{C}_{11}\,Z_1(t) + \tilde{C}_{12}\,Z_2(t) + \ldots + \tilde{C}_{1n}\,Z_n(t)$$

Due to the canonical form, all the states are decoupled and not linked to each other. Hence for the system to be observable, each term corresponding to each state must be observed in the output. Hence **none of the coefficient of $\tilde{C}$ must be zero.**

Thus the **system is completely observable if all the coefficients of $\tilde{C}$ are nonzero, none of the coefficient is zero**. If any element is zero, the corresponding state remains unobserved i.e. shielded from observation.

Example 4.15 : *Evaluate the observability of the system with*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } C = [3 \quad 4 \quad 1]$$

Using Gilbert's test.

Solution : For Gilbert's test, find the eigen values.

$$[\lambda I - A] = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 2 & \lambda+3 \end{vmatrix} = 0$$

$$\therefore \quad \lambda^3 + 3\lambda^2 + 2\lambda = 0$$

$$\therefore \quad \lambda(\lambda^2 + 3\lambda + 2) = 0$$

$$\therefore \quad \lambda(\lambda+1)(\lambda+2) = 0$$

$$\therefore \lambda_1 = 0, \quad \lambda_2 = -1, \quad \lambda_3 = -2$$

As the eigen values are distinct, the modal matrix M is a Vander Monde matrix.

$$\therefore \quad M = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$$

When the modal is transformed to canonical form,

$$\tilde{C} = CM = [3 \quad 4 \quad 1] \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$= [3 \quad 0 \quad -1]$$

As there is one zero element in $\tilde{C}$, the system is **not completely observable.**

Examples with Solutions

Example 4.16 : *Find the state transition Matrix of the state equation.*

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} U$$

Using the inverse transform method. **(Bangalore Univ., Aug.-96)**

Solution :

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$e^{At} = L^{-1}[sI - A]^{-1}$$

Now $$sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{\text{Adj}\,[sI - A]}{|sI - A|}$$

Adjoint $$[sI - A] = [\text{cofactor matrix}]^T = \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$$

$$|sI - A| = (s-1)(s-1)$$

$$[sI - A]^{-1} = \frac{\begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}}{(s-1)(s-1)} = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)(s-1)} & \frac{1}{s-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

Taking inverse Laplace transform,

$$\therefore \quad \phi(t) = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$ ... Required state transition matrix

Example 4.17 : *For a certain system, when*

$$X(0) = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \text{ then } X(t) = \begin{bmatrix} e^{-3t} \\ -3e^{-3t} \end{bmatrix}$$

$$\text{while } X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ then } X(t) = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$$

Determine the system matrix A. **(Bangalore Univ. Aug.-95, April-99)**

Solution : The solution of the equation is,

$$X(t) = e^{At}\, X(0)$$

Let $$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

and the equation is,

$$\dot{X}(t) = A\, X(t)$$

Now $$X(t) = \begin{bmatrix} e^{-3t} \\ -3\, e^{-3t} \end{bmatrix} \quad \text{hence} \quad \dot{X}(t) = \begin{bmatrix} -3e^{-3t} \\ 9\, e^{-3t} \end{bmatrix}$$

and $$X(0) = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \dot{X}(0) = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

Now $$\dot{X}(0) = A\, X(0)$$

$$\therefore \quad \begin{bmatrix} -3 \\ 9 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\therefore \quad A_1 - 3\, A_2 = -3 \qquad \text{... (1)}$$

$$A_3 - 3\, A_4 = 9 \qquad \text{... (2)}$$

Similarly $$X(t) = \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix} \qquad \dot{X}(t) = \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix}$$

$$\therefore \quad X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \dot{X}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad \dot{X}(0) = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} X(0)$$

$$\therefore \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad A_1 + A_2 = 1 \qquad \text{... (3)}$$

$$\therefore \quad A_3 + A_4 = 1 \qquad \text{... (4)}$$

Subtracting (3) from (1),

$$-4\, A_2 = -4$$

$$\therefore \quad A_2 = 1$$

$$\therefore \quad A_1 = 0$$

Subtracting (4) from (2),

$$-4\,A_4 = 8 \qquad \text{i.e. } A_4 = -2$$

$$\therefore \qquad A_3 = 3$$

$$\therefore \qquad A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$$

Example 4.18 : *Obtain the solution of the homogeneous state equation $\dot{X} = AX$ where*

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -4 \end{bmatrix} \text{ and } X(0) = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

(Bangalore Univ. Aug.-97)

Solution : The state transition matrix is,

$$e^{At} = L^{-1}\left\{(sI - A)^{-1}\right\}$$

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} s-1 & +2 \\ -1 & s+4 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{\text{Adj}\,[sI - A]}{|sI - A|}$$

$$\text{Adj}\,[sI - A] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} s+4 & 1 \\ -2 & s-1 \end{bmatrix}^T = \begin{bmatrix} s+4 & -2 \\ 1 & s-1 \end{bmatrix}$$

$$|sI - A| = (s-1)(s+4) + 2 = s^2 - s + 4s - 4 + 2$$

$$= s^2 + 3s - 2 = (s - 0.561)(s + 3.561)$$

$$[sI - A]^{-1} = \begin{bmatrix} \dfrac{s+4}{(s-0.561)(s+3.561)} & \dfrac{-2}{(s-0.561)(s+3.561)} \\ \dfrac{1}{(s-0.561)(s+3.561)} & \dfrac{(s-1)}{(s-0.561)(s+3.561)} \end{bmatrix}$$

$$L^{-1}\left[\frac{s+4}{(s-0.561)(s+3.561)}\right] = L^{-1}\left[\frac{1.106}{(s-0.561)} - \frac{0.106}{(s+3.561)}\right]$$

$$= 1.106\,e^{+0.561\,t} - 0.106\,e^{-3.561\,t}$$

$$L^{-1}\left[\frac{-2}{(s-0.561)(s+3.561)}\right] = L^{-1}\left[\frac{-0.485}{(s-0.561)} + \frac{0.485}{(s+3.561)}\right]$$

$$= -0.485\,e^{+0.561\,t} + 0.485\,e^{-3.561\,t}$$

$$L^{-1}\left[\frac{1}{(s-0.561)(s+3.561)}\right] = L^{-1}\left[\frac{0.242}{(s-0.561)} - \frac{0.242}{(s+3.561)}\right]$$

$$= 0.242\, e^{+0.561\,t} + 0.242\, e^{-3.561\,t}$$

$$L^{-1}\left[\frac{(s-1)}{(s-0.561)(s+3.561)}\right] = L^{-1}\left[\frac{-0.106}{(s-0.561)} + \frac{1.106}{(s+3.561)}\right]$$

$$= -0.106\, e^{+0.561\,t} + 1.106\, e^{-3.561\,t}$$

$$\therefore\ e^{At} = \begin{bmatrix} 1.106\, e^{0.561\,t} - 0.106\, e^{-3.561\,t} & -0.485\, e^{0.561\,t} + 0.485\, e^{-3.561\,t} \\ 0.242\, e^{0.561\,t} - 0.242\, e^{-3.561\,t} & -0.106\, e^{0.561\,t} + 0.106\, e^{-3.561\,t} \end{bmatrix}$$

$$\therefore \quad X(t) = e^{At}\, X(0) = e^{At}\begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.068\, e^{0.561\,t} + 0.432\, e^{-3.561\,t} \\ 0.015\, e^{0.561\,t} + 0.985\, e^{-3.561\,t} \end{bmatrix}$$

Example 4.19 : *For a system represented by $\dot{X} = AX$, the response is*

$$X(t) = \begin{bmatrix} 2e^{-4t} \\ e^{-4t} \end{bmatrix} \text{ when } X(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } \dot{X}(t) = \begin{bmatrix} 4e^{-2t} \\ e^{-2t} \end{bmatrix} \text{ when } X(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Determine the system matrix A and the state transition matrix.

(VTU : Jan./Feb.-2005)

Solution : Let the solution is,

$$X(t) = e^{At} X(0)$$

Let $$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

Now $$X(t) = \begin{bmatrix} 2e^{-4t} \\ e^{-4t} \end{bmatrix} \qquad \dot{X}(t) = \begin{bmatrix} -8\,e^{-4t} \\ -4\,e^{-4t} \end{bmatrix}$$

At t = 0, $$X(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \dot{X}(0) = \begin{bmatrix} -8 \\ -4 \end{bmatrix}$$

And $$X(t) = \begin{bmatrix} 4e^{-2t} \\ e^{-2t} \end{bmatrix} \qquad \dot{X}(t) = \begin{bmatrix} -8\,e^{-2t} \\ -2\,e^{-2t} \end{bmatrix}$$

At t = 0 $$X(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad \dot{X}(0) = \begin{bmatrix} -8 \\ -2 \end{bmatrix}$$

$$\dot{X}(t) = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} X(t)$$

$$\therefore \quad \dot{X}(0) = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} X(0)$$

$$\therefore \quad \begin{bmatrix} -8 \\ -4 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\therefore \quad -8 = 2A_1 + A_2 \qquad \dots (1)$$

$$-4 = 2A_3 + A_4 \qquad \dots (2)$$

and $$\begin{bmatrix} -8 \\ -2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\therefore \quad -8 = 4A_1 + A_2 \qquad \dots (3)$$

$$-2 = 4A_3 + A_4 \qquad \dots (4)$$

Solving (1), (3) and (2), (4) simultaneously we get,

$$A = \begin{bmatrix} 0 & -8 \\ 1 & -6 \end{bmatrix}$$

State transition matrix = $e^{At} = L^{-1}[sI - A]^{-1}$

$$\therefore \quad [sI - A] = \begin{bmatrix} s & s \\ -1 & s+6 \end{bmatrix}$$

$$\therefore \quad \text{Adj}\,[sI - A] = \begin{bmatrix} s+6 & -8 \\ 1 & s \end{bmatrix}$$

$$|sI - A| = s^2 + 6s + 8 = (s+2)\,(s+4)$$

$$\therefore \quad e^{At} = L^{-1}\left\{ \frac{\begin{bmatrix} s+6 & -8 \\ 1 & s \end{bmatrix}}{(s+2)\,(s+4)} \right\}$$

$$= L^{-1}\begin{bmatrix} \dfrac{s+6}{(s+2)(s+4)} & \dfrac{-8}{(s+2)(s+4)} \\ \dfrac{1}{(s+2)(s+4)} & \dfrac{s}{(s+2)(s+4)} \end{bmatrix}$$

$$= L^{-1}\begin{bmatrix} \frac{2}{s+2}-\frac{1}{s+4} & \frac{-4}{s+2}+\frac{4}{s+4} \\ \frac{0.5}{s+2}-\frac{0.5}{(s+4)} & \frac{1}{(s+2)}+\frac{2}{(s+4)} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-2t}-e^{-4t} & -4e^{-2t}+4e^{-4t} \\ 0.5e^{-2t}-0.5e^{-4t} & -e^{-2t}+2e^{-4t} \end{bmatrix}$$

Example 4.20 : *A linear time invariant system is characterised by the homogeneous state equation :*

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Compute the solution of homogeneous equation, assume the initial state vector :

$$X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution : From the given model,

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$$

$$\text{Adj }[sI - A] = \begin{bmatrix} s-1 & 1 \\ 0 & s-1 \end{bmatrix}^T=\begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}$$

$$|sI - A| = (s-1)^2$$

$$\therefore \quad [sI-A]^{-1} = \frac{\text{Adj }[sI-A]}{|sI-A|}=\frac{\begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}}{(s-1)^2}$$

$$= \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

$$e^{At} = L^{-1}[sI-A]^{-1} = L^{-1}\begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

$$\therefore \quad X(t) = e^{At}\, X(0) = \text{zero input response}$$

$$= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^t \\ t\,e^t \end{bmatrix}$$

This is the required solution.

Example 4.21 : *Obtain the time response of the following system :*

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} U(t)$$

where $U(t)$ is the unit step occurring at $t = 0$ and $X^T(0) = [1\ 0]$.

Solution : From the given model,

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Now as matrix A is same as considered in Ex. 4.20 while $X(0)$ is also same as consider in Ex. 4.20, the zero input response is same as obtained in Ex. 4.20.

$$\text{ZIR} = X(t) = \begin{bmatrix} e^t \\ t\,e^t \end{bmatrix}$$

The zero state response which depends on U(t) and matrix B is given by,

$$\text{ZSR} = X(t) = L^{-1}\left\{\phi(s)\, B\, U(s)\right\}$$

where

$$\phi(s) = L\{\phi(t)\} = L\left\{e^{AT}\right\} = [s\,I - A]^{-1}$$

$$= \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{(s-1)} \end{bmatrix} \text{ as obtained in Ex. 4.20.}$$

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } U(s) = \begin{bmatrix} \frac{1}{s} \end{bmatrix} \text{ as unit step}$$

$$\phi(s)\ B\ U(s) = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{(s-1)} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} \frac{1}{s} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{(s-1)} \\ \frac{1}{(s-1)^2} - \frac{1}{(s-1)} \end{bmatrix} \begin{bmatrix} \frac{1}{s} \end{bmatrix} = \begin{bmatrix} \frac{1}{s(s-1)} \\ \frac{1}{s}\left\{\frac{1}{(s-1)^2} - \frac{1}{(s-1)}\right\} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s-1} - \frac{1}{s} \\ \frac{1}{(s-1)^2} \end{bmatrix} \text{ ... using partial fraction}$$

$$\text{ZSR} = L^{-1}\left[\phi(s)\,B\,U(s)\right] = L^{-1}\begin{bmatrix} \frac{1}{s-1} - \frac{1}{s} \\ \frac{1}{(s-1)^2} \end{bmatrix} = \begin{bmatrix} e^t - 1 \\ t\,e^t \end{bmatrix}$$

Total response = X(t) = ZIR + ZSR

$$= \begin{bmatrix} e^t \\ t\,e^t \end{bmatrix} + \begin{bmatrix} e^t - 1 \\ t\,e^t \end{bmatrix} = \begin{bmatrix} 2e^t - 1 \\ 2\,t\,e^t \end{bmatrix}$$

Example 4.22 : *Find and sketch the response of the system with the following transfer function, input and initial conditions.*

$$T(s) = \frac{-4s+20}{s+300}$$

$$r(t) = 10\,u(t) \text{ ... (input)}$$

$$y(0) = 0$$

Solution : r(t) = input and y(t) = output

$$T(s) = \frac{Y(s)}{R(s)} = \frac{-4s+20}{s+300}$$

Dividing numerator by denominator as are of same order,

$$\frac{Y(s)}{R(s)} = -4 + \frac{1220}{s+300}$$

The corresponding state diagram is,

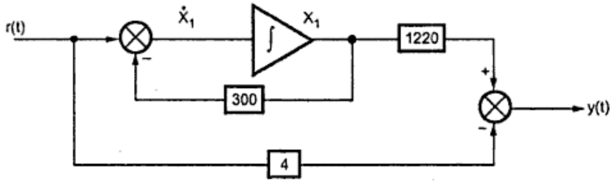


Fig. 4.1

$$\dot{X}_1 = r(t) - 300\,X_1$$

and $$y(t) = 1220\,X_1 - 4\,r(t)$$

$A = [-300]$, $B = [1]$, $C = [1220]$, $D = [-4]$

$$[sI - A] = [s + 300]$$

$$[sI - A]^{-1} = \left[\frac{1}{s+300}\right]$$

$$e^{AT} = \left[e^{-300\,t}\right]$$

Now $$y(0) = 0 \quad \text{hence substituting in } y(t)$$

$$y(0) = 1220X_1(0) - 4\,r(0), \quad r(0) = 10 \text{ as given to be } 10\,u(t)$$

$$0 = 1220\,X_1(0) - 40$$

$$X_1(0) = \frac{40}{1220}$$

$$ZIR = e^{AT}\;X_1(0) = \frac{40}{1220}\;e^{-300\,t}$$

To find ZSR use,

$$ZSR = L^{-1}\;[\phi(s)\,B\,U(s)]$$

$$= L^{-1}\left[\frac{1}{(s+300)}\cdot 1\cdot\frac{10}{s}\right] \text{as } U(s) = R(s) = \frac{10}{s}$$

$$= L^{-1}\left[\frac{10}{s\,(s+300)}\right] = L^{-1}\left[\frac{1/30}{s} - \frac{1/30}{s+300}\right]$$

$$ZSR = \frac{1}{30} - \frac{1}{30}\;e^{-300\,t}$$

$$X(t) = ZIR + ZSR = \frac{40}{1220} e^{-300t} + \frac{1}{30} - \frac{1}{30} e^{-300t}$$

$$X(t) = \frac{1}{30} - 5.4\times 10^{-4} e^{-300t}$$

$$y(t) = 1220\, X(t) - 4\, r(t)$$

$$= 1220 \left[\frac{1}{30} - 5.4\times 10^{-4} e^{-300t}\right] - 4\ (10)$$

$$= 0.667 - 0.667\ e^{-300t}$$

The response y(t) is,

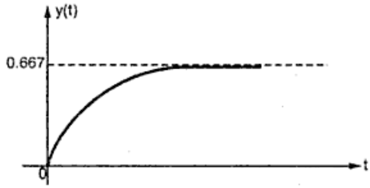


Fig. 4.2

Example 4.23 : *A linear dynamic time invariant system is represented by*

$$\dot{X} = A\,X(t) + B\,U(t)$$

where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$

Find if the system is completely controllable.

Solution : For the system, n = 3

$$\therefore \quad Q_c = [B : AB : A^2B]$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -3 & 0 \end{bmatrix}$$

$$A^2B = A\,[A\,B] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 0 \\ 7 & 0 \end{bmatrix}$$

$$\therefore \qquad Q_c = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 \\ 1 & 0 & -3 & 0 & 7 & 0 \end{bmatrix}$$

Consider the determinant,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -3 \end{bmatrix} = 1 = \text{nonzero}$$

Hence rank of $Q_c = 3 = n$

Thus the given system is **completely controllable**

Example 4.24 : *Consider the system represented by,*

$$\dot{X} = \begin{bmatrix} -0.2 & 0.4 \\ 0.1 & -0.1 \end{bmatrix} X(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} U(t)$$

$$Y(t) = [1 \quad 0]\; X(t)$$

Find the complete observability of the system.

Solution : For the system, $n = 2$

$$A = \begin{bmatrix} -0.2 & 0.4 \\ 0.1 & -0.1 \end{bmatrix} \qquad C = [1 \quad 0]$$

$$\therefore \qquad A^T = \begin{bmatrix} -0.2 & 0.1 \\ 0.4 & -0.1 \end{bmatrix} \qquad C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Q_o = [C^T \quad A^T C^T]$$

$$\therefore \qquad A^T C^T = \begin{bmatrix} -0.2 & 0.1 \\ 0.4 & -0.1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.2 \\ 0.4 \end{bmatrix}$$

$$\therefore \qquad Q_o = \begin{bmatrix} 1 & -0.2 \\ 0 & 0.4 \end{bmatrix}$$

Consider the determinant,

$$\begin{vmatrix} 1 & -0.2 \\ 0 & 0.4 \end{vmatrix} = 0.4 = \text{Nonzero}$$

$\therefore \quad$ Rank of $Q_o = 2 = n$

Hence the system is **completely observable.**

Example 4.25 : *Using similarity transformation find e^{At} for,*

$$A = \begin{bmatrix} -4 & 3 \\ -6 & 5 \end{bmatrix}$$

Solution : Find the eigen values.

$$|\lambda I - A| = \begin{vmatrix} \lambda+4 & -3 \\ 6 & \lambda-5 \end{vmatrix} = 0$$

$$\therefore \quad \lambda^2 - \lambda - 2 = 0$$

$$\therefore \quad (\lambda + 1)(\lambda - 2) = 0$$

$$\lambda_1 = -1, \ \lambda_2 = 2$$

Find the eigen vectors.

For $\lambda_1 = -1$, $\quad [\lambda_1 I - A] = \begin{bmatrix} 3 & -3 \\ 6 & -6 \end{bmatrix}$

$$\therefore \quad M_1 = \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} = \begin{bmatrix} -6 \\ -6 \end{bmatrix} = \begin{bmatrix} +1 \\ +1 \end{bmatrix}$$

For $\lambda_2 = -2$, $[\lambda_2 I - A] = \begin{bmatrix} 6 & -3 \\ 6 & -3 \end{bmatrix}$

$$\therefore \quad M_2 = \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} = \begin{bmatrix} -3 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\therefore \quad M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\therefore \quad M^{-1} = \frac{\text{Adj}[M]}{|M|} = \frac{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}{1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

$$\therefore \quad e^{At} = Me^{\Lambda t}M^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{2t} & -e^{-t} + e^{2t} \\ 2e^{-t} - 2e^{2t} & -e^{-t} + 2e^{2t} \end{bmatrix}$$

Example 4.26 : *Using Laplace transform method, find e^{At} for*

a) $A = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix}$ *b)* $A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$

Solution : a) $A = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix}$

$$[sI - A] = \begin{bmatrix} s & 3 \\ -1 & s+4 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{\text{Adj}\,[sI - A]}{|sI - A|} = \frac{\begin{bmatrix} s+4 & -3 \\ 1 & s \end{bmatrix}}{s(s+4)+3} = \frac{\begin{bmatrix} s+4 & -3 \\ 1 & s \end{bmatrix}}{(s+1)(s+3)}$$

$$\therefore \quad \phi(s) = \begin{bmatrix} \frac{s+4}{(s+1)(s+3)} & \frac{-3}{(s+1)(s+3)} \\ \frac{1}{(s+1)(s+3)} & \frac{s}{(s+1)(s+3)} \end{bmatrix}$$

$$\therefore \quad \phi(t) = e^{At} = L^{-1} \begin{bmatrix} \frac{s+4}{(s+1)(s+3)} & \frac{-3}{(s+1)(s+3)} \\ \frac{1}{(s+1)(s+3)} & \frac{s}{(s+1)(s+3)} \end{bmatrix}$$

$$= L^{-1} \begin{bmatrix} \frac{1.5}{s+1} - \frac{0.5}{s+3} & \frac{-1.5}{s+1} + \frac{1.5}{s+3} \\ \frac{0.5}{s+1} - \frac{0.5}{s+3} & \frac{-0.5}{s+1} + \frac{1.5}{s+3} \end{bmatrix}$$

$$\therefore \quad e^{At} = \begin{bmatrix} 1.5e^{-t} - 0.5e^{-3t} & -1.5e^{-t} + 1.5e^{-3t} \\ 0.5e^{-t} - 0.5e^{-3t} & -0.5e^{-t} + 1.5e^{-3t} \end{bmatrix}$$

b) $A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$

$$[sI - A] = \begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix}$$

$$\therefore \quad [sI - A]^{-1} = \frac{\text{Adj}[sI-A]}{|sI-A|} = \frac{\begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}}{s^2+4s+3} = \frac{\begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}}{(s+1)(s+3)}$$

$$\therefore \quad \phi(s) = \begin{bmatrix} \frac{s+4}{(s+1)(s+3)} & \frac{1}{(s+1)(s+3)} \\ \frac{-3}{(s+1)(s+3)} & \frac{s}{(s+1)(s+3)} \end{bmatrix}$$

$$\therefore \quad e^{At} = L^{-1}\begin{bmatrix} \frac{s+4}{(s+1)(s+3)} & \frac{1}{(s+1)(s+3)} \\ \frac{-3}{(s+1)(s+3)} & \frac{s}{(s+1)(s+3)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1.5}{s+1} - \frac{0.5}{s+3} & \frac{0.5}{s+1} - \frac{0.5}{s+3} \\ \frac{-1.5}{s+1} + \frac{1.5}{s+3} & \frac{-0.5}{s+1} + \frac{1.5}{s+3} \end{bmatrix}$$

$$\therefore \quad e^{At} = \begin{bmatrix} 1.5e^{-t} - 0.5e^{-3t} & 0.5e^{-t} - 0.5e^{-3t} \\ -1.5e^{-t} + 1.5e^{-3t} & -0.5e^{-t} + 1.5e^{-3t} \end{bmatrix}$$

Example 4.27 : *Using Cayley Hamilton method, find e^{At} for,*

a) $A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$ *b)* $A = \begin{bmatrix} 0 & 2 \\ -2 & -4 \end{bmatrix}$

Solution : a) $A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$

$$f(A) = e^{At} \quad \text{hence} \quad p(\lambda) = e^{\lambda t}$$

Step 1 : Find eigen values

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 6 & \lambda+5 \end{vmatrix} = 0$$

$$\therefore \quad \lambda^2 + 5\lambda + 6 = 0 \quad \text{i.e.} \quad (\lambda + 2)(\lambda + 3) = 0$$

$$\lambda_1 = -2 \quad \text{and} \quad \lambda_2 = -3$$

Step 2 : n = 2

$\therefore$ $$R(\lambda) = \alpha_0 + \alpha_1\lambda = p(\lambda)$$

i.e. $$\alpha_0 + \alpha_1\lambda = e^{\lambda t}$$

Step 3 : Substitute values of λ.

$\therefore$ $$\alpha_0 - 2\alpha_1 = e^{-2t} \quad \text{... (1)}$$

and $$\alpha_0 - 3\alpha_1 = e^{-3t} \quad \text{... (2)}$$

Step 4 : Solving equations (1) and (2),

$$\alpha_1 = e^{-2t} - e^{-3t}, \quad \alpha_0 = 3e^{-2t} - 2e^{-3t}$$

Step 5 : $$f(A) = R(A) = \alpha_0 I + \alpha_1 A$$

$\therefore$ $$e^{At} = (3e^{-2t} - 2e^{-3t})\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + (e^{-2t} - e^{-3t})\begin{bmatrix}0 & 1\\-6 & -5\end{bmatrix}$$

$\therefore$ $$e^{At} = \begin{bmatrix}3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t}\\-6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t}\end{bmatrix}$$

b) $$A = \begin{bmatrix}0 & 2\\-2 & -4\end{bmatrix}$$

$$f(A) = e^{At} \quad \text{hence} \quad p(\lambda) = e^{\lambda t}$$

Step 1 : Find eigen values.

$$|\lambda I - A| = \begin{vmatrix}\lambda & -2\\2 & \lambda+4\end{vmatrix} = 0$$

$\therefore$ $$\lambda^2 + 4\lambda + 4 = 0$$

$\therefore$ $$(\lambda + 2)^2 = 0$$

$\therefore$ $$\lambda_1 = -2, \quad \lambda_2 = -2$$

Step 2 : n = 2

$\therefore$ $$R(\lambda) = \alpha_0 + \alpha_1\lambda = p(\lambda)$$

i.e. $$\alpha_0 + \alpha_1\lambda = e^{\lambda t}$$

Step 3 : Substitute values of λ.

$$\therefore \quad \alpha_0 - 2\alpha_1 = e^{-2t} \qquad \ldots (1)$$

But as λ is repeated, second equation must be obtained as,

$$\frac{dR(\lambda)}{d\lambda} = \frac{dp(\lambda)}{d\lambda}$$

$$\therefore \quad \alpha_1 = t\, e^{\lambda t} \text{ at } \lambda = -2 \qquad \ldots (2)$$

$$\therefore \quad \alpha_1 = t\, e^{-2t}$$

Step 4 : $\alpha_0 = e^{-2t} + 2\alpha_1 = e^{-2t} + 2te^{-2t}$ ... from (1)

Step 5 : $f(A) = R(A) = \alpha_0 I + \alpha_1 A$

$$\therefore \quad e^{At} = (e^{-2t} + 2t\,e^{-2t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{-2t}\begin{bmatrix} 0 & 2 \\ -2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t}(1+2t) & 2t\,e^{-2t} \\ -2t\,e^{-2t} & e^{-2t}(1-2t) \end{bmatrix}$$

Example 4.28 : *Find the response of the system,*

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} X + \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} U(t), \quad X(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and $Y(t) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} X$ *to the following input,*

$$U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix} = \begin{bmatrix} u(t) \\ e^{-3t}u(t) \end{bmatrix} \text{ where } u(t) = \text{unit step function.}$$

Solution : Find e^{At} for $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ thus $[sI - A] = \begin{bmatrix} s & -1 \\ 2 & s+2 \end{bmatrix}$

$$[sI - A]^{-1} = \frac{\text{Adj}\,[sI - A]}{|sI - A|} = \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{s^2 + 3s + 2} = \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{(s+1)(s+2)}$$

$$\therefore \quad \phi(s) = [sI - A]^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$\therefore \quad e^{At} = L^{-1}[\phi(s)] = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\therefore \quad ZIR = e^{At} X(0) = e^{At} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now $\quad ZSR = L^{-1}\{\phi(s)\, B\, U(s)\}$

$$U(s) = \begin{bmatrix} \frac{1}{s} \\ \frac{1}{s+2} \end{bmatrix}$$

$$\therefore \quad ZSR = L^{-1}\left\{ \begin{bmatrix} \frac{s+3}{(s+1)(s+3)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s} \\ \frac{1}{s+3} \end{bmatrix} \right\}$$

$$= L^{-1}\left\{ \begin{bmatrix} \frac{s+3}{(s+1)(s+3)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} \frac{2}{s} + \frac{1}{s+3} \\ \frac{1}{s+3} \end{bmatrix} \right\}$$

$$= L^{-1}\left\{ \begin{bmatrix} \frac{s+3}{(s+1)(s+3)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} \frac{3(s+2)}{s(s+3)} \\ \frac{1}{s+3} \end{bmatrix} \right\}$$

$$= L^{-1} \begin{bmatrix} \frac{3}{s(s+1)} + \frac{1}{(s+1)(s+2)(s+3)} \\ \frac{-6}{s(s+1)(s+3)} + \frac{5}{(s+1)(s+2)(s+3)} \end{bmatrix}$$

$$= L^{-1} \begin{bmatrix} \frac{3}{s} - \frac{3}{s+1} + \frac{0.5}{s+1} - \frac{1}{s+2} + \frac{0.5}{s+3} \\ \frac{-2}{s} + \frac{3}{s+1} - \frac{1}{s+3} - \frac{0.5}{s+1} + \frac{2}{s+2} - \frac{1.5}{s+3} \end{bmatrix}$$

$$= \begin{bmatrix} 3-2.5e^{-t}-e^{-2t}+0.5e^{-3t} \\ -2+2.5e^{-t}+2e^{-2t}-2.5e^{-3t} \end{bmatrix}$$

$$\therefore \quad X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \text{ZIR} + \text{ZSR} = \begin{bmatrix} 3-2.5e^{-t}-e^{-2t}+0.5e^{-3t} \\ -2+2.5e^{-t}+2e^{-2t}-2.5e^{-3t} \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}$$

$$\therefore \quad Y_1(t) = X_1(t) = \mathbf{3 - 2.5\ e^{-t} - e^{-2t} + 0.5\ e^{-3t}}$$

and $$Y_2(t) = X_1(t) + X_2(t) = \mathbf{1 + e^{-2t} - 2\ e^{-3t}}$$

Example 4.29 : *The Fig. 4.3 shows the block diagram of a process control system with state variable feedback and feed forward control. The state model of process is,*

$$\dot{X} = \begin{bmatrix} -3 & 2 \\ 4 & -5 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U$$

and $Y = [0 \quad 1]\ X$

a) Derive state model for the entire system.

b) Find the response if input r(t) is unit step assuming $X(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

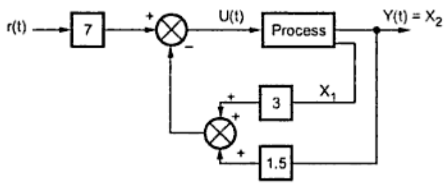


Fig. 4.3

Solution : a) From the block diagram,

$$U(t) = +7\ r(t) - 3X_1 - 1.5X_2 \quad \dots (1)$$

From the given state model of the system,

$$\dot{X}_1 = -3X_1 + 2X_2 + U(t) \quad \dots (2)$$

and $$\dot{X}_2 = 4X_1 - 5X_2 \quad \dots (3)$$

Using (1) in (2),

$$\dot{X}_1 = -3\,X_1 + 2X_2 + 7\,r(t) - 3X_1 - 1.5\,X_2$$

$$\therefore \quad \dot{X}_1 = -6\,X_1 + 0.5\,X_2 + 7\,r(t) \qquad \ldots (4)$$

The equation (3) remain unchanged. Hence state model of the entire system is,

$$\dot{X} = \begin{bmatrix} -6 & 0.5 \\ 4 & -5 \end{bmatrix} X + \begin{bmatrix} 7 \\ 0 \end{bmatrix} r(t)$$

The output equation remains unchanged.

$$Y(t) = [0 \;\; 1] \; X(t)$$

Thus, $A = \begin{bmatrix} -6 & 0.5 \\ 4 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$, $C = [0 \;\; 1]$

b) Find $\phi(s) = [sI - A]^{-1}$

$$[sI - A] = \begin{bmatrix} s+6 & -0.5 \\ -4 & s+5 \end{bmatrix}$$

$$\therefore \quad [sI - A]^{-1} = \frac{\text{Adj}\,[sI - A]}{|sI - A|} = \frac{\begin{bmatrix} s+5 & 0.5 \\ 4 & s+6 \end{bmatrix}}{(s+6)\,(s+5)-2} = \frac{\begin{bmatrix} s+5 & 0.5 \\ 4 & s+6 \end{bmatrix}}{(s^2+11s+28)}$$

$$= \frac{\begin{bmatrix} s+5 & 0.5 \\ 4 & s+6 \end{bmatrix}}{(s+4)(s+7)}$$

$$\therefore \quad \phi(s) = \begin{bmatrix} \dfrac{s+5}{(s+4)\,(s+7)} & \dfrac{0.5}{(s+4)\,(s+7)} \\ \dfrac{4}{(s+4)\,(s+7)} & \dfrac{s+6}{(s+4)\,(s+7)} \end{bmatrix}$$

No need to calculate e^{At} as $X(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ the ZIR is null matrix i.e. zero.

ZSR = L^{-1} {ϕ (s) B U(s)} but here input is r(t) and not U(t) to the entire system.

$$\therefore \quad \text{ZSR} = L^{-1}\{\phi(s)\,B\,R(s)\} \text{ and } R(s) = \frac{1}{s}$$

$$= L^{-1}\left\{\phi(s)\begin{bmatrix}7\\0\end{bmatrix}\left[\frac{1}{s}\right]\right\} = L^{-1}\left\{\phi(s)\begin{bmatrix}\frac{7}{s}\\0\end{bmatrix}\right\}$$

$$= L^{-1}\begin{bmatrix}\dfrac{7(s+5)}{s(s+4)(s+7)}\\ \dfrac{28}{s(s+4)(s+7)}\end{bmatrix} = L^{-1}\begin{bmatrix}\dfrac{1.25}{s}-\dfrac{05833}{s+4}-\dfrac{0.666}{s+7}\\ \dfrac{1}{s}-\dfrac{2.33}{s+4}+\dfrac{1.333}{s+7}\end{bmatrix}$$

$$\therefore \qquad ZSR = \begin{bmatrix}1.25-05833\,e^{-4t}-0.667\,e^{-7t}\\ 1-2.33\,e^{-4t}+1.333\,e^{-7t}\end{bmatrix} = \begin{bmatrix}X_1(t)\\X_2(t)\end{bmatrix}$$

$$\therefore \qquad Y(t) = [0 \quad 1]\begin{bmatrix}X_1\\X_2\end{bmatrix} = X_2(t)$$

$$= \mathbf{1 - 2.33\ e^{-4t} + 1.333\ e^{-7t}}$$

This is the required response.

Example 4.30 : *Evaluate controllability and observability of the following state models,*

a) $A = \begin{bmatrix}-2 & 1\\1 & -2\end{bmatrix}, \quad B = \begin{bmatrix}1\\0\end{bmatrix}, \quad C = [1 \quad -1]$

b) $A = \begin{bmatrix}-1 & 0 & 0\\0 & -2 & 0\\0 & 0 & -3\end{bmatrix}, \quad B = \begin{bmatrix}1 & 0\\1 & 2\\2 & 1\end{bmatrix}, \quad C = \begin{bmatrix}1 & 1 & 2\\3 & 1 & 5\end{bmatrix}$

c) $A = \begin{bmatrix}0 & 0 & 0\\1 & 0 & -3\\0 & 1 & -4\end{bmatrix}, \quad B = \begin{bmatrix}40\\10\\0\end{bmatrix}, \quad C = [0 \quad 0 \quad 1]$

Solution : a) For controllability, $Q_c = [B : AB]$ as $n = 2$

$$AB = \begin{bmatrix}-2 & 1\\1 & -2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix}$$

$$\therefore \qquad Q_c = \begin{bmatrix}1 & -2\\0 & 1\end{bmatrix}$$

$$\begin{vmatrix}1 & -2\\0 & 1\end{vmatrix} = 1 \text{ hence rank of } Q_c = 2 = n$$

Hence system is **completely controllable.**

For observability, $Q_o = [C^T : A^T C^T]$ as n = 2

$$C^T = [1 - 1]^T = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A^T C^T = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

$$\therefore \quad Q_o = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix} \text{ and } \begin{vmatrix} 1 & -3 \\ -1 & 3 \end{vmatrix} = 0$$

$\therefore$ Hence the rank of $Q_o = 1 < n$

Thus system is **not completely observable.**

b) For controllability, $Q_c = [B : AB : A^2B]$ as n = 3.

$$AB = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -2 & -4 \\ -6 & -3 \end{bmatrix}$$

$$A^2B = A\,[AB] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ -2 & -4 \\ -6 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 4 & 8 \\ 18 & 9 \end{bmatrix}$$

$$\therefore \quad Q_c = \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 2 & -2 & -4 & 4 & 8 \\ 2 & 1 & -6 & -3 & 18 & 9 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & -2 \\ 2 & 1 & -6 \end{vmatrix} = -7 \neq 0 \text{ hence rank of } Q_c = 3 = n$$

Thus the system **is completely controllable.**

For observability, $Q_o = [C^T : A^T C^T : A^{T^2}\, C^T]$ as n = 3

$$A^T C^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ -2 & -2 \\ -6 & -15 \end{bmatrix}$$

$$A^T C^T = A^T [A^T C^T] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ -1 & -2 \\ -6 & -15 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 4 \\ 18 & 45 \end{bmatrix}$$

$$\therefore \qquad Q_o = \begin{bmatrix} 1 & 3 & -1 & -3 & 1 & 3 \\ 1 & 1 & -2 & -2 & 4 & 4 \\ 2 & 5 & -6 & -15 & 18 & 45 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 3 & -1 \\ 1 & 1 & -2 \\ 2 & 5 & -6 \end{vmatrix} = +7 \neq 0 \text{ hence rank of } Q_o = 3 = n$$

Thus the system is **completely observable.**

c) For controllability, $Q_c = [B \ : \ AB \ : \ A^2B]$ as $n = 3$

$$AB = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} 40 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 40 \\ 10 \end{bmatrix}$$

$$A^2B = A\,[AB] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ 40 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ -30 \\ 0 \end{bmatrix}$$

$$\therefore \qquad Q_c = \begin{bmatrix} 40 & 0 & 0 \\ 10 & 40 & -30 \\ 0 & 10 & 0 \end{bmatrix} = -12000 \neq 0 \text{ hence rank of } Q_c = 3 = n$$

Thus the system is **completely controllable.**

For observability, $Q_o = [C^T : A^TC^T : A^{T^2}\ C^T]$ as $n = 3$

$$A^TC^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}$$

$$A^{T^2}\ C^T = A^T\,[A^T\ C^T] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 13 \end{bmatrix}$$

$$Q_o = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 13 \end{bmatrix}$$

$$\therefore \qquad \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 13 \end{vmatrix} = 1 \neq 0 \text{ hence rank of } Q_o = 3 = n$$

Thus the system is **completely observable.**

Example 4.31 : *Consider the state model,*

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} X(t), \quad Y(t) = [1 \;\; 2] \;\; X(t)$$

Find the set of initial conditions such that the mode e^{2t} is suppressed in Y(t).

Solution : Find e^{At} for $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ hence $[sI - A] = \begin{bmatrix} s & -1 \\ -2 & s-1 \end{bmatrix}$

$$\therefore \quad \phi(s) = [sI - A]^{-1} = \frac{\text{Adj}\,[sI-A]}{|sI-A|} = \frac{\begin{bmatrix} s-1 & 1 \\ 2 & s \end{bmatrix}}{s^2 - s - 2}$$

$$= \begin{bmatrix} \frac{s-1}{(s+1)(s-2)} & \frac{1}{(s+1)(s-2)} \\ \frac{2}{(s+1)(s-2)} & \frac{s}{(s+1)(s-2)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2/3}{(s+1)} + \frac{1/3}{(s-2)} & \frac{-1/3}{(s+1)} + \frac{1/3}{(s-2)} \\ \frac{-2/3}{(s+1)} + \frac{2/3}{(s-2)} & \frac{1/3}{(s+1)} + \frac{2/3}{(s-2)} \end{bmatrix}$$

$$\therefore \quad e^{At} = L^{-1}[\phi(s)]$$

$$\therefore \quad e^{At} = \begin{bmatrix} \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t} & -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} \\ \frac{-2}{3}e^{-t} + \frac{2}{3}e^{2t} & \frac{1}{3}e^{-t} + \frac{2}{3}e^{2t} \end{bmatrix}$$

The equation is homogeneous hence the solution is,

$$X(t) = e^{At} X(0), \quad \text{Let } X(0) = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t} & -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} \\ \frac{-2}{3}e^{-t} + \frac{2}{3}e^{2t} & \frac{1}{3}e^{-t} + \frac{2}{3}e^{2t} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}e^{-t}(2a-b)+\frac{1}{3}e^{2t}(a+b) \\ \frac{1}{3}e^{-t}(-2a+b)+\frac{1}{3}e^{2t}(2a+2b) \end{bmatrix}$$

And $$Y(t) = [1 \quad 2]\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = X_1 + 2X_2$$

$$= \frac{1}{3}e^{-t}(-2a+b) + \frac{1}{3}e^{2t}(5a+5b)$$

To suppress the mode e^{2t} from Y(t),

$$5a + 5b = 0$$

i.e. $$a = -b$$

Hence the initial state must be,

$$X(0) = \begin{bmatrix} +m \\ -m \end{bmatrix} \quad \text{where } m \neq 0$$

Example 4.32 : *Write the state equations of the system shown and determine its state controllability and observability.* **(VTU: Jan./Feb.-2006)**

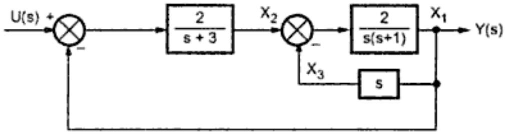


Fig. 4.4

Solution : Consider the block $\frac{2}{s(s+1)}$. Its state diagram is,

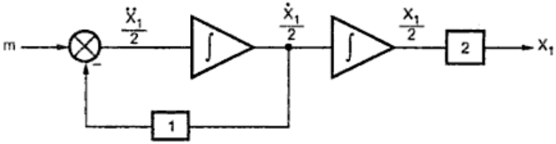


Fig. 4.5

From the state diagram,

$$\frac{\ddot{X}_1}{2} = m - \frac{\dot{X}_1}{2} \qquad \text{... (1)}$$

Now $s = \frac{d}{dt}$ which is a differentiator and input to it is X_1.

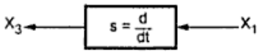


Fig. 4.6

Thus, $$\frac{d}{dt} X_1 = X_3$$

$$\therefore \quad \dot{X}_1 = X_3 \qquad \text{... (2)}$$

Now the block $\frac{2}{s+3}$ has state diagram,

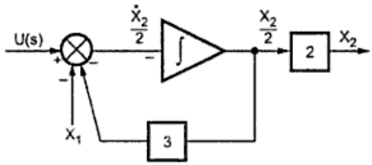


Fig. 4.7

$$\frac{\dot{X}_2}{2} = U(t) - X_1 - \frac{3X_2}{2}$$

$$\therefore \quad \dot{X}_2 = -2X_1 - 3X_2 + 2U(t) \qquad \text{... (3)}$$

Now $m = X_2 - X_3$ hence using in (1),

$$\therefore \quad \frac{\ddot{X}_1}{2} = X_2 - X_3 - \frac{\dot{X}_1}{2}$$

$$\therefore \quad \ddot{X}_1 = 2X_2 - 2X_3 - \dot{X}_1 \qquad \text{...(4)}$$

but $\dot{X}_1 = X_3$ hence $\ddot{X}_1 = \dot{X}_3$ and using in (4),

$$\dot{X}_3 = 2X_2 - 2X_3 - X_3 = 2X_2 - 3X_3 \qquad \text{...(5)}$$

The equations (2), (3), (5) give us required state model.

$$\dot{X}(t) = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} U(t)$$

and $$Y(t) = [1 \quad 0 \quad 0]\, X(t)$$

For conrollability, $Q_c = [B : AB : A^2 B]$ as $n = 3]$

$$AB = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix}$$

$$A^2 B = A\,[AB] = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 18 \\ -24 \end{bmatrix}$$

$$\therefore \quad Q_c = \begin{bmatrix} 0 & 0 & 4 \\ 2 & -6 & 18 \\ 0 & 4 & -24 \end{bmatrix}$$

$$\therefore \quad \begin{vmatrix} 0 & 0 & 4 \\ 2 & -6 & 18 \\ 0 & 4 & -24 \end{vmatrix} = +\,32 \neq 0 \text{ hence rank of } Q_c = 3 = n$$

Thus system is **completely controllable** :

For observability, $Q_o = [C^T : A^T C^T : A^{T^2} C^T]$ as $n = 3$

$$A^T C^T = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A^{T^2} C^T = A^T\,[A^T C^T] = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

$$Q_o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

$$\therefore \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -3 \end{vmatrix} = -\,2 \neq 0 \text{ hence rank of } Q_o = 3 = n$$

Thus the system is **completely observable.**

Example 4.33: *Use controllability and observbility matrices to determine whether the system represented by the flow graph shown in Fig. 1 is completely controllable and completely observable.* **(VTU: Jan./Feb.-2005)**

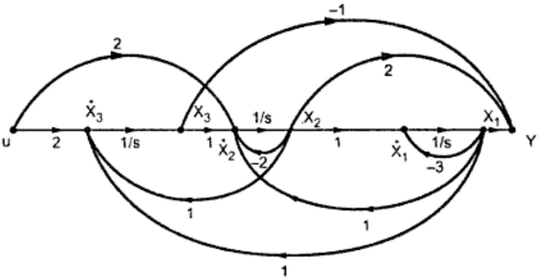


Fig. 4.8

Solution : The value of the variable at the node is an algebraic sum of all the signals entering at that node.

$$\therefore \quad \dot{X}_3 = X_1 + X_2 + 2u, \qquad \dot{X}_2 = X_1 - 2X_2 + X_3 + 2u$$

$$\dot{X}_1 = -3X_1 + X_2, \qquad Y = X_1 + 2X_2 - X_3$$

$$\therefore \quad A = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, C = [1 \ \ 2 \ -1]$$

For controllability, $Q_c = [B : AB : A^2B]$ ...n = 3

$$AB = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 10 & -5 & 1 \\ -4 & 6 & -2 \\ -2 & -1 & 1 \end{bmatrix}$$

$$\therefore \quad A^2B = \begin{bmatrix} 10 & -5 & 1 \\ -4 & 6 & -2 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ 8 \\ 0 \end{bmatrix}$$

$$\therefore \quad Q_c = \begin{bmatrix} 0 & 2 & -8 \\ 2 & -2 & 8 \\ 2 & 2 & 0 \end{bmatrix}$$

$$\therefore \qquad |Q_c| = -32 \neq 0 \text{ hence rank of } Q_c = 3 = n$$

The system is **completely** controllable.

For observability, $Q_o = \left[C^T : A^T C^T : A^{T^2} C^T\right]$ ...n = 3

$$A^T C^T = \begin{bmatrix} -3 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}$$

$$A^{T^2} C^T = A^T\left[A^T C^T\right] = \begin{bmatrix} -3 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ -4 \end{bmatrix}$$

$$\therefore \qquad Q_o = \begin{bmatrix} 1 & -2 & 4 \\ 2 & -4 & 8 \\ -1 & 2 & -4 \end{bmatrix}$$

$$|Q_o| = 0 \text{ and } \begin{vmatrix} 2 & -4 \\ -1 & 2 \end{vmatrix} = 0 \text{ thus } 2 \times 2 \text{ determinant is zero}$$

Thus rank of Q_o= 1 hence system **is not observable.**

Example 4.34 : *Given the time invariant system :*

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ 0 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \; : \; y = [1 \;\; 0] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

and that $u(t) = e^{-1}$ *and* $y(t) = 2 - \alpha te^{-1}$ *, find* $X_1(t)$ *and* $X_2(t)$.

Find also $X_1(0)$ *and* $X_2(0)$. *What happens if* $\alpha = 0$ *?* **(VTU : Jan./Feb.-2005)**

Solution : $A = \begin{bmatrix} 0 & \alpha \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \;\; 0]$

$$[sI - A] = \begin{bmatrix} s & -\alpha \\ 0 & s+1 \end{bmatrix}, \; \text{Adj}\,[sI - A] = \begin{bmatrix} s+1 & \alpha \\ 0 & s \end{bmatrix}$$

$$\therefore \qquad [sI - A]^{-1} = \frac{\text{Ad}_j\;[sI - A]}{|sI - A|} = \frac{\begin{bmatrix} s+1 & \alpha \\ 0 & s \end{bmatrix}}{s^2 + s} = \begin{bmatrix} \frac{1}{s} & \frac{\alpha}{s(s+1)} \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

$$e^{At} = L^{-1}[sI - A]^{-1} = \begin{bmatrix} 1 & \alpha\left(1-e^{-t}\right) \\ 0 & e^{-t} \end{bmatrix}$$

$$\therefore \quad ZIR = e^{At}\, X(0) = \begin{bmatrix} 1 & \alpha\left(1-e^{-t}\right) \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} = \begin{bmatrix} X_1(0) + X_2(0)\alpha\left(1-e^{-t}\right) \\ e^{-t} X_2(0) \end{bmatrix}$$

$$ZSR = L^{-1}\ \{\phi(s)BU(s)\} = L^{-1} \left\{ \begin{bmatrix} \frac{1}{s} & \frac{\alpha}{s(s+1)} \\ 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left[\frac{e^{-1}}{s}\right] \right\} \quad \ldots U(s) = \frac{e^{-1}}{s}$$

$$L^{-1}\{\phi(s)BU(s)\} = L^{-1} \begin{bmatrix} \frac{\alpha e^{-1}}{s^2(s+1)} \\ \frac{e^{-1}}{s(s+1)} \end{bmatrix} = e^{-1}\, L^{-1} \begin{bmatrix} \frac{\alpha}{s^2} - \frac{\alpha}{s} + \frac{\alpha}{s+1} \\ \frac{1}{s} - \frac{1}{s+1} \end{bmatrix} \quad \ldots \text{Using partial fractions}$$

$$= e^{-1} \begin{bmatrix} \alpha t - \alpha + \alpha e^{-t} \\ 1 - e^{-t} \end{bmatrix}$$

$$\therefore \quad X(t) = ZIR + ZSR = \begin{bmatrix} X_1(0) + X_2(0)\alpha(1-e^{-t}) + e^{-1}\alpha t - e^{-1}\alpha + e^{-1}\alpha e^{-t} \\ e^{-t} X_2(0) + e^{-1} - e^{-1}e^{-t} \end{bmatrix}$$

$$\therefore \quad y(t) = X_1(t) = X_1(0) + \alpha X_2(0) - \alpha X_2(0)e^{-t} + e^{-1}\alpha t - e^{-1}\alpha + e^{-1}\alpha e^{-t}$$

But given $\quad y(t) = 2 - \alpha t e^{-1}$

At $t = 0$, $y(t) = 2 = X_1(t)\big|_{t=0} = X_1(0)$ and $X_2(0) = 0$

$$\boxed{\begin{aligned} \therefore \quad X_1(t) &= 2 + \alpha t e^{-1} + e^{-1}\alpha(e^{-t} - 1) \\ \text{and} \quad X_2(t) &= e^{-1}\left(1 - e^{-t}\right) \end{aligned}}$$

If $\quad \alpha = 0, X_1(t) = \mathbf{2}$

Example 4.35 : *Given the state model of a system :* **(VTU: July /Aug.- 2005)**

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$Y = [1 \ \ 0]\ X$$

with initial conditions $X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Determine :

i) The state transition matrix.

ii) The state transition equation X(t) and output Y(t) for an unit step input.

iii) Inverse state transition matrix.

Solution : i)

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}$$

$$\therefore \quad [sI - A] = \begin{bmatrix} s & -1 \\ 4 & s+5 \end{bmatrix} \quad \therefore \text{Adj } [sI - A] = \begin{bmatrix} s+5 & 1 \\ -4 & s \end{bmatrix}$$

$$\therefore \quad [sI - A]^{-1} = \frac{\text{Adj } [sI - A]}{|sI - A|} = \frac{\begin{bmatrix} s+5 & 1 \\ -4 & s \end{bmatrix}}{s^2 + 5s + 4} = \frac{\begin{bmatrix} s+5 & 1 \\ -4 & s \end{bmatrix}}{(s+1)(s+4)}$$

$$\therefore \quad \phi(s) = [sI - A]^{-1} = \begin{bmatrix} \frac{s+5}{(s+1)(s+4)} & \frac{1}{(s+1)(s+4)} \\ \frac{-4}{(s+1)(s+4)} & \frac{s}{(s+1)(s+4)} \end{bmatrix}$$

$$\therefore \quad \phi(s) = \begin{bmatrix} \frac{4/3}{s+1} - \frac{1/3}{s+4} & \frac{1/3}{s+1} - \frac{1/3}{s+4} \\ \frac{-4/3}{s+1} + \frac{4/3}{s+4} & \frac{-1/3}{s+1} + \frac{4/3}{s+4} \end{bmatrix} \quad \text{... Partial fractions}$$

$$\therefore \quad e^{At} = L^{-1}[\phi(s)] = \begin{bmatrix} \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t} & \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t} \\ \frac{-4}{3}e^{-t} + \frac{4}{3}e^{-4t} & \frac{-1}{3}e^{-t} + \frac{4}{3}e^{-4t} \end{bmatrix}$$

ii)

$$ZSR = e^{At} X(0) = e^{At}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t} \\ \frac{-5}{3}e^{-t} + \frac{8}{3}e^{-4t} \end{bmatrix}$$

$$ZIR = L^{-1}\{\phi(s)\, BU(s)\} = L^{-1}\left\{\begin{bmatrix} \frac{s+5}{(s+1)\,(s+4)} & \frac{1}{(s+1)\,(s+4)} \\ \frac{-4}{(s+1)\,(s+4)} & \frac{s}{(s+1)\,(s+4)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s} \end{bmatrix}\right\}$$

$$= L^{-1}\begin{bmatrix} \dfrac{1}{s(s+1)\ (s+4)} \\ \dfrac{1}{(s+1)\ (s+4)} \end{bmatrix} = L^{-1}\begin{bmatrix} \dfrac{1/4}{s}-\dfrac{1/3}{s+1}+\dfrac{1/12}{s+4} \\ \dfrac{1/3}{s+1}-\dfrac{1/3}{s+4} \end{bmatrix}$$

$$\therefore \qquad ZIR = \begin{bmatrix} \frac{1}{4}-\frac{1}{3}e^{-t}+\frac{1}{12}e^{-4t} \\ \frac{1}{3}e^{-t}-\frac{1}{3}e^{-4t} \end{bmatrix}$$

$$\therefore \qquad X(t) = ZSR + ZIR = \begin{bmatrix} \frac{1}{4}+\frac{4}{3}e^{-t}-\frac{7}{12}e^{-4t} \\ \frac{-4}{3}e^{-t}+\frac{7}{3}e^{-4t} \end{bmatrix}$$

$$\therefore \qquad Y(t) = [1 \quad 0]\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = x_1(t)$$

$$\therefore \qquad Y(t) = \frac{1}{4}+\frac{4}{3}e^{-t}-\frac{7}{12}e^{-4t}$$

iii) The property of state transition matrix is,

$$\phi^{-1}(t) = \phi(-t)$$

$$\phi^{-1}(t) = \begin{bmatrix} \frac{4}{3}e^{t}-\frac{1}{3}e^{4t} & \frac{1}{3}e^{t}-\frac{1}{3}e^{4t} \\ \frac{-4}{3}e^{t}+\frac{4}{3}e^{4t} & -\frac{1}{3}e^{t}+\frac{4}{3}e^{4t} \end{bmatrix}$$

➡ **Example 4.36 :** *Determine the controllability and observability of the following state model.*

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$Y = [10 \quad 5 \quad 1] X$$

(VTU : July/Aug.-2005)

Solution : From the given model,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = [10 \ 5 \ 1]$$

For controllability, $Q_c = \left[B : AB : A^2B\right]$ .....n = 3

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -12 \end{bmatrix}$$

$$A^2B = A\,[AB] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -12 \end{bmatrix} = \begin{bmatrix} 1 \\ -12 \\ 61 \end{bmatrix}$$

$$\therefore \qquad Q_c = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -12 \\ 1 & -12 & 61 \end{bmatrix}$$

Now $|Q_c| = -84 \neq 0$ hence rank of $Q_c = n = 3$.

The system is **completely controllable.**

For observability, $Q_0 = \left[C^T \;:\; A^TC^T \;:\; A^{T^2}C^T\right]$

$$A^TC^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -1 \\ -1 \end{bmatrix}$$

$$A^{T^2}C^T = A^T\left[A^TC^T\right] = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} -6 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 5 \end{bmatrix}$$

$$\therefore \qquad Q_o = \begin{bmatrix} 10 & -6 & 6 \\ 5 & -1 & 5 \\ 1 & -1 & 5 \end{bmatrix}$$

Now $|Q_o| = 96 \neq 0$ hence rank of $Q_o = n = 3$

The system **is completely observable.**

Example 4.37 : *A system represented by following state model is controllable but not observable. Show that the non-observability is due to a pole-zero cancellation in $C[sI-A]^{-1}B$.*

(VTU: July/Aug.-2005)

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$Y = [1 \;\; 1 \;\; 0]\, X$$

Solution : From the given model,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix},\; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\; C = [1 \;\; 1 \;\; 0]$$

$$[sI-A] = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s+6 \end{bmatrix}, \ \text{Adj}\ [sI-A] = \begin{bmatrix} s^2+6s+11 & -6 & -6s \\ s+6 & +s(s+6) & -(11s+6) \\ 1 & s & s^2 \end{bmatrix}^T$$

$$\therefore \quad \text{Adj}\ [sI-A] = \begin{bmatrix} s^2+6s+11 & s+6 & 1 \\ -6 & s(s+6) & s \\ -6s & -(11s+6) & s^2 \end{bmatrix}$$

$$\therefore \quad [sI-A]^{-1} = \frac{\text{Adj}\ [sI-A]}{|sI-A|} = \frac{\text{Adj}\ [sI-A]}{s^3+6s^2+11s+6} = \frac{\text{Adj}\ [sI-A]}{(s+1)\ (s+2)\ (s+3)}$$

$$C[sI-A]^{-1}B = [1 \ \ 1 \ \ 0]\ \frac{\text{Adj}\ [sI-A]}{(s+1)\ (s+2)(s+3)} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(s+1)(s+2)(s+3)} \left\{ [1 \ \ 1 \ \ 0] \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} \right\} = \frac{s+1}{(s+1)(s+2)(s+3)}$$

Thus the factor (s+1) gets cancelled from the numerator and denominator.

This shows that as there is pole-zero cancellation in $C[sI-A]^{-1}$ B which is the transfer function of the system, the system is not observable.

Example 4.38 : *Consider the homogeneous equation* $\dot{X} = AX$ *where A is a* 3×3 *matrix. The following three solution for three different initial conditions are available,*

$$\begin{bmatrix} e^{-t} \\ -e^{-t} \\ 2e^{-t} \end{bmatrix}, \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \\ 0 \end{bmatrix}, \begin{bmatrix} 2e^{-3t} \\ -6e^{-3t} \\ 0 \end{bmatrix}$$

i) Identify the initial conditions

ii) Find the state transition matrix

iii) Hence or otherwise find the system matrix A. **(VTU: Jan./Feb.-2006)**

Solution : i) For initial conditions, put t = 0.

$$\therefore \quad X(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \\ 2e^{-t} \end{bmatrix} \quad X(0) = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$X(t) = \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \\ 0 \end{bmatrix} \quad X(0) = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$X(t) = \begin{bmatrix} 2e^{-3t} \\ -6e^{-3t} \\ 0 \end{bmatrix} \quad X(0) = \begin{bmatrix} 2 \\ -6 \\ 0 \end{bmatrix}$$

ii)

$$\dot{X}(t) = \begin{bmatrix} -e^{-t} \\ e^{-t} \\ -2e^{-t} \end{bmatrix} \qquad \dot{X}(0) = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

$$\dot{X}(t) = \begin{bmatrix} -2e^{-2t} \\ +4e^{-2t} \\ 0 \end{bmatrix} \qquad \dot{X}(0) = \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}$$

$$\dot{X}(t) = \begin{bmatrix} -6e^{-3t} \\ +18e^{-3t} \\ 0 \end{bmatrix} \qquad \dot{X}(0) = \begin{bmatrix} -6 \\ 18 \\ 0 \end{bmatrix}$$

$$\dot{X}(0) = \begin{bmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{bmatrix} X(0)$$

$$\therefore \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

i.e

$$A_1 - A_2 + 2A_3 = -1 \qquad ...(1)$$

$$A_4 - A_5 + 2A_6 = 1 \qquad ...(2)$$

$$A_7 - A_8 + 2A_9 = -2 \qquad ...(3)$$

$$\begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

i.e.

$$A_1 - 2A_2 = -2 \qquad ...(4)$$

$$A_4 - 2A_5 = 4 \qquad ...(5)$$

$$A_7 - 2A_8 = 0 \qquad ...(6)$$

$$\begin{bmatrix} -6 \\ 18 \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{bmatrix} \begin{bmatrix} 2 \\ -6 \\ 0 \end{bmatrix} \quad \text{i.e} \quad \begin{aligned} 2A_1 - 6A_2 &= -6 \quad &\ldots(7) \\ 2A_4 - 6A_5 &= 18 \quad &\ldots(8) \\ 2A_7 - 6A_8 &= 0 \quad &\ldots(9) \end{aligned}$$

Solving the equations (1) to (9), the system matrix A is given by,

$$\therefore \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ -6 & -5 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

iii) State transition matrix

$$[sI - A] = \begin{bmatrix} s & -1 & 0 \\ 6 & s+5 & -1 \\ 0 & 0 & s+1 \end{bmatrix} \quad \text{Adj } [sI-A] = \begin{bmatrix} (s+1)\ (s+5) & -6(s+1) & 0 \\ (s+1) & s(s+1) & 0 \\ 1 & s & s^2+5s+6 \end{bmatrix}^T$$

$$\therefore \qquad [sI-A]^{-1} = \frac{\text{Adj } [sI-A]}{|sI-A|} = \frac{\begin{bmatrix} s^2+6s+5 & (s+1) & 1 \\ -6(s+1) & s(s+1) & s \\ 0 & 0 & s^2+5s+6 \end{bmatrix}}{s^3+6s^2+11s+6}$$

$$s^3+6s^2+11s+6 = (s+1)\ (s+2)\ (s+3)$$

$$\therefore \qquad [sI-A]^{-1} = \begin{bmatrix} \dfrac{s+5}{(s+2)(s+3)} & \dfrac{1}{(s+2)(s+3)} & \dfrac{1}{(s+1)(s+2)(s+3)} \\ \dfrac{-6}{(s+2)(s+3)} & \dfrac{s}{(s+2)(s+3)} & \dfrac{s}{(s+1)(s+2)(s+3)} \\ 0 & 0 & \dfrac{1}{s+1} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{3}{(s+2)} - \dfrac{2}{(s+3)} & \dfrac{1}{(s+2)} - \dfrac{1}{(s+3)} & \dfrac{1/2}{(s+1)} - \dfrac{1}{(s+2)} + \dfrac{1/2}{(s+3)} \\ \dfrac{-6}{(s+2)} + \dfrac{6}{(s+3)} & \dfrac{-2}{(s+2)} + \dfrac{3}{(s+3)} & \dfrac{1/2}{(s+1)} + \dfrac{2}{(s+2)} - \dfrac{3/2}{(s+3)} \\ 0 & 0 & \dfrac{1}{(s+1)} \end{bmatrix} \quad \ldots\text{Partial fractions}$$

$$\therefore \quad e^{At} = L^{-1}[sI - A]^{-1} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} & \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t} \\ -6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t} & -\frac{1}{2}e^{-t} + 2e^{-2t} - \frac{3}{2}e^{-3t} \\ 0 & 0 & e^{-t} \end{bmatrix}$$

Example 4.39: *Obtain the time response y(t) of the system given below by first transforming the state model into a 'Canonical model'.* **(VTU: Jan. /Feb.-2006)**

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u, \; y = [1 \; 0 \; 0]X$$

u is a unit step function and $X^T(0) = [0 \; 0 \; 2]$

Solution : First to transform given model into canonical model, find modal matrix M.

$$[\lambda I - A] = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{bmatrix}$$

$$[\lambda I - A] = \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

i.e. $$(\lambda+1)(\lambda+2)(\lambda+3) = 0$$

$$\therefore \quad \lambda_1 = -1, \; \lambda_2 = -2, \; \lambda_3 = -3$$

For $\lambda_1 = -1$, $[\lambda I - A] = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 6 & 11 & 5 \end{bmatrix}$, $M_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

For $\lambda_2 = -2$, $[\lambda I - A] = \begin{bmatrix} -2 & -1 & 0 \\ 0 & -2 & -1 \\ 6 & 11 & 4 \end{bmatrix}$, $M_2 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

For $\lambda_3 = -3$, $[\lambda I - A] = \begin{bmatrix} -3 & -1 & 0 \\ 0 & -3 & -1 \\ 6 & 11 & 3 \end{bmatrix}$, $M_3 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 18 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}$

$$\therefore \quad M = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} = \text{Vander monde matrix}$$

$$\text{Adj M} = \begin{bmatrix} -6 & 6 & -2 \\ -5 & 8 & -3 \\ -1 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} -6 & -5 & -1 \\ 6 & 8 & 2 \\ -2 & -3 & -1 \end{bmatrix}$$

$$\therefore \quad M^{-1} = \frac{\text{Adj} M}{|M|} = \frac{\text{Adj M}}{-2} = \begin{bmatrix} 3 & 2\cdot5 & 0\cdot5 \\ -3 & -4 & -1 \\ 1 & 1\cdot5 & 0\cdot5 \end{bmatrix}$$

$$\therefore \quad M^{-1}AM = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \Lambda$$

$$\tilde{B} = M^{-1}B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \qquad \tilde{C} = CM = [1 \quad 1 \quad 1]$$

The canonical state model is,

$$\dot{Z}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} Z(t) + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} u$$

$$Y(t) = [1 \quad 1 \quad 1]\, Z(t)$$

$X(t) = MZ(t)$ i.e. $Z(t) = M^{-1}X(t)$

$$\therefore \quad Z(0) = M^{-1}X(0) = \begin{bmatrix} 3 & 2\cdot5 & 0\cdot5 \\ -3 & -4 & -1 \\ 1 & 1\cdot5 & 0\cdot5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix}$$

$$\therefore \quad \text{ZIR} = e^{\Lambda t} Z(0) = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ -2e^{-2t} \\ e^{-3t} \end{bmatrix}$$

$$\text{ZSR} = L^{-1}\left\{\phi(s)\tilde{B}U(s)\right\}$$

$$= L^{-1}\left\{\begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s+2} & 0 \\ 0 & 0 & \frac{1}{s+3} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} [1/s]\right\}$$

$$= L^{-1}\begin{bmatrix} \frac{1}{s(s+1)} \\ \frac{-2}{s(s+2)} \\ \frac{1}{s(s+3)} \end{bmatrix} = L^{-1}\begin{bmatrix} \frac{1}{s} - \frac{1}{s+1} \\ -\frac{1}{s}+\frac{1}{s+2} \\ \frac{1/3}{s}-\frac{1/3}{s+3} \end{bmatrix}$$

$$\therefore \quad ZSR = \begin{bmatrix} 1-e^{-t} \\ -1+e^{-2t} \\ \frac{1}{3}-\frac{1}{3}e^{-3t} \end{bmatrix}$$

$$\therefore \quad Z(t) = ZIR + ZSR = \begin{bmatrix} 1 \\ -1-e^{-2t} \\ \frac{1}{3}+\frac{2}{3}e^{-3t} \end{bmatrix}$$

$$\therefore \quad Y(t) = [1 \quad 1 \quad 1]\, Z(t) = 1-1-e^{-2t}+\frac{1}{3}+\frac{2}{3}e^{-3t}$$

$$= \frac{1}{3}+\frac{2}{3}e^{-3t}-e^{-2t}$$

Example 4.40 : *Given the system* $\dot{X} = \begin{bmatrix} -3 & 0 \\ 2 & -1 \end{bmatrix} X + \begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix} U$. *Find the input vector U(t) to give the following time response :* **(VTU : July/Aug.- 2006)**

$$X_1(t) = 6\left(1-e^{-t}\right)$$

$$X_2(t) = 3e^{-3t}-2e^{-4t}+6\left(1-e^{-t}\right)$$

Solution : The given time response must satisfy the state equation.

$$\dot{X}_1(t) = \frac{d}{dt}\left[6\left(1-e^{-t}\right)\right] = +6e^{-t}$$

$$\dot{X}_2(t) = \frac{d}{dt}\left[3e^{-3t}-2e^{-4t}+6\left(1-e^{-t}\right)\right] = -9e^{-3t}+8e^{-4t}+6e^{-t}$$

$$\begin{bmatrix} 6e^{-t} \\ -9e^{-3t}+8e^{-4t}+6e^{-t} \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 6\left(1-e^{-t}\right) \\ 3e^{-3t}-2e^{-4t}+6(1-e^{-t}) \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix} U$$

$$\begin{bmatrix} 6e^{-t} \\ -9e^{-3t}+8e^{-4t}+6e^{-t} \end{bmatrix} = \begin{bmatrix} -18+18e^{-t} \\ 6-6e^{-t}-3e^{-3t}+2e^{-4t} \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix} U$$

$$\begin{bmatrix} 18-12e^{-t} \\ -6+12e^{-t}-6e^{-3t}+6e^{-4t} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix} U$$

$$\begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 18-12e^{-t} \\ -6+12e^{-t}-6e^{-3t}+6e^{-4t} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix} U$$

$$\begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix} = I$$

$$\begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix}^{-1} = \frac{\text{Adj}\begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix}}{\begin{vmatrix} 3 & 0 \\ 3 & 2 \end{vmatrix}} = \frac{\begin{bmatrix} 2 & 0 \\ -3 & 3 \end{bmatrix}}{6} = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\therefore \qquad U = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 18-12e^{-t} \\ -6+12e^{-t}-6e^{-3t}+6e^{-4t} \end{bmatrix}$$

$$U(t) = \begin{bmatrix} 6-4e^{-t} \\ -12+12e^{-t}-3e^{-3t}+3e^{-4t} \end{bmatrix} \quad \text{... Required input vector}$$

Example 4.41 : *The following is the state space representation of a linear system whose eigen values are –3, –2, –1.* **(VTU : July/Aug.-2006)**

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} [X] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u$$

Given that $u = 0$, $X(0) = [0\ 0\ 1]^T$. Find X(t).

Solution :

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

$$\therefore \quad [sI-A] = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s+6 \end{bmatrix}, \text{ Adj } [sI-A] = \begin{bmatrix} s^2+6s+11 & -6 & -6s \\ s+6 & s(s+6) & -(11s+6) \\ 1 & s & s^2 \end{bmatrix}^T$$

$$\text{Adj } [sI-A] = \begin{bmatrix} s^2+6s+11 & s+6 & 1 \\ -6 & s(s+6) & s \\ -6s & -(11s+6) & s^2 \end{bmatrix}$$

$$[sI-A] = s^3+6s^2+11s+6 = (s+1)(s+2)(s+3)$$

$$\therefore [sI-A]^{-1} = \frac{\text{Adj}[sI-A]}{|sI-A|} = \begin{bmatrix} \frac{s^2+6s+11}{(s+1)(s+2)(s+3)} & \frac{s+6}{(s+1)(s+2)(s+3)} & \frac{1}{(s+1)(s+2)(s+3)} \\ \frac{-6}{(s+1)(s+2)(s+3)} & \frac{s(s+6)}{(s+1)(s+2)(s+3)} & \frac{s}{(s+1)(s+2)(s+3)} \\ \frac{-6s}{(s+1)(s+2)(s+3)} & \frac{-(11s+6)}{(s+1)(s+2)(s+3)} & \frac{s^2}{(s+1)(s+2)(s+3)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{s+1}-\frac{3}{s+2}+\frac{1}{s+3} & \frac{2.5}{s+1}-\frac{4}{s+2}+\frac{1.5}{s+3} & \frac{0.5}{s+1}-\frac{1}{s+2}+\frac{0.5}{s+3} \\ \frac{-3}{s+1}+\frac{6}{s+2}-\frac{3}{s+3} & \frac{-2.5}{s+1}+\frac{8}{s+2}-\frac{4.5}{s+3} & \frac{-0.5}{s+1}+\frac{2}{s+2}-\frac{1.5}{s+3} \\ \frac{3}{s+1}-\frac{12}{s+2}+\frac{9}{s+3} & \frac{2.5}{s+1}-\frac{16}{s+2}+\frac{13.5}{s+3} & \frac{0.5}{s+1}-\frac{4}{s+2}+\frac{4.5}{s+3} \end{bmatrix}$$

$$\therefore \quad e^{At} = L^{-1}[sI-A]^{-1}$$

$$= \begin{bmatrix} 3e^{-t}-3e^{-2t}+e^{-3t} & 2.5e^{-t}-4e^{-2t}+1.5e^{-3t} & 0.5e^{-t}-e^{-2t}+0.5e^{-3t} \\ -3e^{-t}+6e^{-2t}-3e^{-3t} & -2.5e^{-t}+8e^{-2t}-4.5e^{-3t} & -0.5e^{-t}+2e^{-2t}-1.5e^{-3t} \\ 3e^{-t}-12e^{-2t}+9e^{-3t} & 2.5e^{-t}-16e^{-2t}+13.5e^{-3t} & 0.5e^{-t}-4e^{-2t}+4.5e^{-3t} \end{bmatrix}$$

$$\therefore \quad X(t) = e^{At}X(0) = e^{At}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5e^{-t}-e^{-2t}+0.5e^{-3t} \\ -0.5e^{-t}+2e^{-2t}-1.5e^{-3t} \\ 0.5e^{-t}-4e^{-2t}+4.5e^{-3t} \end{bmatrix}$$

Example 4.42 : *Find the transition matrix $\phi(t)$ for a system whose system matrix is given by* $A = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix}$ *by the following techniques :*

i)Laplace transform ii) Infinite series iii) Cayley-Hamilton **(VTU: July/Aug.-2006)**

Solution : $A = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix}$

i) Laplace transform method

$$[sI-A] = \begin{bmatrix} s+5 & 1 \\ -3 & s+1 \end{bmatrix} \quad \text{Adj}[sI-A] = \begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix}$$

$$\therefore \quad [sI-A] = (s+5)(s+1)+3 = s^2+6s+8 = (s+2)(s+4)$$

$$\therefore \quad [sI-A]^{-1} = \frac{\text{Adj}[sI-A]}{|sI-A|} = \begin{bmatrix} \dfrac{s+1}{(s+2)(s+4)} & \dfrac{-1}{(s+2)(s+4)} \\ \dfrac{3}{(s+2)(s+4)} & \dfrac{(s+5)}{(s+2)(s+4)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{-0.5}{s+2}+\dfrac{1.5}{s+4} & \dfrac{-0.5}{s+2}+\dfrac{0.5}{s+4} \\ \dfrac{1.5}{s+2}-\dfrac{1.5}{s+4} & \dfrac{1.5}{s+2}-\dfrac{0.5}{s+4} \end{bmatrix}$$

$$\therefore \quad e^{At} = L^{-1}[sI-A]^{-1} = \begin{bmatrix} -0.5e^{-2t}+1.5e^{-4t} & -0.5e^{-2t}+0.5e^{-4t} \\ 1.5e^{-2t}-1.5e^{-4t} & 1.5e^{-2t}-0.5e^{-4t} \end{bmatrix}$$

ii) Power series method

$$A^2 = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 22 & 6 \\ -18 & -2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 22 & 6 \\ -18 & -2 \end{bmatrix}\begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -92 & -28 \\ 84 & 20 \end{bmatrix}$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \ldots$$

$$= \begin{bmatrix} 1-5t+11t^2-\dfrac{92}{6}t^3+\ldots & -t+3t^2-\dfrac{28}{6}t^3+\ldots \\ 3t-9t^2+14t^3-\ldots & 1-t-t^2+\dfrac{20}{6}t^3\ldots \end{bmatrix}$$

iii) Cayley - Hamilton method

Step 1 : Find eigen values.

$|\lambda I - A| = 0$ i.e. $\begin{vmatrix} \lambda+5 & 1 \\ -3 & \lambda+1 \end{vmatrix} = 0$

i.e. $\lambda^2 + 6\lambda + 8 = 0$ i.e. $\lambda_1 = -2,\ \lambda_2 = -4$

Step 2: Construct $R(\lambda)$

$R(\lambda) = \alpha_0 + \alpha_1\lambda = p(\lambda)$ Where $p(\lambda) = e^{\lambda t}$

$\therefore \quad \alpha_0 + \alpha_1\lambda = e^{\lambda t}$

Step 3 : Substituting values of λ_1 and λ_2

$\alpha_0 - 2\alpha_1 = e^{-2t}$ and $\alpha_0 - 4\alpha_1 = e^{-4t}$

Step 4 : Solving above equations,

$\alpha_1 = \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t}$, $\alpha_0 = 2e^{-2t} - e^{-4t}$

Step 5 : $f(A) = R(A) = \alpha_0 I + \alpha_1 A$

$$\therefore e^{At} = 2e^{-2t} - e^{-4t}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(\frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t}\right)\begin{bmatrix} -5 & -1 \\ +3 & -1 \end{bmatrix}$$

$$\therefore \quad e^{At} = \begin{bmatrix} -0.5e^{-2t} + 1.5e^{-4t} & -0.5e^{-2t} + 0.5e^{-4t} \\ 1.5e^{-2t} - 1.5e^{-4t} & 1.5e^{-2t} - 0.5e^{-4t} \end{bmatrix}$$

Example 4.43 : *Obtain the state transition matrix using :* **(Jan./Feb.-2007)**

i) Laplace transformation method and

ii) Cayley-Hamilton method.

For the system describe by,

$$X(t) = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} X(0)$$

Solution :

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} X(0)$$

i) Laplace tansform method

$$[sI - A] = \begin{bmatrix} s & -1 \\ 4 & s+4 \end{bmatrix},\ \text{Adj}\,[sI - A] = \begin{bmatrix} s+4 & 1 \\ -4 & s \end{bmatrix}$$

$$[sI - A] = s^2 + 4s + 4 = (s+2)^2$$

$$\therefore \quad [sI-A]^{-1} = \frac{Adj[sI-A]}{|sI-A|} = \begin{bmatrix} \frac{s+4}{(s+2)^2} & \frac{1}{(s+2)^2} \\ \frac{-4}{(s+2)^2} & \frac{s}{(s+2)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+2}+\frac{2}{(s+2)^2} & \frac{1}{(s+2)^2} \\ \frac{-4}{(s+2)^2} & \frac{1}{s+2}-\frac{2}{(s+2)^2} \end{bmatrix} \quad \text{... Partial fractions}$$

$$\therefore \quad e^{At} = L^{-1}[sI-A]^{-1} = \begin{bmatrix} e^{-2t}+2te^{-2t} & te^{-2t} \\ -4te^{-2t} & e^{-2t}-2te^{-2t} \end{bmatrix}$$

ii) Cayley - Hamilton method

Step 1 : Find eigen values

$$|\lambda I-A| = 0 \text{ i.e.} \begin{vmatrix} \lambda & -1 \\ 4 & \lambda+4 \end{vmatrix} = 0 \qquad \text{i.e. } \lambda^2+4\lambda+4=0$$

$$\therefore \quad (\lambda+2)^2 = 0 \text{ i.e. } \lambda_1 = -2, \ \lambda_2 = -2$$

Step 2 : Construct $R(\lambda)$

$$R(\lambda) = \alpha_0+\alpha_1\lambda = p(\lambda) \text{ where } p(\lambda) = e^{\lambda t}$$

$$\therefore \quad \alpha_0+\alpha_1\lambda = e^{\lambda t}$$

Step 3 : Substitute λ_1

$$\alpha_0-2\alpha_1 = e^{-2t}$$

For λ_2, as it is repeated use,

$$\frac{d}{d\lambda}p(\lambda)\Big|_{\lambda=\lambda_2} = \frac{d}{d\lambda}R(\lambda)\Big|_{\lambda=\lambda_2}$$

$$\therefore \quad \frac{d}{d\lambda}e^{\lambda t}\Big|_{\lambda=\lambda_2} = \frac{d}{d\lambda}[\alpha_0+\alpha_1\lambda]\Big|_{\lambda=\lambda_2}$$

$$\therefore \quad te^{\lambda t}\Big|_{\lambda=\lambda_2} = \alpha_1$$

$$\therefore \quad \alpha_1 = te^{-2t} \quad \text{and} \quad \alpha_0 = 2te^{-2t}+e^{-2t}$$

Step 5 : $f(A) = R(A) = \alpha_0 I + \alpha_1 A$

$$\therefore e^{At} = 2te^{-2t} + e^{-2t}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + te^{-2t}\begin{bmatrix}0 & 1\\-4 & -4\end{bmatrix}$$

$$= \begin{bmatrix}2te^{-2t}+e^{-2t} & te^{-2t}\\-4te^{-2t} & e^{-2t}-2te^{-2t}\end{bmatrix}$$

Example 4.44 : *Compute e^{At} for the given matrix :* **(VTU : July/Aug.-2007)**

$$A_1 = \begin{bmatrix}6 & 0\\0 & 6\end{bmatrix};\ A_2 = \begin{bmatrix}0 & \omega\\-\omega & 0\end{bmatrix};\ A = \begin{bmatrix}6 & \omega\\-\omega & 6\end{bmatrix}$$

Solution : If $A = A_1 + A_2$,

then $$e^{At} = e^{(A_1+A_2)t} = e^{A_1 t} \times e^{A_2 t} \qquad ...(1)$$

This is the property of state transition matrix if $A_1A_2 = A_2A_1$. So verify this condition.

$$A_1A_2 = \begin{bmatrix}6 & 0\\0 & 6\end{bmatrix}\begin{bmatrix}0 & \omega\\-\omega & 0\end{bmatrix} = \begin{bmatrix}0 & 6\omega\\-6\omega & 0\end{bmatrix}$$

$$A_2A_1 = \begin{bmatrix}0 & \omega\\-\omega & 0\end{bmatrix}\begin{bmatrix}6 & 0\\0 & 6\end{bmatrix} = \begin{bmatrix}0 & 6\omega\\-6\omega & 0\end{bmatrix}$$

As $A_1A_2 = A_2A_1$, the above property holds good. So calculate $e^{A_1 t}$ and $e^{A_2 t}$ and then multiply to obtain e^{At}.

$$e^{A_1 t} = L^{-1}[sI - A_1]^{-1}$$

$$[sI - A_1] = \begin{bmatrix}s-6 & 0\\0 & s-6\end{bmatrix},\ \text{Adj}\,[SI - A_2] = \begin{bmatrix}s-6 & 0\\0 & s-6\end{bmatrix}$$

$$|sI - A_1| = (s-6)^2$$

$$[sI - A_1]^{-1} = \frac{\text{Adj}[sI - A_1]}{|sI - A_1|} = \begin{bmatrix}\frac{1}{s-6} & 0\\0 & \frac{1}{s-6}\end{bmatrix}$$

$$\therefore \quad e^{A_1 t} = L^{-1}[sI - A_1]^{-1} = \begin{bmatrix}e^{6t} & 0\\0 & e^{6t}\end{bmatrix}$$

$$[sI - A_2] = \begin{bmatrix}s & -\omega\\\omega & s\end{bmatrix},\ \text{Adj}[sI - A_2] = \begin{bmatrix}s & \omega\\-\omega & s\end{bmatrix}$$

$$|sI - A_2| = s^2 + \omega^2$$

$$[sI - A_2]^{-1} = \frac{\text{Adj}[sI - A_2]}{|sI - A_2|} = \begin{bmatrix} \dfrac{s}{s^2+\omega^2} & \dfrac{\omega}{s^2+\omega^2} \\ \dfrac{-\omega}{s^2+\omega^2} & \dfrac{s}{s^2+\omega^2} \end{bmatrix}$$

$$\therefore \quad e^{A_2 t} = L^{-1}[sI - A_2]^{-1} = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}$$

$$\therefore \quad e^{At} = \begin{bmatrix} e^{6t} & 0 \\ 0 & e^{6t} \end{bmatrix} \times \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} = \begin{bmatrix} e^{6t}\cos\omega t & e^{6t}\sin\omega t \\ -e^{6t}\sin\omega t & e^{6t}\cos\omega t \end{bmatrix}$$

Example 4.45 : *Obtain the time response of the following vector matrix differential equation.* $\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ *and* $y = [1 \quad 0]\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, *where u(t) an unit step input and the initial conditions are* $X_1(0) = X_2(0) = 0$. **(VTU: July/Aug.-2007)**

Solution :

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \quad 0]$$

$$[sI - A] = \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix}, \text{Adj}\ [sI - A] = \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}$$

$$[sI - A] = s^2 + 5s + 6 = (s+2)(s+3)$$

$$[sI - A]^{-1} = \frac{\text{Adj}[sI - A]}{|sI - A|} = \begin{bmatrix} \dfrac{s+5}{(s+2)(s+3)} & \dfrac{1}{(s+2)(s+3)} \\ \dfrac{-6}{(s+2)(s+3)} & \dfrac{s}{(s+2)(s+3)} \end{bmatrix} = \phi(s)$$

As $X(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, e^{At} is not required.

Thus $\quad \text{ZIR} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\text{ZSR} = L^{-1}\{\phi(s)BU(s)\} \quad \ldots U(s) = \frac{1}{s} \text{ for unit step}$$

$$= L^{-1}\left\{\begin{bmatrix} \dfrac{s+5}{(s+2)(s+3)} & \dfrac{1}{(s+2)(s+3)} \\ \dfrac{-6}{(s+2)(s+3)} & \dfrac{s}{(s+2)(s+3)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{s} \end{bmatrix}\right\}$$

$$= L^{-1}\begin{bmatrix} \dfrac{1}{s(s+2)(s+3)} \\ \dfrac{1}{(s+2)(s+3)} \end{bmatrix} = L^{-1}\begin{bmatrix} \dfrac{1/6}{s} - \dfrac{1/2}{s+2} + \dfrac{1/3}{s+3} \\ \dfrac{1}{s+2} - \dfrac{1}{s+3} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{6} - \dfrac{1}{2}e^{-2t} + \dfrac{1}{3}e^{-3t} \\ e^{-2t} - e^{-3t} \end{bmatrix}$$

$$\therefore \quad X(t) = ZIR + ZSR = \begin{bmatrix} \dfrac{1}{6} - \dfrac{1}{2}e^{-2t} + \dfrac{1}{3}e^{-3t} \\ e^{-2t} - e^{-3t} \end{bmatrix}$$

$$Y(t) = [1 \quad 0]\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = X_1(t)$$

$$\therefore \quad Y(t) = \frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t}$$

Example 4.46 : *Obtain the observable phase variable state model of T.F.*

$$T(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}.$$ *Draw the signal flow graph of T(s).*

(VTU : July/Aug.-2007)

Solution : The given transfer function is

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$\therefore \left[s^3 + a_1 s^2 + a_2 s + a_3\right] Y(s) = b_0 s^3 + b_1 s^2 + b_2 s + b_3 [U(s)]$$

$$\therefore s^3[Y(s) - b_0 U(s)] + s^2[a_1 Y(s) - b_1 U(s)] + s[a_2 Y(s) - b_2 U(s)] + [a_3 Y(s) - b_3 U(s)] = 0$$

Divide the equation by s^3,

$$Y(s) - b_0 U(s) + \frac{1}{s}[a_1 Y(s) - b_1 U(s)] + \frac{1}{s^2}[a_2 Y(s) - b_2 U(s)] + \frac{1}{s^3}[a_3 Y(s) - b_3 U(s)] = 0$$

$$Y(s) = b_0 U(s) + \frac{1}{s}[-a_1 Y(s) + b_1 U(s)] + \frac{1}{s^2}[-a_2 Y(s) + b_2 U(s)] + \frac{1}{s^3}[-a_3 Y(s) + b_3 U(s)] \quad ...(1)$$

Now define the state variables as,

$$X_3(s) = \frac{1}{s}[b_1 U(s) - a_1 Y(s) + X_2(s)] \quad ...(2)$$

$$X_2(s) = \frac{1}{s}[b_2 U(s) - a_2 Y(s) + X_1(s)] \quad ...(3)$$

Example 4.47: *Obtain the controllable phase variable form of the transfer function,*

$$T(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

(VTU: July/Aug.-2007)

Solution: The given transfer function can be written as,

$$\frac{Y(s)}{U(s)} = b_0 + \frac{(b_1 - a_1 b_0)s^2 + (b_2 - a_2 b_0)s + (b_3 - a_3 b_0)}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$\therefore \quad Y(s) = b_0 U(s) + \hat{Y}(s)$$

Where
$$\hat{Y}(s) = \frac{(b_1 - a_1 b_0)s^2 + (b_2 - a_2 b_0)s + (b_3 - a_3 b_0)}{s^3 + a_1 s^2 + a_2 s + a_3} U(s)$$

$$\therefore \quad \frac{\hat{Y}(s)}{(b_1 - a_1 b_0)s^2 + (b_2 - a_2 b_0)s + (b_3 - a_3 b_0)} = \frac{U(s)}{s^3 + a_1 s^2 + a_2 s + a_3} = Q(s)$$

$$\therefore \quad U(s) = s^3 Q(s) + a_1 s^2 Q(s) + a_2 s Q(s) + a_3 Q(s)$$

$$s^3 Q(s) = -a_1 s^2 Q(s) - a_2 s Q(s) - a_3 Q(s) + U(s) \quad \ldots(1)$$

$$\hat{Y}(s) = (b_1 - a_1 b_0)s^2 Q(s) + (b_2 - a_2 b_0)s Q(s) + (b_3 - a_3 b_0)Q(s) \quad \ldots(2)$$

Define the state variables as,

$$X_1(s) = Q(s), \qquad X_2(s) = sQ(s), \qquad X_3(s) = s^2 Q(s)$$

$$\therefore \quad sX_1(s) = X_2(s)$$

$$sX_2(s) = X_3(s)$$

Taking inverse Laplace transform,

$$\dot{X}_1 = X_2, \quad \dot{X}_2 = X_3,$$

Substituting selected state variables in the equation (1),

$$sX_3(s) = -a_1 X_3(s) - a_2 X_2(s) - a_3 X_1(s) + U(s)$$

i.e.
$$\dot{X}_3 = -a_3 X_1 - a_2 X_2 - a_1 X_3 + U$$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U$$

From (2), $\hat{Y}(s) = (b_1 - a_1 b_0)X_3(s) + (b_2 - a_2 b_0)X_2(s) + (b_3 - a_3 b_0)X_1(s)$

$\therefore \quad Y(s) = b_0 U(s) + (b_1 - a_1 b_0)X_3(s) + (b_2 - a_2 b_0)X_2(s) + (b_3 - a_3 b_0)X_1(s)$

Taking inverse Laplace transform,

$$Y = [b_3 - a_3 b_0]X_1 + [b_2 - a_2 b_0]X_2 + [b_1 - a_1 b_0]X_3 + b_0 U$$

$$\therefore \quad C = [(b_3 - a_3 b_0)\ (b_2 - a_2 b_0)\ (b_1 - a_1 b_0)],\ D = [b_0]$$

This is controllable canonical form of the given transfer function.

Review Questions

1. *What is homogeneous and nonhomogeneous state equation ?*
2. *Define state transition matrix using classical method of obtaining solution.*
3. *What is zero input response and zero state response ?*
4. *Obtain the complete solution of nonhomogeneous state equation using time domain method.*
5. *State the importance of state transition matrix.*
6. *State the various properties of state transition matrix.*
7. *Obtain the solution of nonhomogeneous state equation using Laplace transform method.*
8. *What is resolvant matrix ?*
9. *Explain Laplace transform method of obtaining e^{At}.*
10. *Explain power series method of obtaining e^{At}.*
11. *Explain Cayley Hamilton method of obtaining e^{At}.*
12. *Explain similarity transformation method of obtaining e^{At}.*
13. *Obtain e^{At} for following matrices using all the methods,*

a. $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Ans. : $\begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Ans. : $\begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$

Ans. : $\begin{bmatrix} \cos\sqrt{2}t & \frac{1}{\sqrt{2}}\sin\sqrt{2}t \\ -\sqrt{2}\sin\sqrt{2}t & \cos\sqrt{2}t \end{bmatrix}$

14. *Obtain the homogeneous solution of the equation* $\dot{X}(t) = A\,X(t)$ *where*

a. $A = \begin{bmatrix} -9 & 1 \\ -14 & 0 \end{bmatrix}$ *and* $X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Ans. : $\begin{bmatrix} -\frac{1}{5}e^{-2t} + \frac{6}{5}e^{-7t} \\ -\frac{7}{5}e^{-2t} + \frac{12}{5}e^{-7t} \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & -1 \\ 2 & -4 \end{bmatrix}$ *and* $X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Ans. : $\begin{bmatrix} 1.11\,e^{0.56t} - 0.11\,e^{-3.56t} \\ 0.48\,e^{0.56t} - 0.48\,e^{-3.56t} \end{bmatrix}$

15. *Find the output of the system having state model,*

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} X(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} U(t)$$

and $Y(t) = [1 \quad -1]\; X(t)$

The input $U(t)$ *is unit step and* $X(0) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$

(Ans.: $Y(t) = \frac{5}{3} + \frac{460}{3}e^{-3t} - 145e^{-4t}$**)**

16. *Show that the following system is completely state controllable and observable.*

$$\dot{X}(t) = \begin{bmatrix} -1 & -2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} X(t) + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} U(t)$$

and $Y(t) = [1 \quad 1 \quad 0]\; X(t)$

17. *Find the solution is,*

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} X(t) \text{ with } X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(Ans. : $X(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$**)**

18. *Find the solution of,*

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} X(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U(t)$$

and $Y(t) = [1 \quad 0]\; X(t)$

if $X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and system is subjected to the unit step-input.*

(Ans. : $Y(t) = -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t}$**)**

19. *Define and explain state controllability.*

20. *Explain Kalman's test for determining state controllability.*

21. *Explain Gilbert's test for determining state controllability.*

22. *What is output controllability ? How to evaluate it for the system ?*

23. *State the relation between transfer function in s domain and controllability, obesrvability of a system.*

24. *Define and explain concept of observability.*

25. *Explain Kalman's test for determining complete observability.*

26. *Explain Gilbert's test for determining complete observability.*

27. *Evaluate the controllability of the following systems with,*

a. $A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

b. $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

c. $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 4 & 2 \\ 0 & 0 \\ 3 & 0 \end{bmatrix}$ **(Ans. : a. Controllable, b. Controllable c. Not controllable)**

28. *Evaluate the observability of the following systems with,*

a. $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, c = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}$

b. $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, c = [0 \quad 1]$

c. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, c = [4 \ \ 5 \ \ 1]$

(Ans. : a. Observable b. and c. Not observable)

❑❑❑

5 Pole Placement Technique

5.1 Introduction

This chapter deals with the problem of pole placement. The pole placement technique or pole assignment technique is a method for design of control system which is discussed in this chapter.

If it is assumed that all the state variables for a given system are measurable and available for the feedback then it can be proved that for a completely state controllable such system, its closed loop system poles can be placed at any desired location through state feedback by means of an appropriate state feedback gain matrix.

The desired closed loop poles are firstly evaluated based on transient response and/or frequency response requirements such as say damping ratio, speed, bandwidth as well as the steady state requirements. Let us consider that the desired closed loop poles are to be located at $s = \alpha_1$, $s = \alpha_2$ ....$s = \alpha_n$. Then by properly selecting gain matrix for state feedback, it is possible to locate the closed loop poles at the desired locations provided that the original system is completely state controllable.

In the following sections, we will consider the control signal to be scalar. It can then be proved that the necessary and sufficient condition that the closed loop poles can be placed at any arbitrary locations in the s-plane is that the system is completely state controllable i.e. it is possible to transfer the system state from any initial state $X(t_0)$ to any other desired state $X(t_f)$ in a specified finite time interval (t_f) by a control vector U(t). The required state feedback gain matrix can then be evaluated using various methods.

In this chapter, it is considered that the control signal is a scalar quantity and not a vector quantity. Under such case, the mathematical aspects of the pole placement technique is complicated as the state feedback gain matrix is not unique.

5.2 Pole Placement Design

The conventional method of design of single input single output control system consists of design of a suitable controller or compensator in such a way that the dominant closed loop poles will have a desired damping ratio ξ and undamped natural frequency ω_n. The order of the system in this case is increased by 1 or 2 if there are no pole zero cancellation taking place. It is assumed in this method that the effects on the responses of non dominant closed loop poles to be negligible.

Instead of specifying only the dominant closed loop poles in the conventional method of design, the pole placement technique describes all the closed loop poles which requires measurements of all state variables or inclusion of a state observer in the system. The system closed loop poles can be placed at arbitrarily chosen locations with the condition that the system is completely state controllable. This condition can be proved and the proof is given below.

Consider a control system described by following state equation

$$\dot{x} = Ax + Bu \qquad \text{... (1)}$$

Here x is a state vector, u is a control signal which is scalar, A is $n \times n$ state matrix. B is $n \times 1$ constant matrix.

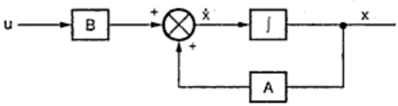


Fig. 5.1 Open loop control system

The system defined by above equation represents open loop system. The state x is not fed back to the control signal u.

Let us select the control signal to be u = – Kx state. This indicates that the control signal is obtained from instantaneous state. This is called state feedback. The k is a matrix of order $1 \times n$ called state feedback gain matrix. Let us consider the control signal to be unconstrained. Substituting value of u in equation 1.

$$\dot{x} = Ax + B(-Kx) = (A - BK)x \qquad \text{... (2)}$$

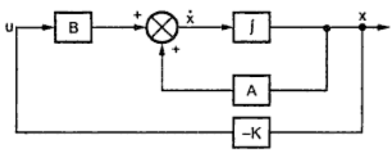


Fig. 5.2

The system defined by above equation is shown in the Fig. 5.2. It is a closed loop control system as the system state x is fed back to the control system as the system state x is fed back to control signal u. Thus this a system with state feedback.

The solution of equation 2 is say

$$x(t) = e, \qquad x(0) \text{ is the initial state} \qquad \dots (3)$$

The stability and the transient response characteristics are determined by the eigen values of matrix A – BK.

Depending on the selection of state feedback gain matrix K, the matrix A – BK can be made asymptotically stable and it is possible to make x(t) approaching to zero as time t approaches to infinity provided $x(0) \neq 0$.

The eigen values of matrix A – BK are called regulator poles. These regulator poles when placed in left half of s plane then x(t) approaches zero as time t approaches infinity. The problem of placing the closed loop poles at the desired location is called a pole placement problem.

5.3 Necessary and Sufficient Condition for Arbitrary Pole Placement

Consider a control system defined by following state equation.

$$\dot{x} = Ax + Bu \qquad \dots (1)$$

Let the control signal selected be u = – Kx

$$\therefore \quad \dot{x} = Ax - BKx = (A - BK)\, x \qquad \dots (2)$$

The solution of the above equation is given as,

$$x(t) = e^{(A-BK)t} \cdot x(0) \qquad \dots (3)$$

The eigen values of the matrix (A – BK) are nothing but desired closed loop poles.

The necessary and sufficient condition for arbitrary pole placement is that the system is completely state controllable.

If suppose the system is not completely state controllable then there are eigen values of matrix A – BK that can not be controlled by state feedback.

According to Kalman's test for state controllability, the system is said to be completely state controllable if the rank of the composite matrix Q_c is n.

$$Q_c = \left[B : AB : A^2B : \dots\dots\dots\dots A^{n-1}B\right]$$

Let us suppose that the system is not completely state controllable. The rank of the controllability matrix is less than n.

$$\text{rank}\left[B : AB : A^2B : \dots\dots\dots\dots : A^{n-1}B\right] = s < n$$

Thus there are s linearly independent column vectors in the controllability matrix. Let these s linearly independent column vectors as c_1, c_2, ... c_s. Also let us select n – s additional vectors t_{s+1}, t_{s+2},....... t_n such that

$$P = [c_1 : c_2 : \dots\dots : c_s : t_{s+1} : t_{s+2} : \dots\dots : t_n]$$

The above matrix will have rank n

By using matrix P as the transfomation matrix, let us define $P^{-1}AP = \hat{A}$, $P^{-1}B = \hat{B}$

Now we have $AP = P\hat{A}$

$$A[c_1 : c_2 : \dots\dots : c_s : t_{s+1} : t_{s+2} : \dots\dots : t_n] = [c_1 : c_2 : \dots\dots : c_s : t_{s+1} : t_{s+2} : \dots\dots : t_n]\,\hat{A}$$

$$[Ac_1 : Ac_2 : \dots\dots : Ac_s : At_{s+1} : At_{s+2} : \dots\dots : At_n] = [c_1 : c_2 : \dots\dots : c_s : t_{s+1} : t_{s+2} : \dots\dots : t_n]\;\hat{A}$$

As there are s linearly independent column vectors $c_1, c_2, \dots\dots c_s$ using Cayley-Hamilton theorem vectors $Ac_1, Ac_2 \dots\dots Ac_s$ can be expressed in terms of these s vectors.

$$Ac_1 = a_{11}c_1 + a_{21}c_2 + \dots\dots + a_{s1}c_s$$

$$Ac_2 = a_{12}c_1 + a_{22}c_2 + \dots\dots + a_{s2}c_s$$

$$\vdots$$

$$Ac_s = a_{1s}c_1 + a_{2s}c_2 + \dots\dots + a_{ss}c_s$$

Thus we can write

$$[Ac_1 : Ac_2 : \dots : Ac_s : At_{s+1} : At_{s+2} : \dots\dots : At_n] = [c_1 : c_2 : \dots : c_s : t_{s+1} : t_{s+2} : \dots t_n]\left[\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1s} & a_{1s+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2s} & a_{2s+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{s1} & \cdots & a_{ss} & a_{ss+1} & \cdots & a_{sn} \\ \hline 0 & \cdots & 0 & a_{s+1s+1} & \cdots & a_{s+1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{ns+1} & \cdots & a_{nn} \end{array}\right]$$

Let us define

$$A_{11} = \begin{bmatrix} a_{11} & \cdots & a_{1s} \\ a_{21} & \cdots & a_{2s} \\ \vdots & & \vdots \\ a_{s1} & \cdots & a_{ss} \end{bmatrix} ; A_{12} = \begin{bmatrix} a_{1s+1} & \cdots & a_{1n} \\ a_{2s+1} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{ss+1} & \cdots & a_{sn} \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \begin{matrix} (n-s)\times s \\ \text{zero matrix} \end{matrix} ; A_{22} = \begin{bmatrix} a_{s+1s+1} & \cdots & a_{s+1n} \\ \vdots & & \vdots \\ a_{ns+1} & \cdots & a_{nn} \end{bmatrix}$$

We have,

$$\left[Ac_1 : Ac_2 \cdots : Ac_s : At_{s+1} : \cdots : At_n\right] = \left[c_1 : c_2 : \cdots : c_s : t_{s+1} : t_{s+2} : \cdots : t_n\right] \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right]$$

$$\therefore A\left[c_1 : c_2 : \cdots : c_s : t_{s+1} : t_{s+2} : \cdots : t_n\right] = \left[c_1 : c_2 : \cdots : c_s : t_{s+1} : t_{s+2} : \cdots : t_n\right] \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right]$$

$$\therefore \qquad AP = P \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right]$$

Hence we have, $$P^{-1} AP = \hat{A} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right]$$

Now we have, $B = P\hat{B}$

$$\therefore \quad B = [c_1 : c_2 : \ldots\ldots : c_s : t_{s+1} : t_{s+2} : \ldots\ldots : t_n]\hat{B}$$

Now B can be expressed in terms of s linearly independent column vectors $c_1, c_2, \ldots\ldots\ldots c_s$. Therefore we get,

$$B = b_{11}c_1 + b_{21}c_2 + \ldots\ldots\ldots b_{s1}c_s$$

$$\therefore b_{11}c_1 + b_{21}c_2 + \ldots\ldots\ldots + bs_1 c_s = [c_1 : c_2 : \ldots\ldots : c_s : t_{s+1} : t_{s+2} : \ldots\ldots : t_n] \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ \vdots \\ b_{s1} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

Let us define $B_{11} = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ \vdots \\ bs_1 \end{bmatrix}$

$\therefore$ We have $\hat{B} = \begin{bmatrix} B_{11} \\ \cdots\cdots \\ 0 \end{bmatrix}$ or $\hat{B} = P^{-1}B = \begin{bmatrix} B_{11} \\ \cdots\cdots \\ 0 \end{bmatrix}$

Now we define $\hat{K} = KP = [k_1 : k_2]$

We have

$$|sI - A + BK| = \left|P^{-1}(sI - A + BK)P\right|$$

$$= \left|sI - P^{-1}AP + P^{-1}BKP\right|$$

But $P^{-1}AP = \hat{A} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right]$

$$P^{-1}B = \hat{B} = \begin{bmatrix} B_{11} \\ \cdots\cdots \\ 0 \end{bmatrix}$$

$$KP = [K_1 : K_2]$$

Substituting

$$|sI - A + BK| = |sI - \hat{A} + \hat{B} \times \hat{K}|$$

$$= \left| sI - \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right] + \left[\begin{array}{c} B_{11} \\ \hline 0 \end{array}\right] \left[K_1 : K_2\right] \right|$$

In the Identity matrix I, we have n rows and the diagonal elements as 1. This matrix can be split as we have the matrix $\hat{A}$.

$$I = \left[\begin{array}{ccc|ccc} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\ \vdots\\ 0\end{array}}\right\} s \text{ rows} \\ \left.\vphantom{\begin{array}{c}1\\ \vdots\\ 0\end{array}}\right\} (n-s) \text{ rows}\end{array}$$

$$|sI - A + BK| = \left| S \left[\begin{array}{ccc|ccc} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{array}\right] - \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right] + \begin{bmatrix} B_{11}K_1 & B_{11}K_2 \\ 0 & 0 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} sI_s - A_{11} & -A_{12} \\ 0 & sI_{n-s} - A_{22} \end{bmatrix} + \begin{bmatrix} B_{11}K_1 & B_{11}K_2 \\ 0 & 0 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} sI_s - A_{11} + B_{11}K_1 & -A_{12} + B_{11}K_2 \\ 0 & sI_{n-s} - A_{22} \end{bmatrix} \right|$$

$$|sI_s - A_{11} + B_{11}K_1| \cdot |sI_{n-s} - A_{22}| = 0$$

The above equation can be rewritten as,

$$B = T\begin{bmatrix}0\\0\\1\end{bmatrix} = Q_c W\begin{bmatrix}0\\0\\1\end{bmatrix} \qquad \text{But } Q_c = \left[B : AB : A^2B\right]$$

$$\therefore \quad T\begin{bmatrix}0\\0\\1\end{bmatrix} = \left[B : AB : A^2B\right]\begin{bmatrix}a_2 & a_1 & 1\\a_1 & 1 & 0\\1 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= \left[B : AB : A^2B\right]\begin{bmatrix}1\\0\\0\end{bmatrix} = B$$

$$\therefore \quad T\begin{bmatrix}0\\0\\1\end{bmatrix} = B$$

$$T^{-1}T\cdot\begin{bmatrix}0\\0\\1\end{bmatrix} = T^{-1}B$$

$$\therefore \quad T^{-1}B = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

The results which are proved for n = 3 can be extended and generalised so that we have equations (2) and (3) as given below.

$$T^{-1}AT = \begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_n & -a_{n-1} & -a_{n-2} & & -a_1\end{bmatrix}$$

$$T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$$

Now consider the following equation

$$\dot{\hat{x}} = T^{-1}AT\hat{x} + T^{-1}Bu$$

The above equation can be transformed into the controllable canonical form if the system is completely state controllable. The state vector x can be transformed into state vector $\hat{x}$ by use of transformation matrix T = MW.

Now we will select a set of desired eigen values as $\alpha_1, \alpha_2, \ldots\ldots\ldots\alpha_n$. Then the desired characteristic equation becomes,

$$(s-\alpha_1)(s-\alpha_2)\ldots\ldots\ldots(s-\alpha_n) = s^n + \delta_1 s^{n-1} + \ldots\ldots\delta_{n-1}s + \delta_n = 0. \quad \ldots(7)$$

Let $\hat{K} = KT = [\mu_n \; \mu_{n-1} \ldots\ldots\ldots\mu_1]$ ... (8)

When $u = -\hat{K}\hat{x} = -\hat{K}T\hat{x}$ is used to control the system having the equation

$$\dot{\hat{x}} = T^{-1}AT\hat{x} + T^{-1}Bu$$

$$\therefore \quad \dot{\hat{x}} = T^{-1}AT\hat{x} + T^{-1}B\left(-\hat{K}\hat{x}\right)$$

$$\therefore \quad \dot{\hat{x}} = T^{-1}AT\hat{x} + T^{-1}B\left(-\hat{K}T\hat{x}\right)$$

$$\therefore \quad \dot{\hat{x}} = T^{-1}AT\hat{x} - T^{-1}BKT\hat{x}$$

$$\therefore \quad \dot{\hat{x}} = \left(T^{-1}AT - T^{-1}BKT\right)\hat{x}$$

The characteristic equation is,

$$\left|sI - T^{-1}AT + T^{-1}BKT\right| = 0$$

This characteristic equation is the same as the characteristic equation for the system defined by $\dot{x} = Ax + Bu$ when $u = -Kx$ is used as a control signal. This is proved below.

$$\dot{x} = Ax + Bu = Ax + B(-Kx) = Ax - BKx = (A - BK)\,x$$

The characteristic equation for this system is,

$$|sI - A + BK| = |T^{-1}(sI - A + BK)T| = |sI - T^{-1}AT + T^{-1}BKT|$$

Now we will simplify the characteristic equation of the system in the controllable canonical form.

$$|sI - T^{-1}AT - T^{-1}BKT| = \left| sI - \begin{bmatrix} 0 & 1 & \cdots\cdots & \cdots\cdots & 0 \\ 0 & 0 & & & 0 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & & & 1 \\ -a_n & -a_{n-1} & \cdots\cdots & \cdots\cdots & -a_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{bmatrix} [\mu_n \mu_{n-1} \cdots\cdots \mu_1] \right|$$

$$= \begin{vmatrix} s & -1 & \cdots\cdots & \cdots\cdots & 0 \\ 0 & s & & & 0 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ a_n + \mu_n & a_{n-1} + \mu_{n-1} & & & s + a_1 + \mu_1 \end{vmatrix}$$

$$= s^n + (a_1 + \mu_1)\, s^{n-1} + \cdots\cdots + (a_{n-1} + \mu_{n-1})s + (a_n + \mu_n) = 0$$

This is the characteristic equation for the system with state feedback. It is same as equation (7).

$$\begin{aligned} \delta_1 &= a_1 + \mu_1 \\ \delta_2 &= a_2 + \mu_2 \\ &\vdots \\ &\vdots \\ \delta_n &= a_n + \mu_n \end{aligned}$$

The above equations can be solved for μ_i's and substituting in equation (8)

$$KT = [\mu_n \mu_{n-1} \cdots\cdots \mu_1]$$

$$\therefore \quad K = [\delta_n - a_n \;\; \delta_{n-1} - a_{n-1} \cdots\cdots \delta_1 - a_1]\, T^{-1} \qquad \text{... (9)}$$

If the system is completely state controllable, all eigen values can be arbitrarily placed by choosing matrix K according to above equation no (9). This is the sufficient condition.

Thus the necessary and sufficient condition for arbitrary pole placement is proved and it requires the system to be completely state controllable.

For a system which is not completely state controllable but is stabilizable, then the total system can be made stable by placement of closed loop poles at the desired locations for s controllable modes. The rest n – s uncontrollable modes are stable which shows that the total system can be made stable.

5.4 Evaluation of State Feedback Gain Matrix K

Following are the different methods for evaluation of state feedback gain matrix K

i) Using Transformation Matrix T

ii) Using Direct Substitution Method

iii) Using Ackermann's Formula

5.4.1 Use of Transformation Matrix T

Consider a system defined by a state equation $\dot{x} = Ax + Bu$. Let the control signal be $u = -Kx$.

$$\therefore \quad \dot{x} = Ax + B(-Kx) = Ax - BKx = (A - BK)x$$

The state feedback gain matrix K forces the eigen values of (A – BK) to be $\alpha_1, \alpha_2 \ldots\ldots\ldots\ldots \alpha_n$. Then the desired characteristic equation is

$$(s-\alpha_1)(s-\alpha_2)\ldots\ldots\ldots\ldots(s-\alpha_n) = s^n + \delta_1 s^{n-1} + \delta_2 s^{n-2} + \ldots\ldots\ldots + \delta_{n-1}s + \delta_n = 0$$

The desired eigen values $\alpha_1, \alpha_2 \ldots\ldots\ldots\ldots \alpha_n$ can be obtained from following steps.

Step I : The controllability condition is checked for the system. For the system which is completely state controllable following steps can be used.

Step II : The characteristic polynomial for matrix A is given by

$$|sI - A| = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots\ldots\ldots + a_{n-1}s + a_n$$

From this values $a_1, a_2, \ldots\ldots\ldots\ldots a_n$ can be obtained.

Step III : The transformation matrix T is determined which converts the system state equations into controllable canonical form. For system equations which are already in controllable canonical form T = I. The system state equations are not required to be written in controllable canonical form but matrix T is found from.

$$T = Q_c W \qquad \text{where } Q_c \text{ is controllability matrix}$$

$$Q_c = \left[B : AB : A^2B : \ldots\ldots : A^{n-1}B\right]$$

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots\cdots & \cdots\cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & & & 1 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ \vdots & \vdots & & & \vdots & \vdots \\ \vdots & \vdots & & & \vdots & \vdots \\ a_1 & 1 & \cdots\cdots & \cdots\cdots & 0 & 0 \\ 1 & 0 & \cdots\cdots & \cdots\cdots & 0 & 0 \end{bmatrix}$$

Step IV : Using the desired eigen values which are desired closed loop poles, the desired characteristic equation can be obtained $(s-\alpha_1)(s-\alpha_2)........(s-\alpha_n) = s^n + \delta_1 s^{n-1} + \delta_2 s^{n-2} + ... \delta_{n-1} s + \delta_n$.

The values of $\delta_1, \delta_2, ... \delta_n$ can be obtained.

Step V : The state feedback gain matrix K can be obtained from following equation

$$K = [\delta_n - a_n \quad \delta_{n-1} - a_{n-1} .... \delta_1 - a_1]\, T^{-1}$$

5.4.2 Direct Substitution Method

For low order system with n less than or equal to 3 matrix K can be directly substituted into the desired characteristic equation. If suppose n = 3 then the state feedback gain matrix K is

$$K = [K_1 \quad K_2 \quad K_3]$$

The above matrix is directly substituted in the desired characteristic equation $|sI - A + BK|$ and equated to $(s-\alpha_1)(s-\alpha_2)(s-\alpha_3)$ i.e.

$$|sI - A + BK| = (s-\alpha_1)(s-\alpha_2)(s-\alpha_3)$$

The both sides of the characteristic equation are polynomials in s. Hence the co-efficient of similar powers in s on both sides can be equated so as to get the values of K_1, K_2 and K_3. This method is restricted for n = 2 or 3. For higher orders of n the calculations become complex. It is not possible to determine matrix K if the system is not completely controllable.

5.4.3 Ackermann's Formula

Consider a system defined by state equation $\dot{x} = Ax + Bu$. Let u = –Kx be state feedback control. Assuming the system to be completely state controllable and the desired closed loop poles are assumed at $s = \alpha_1$, $s = \alpha_2$, ... $s = \alpha_n$.

Using u = – Kx in state equation

$$\dot{x} = Ax + B(-Kx) = (A - BK)x$$

Let $\qquad \tilde{A} = A - BK$

$\therefore \qquad \dot{x} = \tilde{A}x$

The desired characteristic equation is,

$$|sI - A + BK| = |sI - \tilde{A}| = (s-\alpha_1)(s-\alpha_2)......(s-\alpha_n)$$

$$= s^n + \delta_1 s^{n-1} + ... + \delta_{n-1} s + \delta_n$$

By cayley-Hamilton theorem, every matrix satisfies its own characteristic equation.

$$\therefore \qquad \phi(\tilde{A}) = \tilde{A}^n + \delta_1 \tilde{A}^{n-1} + ... + \delta_{n-1} A + \delta_n I = 0$$

The above equation can be used for deriving Ackermann's formula. Let us consider the case of n = 3 for simplification of derivation.

We have following identities

$$\tilde{A} = A - BK$$

$$\tilde{A}^2 = (A - BK)^2 = A^2 - 2ABK + B^2K^2$$

$$= A^2 - ABK - ABK + B^2K^2 = A^2 - ABK - BK(A - BK)$$

$$= A^2 - ABK - BK\tilde{A}$$

$$\tilde{A}^3 = (A - BK)^3 = A^3 - A^2BK - ABK\tilde{A} - BK\tilde{A}^2$$

$$I = I$$

Now consider

$$\tilde{A}^3 + \delta_1 \tilde{A}^2 + \delta_2 \tilde{A} + \delta_3 I = A^3 - A^2BK - ABK\tilde{A} - BK\tilde{A}^2 + \delta_1\left(A^2 - ABK - BK\tilde{A}\right)$$

$$+ \delta_2(A - BK) + \delta_3 I$$

$$= A^3 - A^2BK - ABK\tilde{A} - BK\tilde{A}^2 + \delta_1 A^2 - \delta_1 ABK$$

$$- \delta_1 BK\tilde{A} + \delta_2 A - \delta_2 BK + \delta_3 I$$

$$= \delta_3 I + \delta_2 A + \delta_1 A^2 + A^3 - \delta_2 BK - \delta_1 ABK$$

$$- \delta_1 BK\tilde{A} - A^2BK - ABK\tilde{A} - BK\tilde{A}^2$$

We have, $\qquad \phi(\tilde{A}) = \delta_3 I + \delta_2 A + \delta_1 A^2 + A^3 = 0$ (for n = 3)

and $\qquad \delta_3 I + \delta_2 A + \delta_1 A^2 + A^3 = \phi(A) \neq 0$

substituting the above equations,

$$\phi(\tilde{A}) = \phi(A) - \delta_2 BK - \delta_1 ABK - \delta_1 BK\tilde{A} - A^2BK - ABK\tilde{A} - BK\tilde{A}^2$$

But $$\phi(\tilde{A}) = 0$$

$$\phi(A) = \delta_2 BK + \delta_1 BK\tilde{A} + BK\tilde{A}^2 + \delta_1 ABK + ABK\tilde{A} + A^2BK$$

$$= B\left(\delta_2 K + \delta_1 K\tilde{A} + K\tilde{A}^2\right) + AB\left(\delta_1 K + K\tilde{A}\right) + A^2BK$$

$$\therefore \qquad \phi(A) = \left[B : AB : A^2B\right] \begin{bmatrix} \delta_2 K + \delta_1 K\tilde{A} + K\tilde{A}^2 \\ \delta_1 K + K\tilde{A} \\ K \end{bmatrix}$$

As the system is completely state controllable, the inverse of the controllability matirx $Q_c = \left[B : AB : A^2B\right]$ exists. Multiplying bothsides by inverse of controllability matrix.

$$\left[B : AB : A^2B\right]^{-1} \phi(A) = \begin{bmatrix} \delta_2 K + \delta_1 K\tilde{A} + K\tilde{A}^2 \\ \delta_1 K + K\tilde{A} \\ K \end{bmatrix}$$

Premultiplying both sides of above equation by [0 0 1] we get

$$[0\ 0\ 1]\left[B : AB : A^2B\right]^{-1} \phi(A) = [0 \quad 0 \quad 1] \begin{bmatrix} \delta_2 K + \delta_1 K\tilde{A} + K\tilde{A}^2 \\ \delta_1 K + K\tilde{A} \\ K \end{bmatrix}$$

$$= K$$

$$\therefore \qquad K = [0 \quad 0 \quad 1]\left[B : AB : A^2B\right] \phi(A)$$

The above equation gives the required state feedback gain matrix K for arbitrary positive integer n,

$$K = [0 \quad 0 \quad \ldots\ldots\ldots \quad 0 \quad 1]\left[B : AB : A^2B : \ldots\ldots\ldots\ldots : A^{n-1}B\right]^{-1} \phi(A)$$

This equation is called Ackermann's formula for the evaluation of state feedback gain matrix K.

5.5 Selection of Location of Desired Closed Loop Poles

The locations of desired closed loop poles must first be chosen in the pole placement design technique. The commonly used method for such selection of poles is based on the experiences obtained from root locus design of placing a dominant pair of closed loop poles. The other poles are selected in such a way that they are away from dominant closed loop poles in left half of s plane.

If the dominant closed loop poles are placed away from jω or imaginary axis then the response obtained from the system is very quick but at the cost of system becoming nonlinear which should be avoided.

The other method of selection of poles is the quadratic optimal control technique. In this method, the desired closed loop poles are determined in such a way that it balances between the acceptable response and the amount of control energy required.

The high speed response indicates that large amount of control energy is required. In addition to this a larger and heavier actuator is required which increases the cost.

Example 5.1 : *Consider the system defined by*

$$\dot{x} = Ax + Bu$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

By using the state feedback control u = – Kx, it is desired to have the closed loop poles at s = – 1 ±j2, s = – 10. Determine the state feedback gain matrix K.

Solution : Let us first check the controllability matrix of the system. The controllability matrix, $Q_c = [B : AB : A^2B]$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix}$$

$$A^2 = A.A. = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -5 & -6 \\ +6 & 29 & 31 \end{bmatrix}$$

$$\therefore \quad A^2B = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -5 & -6 \\ +6 & +29 & 31 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 31 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}; \quad |Q_c| = 1(0-1) = -1$$

The rank of matrix Q_c is 3. The system is controllable and hence it is possible to place the poles arbitrarily.

Now the state feedback gain matrix K can be obtained by using any of the three methods discussed previously.

Method 1 : Using transformation matrix T

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}$$

The characteristic equation for the system is,

$$|sI - A| = \left| s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \right| = \left| \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 5 & s+6 \end{bmatrix} \right|$$

$$= s[s(s+6)+5]+1[1]$$

$$= s\left[s^2+6s+5\right]+1$$

$$= s^3+6s^2+5s+1 = 0$$

Comparing this equation with $s^3 + a_1 s^2 + a_2 s + a_3 = 0$

$\therefore \; a_1 = 6, \; a_2 = 5, \; a_3 = 1$

The desired characteristic equation is ,

$$(s+1-j2)(s+1+j2)(s+10) = \left[(s+1)^2+4\right](s+10) = \left[s^2+2s+5\right][s+10]$$

$$= s^3+10s^2+20s+2s^2+5s+50$$

$$= s^3+12s^2+25s+50$$

$\therefore \; \delta_1 = 12, \; \delta_2 = 25, \; \delta_3 = 50$

$$K = [\delta_3 - a_3 \quad \delta_2 - a_2 \quad \delta_1 - a_1]\, T^{-1}$$

As the given state equation is given in controllable canonical from T = I.

$$\therefore \quad T^{-1} = I$$

$$\therefore \quad K = [\delta_3 - a_3 \quad \delta_2 - a_2 \quad \delta_1 - a_1] = [50-1 \quad 25-5 \quad 12-6]$$

$$K = [49 \quad 20 \quad 6]$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ -1 & -5 & -6 \\ 6 & 29 & 31 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -5 & -6 \\ 6 & 29 & 31 \\ -31 & -149 & -157 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -5 & -6 \\ 6 & 29 & 31 \end{bmatrix}$$

$$\therefore \quad \phi(A) = \begin{bmatrix} -1 & -5 & -6 \\ 6 & 29 & 31 \\ -31 & -149 & -157 \end{bmatrix} + 12 \begin{bmatrix} 0 & 0 & 1 \\ 1 & -5 & -6 \\ -6 & 29 & 31 \end{bmatrix} + 25 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + 50 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 49 & 20 & 6 \\ 18 & 19 & -16 \\ -128 & 74 & 115 \end{bmatrix}$$

$$\left[B : AB : A^2 B\right]^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

$$K = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}^{-1} \begin{bmatrix} 49 & 20 & 6 \\ 18 & 19 & 16 \\ -128 & 74 & 115 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 49 & 20 & 6 \\ 18 & 19 & -16 \\ -128 & 74 & 115 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 49 & 20 & 6 \\ 18 & 19 & -16 \\ -128 & 74 & 115 \end{bmatrix}$$

$$= \begin{bmatrix} 49 & 20 & 6 \end{bmatrix}$$

Example 5.2 : *It is desired to place the closed loop poles of the following system at s = –3 and s = – 4 by a state feedback controller with the control u = – Kx. Determine the state feedback gain matrix K and the control signal.* **(VTU : Aug.-2005)**

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

$$y = [1 \quad 0] x$$

Solution : Equating the state equations with standard equations with standard form $\dot{x} = Ax + Bu$ and $y = Cx$ we get

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad ; \quad C = [1 \quad 0]$$

Let us check the controllability matrix of the system

$$Q_c = [B \; : AB]$$

$$AB = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

$$\therefore \quad Q_c = \begin{bmatrix} 0 & 2 \\ 2 & -6 \end{bmatrix}$$

The rank of above matrix is 2. Hence the given system is completely controllable.

The given equation is not in the controllable canonical form because B is other than $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we use method of transformation matrix T to get state feedback gain matrix.

$$T = Q_c \, . \, W$$

$$Q_c = \begin{bmatrix} 0 & 2 \\ 2 & -6 \end{bmatrix} \quad ; \quad W = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{consider } A = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} ; \; |sI - A| = \begin{vmatrix} s & -1 \\ 1 & s+3 \end{vmatrix} = s(s + 3) + 1 = s^2 + 3s + 1$$

Equating with $s^2 + a_1 s + a_2$ we get

$$\therefore \quad a_1 = 3, a_2 = 1$$

$$\therefore \quad T = \begin{bmatrix} 0 & 2 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

5.7 Concept of State Observer

In case of state observer, the state variables are estimated based on the measurements of the output and control variables. The concept of observability plays important part here in case of state observer.

Consider a system defined by following state equations

$$\dot{x} = Ax+Bu$$

$$y = Cx$$

Let us consider $\tilde{x}$ as the observed state vectors. The observer is basically a subsystem which reconstructs the state vector of the system. The mathematical model of the observer is same as that of the plant except the inclusion of additional term consisting of estimation error to compensate for inaccuracies in matrices A and B and the lack of the initial error.

The estimation error or the observation error is the difference between the measured output and the estimated output. The initial error is the difference between the initial state and the initial estimated state. Thus the mathematical model of the observer can be defined as,

$$\begin{aligned}\dot{\tilde{x}} &= A\tilde{x}+Bu+K_e(y-C\tilde{x}) \\ &= A\tilde{x}+Bu+K_ey+KeC\tilde{x} \\ &= (A-K_eC)\tilde{x}+Bu+K_ey\end{aligned}$$

Here $\tilde{x}$ is the estimated state and $C\tilde{x}$ is the estimated output. The observer has inputs of output y and control input u. Matrix K_e is called the observer gain matrix. It is nothing but weighting matrix for the correction term which contains the difference between the measured output y and the estimated output $C\tilde{x}$.

This additional term continuously corrects the model output and the performance of the observer is improved.

Full order state observer

The system equations are already defined as

$$\dot{x} = Ax+Bu$$

$$y = Cx$$

The mathematical model of the state observer is taken as,

$$\dot{\tilde{x}} = A\tilde{x}+Bu+K_e(y-C\tilde{x})$$

To determine the observer error equation, subtracting equation of $\dot{\tilde{x}}$ from $\dot{x}$ we get

$$\begin{aligned} \dot{x}-\dot{\tilde{x}} &= [Ax+Bu]-[A\tilde{x}+Bu+K_e(y-C\tilde{x})] \\ &= Ax-A\tilde{x}-K_e(y-C\tilde{x}) \\ &= A(x-\tilde{x})-K_e(Cx-C\tilde{x}) \\ &= A(x-\tilde{x})-K_e\,C(x-\tilde{x}) \end{aligned}$$

$$\therefore \quad \dot{x}-\dot{\tilde{x}} = (A-K_eC)\,(x-\tilde{x})$$

Let $\quad e = x-\tilde{x} \qquad \therefore \dot{e}=\dot{x}-\dot{\tilde{x}}$

$$\therefore \quad \dot{e} = (A-K_eC)\,e$$

The block diagram of the system and full order state observer is shown in the Fig. 5.3.

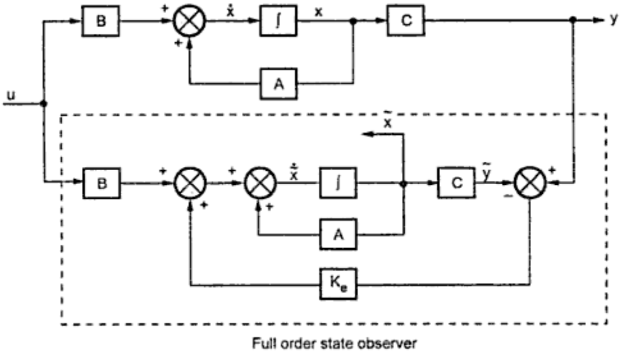


Fig. 5.3

The dynamic behavior of the error vector is obtained from the eigen values of matrix $A-K_eC$. If matrix $A-K_eC$ is a stable matrix then the error vector will converge to zero for any initial error vector e(0). Hence $\tilde{x}(t)$ will converge to x(t) irrespective of values of x(0) and $\tilde{x}(0)$.

If the eigen values of matrix $A-K_eC$ are selected in such a manner that the dynamic behaviour of the error vector is asymptotically stable and is sufficiently fast then any of the error vector will tend to zero with sufficient speed.

If the system is completely observable then it can be shown that it is possible to select matrix K_e such that $A-K_eC$ has arbitrarily desired eigen values. i.e. observer gain matrix K_e can be obtained to get the desired matrix $A-K_eC$.

Here $$T = Q_c.W = [\ C^* : A^*C^* : (A^*)^2\ C^* : .... : (A^*)^{n-1}\ C^*\ ]\ W$$

For the original system the observability matrix is,

$$[\ C^* : A^*C^* \ .... : (A^*)^{n-1}\ C^*\ ] = N$$

$$\therefore \quad T = NW$$

$$T^* = W^*N^* = WN^* \qquad \because (AB)^T = B^T \cdot A^T$$

As $$W = W^*$$

$$\therefore \quad (T^*)^{-1} = (WN^*)^{-1}$$

Taking conjugate transpose of both sides of equation I

$$K^* = \left(T^{-1}\right)^* \begin{bmatrix} \delta_n - a_n \\ \delta_{n-1} - a_{n-1} \\ \vdots \\ \vdots \\ \delta_1 - a_1 \end{bmatrix} = (T^*)^{-1} \begin{bmatrix} \delta_n - a_n \\ \delta_{n-1} - a_{n-1} \\ \vdots \\ \vdots \\ \delta_1 - a_1 \end{bmatrix} = (WN^*)^{-1} \begin{bmatrix} \delta_n - a_n \\ \delta_{n-1} - a_{n-1} \\ \vdots \\ \vdots \\ \delta_1 - a_1 \end{bmatrix}$$

Here $$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots\cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & & 1 & 0 \\ \vdots & & & \vdots & \vdots \\ \vdots & & & \vdots & \vdots \\ a_1 & 1 & & 0 & 0 \\ 1 & 0 & & 0 & 0 \end{bmatrix}$$

$$N = [\ C^* : A^*C^* : (A^*)^2\ C^* : .... : (A^*)^{n-1}\ C^*\ ]$$

The second method is based on direct substitution method to get the state observer gain matrix K_e. This is applicable for system having low order. The matrix K_e is directly substituted in the desired characteristic polynomial.

Let us consider $n = 3$

$$\therefore \quad K_e = \begin{bmatrix} K_{e1} \\ K_{e2} \\ K_{e3} \end{bmatrix}$$

Substitute this matrix into desired characteristic polynomial.

$$|sI - (A - K_eC)| = (s - \alpha_1)(s - \alpha_2)..............(s - \alpha_n)$$

By equating the like power coefficients of s on both sides, the values of K_{e1}, K_{e2} and K_{e3} can be determined. This method is restricted for value of n at the most 3.

The third method is use of Ackermann's formula. For the original system defined as $\dot{x} = Ax + Bu$ and $y = Cx$ we have already derived the formula known as Ackermann's formula given as,

$$K = [0 \ \ 0 \ \ \ldots\ldots \ \ 1] \ \ \left[B : AB : \ldots\ldots\ldots\ldots A^{n-1}B\right]^{-1} \phi(A)$$

Now let us consider the dual system

$$\dot{z} = A^*z + C^*V$$

$$y = B^*z$$

The Ackermann's formula for dual system is modified as

$$K = [0 \ \ 0 \ \ \ldots\ldots \ \ 1] [\, C^* : A^*C^* : \ldots\ldots\ldots\ldots : (A^*)^{n-1} \, C^* \,]^{-1} \, \phi(A^*)$$

Both observer gain matrix, $K_e = K^*$.

$$\therefore K_e = K^* = \phi(A^*)^* \begin{bmatrix} C \\ CA \\ \vdots \\ \vdots \\ CA^{n-2} \\ CA^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} = \phi(A) \begin{bmatrix} C \\ CA \\ \vdots \\ \vdots \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} \qquad \ldots \text{(II)}$$

The desired characteristic polynomial for state observer is

$$\phi(s) = (s-\alpha_1)(s-\alpha_2)\ldots\ldots\ldots\ldots(s-\alpha_n)$$

Here $\alpha_1, \ \alpha_2 \ldots\ldots\ldots\ldots \alpha_n$ are desired eigen values.

Equation (II) is Ackermann's formula for determination of observer gain matrix K_e.

5.11 Selection of Suitable Value of Observer Gain Matrix K_e

The feedback signal through the observer gain matrix K_e acts as a correction signal to the plant model to account for the unknowns in the plant.

If the unknowns involved are significant then the feedback signal through matrix K_e is comparatively large. But if the output signal is affected by disturbances and measurement noises then output y is not reliable and feedback signal through matrix K_e should be small. The effects of disturbances and noises involved in output y should be neatly studied while evaluating matrix K_e.

The observer gain matrix is dependent on the desired characteristic equation $(s-\alpha_1)(s-\alpha 2)\ldots\ldots\ldots\ldots(s-\alpha_n) = 0$. The selection of $\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_n$ is many times not unique. But normally the observer poles should be two to five times faster than the controller poles. This makes the observation or estimation error converging to zero quickly. The observer estimation error decays two to five times faster than does the state vector x. Due to this faster decay of observer error, the system response is dominated by controller poles.

With the appreciable noise produced by sensor then the observer poles can be selected to be slower than two times the controller poles. The bandwidth of the system will then be less and noise will be smoothened. The system response is dependent on observer poles in such case. The observer poles are located to the right of controller poles in left half of s-plane so that the system response is influenced by observer poles.

While designing the state observer, it is required to evaluate various observer gain matrices K_e based on number of various desired characteristic equations. For each of these different matrices K_e, the simulation tests can be run to check the overall system performance. Based on the performance, the best possible K_e can be selected. In many systems, the best value of K_e is obtained after a compromise between fast response and sensitivity to disturbances and noise signals.

Example 5.3 *Consider the system represented by*

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0]\, x$$

Design a full order observer such that the observer eigen values are at $-2 \pm j2\sqrt{3}$ and -5. **(VTU: Aug.-2005)**

Solution :

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad ; \quad C = [1 \quad 0 \quad 0]$$

Let us check the rank of dual system of the original system

$$[\, C^* : A^*C^* : (A^*)^2 C^* : \ldots\ldots\ldots\ldots : (A^*)^{n-1} C^* \,]$$

Here n = 3

$$\therefore [\, C^* : A^*C^* : (A^*)^2 C^* \,]$$

$$C^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \; ; \quad A^*C^* = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$(A^*)^2 = A^* \,.\, A^* = 1\begin{bmatrix} 0 & 0 & -6 \\ 0 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} = \begin{bmatrix} 0 & -6 & 36 \\ 0 & -11 & 60 \\ 1 & -6 & 25 \end{bmatrix}$$

$$(A^*)^2 = \begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 6 \\ 0 & 7 & -2 \\ 2 & -1 & 2 \end{bmatrix}$$

$$(A^*)^2 C^* = \begin{bmatrix} 7 & 0 & 6 \\ 0 & 7 & -2 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix}$$

$$\therefore \quad [\, C^* : A^*C^* : (A^*)^2 C^* \,] = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 2 & -2 \\ 1 & 0 & 2 \end{bmatrix}$$

The rank of above matrix 3. Hence the original system is observable.

Let $\quad K_e = \begin{bmatrix} K_{e1} \\ K_{e2} \\ K_{e3} \end{bmatrix}$ be observer gain matrix

The desired characteristic polynomial is,

$$[s-(-3+j1)] \; [s-(-3-j1)] \; [s-(-4)] = \left[(s+3)^2+1\right] \; [s+4]$$

$$= \left(s^2+6s+10\right) \; (s+4) = s^3+6s^2+10s+4s^2+24s+40$$

$$= s^3+10s^2+34s+40$$

Now consider $|sI-(A+K_e\,-C)| =$

$$\left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \left\{ \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} - \begin{bmatrix} K_{e1} \\ K_{e2} \\ K_{e3} \end{bmatrix} [0 \;\; 0 \;\; 1] \right\} \right|$$

$$= \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \left\{ \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & K_{e1} \\ 0 & 0 & K_{e2} \\ 0 & 0 & K_{e3} \end{bmatrix} \right\} \right|$$

$$= \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 2 & -K_{e1} \\ 3 & -1 & 1-K_{e2} \\ 0 & 2 & -K_{e3} \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} s-1 & -2 & K_{e1} \\ -3 & s+1 & -1+K_{e2} \\ 0 & -2 & s+K_{e3} \end{bmatrix} \right|$$

$$= (A - K_e C)\ (x - \tilde{x})$$

$$\therefore \qquad \dot{e} = (A - K_e C)\, e \qquad \dots \text{(II)}$$

Equations (I) and (II) can be combined and rewritten in matrix form as,

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - K_e C \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \qquad \dots \text{(III)}$$

Equation (III) shows the dynamics of observed state feedback control system. The characteristic equation for the system is

$$\begin{vmatrix} sI - A + BK & -BK \\ 0 & sI - A + K_e C \end{vmatrix} = 0$$

$$|sI - A + BK| \cdot |sI - A + K_e C| = 0$$

The closed loop poles of observed state feedback system consists of the poles due to pole placement design alone and poles due to observer design alone. Thus the pole placement design and observer design are independent of each other. The design of each can be done independently and it can be combined to get observed state feedback control system. For n as the order of the plant, the order of the observer is also n while the resulting characteristic equation of the total closed loop system is of order 2n.

To derive the transfer function of observer based controller, let us consider the system defined as

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

Let the plant be completely observable and let the observed state feedback control is $u = -K\tilde{x}$.

$$\dot{\tilde{x}} = A\tilde{x} + Bu + K_e(y - C\tilde{x})$$

$$= A\tilde{x} + B(-K\tilde{x}) + K_e(y - C\tilde{x})$$

$$\therefore \dot{\tilde{x}} = (A - BK - K_e C)\tilde{x} + K_e y$$

Taking Laplace transform of above equation

$$s\tilde{X}(s) - \tilde{x}(0) = (A - BK - K_e C)\tilde{X}(s) + K_e Y(s)$$

Assuming zero initial condition,

$$[sI-(A-BK-K_eC)]\,\tilde{x}(s) = K_e\,Y(s)$$

$$\therefore \quad \tilde{x}(s) = (sI-A+BK+K_eC)^{-1}K_eY(s)$$

Substituting this value of $\tilde{x}(s)$ in equation for u,

$$U(s) = -K\,\tilde{x}(s)$$

$$\therefore \quad U(s) = -K\,(sI-A+BK+K_eC)^{-1}\,K_e\,Y(s)$$

The transfer function is given by,

$$\frac{U(s)}{Y(s)} = -K(sI-A+K_eC+BK)^{-1}K_e$$

$$\therefore \quad \frac{U(s)}{-Y(s)} = K(sI-A+K_eC+BK)^{-1}K_e$$

The block diagram representation of above system is shown in the Fig. 5.5

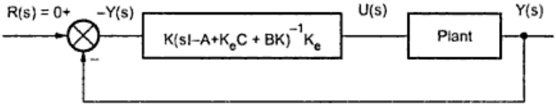


Fig. 5.5

The transfer function obtained above is called observer based controller transfer function. It can be seen that the observer controller matrix A – K_eC – BK may or may not be stable even though A – BK and A – K_eC are selected to be stable. In certain cases, the matrix A – K_eC – BK may be poorly stable or even unstable also.

Examples with Solutions

Example 5.5 : *A system represented by following state model is controllable but not observable. Show that the non-observability is due to a pole-zero cancellation in C[sI – A]$^{-1}$.*
(VTU: July/Aug.-2005)

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$Y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x$$

Solution :

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$Y = [1 \quad 1 \quad 0]\, x$$

Here,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad C = [1 \quad 1 \quad 0]$$

$$[sI - A] = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s+6 \end{bmatrix}$$

$$|sI - A| = s[(s+6)s + 11] + 1[6] = s^3 + 6s^2 + 11s + 6$$

$$= (s+1)(s+2)(s+3)$$

$$[sI - A]^{-1} = \frac{\text{Adj of } [sI - A]}{|sI - A|}$$

$$= \frac{1}{(s+1)(s+2)(s+3)} \begin{bmatrix} s^2 + 6s + 11 & -6 & -6s \\ +s+6 & s^2 + 6s & -11s - 6 \\ 1 & s & s^2 \end{bmatrix}^T$$

$$= \frac{1}{(s+1)(s+2)(s+3)} \begin{bmatrix} s^2 + 6s + 11 & s+6 & 1 \\ -6 & s(s+6) & s \\ -6s & -(11s+6) & s^2 \end{bmatrix}$$

$$\text{T.F.} = \frac{Y(s)}{U(s)} = C[sI - A]^{-1} B + D = C[sI - A]^{-1} B = (\because D = 0)$$

$$= [1 \quad 1 \quad 0] \frac{1}{(s+1)(s+2)(s+3)} \begin{bmatrix} s^2 + 6s + 11 & s+6 & 1 \\ -6 & s(s+6) & s \\ -6s & -(11s+6) & s^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(s+1)(s+2)(s+3)} \left[s^2 + 6s + 11 - 6 \quad (s+6) + s(s+6) \quad 1 + s \right] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(s+1)(s+2)(s+3)} [s+1]$$

$$= \frac{s+1}{(s+1)(s+2)(s+3)}$$

$$\text{T.F.} = \frac{Y(s)}{U(s)} = \frac{s+1}{(s+1)(s+2)(s+3)}$$

Clearly it is seen that the factor (s+1) is cancelled from numerator and denominator.

$$\text{T.F.} = \frac{s+1}{s^3+6s^2+11s+6} = \frac{s+1}{[s(s+6)+11]s+6}$$

Let us draw the simulation of the transfer function.

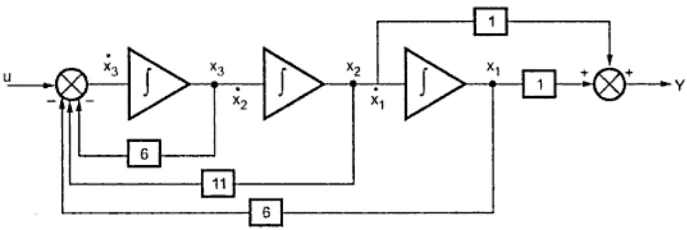


Fig. 5.6

There is no coupling between x_3 and y due to which state x_3 can not be observed from measurement of output y. Thus the system is not completely observable due to pole-zero cancellation as matrix C contains one of the terms as zero.

Example 5.6 : *A regulator system has the plant* **(VTU: Jan./Feb.-2006)**

$$x = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, \quad y = [0 \quad 0 \quad 1]x$$

i) Compute K so that the control law u = – Kx + r(t), r(t) = referance input, places the closed loop poles at $-2 \pm j\sqrt{12}, -5$. **(8)**

ii) Design an observer such that the eigen values of the observer are located at $-2 \pm j\sqrt{12}, -5$. **(6)**

iii) Draw a block diagram implementation of the control configuration. **(3)**

iv) Obtain the state model of the observer based state feed back control system. **(3)**

Solution :

$$\dot{x} = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

$$A = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

i) Let us first check the controllability matrix of the system the controllability matrix for n = 3 is given by,

$$Q_c = \begin{bmatrix} B & : & AB & : & A^2B \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad AB = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} = \begin{bmatrix} 0 & -6 & 36 \\ 0 & -11 & 60 \\ 1 & -6 & 25 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & -6 & 36 \\ 0 & -11 & 60 \\ 1 & -6 & 25 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad |Q_c| = 1(1) = 1 \neq 0$$

$\therefore$ Rank of Q_c = 3

The system is completely state controllable as the rank of controllability matrix $Q_c = n = 3$.

Now the desired characteristics equation is,

$$[s-(-5)]\left[s-\left(-2+j\sqrt{12}\right)\right]\left[s-\left(-2-j\sqrt{12}\right)\right] = (s+5)\left[(s+2)-j\sqrt{12}\right]\left[(s+2)+j\sqrt{12}\right]$$

$$= (s+5)\left[(s+2)^2+12\right]$$

$$= (s+5)\left[s^2+4s+16\right]$$

$$A^2B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -6 & -5 \\ 0 & 30 & 19 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 19 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & -5 & 19 \end{bmatrix}; \quad |Q_c| = 1(0-1) = -1 \neq 0$$

$\therefore$ Rank of $Q_c = 3$

The system is completely state controllable as rank of matrix $Q_c = n$. Hence it is possible to place the poles arbitrarily. Let us now find the state feedback gain matrix by use of transformation matrix T.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix}$$

The characteristics equation for the system is,

$$|sI-A| = \left|\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix}\right| = \left|\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 6 & s+5 \end{bmatrix}\right|$$

$$= s[s(s+5)+6]-(-1)[0]$$

$$= s\left[s^2+5s+6\right] = s^3+5s^2+6s$$

Comparing the above equation with $s^3+a_1s^2+a_2s+a_3=0$

$a_1 = 5$; $a_2 = 6$; $a_3 = 0$

The desired characteristic equation is,

$$[s-(-1+j1)][s-(-1-j1)][s-(-5)] = [(s+1)-j1][(s+1)+j1][s+5]$$

$$= \left[(s+1)^2+1\right][s+5]$$

$$= \left(s^2+2s+2\right)(s+5) = s^3+7s^2+12s+10$$

Comparing above equation with $s^3+\delta_1s^2+\delta_2s+\delta_3$ we have $\delta_1 = 7$; $\delta_2 = 12$; $\delta_3 = 10$

$$K = [\delta_3-a_3 \quad \delta_2-a_2 \quad \delta_1-a_1]\, T^{-1}$$

As the given state equation is given in controllable form T = I

$\therefore$ $T^{-1} = I$

$$\therefore \quad K = [10-0 \quad 12-6 \quad 7-5]\,I$$

$$= [10 \quad 6 \quad 2]\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = [10 \quad 6 \quad 2]$$

$\therefore$ The state feedback gain matrix, $\mathbf{K} = [\mathbf{10} \quad \mathbf{6} \quad \mathbf{2}]$

Example 5.8 : *Consider a linear system described by the transfer function* $\dfrac{Y(s)}{U(s)} = \dfrac{10}{s(s+1)(s+2)}$ *.Design a feedback controller with a state feedback so that closed loop poles are placed at* $-2, -1 \pm j$. **(VTU : July/Aug.-2006)**

Solution : The transfer function is given as ,

$$\frac{Y(s)}{U(s)} = \frac{10}{s(s+1)(s+2)} = \frac{10}{s\left(s^2+3s+2\right)} = \frac{10}{s^2+3s^2+2s}$$

The denominator is decomposed as below,

$$s^3+3s^2+2s = [(s+3)s+2]s$$

Let us simulate this denominator

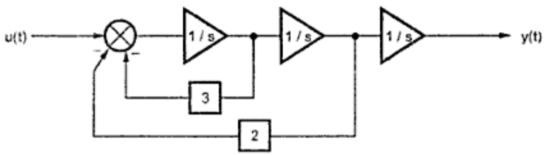


Fig. 5.9

The complete state diagram is as shown in the following Fig. 5.10 after simulating the numerator by shifting take off point.

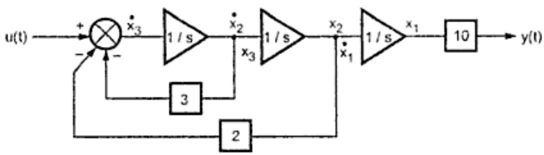


Fig. 5.10

Now the desired characteristics equation is,

$$[s-(-2)][s-(-1+j1)][s-(-1-j1)] = (s+2)(s+1+j1)(s+1-j1)$$

$$= [s+2]\left[(s+1)^2+1\right] = (s+2)\left(s^2+2s+2\right)$$

$$= s^3+2s^2+2s+2s^2+4s+4$$

$$= s^3+4s^2+6s+4$$

Comparing above equation with $s^3+\delta_1 s^2+\delta_2 s+\delta_3$ we have,

$$\delta_1 = 4; \quad \delta_2 = 6; \quad \delta_3 = 4$$

Let us now find state feedback gain matrix by direct substitution method. Let the desired state feedback gain matrix K be,

$$K = [K_1 \quad K_2 \quad K_3]$$

Consider, $$|sI-A+BK| = \left|\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}[K_1 \quad K_2 \quad K_3]\right|$$

$$= \left|\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ K_1 & K_2 & K_3 \end{bmatrix}\right|$$

$$= \left|\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ K_1 & 2+K_2 & s+3+K_3 \end{bmatrix}\right|$$

$$= s\left[s(s+3+K_3)-(-1)(2+K_2)+1[K_1]\right]$$

$$= s\left[s^2+3s+K_3 s+2+1K_2\right]+K_1$$

$$= s^3+(3+K_3)s^2+(2+K_2)s+K_1$$

Equating coefficients of above equation with those from desired equation,

$3 + K_3 = 4$; $K_3 = 1$ $\quad 2 + K_2 = 6$ $\therefore K_2 = 4$ $\quad K_1 = 4$

$\therefore$ State feedback gain matrix, **K = [4 4 1]**

Example 5.9: *Consider the system described by the state model* $\dot{x} = Ax,\ Y = Cx$ *where* $A = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix},\ C = [1 \quad 0]$ *Design a full order state observer. The desired eigen values for the observer matrix are* $u_1 = -5,\ u_2 = -5.$

(VTU:July/Aug.-2006)

Solution :

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix};\ C = [1 \quad 0]$$

$$\therefore \quad A^* = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix};\ C^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let us check the controllability matrix of dual of given system for n = 2

$$\therefore \quad Q_c = \left[C^* \ : \ A^* C^*\right]$$

$$A^* C^* = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\therefore \quad Q_c = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} |Q_c| = 1 \neq 0$$

$\therefore$ Rank of $Q_c = 2$ = Number of state variables

The dual system is completely state controllable

$\therefore$ Original system is observable.

Let $K_e = \begin{bmatrix} K_{e1} \\ K_{e2} \end{bmatrix}$ be observer gain matrix

The desired characteristics equation is,

$$[s-(-5)][s-(-5)] = (s+5)(s+5) = s^2 + 10s + 25$$

Comparing with $s^2 + \delta_1 s + \delta_2$ we have $\delta_1 = 10,\ \delta_2 = 25$

Consider,

$$|sI - A + K_e C| = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} K_{e1} \\ K_{e2} \end{bmatrix} [1 \quad 0] \right|$$

$$= \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} K_{e1} & 0 \\ K_{e2} & 0 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} s+1+K_{e1} & -1 \\ -1+K_{e2} & s+2 \end{bmatrix} \right|$$

$$= \left(s+1+K_{e1}\right)(s+2)-(-1)\left(-1+K_{e2}\right)$$

$$= \left(s+1+K_{e1}\right)(s+2)+1\left(-1+K_{e2}\right)$$

$$= s^2+s+K_{e1}s+2s+2+2K_{e1}-1+K_{e2}$$

$$= s^2+\left(1+K_{e1}+2\right)s+\left(1+2K_{e1}+K_{e2}\right)$$

$$= s^2+\left(3+K_{e1}\right)s+\left(1+2K_{e1}+K_{e2}\right)$$

Comparing the coefficients in above equation with those of desired characteristic equation.

$$3+K_{e1} = 10\ ;\ K_{e1}=7$$

$$1+2K_{e1}+K_{e2} = 25\ ;\ K_{e2}=25-1-2K_{e1}=24-2(7)=10$$

The observer gain matrix K_e is given as,

$$K_e = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$$

Example 5.10 : *Consider the system defined by $\dot{x} = Ax + Bu$, where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. By using state feedback control $u = -Kx$, it is desired to have the closed loop poles at $s = -2 \pm j4$ and $s = -10$. Determine the state feedback gain matrix "K" by any one method.* **(VTU : Jan./Feb.- 2008)**

Solution : We have

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix};\quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let us check the controllability matrix of the system. Here n = 3.

The controllability matrix , $Q_c = \left[B : AB\ :\ A^2B\right]$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};\ AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -5 & -6 \\ 6 & 29 & 31 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -5 & -6 \\ 6 & 29 & 31 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 31 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix} \qquad \therefore |Q_c| = 1[31-(36)] = -5 \neq 0$$

$\therefore$ Rank of matrix Q_c is 3. The system is completely state controllable as rank of matrix $Q_c = n$.

Let us now find state feedback gain matrix by direct substitution method. The desired characteristic equation is,

$$[s-(-2+j4)][s-(-2-j4)][s-(-10)] = [(s+2)-j4][(s+2)+j4][s+10]$$

$$= \left[(s+2)^2+16\right][s+10] = \left[s^2+4s+20\right][s+10]$$

$$= s^3+14s^2+60s+200$$

Comparing above equation with $s^3+\delta_1 s^2+\delta_2 s+\delta_3$ we have

$$\delta_1 = 14, \quad \delta_2 = 60, \quad \delta_3 = 200$$

Let the desired feedback gain matrix K be $K = [\,K_1 \quad K_2 \quad K_3\,]$

Consider,

$$|sI-A+BK| = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [K_1 \quad K_2 \quad K_3] \right|$$

$$= \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ K_1 & K_2 & K_3 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1+K_1 & 5+K_2 & s+6+K_{e3} \end{bmatrix} \right|$$

$$= s\left[s(s+6+K_3)+(5+K_2)\right]+1\left[1+K_1\right]$$

$$= s\left[s^2+(6+K_3)s+(5+K_2)\right]+(1+K_1)$$

$$= s^3+(6+K_3)s^2+(5+K_2)s+(1+K_1)$$

$$K_e = \phi(A)\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 74 & 25 & 3 \\ -18 & 41 & 7 \\ -42 & -95 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$K_e = \begin{bmatrix} 74 & 25 & 3 \\ -18 & 41 & 7 \\ -42 & -95 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix}$$

This is the required observer feedback gain matrix by Ackermann's formula.

Review Questions

1. *What do you mean be pole placement problem?*
2. *Prove the necessary and sufficient condition for arbitrary pole placement?*
3 *Write short note of evaluation of state feedback gain matrix.*
4. *What are the different methods of evaluating state feedback gain matrix? Explain any one method in detail.*
5. *Describe Ackermann's formula method for determining the state feedback gain matrix K.*
6. *Write a note on selection of location of desired closed loop poles.*
7. *What do you mean by state observers?*
8. *Write a note on full order state observer?*
9. *What is dual problem?*
10. *What is necessary and sufficient condition for state observation.*
11. *What are different methods of evaluation of state observer gain matrix* K_e *?*
12. *Write a note on selection of suitable value of observer gain matrix* K_e.
13. *Write a short note on observer based controller.*
14. *Derive the transfer function of observer based controller.*
15. *Consider the system defined by*

$$\dot{x} = Ax + Bu$$

where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

By using the state feedback control $u = -Kx$, *it is desired to have the closed loop poles at* $s = -2 \pm j4$ *and* $s = -10$. *Determine the state feedback gain matrix K.* **[Ans. :** $K = [199 \quad 55 \quad 8]$ **]**

22. *Consider the system represented by*

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

Design a full order observer such that the observer eigen values are at $-2 \pm \rho 2\sqrt{3}$ *and* -5.

(July/Aug.-2005)

6 Controllers

6.1 Background

The concept of a control system is to sense deviation of the output from the desired value and correct it, till the desired output is achieved. The deviation of the actual output form its desired value is called an error. The measurement of error is possible because of feedback. The feedback allows us to compare the actual output with its desired value to generate the error. The error is denoted as e(t). The desired value of the output is also called **reference input** or a **set point**. The error obtained is required to be analysed to take the proper corrective action.

The **controller** is an element which accepts the error in some form and decides the proper corrective action. The output of the controller is then applied to the process or final control element. This brings the output back to its desired set point value. The controller is the heart of a control system. The accuracy of the entire system depends on how sensitive is the controller to the error detected and how it is manipulating such an error. The controller has its own logic to handle the error. Now a days for better accuracy, the digital controllers such as microprocessors, microcontrollers, computers are used. Such controllers execute certain algorithm to calculate the manipulating signal.

This chapter explains the basic discontinuous controllers such as on-off controller, continuous controllers such as proportional, integral etc. and composite controllers such as proportional plus integral, proportional plus derivative etc. Let us study first the general properties of the controller.

6.2 Properties of Controller

Consider a control system shown in the Fig. 6.1 which includes a controller.

The actual output is sensed by a sensor and converted to a proper feedback signal b(t) using a feedback element. The set point value is the reference input r(t). For example the actual output variable may be temperature but using the thermocouple as the feedback element, the feedback signal b(t) is an electrical voltage. This is then compared with reference input which is also an electrical voltage. The thermocouple senses the output temperature and produces the corresponding electrical e.m.f as the feedback signal. Hence actual output variable sensed and the feedback signal may be having different forms.

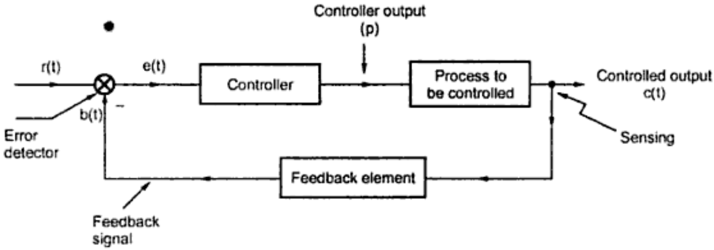


Fig. 6.1 Basic control system

6.2.1 Error

The error detector compares the feedback signal b(t) with the reference input r(t) to generate an error.

$$\therefore \quad \boxed{e(t) = r(t) - b(t)}$$

This gives an absolute indication of an error.

For example if the set point for a range of 5 mV to 20 mV is 12 mV and the feedback signal is 11.8 mV then error is 0.2 mV. But actual vairable to be controlled may be different such as temperature, pressure etc. Hence to obtain correct information from the error, it is expressed in percentage form related to the controller operation. It is expressed as the percentage of the measured variable range. The range of the measured variable b(t) is also called **span**.

Thus $\quad \text{span} = b_{max} - b_{min}$

Hence error can be expressed as percent of span as,

$$\therefore \quad \boxed{e_p = \frac{r-b}{b_{max}-b_{min}} \times 100}$$

Where $\quad e_p$ = error as % of span

Example 6.1 : *The range of measured variable for a certain control system is 2 mV to 12 mV and a setpoint of 7 mV. Find the error as percent of span when the measured variable is 6.5 mV.*

Solution : $b_{max} = 12$ mV, $b_{min} = 2$ mV, $b = 6.5$ mV, $r = 7$ mV

$$\therefore \quad e_p = \frac{r-b}{b_{max}-b_{min}} \times 100 = \frac{7-6.5}{12-2} \times 100$$

$$= \mathbf{5\%}$$

6.2.2 Variable Range

In practical systems, the controlled variable has a range of values within which the control is required to be maintained. This range is specified as the maximum and minimum values allowed for the controlled variable. It can be specified as some nominal values and plus-minus tolerance allowed about this value. Such range is important for the design of contollers.

6.2.3 Controller Output Range

Similar to the controlled variable, a range is associated with a controller output variable. It is also specified interms of the maximum and minimum values.

But often the controller output is expressed as a percentage where minimum controller output is 0% and maximum controller output is 100 %. But 0 % controller output does not mean, zero output. For example it is necessary requirement of the system that a steam flow corresponding to $\frac{1}{4}^{th}$ opening of the valves should be mininum. Thus 0% controller output in such case corresponds to the $\frac{1}{4}^{th}$ opening of the valve.

The controller output as a percent of full scale when the output changes within the specified range is expressed as,

$$\boxed{p = \frac{u - u_{min}}{u_{max} - u_{min}} \times 100}$$

Where p = controller output as a percent of full scale

u = value of the output

u_{max} = maximum value of controlling variable

u_{min} = minimum value of controlling variable

6.2.4 Control Lag

The control system can have a lag associated with it. The control lag is the time required by the process and controller loop to make the necessary changes to obtain the output at its setpoint. The control lag must be compared with the process lag while designing the controllers. For example in a process a valve is required to be open or closed for controlling the output variable. Physically the action of opening or closing of the valve is very slow and is the part of the process lag. In such a case there is no point in designing a fast controller than the process lag.

6.2.5 Dead Zone

Many a times a dead zone is associated with a process control loop. The time corresponding to dead zone is called dead time. The time elapsed between the instant when error occurs and the instant when first corrective action occurs is called dead time.

Nothing happens in the system, during this time though the error occurs. This part is also called dead band. The effect of such dead time must be considered while the design of the controllers.

6.3 Classification of Controllers

The classification of the controllers is based on the response of the controller and mode of operation of the controller. For example in a simple temperature control of a room, the heater is to be controlled. It should be switched on or off by the controller when temperature crosses its setpoint. Such an operation of the controller is called discontinuous operation and the mode of operation is called **discontinuous mode** of controller. But in some process control systems, simple on/off decision is not sufficient. For example, controlling the steam flow by opening or closing the valve. In such case a smooth opening or closing of valve is necessary. The controller in such a case is said to be operating in a **continuous mode**.

Thus the controllers are basically classified as discontinuous controllers and continuous controllers.

The discontinuous mode controllers are further classified as ON-OFF controllers and multiposition controllers.

The continuous mode controllers are further classified as proportional controllers, integral controllers and derivative controllers.

Some continuous mode controllers can be combined to obtain composite controller mode. The examples of such composite controllers are PI, PD and PID controllers.

The most of the controllers are placed in the forward path of control system. But in some cases, input to the controller is controlled through a feedback path. The example of such a controller is rate feedback controller.

In this chapter, only continuous and composite controllers are discussed.

6.4 Continuous Controller Modes

In the discontinuous controller mode, the output of the controller is discontinuous and not smoothly varying. But in the continuous controller mode, the controller output varies smoothly proportional to the error or proportional to some form of the error. Depending upon which form of the error is used as the input to the controller to produce the continuous controller output, these controllers are classified as,

1. Proportional control mode
2. Integral control mode
3. Derivative control mode

Let us discuss these control modes in detail.

6.5 Proportional Control Mode

In this control mode, the output of the controller is simple proportional to the error e(t). The relation between the error e(t) and the controller output p is determined by constant called **proportional gain constant** denoted as K_p. The output of the controller is a linear function of the error e(t). Thus each value of the error has a unique value of the controller output. The range of the error which covers 0 % to 100 % controller output is called **proportional band.**

Now though there exists linear relation between controller output and the error, for a zero error the controller output should not be zero, otherwise the process will come to halt. Hence there exists some controller output p_o for the zero error. Hence mathematically the proportional control mode is expressed as,

$$\boxed{p(t) = K_p e(t) + p_o} \quad ...(1)$$

Where K_p = Proportional gain constant

p_o = Controller output with zero error

The direct and reverse action is possible in the proportional control mode. The error may be positive or negative because error is r–b and b can be less or greater than reference setpoint r.

If the controlled variable i.e. input to the controller increases, causing increase in the controller output, the action is called **direct action.** For example the output valve is to be controlled to maintain the liquid level in a tank. So if the level increases, the valve should be opened more to maintain the level.

If the controlled variable decreases, causing increase in the controller output or increase in the controlled variable, causing decrease in the controller output, the action is called **reverse action.** For example simple heater control for maintaining temperature. If the temperature increases, the drive to the heater must be decreased.

So if e(t) is negative then $K_p e(t)$ gets subtracted from p_o and e(t) is positive, then $K_p e(t)$ gets added to p_o, this is reverse action.

So equation (1) represents the reverse action. But using negative sign to the correction term $K_p e(t)$, the direct action proportional controller can be achieved.

The proportional controller depends on the proper design of the gain K_p. For fixed p_o, if gain K_p is high then large output results for small error but narrow error band can be handled. Beyond these limits of the error, output will be saturated. If the gain is small then the response is smaller but large error band can be handled. This is shown in the Fig. 6.2.

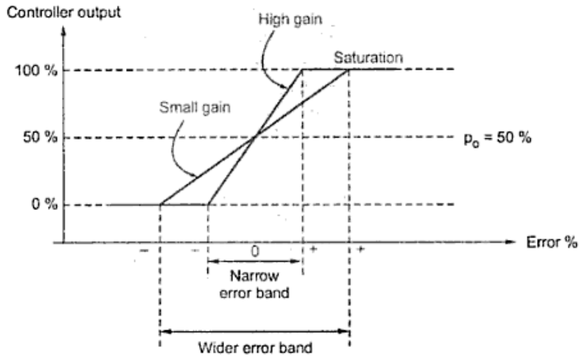


Fig. 6.2

The proportional band is mathematically defined by,

$$\boxed{PB = \frac{100}{K_p}}$$

6.5.1 Characteristic of Proportional Mode

The various characteristics of the proportional mode are,

1. When the error is zero, the controller output is constant equal to p_o.
2. If the error occurs, then for every 1 % of error the correction of K_p % is achieved. If error is positive, K_p% correction gets added to p_o and if error is negative, K_p% correction gets subtracted from p_o.
3. The band of error exists for which the output of the controller is between 0 % to 100 % without saturation.
4. The gain K_p and the error band PB are inversely proportional to each other.

6.5.2 Offset

The major disadvantage of the proportional control mode is that it produces an **offset error** in the output. When the load changes, the output deviates from the setpoint. Such a deviation is called offset error or steady state error. Such an offset error is shown in the Fig. 6.3. The offset error depends upon the reaction rate of the controller. Slow reaction rate produces small offset error while fast reaction rate produces large offset error.

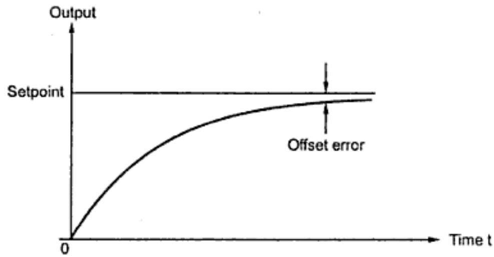


Fig. 6.3 Offset error in porportional mode

The dead time or transfer lag present in the system further worsens the result. It produces not only the large offset at the output but the time required to achieve steady state is also large.

The offset error can be minimized by the large proportional gain K_p which reduces the proportional band. If K_p is made very large, the proportional band becomes so small that it acts as an ON/OFF controller producing oscillations about the setpoint instead of an offset error.

6.5.3 Applications

The proportional controller can be suitable where,

1. Manual reset of the operating point is possible.
2. Load changes are small.
3. The dead time exists in the system is small.

6.6 Integral Control Mode

In the proportional control mode, error reduces but can not go to zero. It finally produces an offset error. It can not adapt with the changing load conditions. To avoid this, another control mode is oftenly used in the control systems which is based on the history of the errors. This mode is called **integral mode** or **reset action controller**.

In such a controller, the value of the controller output p(t) is changed at a rate which is proportional to the actuating error signal e(t). Mathematically it is expressed as,

$$\boxed{\frac{d\,p(t)}{dt} = K_i e(t)}$$

Where K_i = Constant relating error and rate

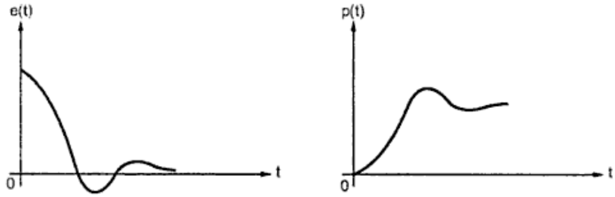
The constant K_i is also called **integral constant**. Integrating the above equation, actual controller output at any time t can be obtained as,

$$\boxed{p(t) = K_i \int e(t)dt + p(0)} \quad \dots (2)$$

Where $p(0)$ = Controller output when integral action starts i.e. at $t = 0$.

The output signal from the controller, at any instant is the area under the actuating error signal curve up to that instant. If the value of the error is doubled, the value of p(t) varies twice as fast i.e. rate of the controller output change also doubles.

If the error is zero, the controller output is not changed. The control signal p(t) can have nonzero value when the error signal e(t) is zero. This is because the output depends on the history of the error and not on the instantaneous value of the error. This is shown in the Fig. 6.4.



(a) Error (b) Controller output

Fig. 6.4 Integral mode

The scale factor or constant K_i expresses the scaling between error and the controller output. Thus a large value of K_i means that a small error produces a large rate of change of p(t) and viceversa. This is shown in the Fig. 6.5.

If there is positive error, the controller output begins to ramp up.

Fig. 6.5

6.6.1 Step Response of Integral Mode

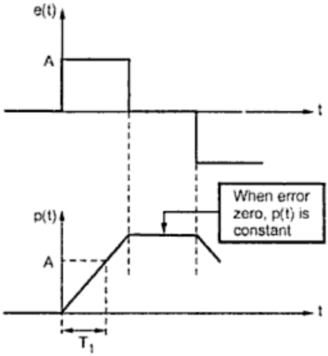


Fig. 6.6 Step response

The step response of the integral control mode is shown in the Fig. 6.6.

The integration time constant is the time taken for the output to change by an amount equal to the input error step. This is shown in the Fig. 6.6.

It can be seen that when error is positive, the output p(t) ramps up. For zero error, there is no change in the output. And when error is negative, the output p(t) ramps down.

6.6.2 Characteristics of Integral Mode

The integrating controller is relatively slow controller. It changes its output at a rate which is dependent on the integrating time constant, until the error signal is cancelled. Compared to the proportional control, the integral control requires time to build up an appreciable output. However it continues to act till the error signal disappears. This corrects the problem of the offset error in the proportional controller.

For example, let us assume that the integral controller is used to control the armature current of a d.c. motor and to keep its value constant at 500 A. As long as the armature current is less than 500 A, the armature voltage, controlled by the controller, will increase. Thus the output of the controller will increase and will continue to do so till the error becomes zero i.e. armature current becomes 500 A. Then the controller output will remain at that value reached. This is possible because the output of the controller can remain at any value within its range, if the input is zero. The controller must not be overdriven as it will not then be effective.

Thus for an integral mode,

1. If error is zero, the output remains at a fixed value equal to what is was, when the error became zero.
2. If the error is not zero, then the output begins to increase or decrease, at a rate K_i % per second for every ± 1 % of error.

In some cases, the inverse of K_i called **integral time** is specified, denoted as T_i.

$$T_i = \frac{1}{K_i} = \text{Integral time}$$

It is expressed in minutes instead of seconds.

6.6.3 Applications

The comparison of proportional and integral mode behaviour at the time of occurrence of an error signal is tabulated below,

Controller	Initial behaviour	Steady state behaviour
P	Acts immediately. Action according to K_p.	Offset error always present. Larger the K_p smaller the error.
I	Acts slowly. It is the time integral of the error signal.	Error signal always becomes zero.

Table 6.1

It can be seen that proportional mode is more favourable at the start while the integral is better for steady state response. In pure integral mode, error can oscillate about zero and can be cyclic. Hence in practice **integral mode is never used alone** but combined with the proportional mode, to enjoy the advantages of both the modes.

6.7 Derivative Control Mode

In practice the error is function of time and at a particular instant it can be zero. But it may not remain zero forever after that instant. Hence some action is required corresponding to the rate at which the error is changing. Such a controller is called derivative controller.

In this mode, the output of the controller depends on the time rate of change of the actual errors. Hence it is also called **rate action mode** or **anticipatory action mode.**

The mathematical equation for the mode is,

$$p(t) = K_d \frac{d\,e(t)}{dt}$$

Where K_d = Derivative gain constant.

The derivative gain constant indicates by how much % the controller output must change for every % per sec rate of change of the error. Generally K_d is expressed in minutes. The important feature of this type of control mode is that for a given rate of change of error signal, there is a unique value of the controller output.

The advantage of the derivative control action is that it responds to the rate of change of error and can produce the significant correction before the magnitude of the actuating error becomes too large. Derivative control thus anticipates the actuating error, initiates an early corrective action and tends to increase stability of the system improving the transient response.

6.7.1 Characteristics of Derivative Control Mode

The Fig. 6.7 shows how derivative mode changes the controller output for the various rates of change of the error.

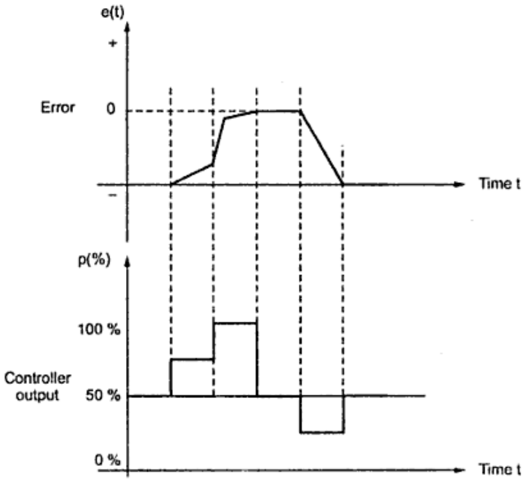


Fig. 6.7

The controller output is 50 % for the zero error. When error starts increasing, the controller output suddenly jumps to the higher value. It further jumps to a higher value for higher rate of increase of error. Then error becomes constant, the output returns to 50 %. When error is decreasing i.e. having negative slope, controller output decreases suddenly to a lower value.

The various characteristics of the derivative mode are,

1. For a given rate of change of error signal, there is a unique value of the controller output.
2. When the error is zero, the controller output is zero.
3. When the error is constant i.e. rate of change of error is zero, the controller output is zero.
4. When error is changing, the controller output changes by K_d % for even 1 % per second rate of change of error.

6.7.2 Applications

When the error is zero or a constant, the derivative controller output is zero. Hence it is never used alone. Its gain should be small because faster rate of change of error can cause very large sudden change of controller output. This may lead to the instability of the system.

6.8 Composite Control Modes

As mentioned earlier, due to offset error proportional mode is not used alone. Similarly integral and derivative modes are also not used individually in practice. Thus to take the advantages of various modes together, the composite control modes are used. The various composite control modes are,

1. Proportional + Integral mode (PI)
2. Proportional + Derivative mode (PD)
3. Proportional + Integral + Derivative mode (PID)

Let us see the characteristics of these three modes.

6.9 Proportional + Integral Mode (PI Control Mode)

This is a composite control mode obtained by combining the proportional mode and the integral mode.

The mathematical expression for such a composite control is,

$$\boxed{p(t) = K_p\, e(t) + K_p K_i \int_0^t e(t)dt + p(0)}$$

Where $p(0)$ = Initial value of the output at $t = 0$

The important advantage of this control is that one to one correspondence of proportional mode is available while the offset gets eliminated due to integral mode, the integral part of such a composite control provides a reset of the zero error output after a load change occurs.

Consider the load change occurring at $t = t_1$ and due to which error varies as shown in the Fig. 6.8. The controller output changes suddenly by amount V_p due to the proportional action. After that the controller output changes linearly with respect to time at a rate K_p / T_i. The reset rate is defined as the reciprocal of T_i.

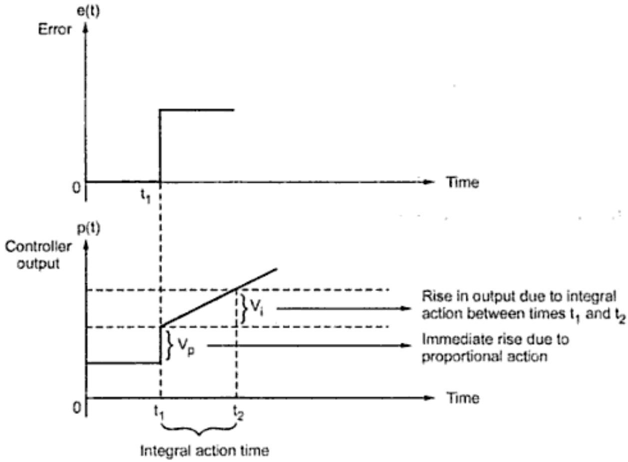


Fig. 6.8 Behaviour of PI controller

The response shown in the Fig. 6.8 is for the direct action of the controller. The response of composite PI control mode for the reverse action is shown in the Fig. 6.9.

In the reverse action, the proportional part is the image of the error. The sum of proportional plus integral action finally leaves the error to zero.

6.9.2 Applications

The composite PI mode completely removes the offset problems of proportional mode. Such a mode can be used in the systems with the frequent or large load changes. But the process must have relatively slow changes in the load, to prevent the oscillations.

6.10 Proportional + Derivative Mode (PD Control Mode)

The series combination of proportional and derivative control modes gives proportional plus derivative control mode. The mathematical expression for the PD composite control is,

$$\boxed{p(t) = K_p e(t) + K_p K_d \frac{d\,e(t)}{dt} + p(0)}$$

The behaviour of such a PD control to a ramp type of the input is shown in the Fig. 6.10.

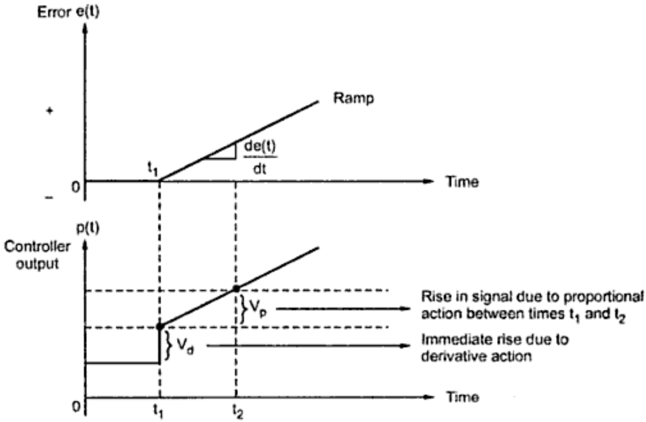


Fig. 6.10 Behaviour of PD controller

The ramp function of error occurs at $t = t_1$. The derivative mode causes a step V_d at t_1 and proportional mode causes a rise of V_p equal to V_d at t_2. This is for direct action PD control.

$$= 0.02[3t+7]+0.02\times 0.04\int_0^t (3t+7)dt+0.5$$

$$= [0.06t]+0.14+8\times 10^{-4}\left[\frac{3t^2}{t}+7t\right]_0^t +0.5$$

$$= [0.06\times 1.5]+[0.14]+8\times 10^{-4}\left[\frac{3}{2}+(1.5)^2+7\times 1.5\right]+0.5$$

$$= 0.09 + 0.14 + 0.0111 + 0.5 = 0.7411$$

i.e. $p(t) = 74.11$ % after t = **1.5 minutes.**

Example 6.3: *Suppose the error shown in the Fig. 6.14 is applied to a PD controller with $K_p = 6$ and $K_d = 0.4$ sec with $p(0) = 25$ %. Draw the graph of the controller output.*

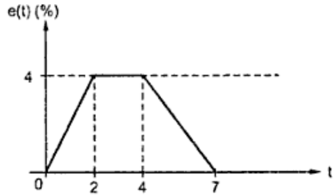


Fig. 6.14

Solution : The output of the PD controller is

$$p(t) = K_p\, e(t)+K_p K_d \frac{d\,e(t)}{dt}+25\%$$

It is necessary to obtain output response, dividing e(t) into three sections,

Section I] $0 \le t \le 2$

$$e(t) = 2t \qquad \text{... slope} = 2$$

$$\frac{d\,e(t)}{dt} = 2$$

$$\therefore \quad p_1(t) = 6\times 2t+6\times 0.4\times 2+25 = 12t + 29.8$$

Thus there is an instantaneous change of 29.8 − 25 = 4.8 % produced by this error in the output at t = 0.

$$\therefore \quad e_{ss} = \frac{A}{1+K_p} = 0 \text{ for the step input}$$

$$K_p = \lim_{s \to 0} s\, G(s)H(s) = \frac{\omega_n}{2\xi}$$

$$\therefore \quad e_{ss} = \frac{A}{K_v} = \frac{2\xi}{\omega_n}$$

So there is a finite error for the ramp input of magnitude A, which depends on ξ and ω_n.

For good time response, the system demands.

1. Less settling time
2. Less overshoot
3. Less rise time
4. Smallest steady state error.

Increasing K_v, the steady state error for the ramp input can be reduced but it increases overshoot and settling time. This may lead to instability of the system. So to keep steady state error and overshoot well within the acceptable limits, the various composite controllers are used. Let us see the effect of PD, PI, PID and rate feedback controller on the time response of the second order system under consideration.

6.14 PD Type of Controller

A controller in the forward path, which changes the controller output corresponding to proportional plus derivative of error signal is called **PD controller.**

i.e. $$\boxed{\text{Output of controller} = K\, e(t) + T_d \frac{de(t)}{dt}}$$

$$\text{Taking Laplace} = K\, E(s) + sT_d\, E(s) = E(s)\, [\, K + sT_d\,]$$

The T.F. of such controller is $[K + sT_d]$. This can be realized as shown in the Fig. 6.17.

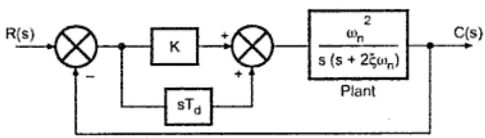


Fig. 6.17

Assuming K = 1, we can write,

$$G(s) = \frac{(1 + sT_d)\,\omega_n^2}{s(s + 2\xi\,\omega_n)}$$

and
$$\frac{C(s)}{R(s)} = \frac{(1 + s\,T_d)\,\omega_n^2}{s^2 + s[2\xi\,\omega_n + \omega_n^2\,T_d] + \omega_n^2}$$

Comparing denominator with standard form, ω_n is same as in the previous P type controller.

and
$$2\,\xi'\,\omega_n = 2\xi\,\omega_n + \omega_n^2\,T_d$$

$\therefore$
$$\boxed{\xi' = \xi + \frac{\omega_n\,T_d}{2}}$$

Because of this controller, damping ratio increases by factor $\frac{\omega_n\,T_d}{2}$.

$$\boxed{\begin{aligned} K_p &= \lim_{s \to 0} G(s)H(s) = \infty \\ K_v &= \lim_{s \to 0} s\,G(s)H(s) = \frac{\omega_n}{2\xi} \end{aligned}}$$

As there is no change in coefficients, error also will remain same.

Key Point: *Hence PD controller has following effects on system.*

i) It increases damping ratio.

ii) 'ω_n' for system remains unchanged.

iii) 'TYPE' of the system remains unchanged.

iv) It reduces peak overshoot.

v) It reduces settling time.

vi) Steady state error remains unchanged.

Key Point : *In general P.D. controller improves transient part without affecting steady state.*

6.15 PI Type of Controller

A controller in the forward path, which changes the controller output corresponding to the proportional plus integral of the error signal is called **PI controller.**

i.e. $$\boxed{\text{Output of controller} = K\, e(t) + K_i \int e(t)\, dt}$$

$$\text{Taking Laplace} = K\,E(s) + \frac{K_i}{s}\,E(s) = E(s)\left[K + \frac{K_i}{s}\right]$$

$\therefore$ The T.F. of such controller is $\left[K + \frac{K_i}{s}\right]$ and can be realised as shown in the Fig. 6.18.

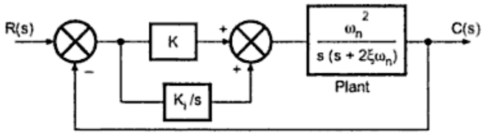


Fig. 6.18

Assuming K = 1, we can write,

$$G(s) = \frac{\left(1 + \frac{K_i}{s}\right)\omega_n^2}{s(s + 2\xi\,\omega_n)}$$

$$= \frac{(K_i + s)\,\omega_n^2}{s^2\,(s + 2\,\xi\,\omega_n)}$$

i.e. system becomes TYPE 2 in nature.

and $$\frac{C(s)}{R(s)} = \frac{(K_i + s)\,\omega_n^2}{s^3 + 2\xi\,\omega_n s^2 + s\,\omega_n^2 + K_i\,\omega_n^2}$$

i.e. it becomes third order.

Now as order increases by one, system relatively becomes less stable as K_i must be designed in such a way that system will remain in stable condition. Second order system is always stable.

Key Point: *Hence transient response gets affected badly if controller is not designed properly.*

While
$$K_p = \lim_{s \to 0} G(s)H(s) = \infty, \quad e_{ss} = 0$$

$$K_v = \lim_{s \to 0} s\,G(s)H(s) = \infty, \; e_{ss} = 0$$

Key Point : *Hence as type is increased by one, error becomes zero for ramp type of inputs i.e. steady state of system gets improved and system becomes more accurate in nature.*

Hence PI controller has following effects :

i) It increases order of the system.

ii) It increases TYPE of the system.

iii) Design of K_i must be proper to maintain stability of system. So it makes system relatively less stable.

iv) Steady state error reduces tremendously for same type of inputs.

Key Point: *In general this controller improves steady state part affecting the transient part.*

6.16 PID Type of Controller

As PD improves transient and PI improves steady state, combination of two may be used to improve overall time response of the system. This can be realized as shown in the Fig. 6.19(a).

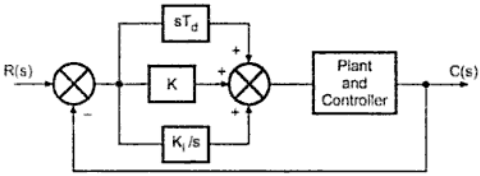


Fig. 6.19(a)

The design of such controller is complicated in practice.

6.17 Rate Feedback Controller (Output Derivative Controller)

This is achieved by feeding back the derivative of output signal internally using a tachogenerator and comparing with signal proportional to error as shown. This is called **minor loop feedback compensation.**

$$\boxed{\text{Output of controller} = K\,E(t) - K_t \frac{dc(t)}{dt}}$$

$\therefore$ Output of the controller = K E(s) − sK_t C(s) ... taking Laplace

This can be realized as shown in the Fig. 6.19(b).

Assuming K = 1, let us study its effect on same system which is considered earlier with $G(s) = \frac{\omega_n^2}{s(s+2\xi\,\omega_n)}$.

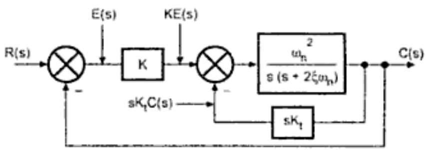


Fig. 6.19 (b)

Key Point: *1 Time constant `T' is the time required by the system output to reach 63.2 % of its final value during the first attempt.*

The equation for the actual response c(t) is,

$$c(t) = 1 - \left\{ \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta) \right\} \quad \text{... For unit step}$$

Where $\omega_d = \omega_n \sqrt{1-\xi^2}$

= Damped frequency of oscillations

and $\theta = \tan^{-1}\left\{ \frac{\sqrt{1-\xi^2}}{\xi} \right\}$ radians.

Note : As M_p is the function of ξ alone, if M_p is known, the equation of M_p can be solved to obtain the corresponding value of damping ratio ξ.

6.19 Steady State Error

Consider a simple closed loop system using negative feedback as shown in the Fig. 6.21.

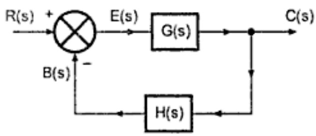


Fig. 6.21

Where E(s) = Error signal, and B(s) = Feedback signal

Now, $E(s) = R(s) - B(s)$

But $B(s) = C(s) \cdot H(s)$

$\therefore$ $E(s) = R(s) - C(s)H(s)$

and $C(s) = E(s) \cdot G(s)$

$\therefore$ $E(s) = R(s) - E(s)G(s)H(s)$

$\therefore$ $E(s) + E(s)G(s)H(s) = R(s)$

$$\therefore \quad \boxed{\begin{aligned} E(s) &= \frac{R(s)}{1+G(s)H(s)} \quad \text{for nonunity feedback} \\ E(s) &= \frac{R(s)}{1+G(s)} \quad \text{for unity feedback} \end{aligned}}$$

This E(s) is the error in Laplace domain and is expression in `s'. We want to calculate the error value. In time domain, corresponding error will be e(t). Now steady state of the system is that state which remains as $t \to \infty$.

$$\text{Steady state error, } e_{ss} = \lim_{t \to \infty} e(t)$$

Now we can relate this in Laplace domain by using **final value theorem** which states that,

$$\lim_{t \to \infty} F(t) = \lim_{s \to 0} s\,F(s) \qquad \text{Where } F(s) = L\{ f(t) \}$$

$$\text{Therefore,} \quad \boxed{e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} s\,E(s) \qquad \text{Where } E(s) \text{ is } L\{ e(t) \}}$$

Substituting E(s) from the expression derived, we can write

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)H(s)}$$

For negative feedback systems use positive sign in denominator while use negative sign in denominator if system uses positive feedback.

From the above expression it can be concluded that steady state error depends on,

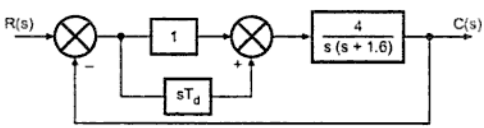
i) R(s) i.e. reference input, its type and magnitude.

ii) G(s)H(s) i.e. open loop transfer function.

iii) Dominant nonlinearities present if any.

Examples with Solutions

Example 6.4: *The figure shows PD controller used for the system. Determine the value of T_d so that system will be critically damped. Calculate its settling time.*



Solution : $$G(s) = \frac{(1+sT_d)\,4}{s\,(s+1.6)} \qquad H(s) = 1$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\dfrac{(1+sT_d)\,4}{s\,(s+1.6)}}{1+\dfrac{(1+sT_d)4}{s(s+1.6)}} = \frac{(1+sT_d)\,4}{s^2+1.65+4T_d\ s+4}$$

Comparing denominator with standard form,

$$\omega_n^2 = 4, \quad \omega_n = 2 \quad \text{and} \quad 2\xi\,\omega_n = 1.6+4\,T_d$$

$$\therefore \quad \xi = \frac{1.6+4T_d}{4}$$

Now system required is critically damped, i.e. $\xi = 1$

$$\therefore \quad 1 = \frac{1.6+4\,T_d}{4}$$

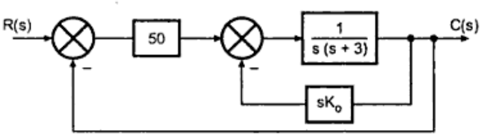
$$\therefore \quad 4 = 1.6+4T_d$$

$$\therefore \quad T_d = \mathbf{0.6} \quad \text{and settling time} = \frac{4}{\xi\,\omega_n}$$

$$T_s = \frac{4}{2\times 1} = \mathbf{2\ sec.}$$

Example 6.5: *A unity feedback system is shown in the following figure.*

i) In the absence of derivative feedback controller ($K_o = 0$). Find ξ and ω_n ii) Find K_o, if ξ is to be modified to 0.5 by use of controller.



Solution : $$G(s) = 50\times\frac{1}{s\,(s+3)} \qquad \text{for } K_o = 0$$

$$H(s) = 1$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{50}{s^2+3s+50}$$

$$\therefore \quad \omega_n^2 = 50 \qquad \therefore\ \omega_n = \sqrt{50} = \mathbf{7.071\ rad/sec.}$$

$$\therefore \quad \omega_n^2 = 14$$

$$\therefore \quad \omega_n = \sqrt{14} = 3.7416 \text{ rad/sec.}$$

$$2\xi\,\omega_n = 14\,K_t + 1.6$$

$$\therefore \quad \xi = \frac{14K_t + 1.6}{2\times\sqrt{14}} = 0.5 \text{ given}$$

$$\therefore \quad K_t = \mathbf{0.1529}$$

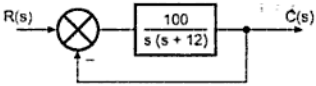
$$\omega_d = \omega_n\sqrt{1-\xi^2} = 3.7416\sqrt{1-(0.5)^2} = 3.2403 \text{ rad/sec}$$

$$T_p = \frac{\pi}{\omega_d} = \mathbf{0.9695\ sec.}$$

$$\%\,M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100 = \mathbf{16.3\ \%}$$

$$T_s = \frac{4}{\xi\,\omega_n} = \mathbf{2.1381\ sec.}$$

Example 6.7: *For the system shown determine % M_p and T_s when it is excited by unit step input. If for the same system, PD controller having constant T_d = 1/30 is used in forward path, determine new values of damping ratio, M_p and T_s. Draw respective waveforms.*



Solution : Without controller,

$$G(s) = \frac{100}{s(s+12)}, \quad H(s) = 1$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{100}{s^2 + 12s + 100}$$

$$\therefore \quad \omega_n^2 = 100 \qquad \therefore\ \omega_n = 10 \text{ rad/sec.}$$

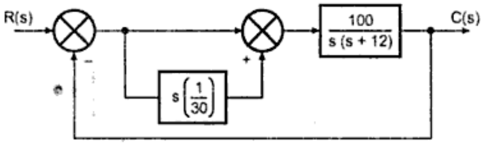
$$2\xi\,\omega_n = 12 \qquad \therefore \quad \xi = 0.6$$

$$\therefore \quad \omega_d = \omega_n\sqrt{1-\xi^2} = 10\times 0.8 = 8 \text{ rad/sec.}$$

$$\therefore \quad \%\,M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = \mathbf{9.47\ \%}$$

$$T_s = \frac{4}{\xi\,\omega_n} = \mathbf{0.666\ sec.}$$

With controller



$$G(s) = \frac{\left(1+\frac{s}{30}\right)100}{s(s+12)} = \frac{(s+30)\times 3.33}{s(s+12)},\ H(s) = 1$$

$$\frac{C(s)}{R(s)} = \frac{\frac{(s+30)\ 3.33}{s(s+12)}}{1+\frac{(s+30)\ 3.33}{s(s+12)}} = \frac{3.33(s+30)}{s^2+12s+3.33s+100} = \frac{3.33(s+30)}{s^2+15.33s+100}$$

$$\therefore \quad \omega_n^2 = 100 \qquad \therefore\ \omega_n = 10 \text{ rad/sec.}$$

$$2\xi\omega_n = 15.33$$

$$\therefore \quad \xi = \frac{15.33}{2\times 10} = \mathbf{0.7665}$$

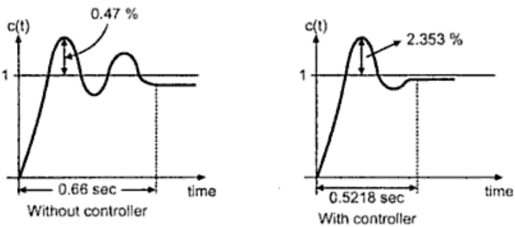
$$\therefore\ \xi \text{ is improved}, \omega_d = \omega_n\sqrt{1-\xi^2} = 10\sqrt{1-(0.7665)^2} = 6.4224 \text{ rad/sec.}$$

$$\%\ M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}\times 100 = \mathbf{2.353\ \%}$$

Overshoot decreased to 2.3 % from 9.47 %.

$$T_s = \frac{4}{\xi\omega_n} = \mathbf{0.5218\ sec.}$$

Comparison : Following figure shows comparison between system with controller and system without controller.



$\therefore \qquad \omega_n = \sqrt{10}$ **rad/sec**

and $\qquad 2\xi\omega_n = 2$

$\therefore \qquad \xi =$ **0.3162**

This is the required damping ratio.

For error calculation :

$$G(s)H(s) = \frac{5}{s\,(1+0.5s)}$$

For ramp input magnitude is unity i.e. A=1

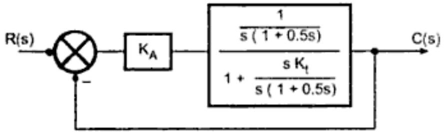
$$\therefore \qquad K_v = \lim_{s \to 0} s\,G(s)H(s) = \lim_{s \to 0} \frac{s.5}{s\,(1+0.5s)} = 5$$

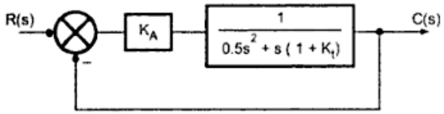
$$\therefore \qquad e_{ss} = \frac{A}{K_v} = \frac{1}{5} = \mathbf{0.2}$$

Case b) The derivative feedback is introduced in the system.

The system becomes,

$$\therefore \qquad G(s) = \frac{K_A}{s\,[0.5s+1+K_t\,]} \quad \text{and} \quad H(s)=1$$





$$\frac{C(s)}{R(s)} = \frac{\dfrac{K_A}{s\,[0.5s+1+K_t\,]}}{1+\dfrac{K_A}{s[0.5s+1+K_t\,]}} = \frac{K_A}{0.5s^2+(1+K_t\,)\,s+K_A}$$

$\therefore \quad \omega_d = 2\pi = \omega_n\sqrt{1-\xi^2}$

$\therefore \quad \omega_n = 6.8676 \text{ rad/sec}$

But $\quad K_i = \omega_n^2 = \mathbf{47.165}$

and $\quad 0.4037 = \dfrac{3+K_p}{2\times\omega_n}$

$\therefore \quad K_p = \mathbf{2.5449}$

Example 6.10 : *An integral controller is used for temperature control within a range 40 – 60 °C. The set point is 48 °C. The controller output is initially 12 % when error is zero. The integral constant K_I = – 0.2 % controller output per second per percentage error. If the temperature increases to 54 °C, calculate the controller output after 2 sec for a constant error.*

Solution : For integral controller,

$$p(t) = K_I\int E_p dt + p(0) \qquad \ldots E_p = \text{error}$$

$$p(0) = 12\,\%, \qquad K_I = -\,0.2\,\%\ /\ \text{sec}\ /\ \%\ \text{error}$$

$$E_p = \text{Constant} = \frac{r-b}{b_{max}-b_{min}}\times 100$$

Now $\quad r$ = Set point = 48 °C, $\quad b$ = Actual temperature = 54 °C

$$b_{max} = 60\,°C, \quad b_{min} = 40\,°C$$

$$\therefore \quad E_p = \frac{48-54}{60-40}\times 100 = -\,30\,\%$$

$$\int E_p dt = E_p\, t \quad \text{as error is constant}$$

$$\therefore \quad p(t) = (-\,0.2)\,(-\,30)\,(t) + 12$$

At t = 2, $\quad p(t) = (0.2 \times 30 \times 2) + 12 = \mathbf{24\,\%}$

This is the controller output after 2 sec.

Example 6.11 : *For a proportional controller, the controlled variable is a temperature with a range of 50 to 130 °C with a set point of 73.5 °C. The controller output is 50 % for zero error. The offset error is corresponding to a load change which causes 55 % controller output. If the proportional gain is 2, find the % controller output if the temperature is 61 °C.*

Solution : For a proportional controller,

$$p(t) = K_P\, E_p(t) + p_0$$

Where $\quad E_p(t)$ = Error, $\quad K_P = 2, \quad p_0 = 50\,\%$

After end of 2 sec, integral term gets accumulated to,

$$P_I(2) = 2\times 2.2\left[\frac{1}{2}t^2\right]_{t=2} + 40 = 48.8\ \%$$

For $2 \le t \le 4$, $E_{p2}(t) = -2.5t + 7$

$$\therefore \quad \int_2^t E_{p2}(t)\,dt = \int_2^t (-2.5t+7)\,dt = \left[\frac{-2.5t^2}{2}+7t\right]_2^t$$

$$= -1.25\,(t^2 - 4) + 7(t - 2)$$

$$\therefore \quad p_2(t) = 2[-2.5t+7]+4.4[-1.25(t^2-4)+7(t-2)]+48.8$$

$$= -5t + 14 - 5.5t^2 + 22 + 30.8t - 61.6 + 48.8$$

$$\therefore \quad p_2(t) = -5.5t^2 + 25.8t + 23.2 \quad \text{for } 2 \le t \le 4 \qquad \ldots (2)$$

This is plotted for 2 – 4 sec.

After end of 4 sec, the integral term gets accumulated to,

$$P_I(4) = -1.25\left(t^2-4\right)+7(t-2)\Big|_{t=4} + 48.8 = 47.8\ \%$$

For $4 \le t \le 6$, $E_{p3}(t) = 1.5t - 9$

$$\therefore \quad \int_4^t E_{p3}(t)\,dt = \int_4^t (1.5t-9)\,dt = \left[\frac{1.5t^2}{2}-9t\right]_4^t$$

$$= 0.75\,(t^2 - 16) - 9\,(t - 4)$$

$$\therefore \quad p_3(t) = 2[1.5t-9]+4.4[0.75(t^2-16)-9(t-4)]+47.8$$

$$= 3t - 18 + 3.3t^2 - 52.8 - 39.6t + 158.4 + 47.8$$

$$= 3.3t^2 - 36.6\,t + 135.4 \quad \text{for } 4 \le t \le 6 \qquad \ldots (3)$$

This is plotted for 4 – 6 sec. After 6 sec as error is zero hence output is accumulated integral response of $p_3(t)\big|_{t=6} = 34.6\ \%$

The graph of p(t) against time is shown in the Fig. 6.23.
(See Fig. 6.23 on next Page)

Example 6.14 : *A proportional controller is employed for the control of temperature in the range 50 °C - 130 °C with a set point of 73.5 °C. The zero error controller output is 50 %. What will be the offset error resulting from a change in the controller output to 55 % ? The proportional gain is 2 % / %. Find the offset in °C.*

Solution : For proportional control mode

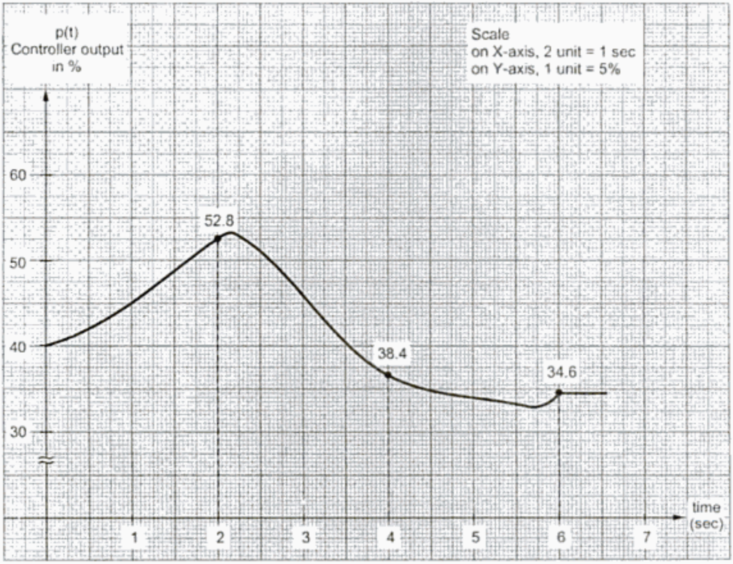


Fig. 6.23

$$p = K_P E_p + p_0$$

Where p_0 = Controller output with no error = 50 %

$$K_P = 2\ \%\ \text{per}\ \%$$

$$\therefore \quad E_p = \frac{p - p_0}{K_P} = \frac{55 - 50}{2} = \mathbf{2.5\ \%}$$

The offset error is 2.5 %.

The range of 50 °C to 130 °C so b_{max} = 130 °C and b_{min} = 50 °C.

$$e = \frac{r - b}{b_{max} - b_{min}} \times 100 \qquad \text{Where } r = \text{set point}$$

$$\therefore \quad 2.5 = \frac{73.5 - b}{130 - 50} \times 100$$

$$\therefore \quad b = \mathbf{71.5\ °C}$$

This is temperature corresponding to offset error.

Example 6.15 : *A PID controller has $K_P = 2.0$, $K_I = 2.2\ sec^{-1}$, $K_D = 2$ sec and $P_I(0) = 40$ %. Draw the plot of controller output for error of Fig. 6.24.*

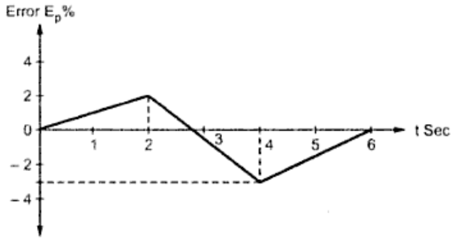


Fig. 6.24

Solution : $K_P = 2$, $K_I = 2.2\ sec^{-1}$, $K_D = 2$ sec, $P_I(0) = 40$ %

From the given error plot,

For 0 – 2 sec, $E_P = m_1 t$ Where $m_1 = \text{Slope} = \frac{2-0}{2-0} = 1$

$$\therefore \quad E_p = t \quad \text{for } 0 - 2 \text{ sec.}$$

For 2 – 4 sec, $E_p = m_2 t + C_2$

Two points on the line are (2, 2) and (4, –3).

$$\therefore \quad m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-2}{4-2} = -2.5$$

$$\therefore \quad E_p = -2.5t + C_2$$

At (2, 2), $2 = -2.5 \times 2 + C_2$, $\therefore C_2 = 7$

$$\therefore \quad E_p = -2.5t + 7 \quad \text{for } 2 - 4 \text{ sec.}$$

For 4 – 6 sec, $E_p = m_3 t + C_3$

Two points on the line are (4, – 3) and (6, 0).

$$\therefore \quad m_3 = \frac{0-(-3)}{6-4} = +1.5$$

$$\therefore \quad E_p = +1.5\,t + C_3$$

At (4, –3), $-3 = 1.5 \times 4 + C_3$, $\therefore C_3 = -9$

$$\therefore \quad E_p = +1.5\,t - 9 \quad \text{for } 4 - 6 \text{ sec.}$$

Three mode equation for PID controller is,

$$p = K_P E_P + K_P K_I \int_0^t E_P dt + K_P K_D \frac{dE_P}{dt} + P_I(0)$$

$$\therefore \quad p = 2E_P + 4.4\int_0^t E_P dt + 4\frac{dE_P}{dt} + 40$$

For 0 – 2 sec, $$p_1 = 2t + 4.4\int_0^t t\,dt + 4\frac{d}{dt}(t) + 40$$

$$= 2t + 2.2t^2 + 44 \quad \ldots (1)$$

This is plotted for 0 – 2 sec. At the end of 2 sec, the integral term has accumulated to,

$$P_I(2) = 4.4\int_0^t t\,dt + 40 = 4.4 \times \left[\frac{t^2}{2}\right]_0^2 + 40 = 48.8\ \%$$

For 2 – 4 sec, $$p_2 = 2(-2.5t+7) + 4.4\int_2^t (-2.5t+7)\,dt + 4\frac{d}{dt}(-2.5t+7) + 48.8$$

$$= -5t + 14 + 4.4\left[-1.25t^2 + 7t\right]_2^t - 4 \times 2.5 + 48.8$$

$$= -5t + 14 + 4.4\left\{-1.25\left(t^2 - 4\right) + 7(t-2)\right\} - 10 + 48.8$$

$$= -5.5t^2 + 25.8t + 13.2 \quad \ldots (2)$$

This is plotted for 2 – 4 sec. At the end of 2 sec, the integral term has accumulated to,

$$P_I(4) = 4.4\left\{-1.25\left(t^2 - 4\right) + 7(t-2)\right\}\Big|_{t=4} + 48.8 = 44.4\ \%$$

For 4 – 6 sec,

$$p_3 = 2(1.5t-9) + 4.4\int_4^t (1.5t-9)\,dt + 4\frac{d}{dt}(1.5t-9) + 44.4$$

$$= 3t - 18 + 4.4\left[0.75t^2 - 9t\right]_4^t + 4 \times 1.5 + 44.4$$

$$= 3t - 18 + 4.4\left\{0.75\left(t^2 - 16\right) - 9(t-4)\right\} + 6 + 44.4$$

$$= 3.3t^2 - 36.6t + 138 \quad \ldots (3)$$

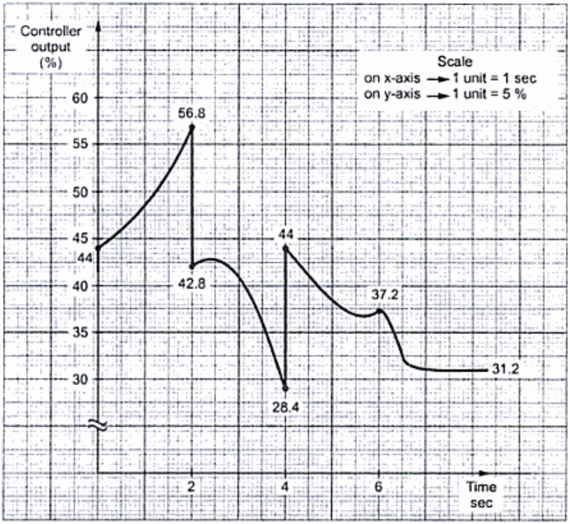
This is plotted for 4 – 6 sec.

After 6 sec, error is zero hence the output will simply be the accumulated integral response providing a constant output.

$$\therefore \quad P_I(6) = 4.4\left\{0.75\left(t^2-16\right)-9(t-4)\right\}\Big|_{t=6} + 44.4$$

$$= 31.2 \ \%$$

The complete graph of controller output is shown in the figure.



Example 6.16 : *A temperature control system has the block diagram given in Fig. 6.25. The input signal is a voltage and represents the desired temperature θ_r is a unit step function and i) D(s) = 1 ii) D(s) = 1 + $\frac{0.1}{s}$ iii) D(s) = 1 + 0.3 s. What is the effect of the integral term in the PI controller and the derivative term in PD controller on the steady state error ?*

(VTU: July/Aug.-2006, 8 Marks)

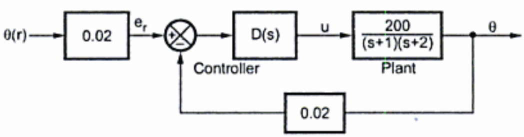


Fig. 6.25

Solution : For the given system,

$$G(s) = \frac{200\,D(s)}{(s+1)(s+2)}, \quad H(s) = 0.02, \quad R(s) = \frac{0.02}{s}$$

$$e_{ss} = \lim_{s\to 0} s\,E(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)}$$

i) $$D(s) = 1$$

$\therefore$ $$e_{ss} = \lim_{s\to 0} \frac{s\times\frac{0.02}{s}}{1+\frac{200}{(s+1)(s+2)}\times 0.02} = \frac{0.02}{1+\frac{200\times 0.02}{2}} = \mathbf{6.66\times 10^{-3}}$$

ii) $$D(s) = 1+\frac{0.1}{s}$$

$\therefore$ $$e_{ss} = \lim_{s\to 0} \frac{s\times\frac{0.02}{s}}{1+\left(\frac{200\left(1+\frac{0.1}{s}\right)}{(s+1)(s+2)}\times 0.02\right)} = \mathbf{0}$$

iii) $$D(s) = 1 + 0.3\,s$$

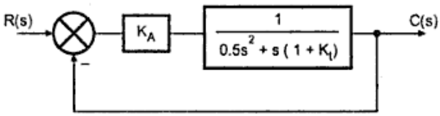
$\therefore$ $$e_{ss} = \lim_{s\to 0} \frac{s\times\frac{0.02}{s}}{1+\left(\frac{200(1+0.3s)}{(s+1)(s+2)}\times 0.02\right)} = \mathbf{6.66\times 10^{-3}}$$

Due to PI controller, the steady state error reduces drastically while PD controller has no effect on the steady state error.

Review Questions

1. *What is controller ? Explain its function in a system.*
2. *Give the classification of controllers.*
3. *State and explain the various properties of controller.*
4. *State the various continuous controller modes.*
5. *Explain the proportional control mode. State its characteristics.*
6. *Explain the integral control mode. State its characteristics.*
7. *Explain the derivative control mode. State its characteristics.*
8. *Explain how constant K_i affect the output of PI control mode.*
9. *Why derivative mode is called anticipatory control mode ?*
10. *State the various composite control modes.*

11. *Explain the PI control mode, stating its characteristics.* **(July-2005)**
12. *Explain the PD control mode, stating its characteristics.*
13. *Write a note on three mode controller.*
14. *Discuss the effect of following controllers on the second order control system,*
    a) PI controller
    b) PD controller
15. *Explain the effect of PI controller on typical second order system.*
16. *Explain the effect of PD controller on typical second order system.* **(July-2005)**
17. *A feedback system which uses a rate feedback controller is shown in the figure.*



a) For $K_A = 10$, *in absence of derivative feedback* $(K_h = 0)$ *determine the damping ratio and natural frequency of oscillations. Also find the s.s error for unit ramp input.*

b) Determine the constant K_h *if the damping factor required is 0.6, with* $K_A = 10$. *With this value of* K_h, *determine s.s error for unit ramp input.* **(Ans. : 0.316, 3.16 rad/sec, 0.2, 1.8, 0.38)**

7

Nonlinear Systems

7.1 Introduction to Nonlinear Systems

It has been mentioned earlier that a control system is said to be linear if it obeys law of superposition. Most of the control systems are nonlinear in nature and are treated to be linear, under certain approximations, from ease of analysis point of view. Let us discuss now the properties of nonlinear systems. In practice nonlinearities may exist in the systems inherently or may be purposely introduced in the systems, to improve the performance. Hence knowledge of properties of nonlinear systems and various nonlinearities is important.

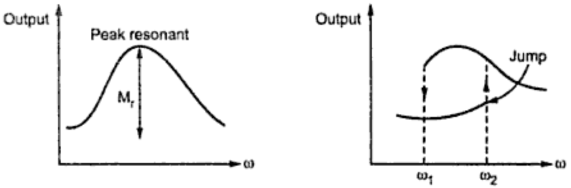
7.2 Properties of Nonlinear Systems

The various characteristics of nonlinear systems are,

1. The most important characteristics of a nonlinear system is that it does not obey the law of superposition. Hence its behaviour with respect to standard test inputs can not be used as base to analyse its behaviour with respect to other inputs. Its response is different for different amplitudes of input signals. Hence while doing the analysis of nonlinear system, alongwith the mathematical model of the system, it is necessary to have information about amplitudes of the probable inputs, initial conditions etc. This makes the analysis of the nonlinear system difficult.
2. Linear system gives sinusoidal output for a sinusoidal input, may be introducing a phase shift. But nonlinear system produces higher harmonics and sometimes the subharmonics. Hence for sinusoidal input, the output of a nonlinear system is generally nonsinusoidal. The output consists of frequencies which are multiples of the input frequency i.e. harmonics. The subharmonics means the presence of frequencies which are lower than the input frequency. The input and output relations are not linear.
3. In linear systems, the sinusoidal oscillations depend on the input amplitude and the initial conditions. But in a nonlinear system, the periodic oscillations may exist

which are not dependent on the applied input and other system parameter variations. In nonlinear system, such periodic oscillations are nonsinusoidal having fixed amplitude and frequency. Such oscillations are called **limit cycles** in case of nonlinear system.

4. Another important phenomenon which exists only in case of nonlinear system is **jump resonance**. This can be explained by considering a frequency response. The Fig. 7.1 (a) shows the frequency response of a linear system which shows that output varies continuously as the frequency changes. Similarly though frequency is increased or decreased, the output travels along the same curve again and again. But in case of a nonlinear system, if frequency is increased, the output shows discontinuity i.e. it jumps at a certain frequency. And if frequency is decreased, it jumps back but at different frequency. This is shown in the Fig. 7.1 (b).



(a) Linear system (b) Nonlinear system

Fig. 7.1 Jump resonance

5. There is no definite criterion for judging the stability of the nonlinear system. The analysis and design techniques of linear systems cannot be applied to the nonlinear system.

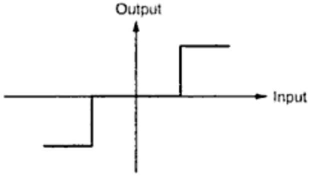
7.3 Classification of Nonlinearities

Basically the nonlinearities are classified based on the fact that whether they are inherent or purposely introduced in the system. Depending upon this the two classes of nonlinearities are,

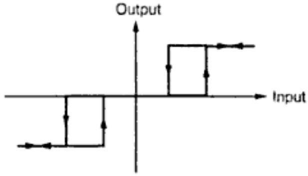
1. Inherent nonlinearities.
2. Intentional nonlinearities.

7.4 Inherent Nonlinearities

The nonlinearities which are unavoidable in the control systems are called inherent nonlinearities. Let us discuss various types of inherent nonlinearities existing in practice.



(a) On-off with deadzone



(b) On-off with deadzone and hysteresis

Fig. 7.7

7.4.5 Nonlinear Friction

In any practical system, where there is a relative motion between the two moving surfaces, there exists a friction. There are various types of such friction. All of them are nonlinear except **viscous friction.** The nonlinear friction is called **Coulomb friction.**

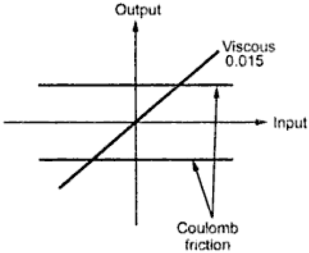


Fig. 7.8 Friction

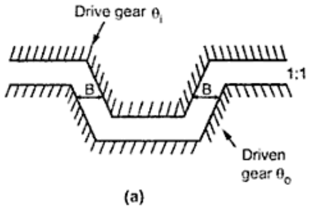
The coulomb friction is a force acting in opposite direction of motion but it is constant in magnitude irrespective of velocity.

The viscous and coulomb friction are shown in the Fig. 7.8. It can be seen that viscous friction is linear while coulomb friction is constant irrespective of the input.

The example of coulomb friction is the friction existing between the brushes resting against the commutator in an electrical motor.

7.4.6 Backlash

This type of nonlinearity is generally found in the mechanical linkage where coupling is not perfect. A common example is the existence of free play between teeth of drive gear and driven gear of a gear train. This is shown in the Fig. 7.9 (a). The driven gear does not work until there is contact between drive and driven gear. So input member has to travel the distance B to have the output, as indicated by input-output characteristics shown in the Fig. 7.9 (b). Once the contact is made output follows the input. If the motion is to be reversed, then input member has to travel distance of 2B in reverse direction, as shown in the Fig. 7.9 (b).



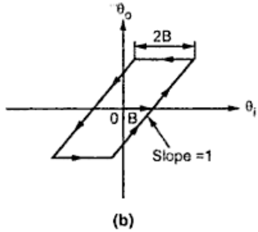


Fig. 7.9 Backlash

This is the most complex nonlinearity from modelling point of view. This is also memory dependent nonlinearity. It is a double valued nonlinearity like on-off with hysteresis. A process control valve with hysteresis is another example of backlash type of nonlinearity.

7.4.7 Nonlinear Spring

A linear spring has a linear equation for its force and displacement.

$$\text{Linear spring force} = kx \qquad \dots (1)$$

where k = positive spring constant

and x = spring displacement

The nonlinear spring is not governed by the equation (1) but its force and displacement are related by the equation,

$$\text{Nonlinear spring force} = kx + k' x^3 \qquad \dots (2)$$

where k = Positive spring constant

and k' = Nonlinear spring constant

The k' can be positive or negative. If k' is positive, the spring is called **hard spring** while if k' is negative, the spring is called **soft spring**. If $k' = 0$, the spring is no longer a nonlinear but behaves in a linear fashion.

The degree of nonlinearity in the spring is totally controlled by the magnitude of k'.

If such a spring is used in the system consisting of mass and friction then it is experimentally observed that the frequency of the free oscillations increases or decreases as the amplitude decreases depending upon the value of k'.

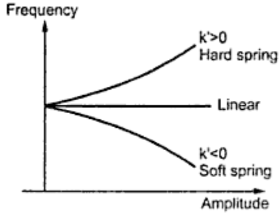


Fig. 7.10 Frequency-amplitude curves for nonlinear spring

The frequency-amplitude curves for a system with nonlinear spring are shown in the Fig. 7.10. For $k' > 0$, as the amplitude decreases, frequency also decreases. This is hard spring. For $k' < 0$, as amplitude decreases, the frequency increases. This is soft spring. For a linear spring such characteristics is a straight line.

These curves are often used to find out whether there exists a nonlinearity in the spring and also to find the degree of nonlinearity.

7.5 Absolute Value Nonlinearity

In some systems, though the direction of the input is changed, the output direction remains same. The value of output is absolute irrespective of the direction of the input applied. The best example of such a nonlinearity is a relay operated by the solenoid. Irrespective of the direction of the current through the solenoid, relay gets operated. This is shown in the Fig. 7.11 (a) and (b).

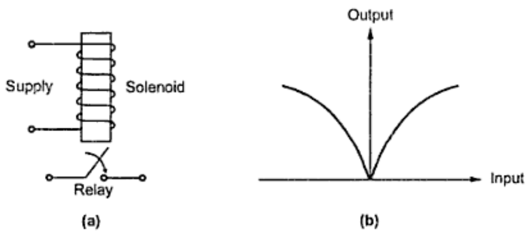


Fig. 7.11 Absolute value nonlinearity

Thus the output is not affected by the direction of the input applied.

8 Phase Plane Method

8.1 Introduction

The phase plane method is an important method used for the analysis of nonlinear systems. The phase plane method is applied normally to first order and second order of linear and nonlinear systems. The system behaviour is qualitatively analysed along with design of system parameters so as to get the desired response from the system. The periodic oscillations in nonlinear systems called limit cycle can be identified with this method which helps in investigating the stability of the system.

8.2 Limit Cycle

In case of linear time invariant systems a periodic oscillation is sinusoidal. The amplitude of such oscillations is a function of excitation applied and initial conditions. If the system parameters are changed slightly i.e. if the system poles are shifted from the imaginary axis of complex s plane, then the oscillations will no longer exist.

In case of nonlinear systems, periodic oscillations are present but are not dependent on amplitude of applied excitation. Also the oscillations are not much sensitive to parameter variations.

A periodic oscillation is case of a nonlinear system is called **limit cycle**. Normally the limit cycles are non-sinusoidal. The limit cycles which are having fixed amplitude and time period can be observed over a finite range of system parameters.

The limit cycle is also called **self excited oscillation** which is observed in certain nonlinear systems. Consider a system described by following equation,

$$M\ddot{x} - \mu(1 - x^2)\dot{x} + kx = 0$$

Here M, μ and k are positive quantities. This is a nonlinear equation. For small values of x, the damping will be negative which indicates the energy is actually put in the system. For large values of x, the damping is positive which removes the energy from the system. Thus it can be expected that such system may exhibit a sustained oscillation. Now as it is not a forced system, these oscillations are called self excited oscillations or limit cycle.

8.3 Jump Resonance

The phenomenon of jump resonance is seen only in case of non linear systems. This can be explained by considering the frequency response plots. The Fig. 8.1 shows frequency response plot for a linear system while the Fig. 8.2 shows frequency response plot for nonlinear system.

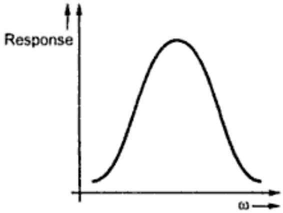


Fig. 8.1 Linear system

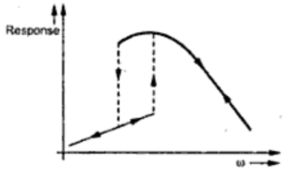


Fig. 8.2 Nonlinear system

Consider a nonlinear system excited by sinusoidal input of constant amplitude. If the input frequency is increased then jump (or discontinuity) occurs in the response amplitude. With decrease in frequency also, jump can be observed but at a different frequency.

A nonlinear system is one which behaves quite differently for various input functions. Consequently the logical design of a nonlinear system requires a complete description of the input signals. This dependence of the system behaviour on the actual input functions is demonstrated by the jump resonance phenomenon which is observed in certain closed loop systems with saturation.

Let us consider a nonlinear equation describing a certain system.

$$M\ddot{x} + f\dot{x} + k_1 x + k_2 x^3 = f \cos \omega t$$

If the magnitude of the response for above system is plotted against frequency, then corresponding frequency response plot for $k_2 > 0$ is shown in the Fig. 8.3.

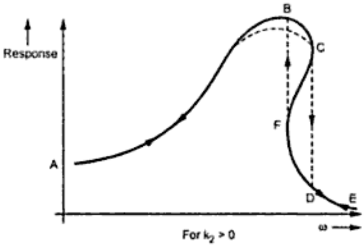


Fig. 8.3 Jump resonance

Due to the presence of a nonlinear term $k_2 x^3$, the resonant peak is bent towards the higher frequencies. As the input frequency is gradually increased from zero with input amplitude fixed, the measured response follows the curve through the points A, B and C. But at C, an increment in frequency results in a discontinuous jump down to the point D, after which with further increase in frequency, the response curve follows through DE.

If the frequency is decreased, the response follows the curve EDF with jump to B occurring at F and then move to A. For certain range of frequency the response function is double valued.

For $k_2 < 0$, the resonant peak bends towards lower frequencies which is shown in the Fig. 8.4. Similar jump resonance takes place with the difference that there is upward jump when frequency is increased and downward jump when frequency is decreased.

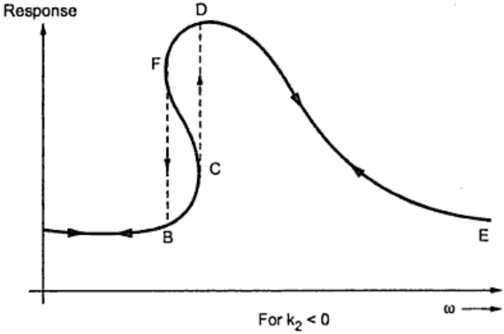


Fig. 8.4 Jump resonance

The overall curve shows the familiar **hysteresis** or **jump resonance** phenomenon. It is well known in nonlinear mechanics and is frequently observed in high gain servo systems. In such cases, the response is multivalued and depends not only on the present value of the input but also on the past history.

8.4 The Phase Plane Method

The equation of a second order nonlinear system having free motion has the following form,

$$\ddot{y} - h(y, \dot{y})\,\dot{y} + g(y, \dot{y})\,y = 0$$

At any particular moment, the state of the system can be represented with the help of a point having coordinates $(y, \dot{y})$ in rectangular coordinate system. This coordinate plane is called a **'phase plane'**.

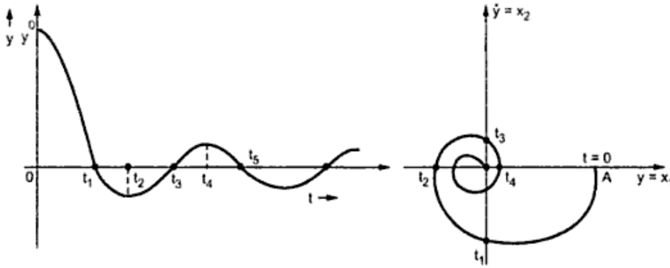


Fig. 8.5

The starting point A with initial condition but no initial velocity, the system comes to equilibrium i.e. to the origin with damped oscillatory behaviour.

8.5 Basic Concept of Phase-Plane Method

Consider the differential equation for a linear second order control system

$$\frac{d^2 y(t)}{dt^2} + \alpha \frac{dy(t)}{dt} + \beta y(t) = u(t)$$

Here u is input while y is output.

The phase plane method is basically a graphical method of solving second order linear as well as non linear system equations. The co-ordinate plane whose axes correspond to dependent variable y(t) and its first derivative y(t) is called phase plane. In phase plane plot y(t) and $\dot{y}(t)$ are plotted in a two axes plane. The trace of the phase plane plot as time t increases is called trajectory. The starting point of the trajectory depends on the initial conditions imposed on the system.

For different sets of initial conditions, various trajectories can be plotted. The complete dynamic behaviour of the system for various initial conditions can be obtained from the trajectories plotted in phase plane.

The dynamic performance and analysis of given second order system differential equation can be made based on destination point of trajectories. The method has limitation of its application restricted to second order as for higher order system equations, three or higher dimensional plane is needed which is not much convenient to construct and visualise. But transient response of second order system excited by step input can be studied by this method. This method is thus applicable to linear time functions and can not be applied to systems excited by sinusoidal signals.

The phase plane method can be considered as a special case of state space. With the use of state variables known as phase variables, the state of second order system can be defined.

In phase plane analysis, the trajectories can be constructed analytically, graphically or experimentally. The analytical method is straight but not useful from practical point of view. The graphical methods are used when it is not possible to solve the given differential equation analytically. The graphical methods are applicable to both linear and non linear equations.

8.6 Isocline Method for Construction of Phase Trajectory

The isocline method is a graphical approach for determination of phase portrait.

Consider the state equation for a second order system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

The slope m of trajectory at any point in the phase plane is given by,

$$\frac{dx_2}{dx_1} = m = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

The point (x_1, x_2) of the phase plane has with it associated slope of the trajectory except at singular point where the slope is indetermiante.

Isoclines are the lines in the phase plane corresponding to slopes of the phase portrait. In order to construct the phase trajectory, it is necessary to integrate the slope equation. This is very difficult except for few simple cases.

We have,

$$f_2\left(\dot{x}_1, x_2\right) = m_1 f_1(x_1, x_2)$$

The above equation gives locus of all points in the phase plane where the slope of trajectory is m_1. Such a locus is called isocline. The direction of phase trajectroy at any point (x_1, x_2) can be obtained from the sign of Δx_1 and Δx_2 for small increment in Δt.

$$\frac{\Delta x_1}{\Delta t} = f_1(x_1, x_2) \qquad \therefore \Delta x_1 = f_1(x_1, x_2)\,\Delta t$$

$$\frac{\Delta x_2}{\Delta t} = f_2(x_1, x_2) \qquad \therefore \Delta x_2 = f_2(x_1, x_2)\,\Delta t$$

Consider once again second order system equation.

$$\frac{d^2 y(t)}{dt^2} + \alpha \frac{dy(t)}{dt} + \beta y(t) = u(t)$$

Subject to initial conditions $y(t_0) = y_0$

Based on various slopes of trajectories, isoclines are drawn as shown in the Fig. 8.7

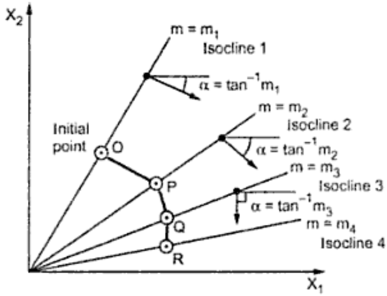


Fig. 8.7

From the given initial conditions, the starting point is O which is the point on isocline 1 corresponding to slope m_1. The phase trajectory which is drawn from point on the isocline whose slope is m_1. The phase trajectory which is drawn from point O on the isocline whose slope is m_1 to that having slope m_2 is a straight line whose slope is the average of m_1 and m_2 i.e. $\frac{m_1 + m_2}{2}$. This line intersects isocline 2 at point P which has a slope of m_2. Thus we get line segment OP. This procedure is continued at point P to get subsequent line segments PQ, QR and so on as shown in the Fig. 8.7

Now consider undamped second order system such that $\xi = 0$

$$\frac{d^2y(t)}{dt^2} + 2\xi\omega n \frac{dy(t)}{dt} + \omega_n^2 \, y(t) = 0$$

For simplicity let us select $\omega_n = 1$ $\therefore$ The above equation is

$$\frac{d^2y(t)}{dt^2} + y(t) = 0$$

Let us now select the state variables

$$x_1 = y(t)$$

$$x_2 = \dot{y}(t)$$

The state equations are,

$$\dot{x}_1 = \dot{y}(t) = x_2$$

$$\dot{x}_2 = \ddot{y}(t) = -y(t) = -x_1$$

Dividing above equations,

$$\frac{dx_2}{dx_1} = \frac{dx_2/dt}{dx_1/dt} = \frac{-x_1}{x_2}$$

$$\therefore \quad x_2 dx_2 = -x_1 dx_1$$

Integrating and rearranging above equation.

$$x_1^2 + x_2^2 = C$$

Here C is arbitrary constant obtained from initial conditions. The phase portrait in phase plane is as shown in the Fig. 8.8. The above equation represents a circle. The phase portrait consists of concentric circles depending on different initial conditions $\left(x_{1_0}^1 \;\; x_{1_0}^2 \;\; x_{1_0}^3 \text{ etc.}\right)$

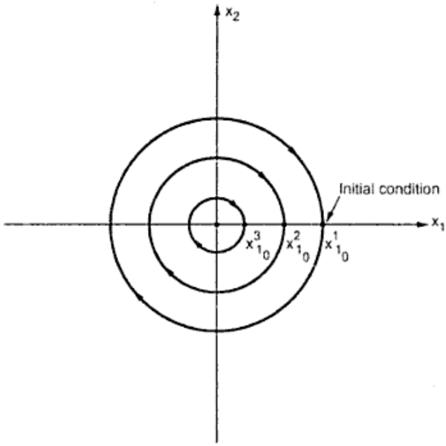


Fig. 8.8

8.7 Application of Phase-Plane Method to Linear Control System

Before studying the application of phase-plane method to non-linear systems, let us see its application in linear systems.

Let us consider second order linear control system as shown in the Fig.8.9

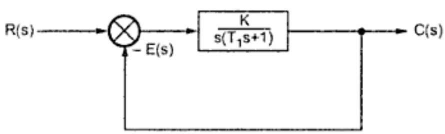


Fig. 8.9

The transfer function is given as,

$$\frac{C(s)}{R(s)} = \frac{\left[\dfrac{K}{s(T_1 s+1)}\right]}{\left[1+\dfrac{K}{s(T_1 s+1)}\right]} = \frac{K}{s(T_1 s+1)+K} = \frac{\left(\dfrac{K}{T_1}\right)}{s^2+\dfrac{1}{T_1}s+\dfrac{K}{T_1}}$$

In the forward path, the transfer function is

$$\frac{C(s)}{E(s)} = \frac{K}{s(T_1 s+1)}$$

$$\left[s(T_1 s+1)\right]C(s) = KE(s)$$

$$T_1 s^2 C(s) + s\,C(s) = KE(s)$$

Taking inverse Laplace transform,

$$T_1 \ddot{c} + \dot{c} = Ke$$

But $e = r - c$

$$\dot{e} = \dot{r} - \dot{c} \quad \therefore \dot{c} = \dot{r} - \dot{e}$$

$$\ddot{e} = \ddot{r} - \ddot{c} \quad \therefore \ddot{c} = \ddot{r} - \ddot{e}$$

$$\therefore T_1\left[\ddot{r} - \ddot{e}\right] + \left[\dot{r} - \dot{e}\right] = Ke$$

$$T_1 \ddot{e} + \dot{e} + Ke = T_1 \ddot{r} + \dot{r}$$

Let the input be step input

$$r(t) = R$$

$$\therefore \quad \dot{r}(t) = \ddot{r}(t) = 0$$

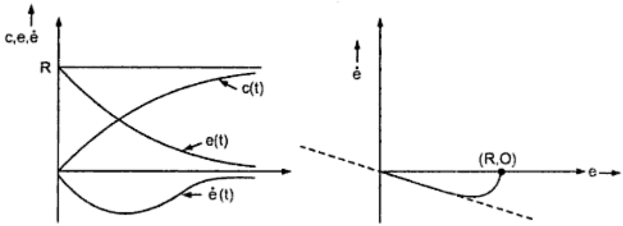
The above equation becomes

$$T_1 \ddot{e} + \dot{e} + Ke = 0 \quad \text{for } t > 0$$

Now $e(0) = r(0) - c(0)$

At $t = 0$, $c = 0$ i.e. $c(0) = 0$ $r(0) = R$

When the value of K is adjusted in such a way as to get the system overdamped then the phase plane trajectory is as shown in the Fig. 8.11



(a) Step response (Overdamped case) (b) Corresponding phase portrait

Fig. 8.11

For the linear systems, where analytical solution can be computed for c and $\dot{c}$ then it is not required to draw phase portrait. It can be effectively applied in case of non-linear systems.

8.8 Second Order Nonlinear System on Phase Plane

Let us consider the Van der Pol's differential equation,

$$\ddot{y} - \mu(1 - y^2)\dot{y} + y = 0$$

This equation describes the behaviour in many nonlinear systems. Let us select the state variables as $x_1 = y$ and $x_2 = \dot{y}$.

The state equations are given as,

$$\dot{x}_1 = \dot{y} = x_2$$

$$\dot{x}_2 = \ddot{y} = \mu(1 - x_1^2)x_2 - x_1$$

The equilibrium point of this system is origin of the phase plane. Now there are two cases viz. $\mu > 0$ and $\mu < 0$. The phase portraits for both these cases is shown in the Fig. 8.12.

When $\mu > 0$ and initial value of x_1 [i.e. $x_1(0)$] is large then the system response is damped. The amplitude of $x_1(t)[= y_1(t)]$ decreases till the system enters the limit cycle. This is shown by the outer trajectory. For small initial values of x_1, the damping is negative.

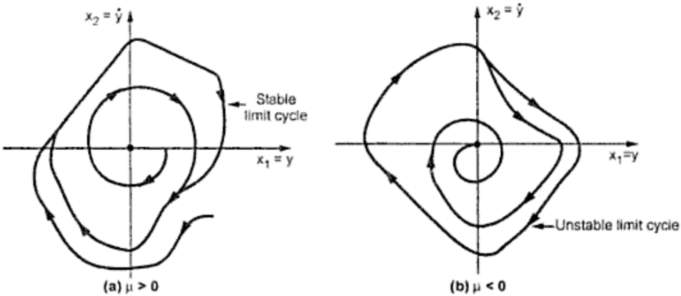


Fig. 8.12

The amplitude of $x_1(t)[= y_1(t)]$ increases till the system state enters the limit cycle which is shown by inner trajectory. The limit cycle is stable as the paths in its neighbourhood are converging. The limit cycle for $\mu < 0$ is unstable.

With increase in order of the system, the complexity is involved in applying phase plane method. The method of phase trajectories is restricted normally to second order systems and it is special case of phase space defined for n^{th} order system.

In case of time invariant systems, the total phase-plane contains trajectories with only one curve passing through every point of the plane except for some critical points. Through these critical points either none of the trajectories or infinite number of trajectories pass.

For time varying systems, two or more trajectories may pass through a single point. The phase portrait then becomes complex and not easy to interpret. Hence the method of phase-plane is mostly applied to second order systems having constant parameters with no or constant input. Some simple time varying systems can be analysed by this method.

8.9 Different Types of Phase Portraits

In this section, the phase portraits of some commonly known linear systems is presented. These portraits can be used for analysis of piecewise linear systems.

8.9.1 Phase Portraits for Type 0 System

Consider a transfer function for a type 0 system.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Let us consider the input r(t) to be zero i.e. **unforced system**

$$\therefore \quad [s^2 + 2\xi\omega_n s + \omega_n^2]C(s) = 0$$

$$\therefore \qquad \ddot{C}+2\xi\omega_n\dot{C}+\omega_n^2 C = 0$$

Let $x_1 = C$ and $x_2 = \dot{C}$. Therefore the state model is given by,

$$\dot{x}_1 = x_2; \quad \dot{x}_2 = -2\xi\omega_n x_2 - \omega_n^2 x_1$$

Eliminating the time variable from the two equations by dividing the equations,

$$\frac{dx_2}{dx_1} = \frac{-2\xi\omega_n x_2 - \omega_n^2 x_1}{x_2}$$

If we consider the point at which $x_1 = x_2 = 0$ then the slope at this point $\frac{dx_2}{dx_1} = \frac{0}{0}$ which is indeterminate. This point is called **singular point**. We will study the behaviour of trajectories which are closed to this singular point.

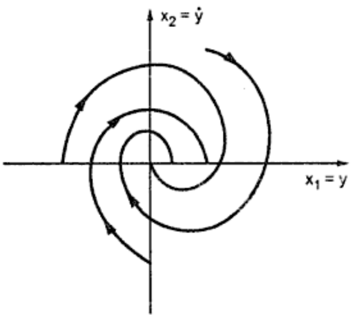
To find out the singular points which are nothing but the roots of characteristic equation $s^2+2\xi\omega_n s+\omega_n^2 = (s-\gamma_1)(s-\gamma_2) = 0$. Depending on the values of γ_1 and γ_2, six different types of singular points are obtained. These are described as below.

8.9.1.1 Stable System with Complex Roots

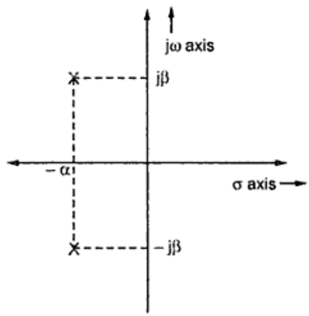
Let $\gamma_1 = -\alpha+j\beta$, $\gamma_2 = -\alpha-j\beta$, $\alpha>0$ and $\beta>0$

The output response in this case is $c(t) = C_1 e^{-\alpha t}\sin(\beta t+C_2)$ C_1 and C_2 are constants whose values can be obtained from initial conditions.

The phase portrait can be obtained from the response equation on (x_1, x_2) plane as $x_1 = C$, $x_2 = \dot{C}$. The typical phase trajectory obtained is a logarithmic spiral into the singular point which is shown in the Fig. 8.13. This type of singular point is called a **stable focus.**

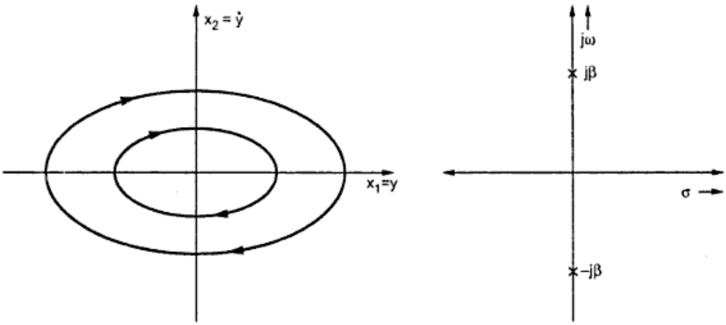


(a) Stable focus



(b) Nature of roots in s plane

Fig. 8.13



(a) Center (b) Nature of roots

Fig. 8.15

8.9.1.4 Stable System with Real Roots

Let γ_1 and γ_2 be two real and distinct roots located in left half of s-plane. The response in this case is given by,

$$C(t) = C_1 e^{\gamma_1 t} + C_2 e^{\gamma_2 t}$$

The response obtained from the system is overdamped. The singular point obtained after plotting the phase trajectory on ($x_1 = y$, $x_2 = \dot{y}$) plane for this case is called a **stable node**. The phase portrait contains two straight line trajectories having equations $x_2(t) = \gamma_1 x_1(t)$; $x_2(t) = \gamma_2 x_1(t)$ which satisfies the differential equation of the given system.

Assuming that the term $e^{\gamma_2 t}$ decays faster in the transient response than the term $e^{\gamma_1 t}$. As time t increases, $x_1 \to C_1 e^{\gamma_1 t} \to 0$ and $x_2 \to \gamma_1 C_1 e^{\gamma_1 t} \to 0$. All the trajectories are tangential at the origin to the straight line trajectory $_2(t) = \gamma_1 x_1(t)$. The other trajectory is given for initial conditions $x_2(0) = \gamma_2 x_1(0)$.

All the trajectories are asymptotic to this straight line trajectory. If the roots are repeated for stable systems then there is a single trajectory whose slope depends on value of root.

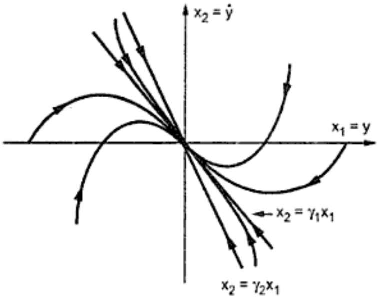


Fig. 8.16 Stable node

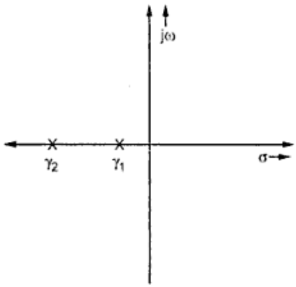


Fig. 8.17 Nature of roots

The phase trajectory and nature of roots is shown in the Fig. 8.16 and Fig. 8.17.

8.9.1.5 Unstable System with Positive Real Roots

Let γ_1 and γ_2 be two real and distinct roots located in right half of s plane γ_1 is a smaller root. The singular point in this case is called **unstable node**. The phase portrait on $(x_1 = y, x_2 = \dot{y})$ plane is shown in the Fig. 8.18.

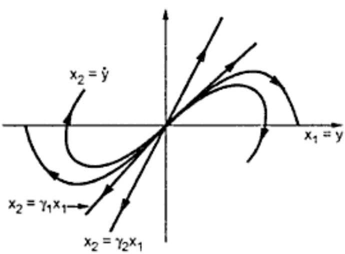


Fig. 8.18 Unstable node

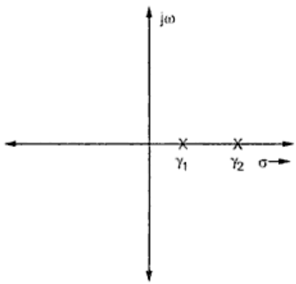


Fig. 8.19 Nature of roots

The trajectories emerge from the singular point and go to the infinity. The trajectories are tangential to the straight line trajectory $x_2(t) = \gamma_1 x_1(t)$ at origin and asymptotic at infinity to other straight line trajectory. For the repeated roots, there is a single trajectory.

8.9.1.6 Unstable System with One Positive and One Negative Real Root

In this case we have two straight line trajectories which have slopes dependent on values of root. The singular point obtained in this case is called a **saddle**. The straightline obtained due to negative root gives trajectory entering into the singular point while due to positive part gives trajectory which leaves the singular point.

The other trajectories approach towards the singular point close to incoming straight line then takes a curve away and leaves the singular point in its vicinity. It approaches the second straight line asymptotically. The phase portrait on $(x_1 = y, x_2 = \dot{y})$ plane is shown in the Fig. 8.20.

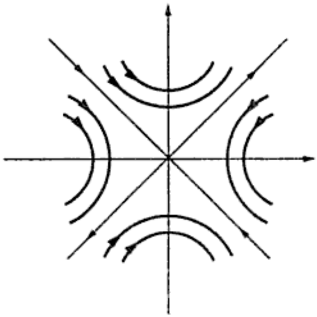


Fig. 8.20 Saddle point

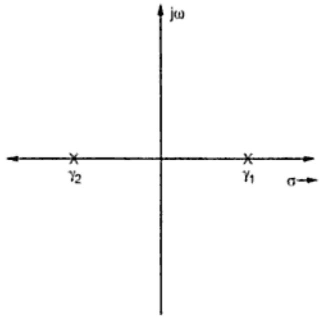


Fig. 8.21 Nature of roots

8.9.2 Forced Second Order Type 0 System

Let us consider a second order system excited by constant input with magnitude A.

$$\therefore \quad C(s)[s^2 + 2\xi\omega_n s + \omega_n^2] = A\omega_n^2$$

$$\therefore \quad \ddot{C} + 2\xi\omega_n \dot{C} + C\omega_n^2 = A\omega_n^2$$

$$\therefore \quad \frac{d^2C}{dt^2} + 2\xi\omega_n \frac{dC}{dt} + \omega_n^2 C = A\omega_n^2$$

The above differential equation can also be written in following form.

$$\frac{d^2}{dt^2}(C-A) + 2\xi\omega_n \frac{d}{dt}(C-A) + \omega_n^2(C-A) = 0$$

The equation is similar to that obtained for unforced system with the difference that the singular point is shifted from origin to point C = A on the Y axis.

8.9.3 Phase Portraits for Type 1 System

Consider the following transfer function which represents Type 1 system. This system is excited by constant input of magnitude A.

∴ The total equation becomes,

$$\frac{1}{\tau}x_1(t) = A - x_2(t) - A\ln[A - x_2(t)] + \frac{1}{\tau}x_1(0) - A + x_2(0) + A\ln[A - x_2(0)]$$

$$\frac{1}{\tau}[x_1 - x_1(0)] = -[x_2 - x_2(0)] - A\ln\left[\frac{A - x_2}{A - x_2(0)}\right]$$

The corresponding phase portrait for A > 0 is shown in the Fig. 8.22.

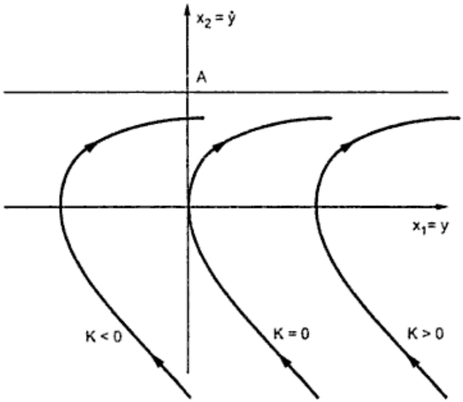


Fig. 8.22 Phase portrait for A > 0

If initial conditions are zero i.e. $x_1(0) = x_2(0) = 0$ then

$$\frac{1}{\tau}x_1 = -x_2 - A\ln\left[\frac{A - x_2}{A}\right]$$

The corresponding phase trajectory is shown in the Fig. 8.22. The trajectory is asymptotic to the line $x_2 = A$.

If the initial point is different $[x_1(0), x_2(0)]$ then the trajectory will have same shape but shifted horizontally by k units so as to pass through the point $[x_1(0), x_2(0)]$.

The equation of trajectory in this case is given by,

$$\frac{1}{\tau}(x_1 - K) = -x_2 - A\ln\left(\frac{A - x_2}{A}\right)$$

The phase portrait for A < 0 is shown in the Fig. 8.23. The value of A = 0 is a special case which is unforced system and the phase portrait consists of number of straight lines having the slope of $-\frac{1}{\tau}$.

Fig. 8.23 Phase portrait for A < 0.

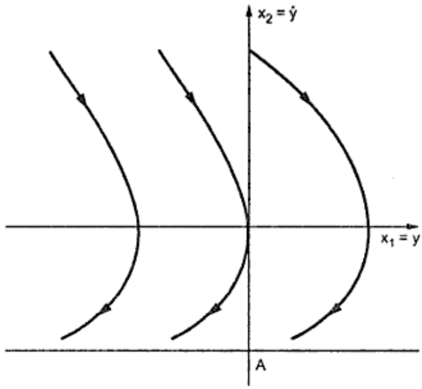


Fig. 8.23 Phase portraits for A < 0

8.9.4 Phase Portraits for Type 2 System

Consider the transfer function of linear system representing type 2 system

$$\frac{C(s)}{R(s)} = \frac{J}{Js^2}$$

Let the input applied be of constant magnitude, r(t) = A. The corresponding differential equation is given by,

$$J\ddot{C} = A$$

Let the state variable be $x_1 = C$, $x_2 = C$

The corresponding state model is given as,

$$\dot{x}_1 = \dot{C} = x_2$$

$$\dot{x}_2 = \ddot{C} = \frac{A}{J}$$

By eliminating the time variable we get,

$$\frac{dx_2}{dx_1} = \frac{A}{Jx_2}$$

$$\therefore \quad Jx_2 dx_2 = A\, dx_1$$

Integrating the above equation,

$$\int Jx_2 dx_2 = \int A dx_1$$

$$J.\frac{x_2^2(t)}{2A}+C = x_1(t)$$

Where C is constant of integration obtained from initial conditions.

$$x_1(t) = \frac{J\, x_2^2(t)}{2\, A}+C$$

$$\therefore \quad x_1(0) = \frac{J\, x_2^2(0)}{2\, A}+C$$

$$\therefore \quad C = x_1(0)-\frac{J\, x_2^2(0)}{2\, A}$$

The nature of trajectory is parabolic for an initial state point $[x_1(0), x_2(0)]$ passing through a point $x_1 = C$ on the x-axis where C is given by above equation. This is shown in the Fig. 8.24 for $A < 0$ and $A > 0$.

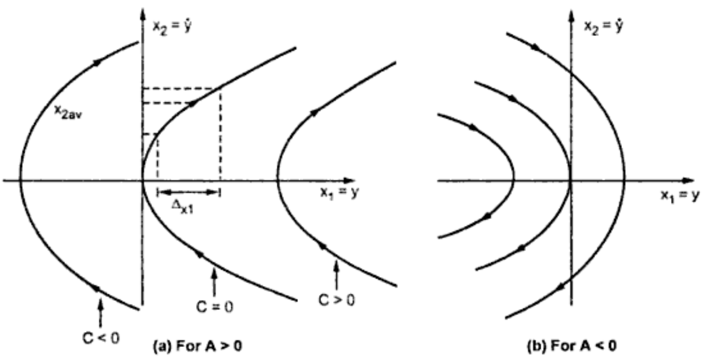


Fig. 8.24

With increase in time t, each trajectory is shown in clockwise direction. The direction of phase trajectory is obtained from the relationship $\dot{x}_1 = x_2$. As x_1 increases with time in upper half of phase plane, the state point hence moves from left to right in the lower half of phase plane. When x_1 decreases with time, the state point moves from right to left.

The time interval between the two points of a trajectory is given by $\Delta t = \Delta x_1 / x_{2av}$. With the aid of this equation the time scale can be given to the trajectories. But the phase portrait is generally used to get an idea about transients in the system.

The phase portrait or $A < 0$ is shown in the Fig. 8.24 (b). The unforced system with $A = 0$ is a special case.

$$\frac{dx_2}{dx_1} = 0 \quad \therefore\ dx_2 = 0$$

Integrating above equation

$$\int dx_2 = 0$$

$$x_2(t) = C$$

Where C is constant of integration obtained from initial conditions.

$$\therefore \qquad C = x_2(0)$$

$\therefore$ We have equation of trajectory as $x_2(t) = x_2(0)$. This equation represents the trajectories of straight lines parallel to x_1 axis.

Example 8.1 : *Determine the kind of singularity for each of the following differential equations.*

i) $\ddot{y} + 3\dot{y} + 2y = 0$

ii) $\ddot{y} - 8\dot{y} + 17y = 34$ **(VTU: Jan/Feb.-2005)**

Solution : i) $\ddot{y} + 3\dot{y} + 2y = 0$

Let the state variables be $x_1 = y$ and $x_2 = \dot{y}$

The corresponding state model is

$\dot{x}_1 = \dot{y} = x_2;\ \dot{x}_2 = \ddot{y} = -3x_2 - 2x_1$

Eliminating the time variable,

$$\frac{dx_2}{dx_1} = \frac{-3x_2 - 2x_1}{x_2}$$

The characteristic equation is given by,

$$s^2+3s+2 = 0$$

$$\therefore \quad (s+2)(s+1) = 0$$

$$s = -2; \; s = -1$$

We have, $\gamma_1 = -1$ and $\gamma_2 = -2$

The nature of phase trajectory is as shown in the Fig. 8.25 and the type of singular point obtained is stable node.

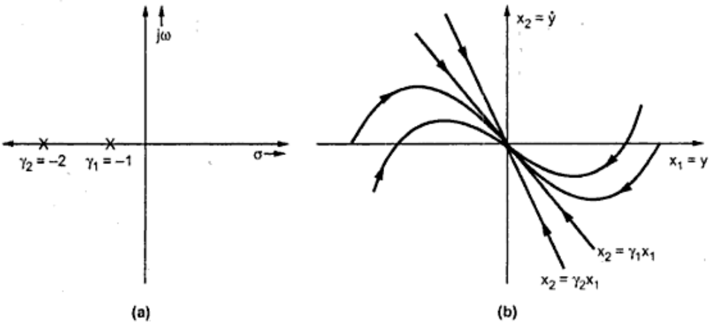


Fig. 8.25

The response in this case is given by $y(t) = C_1e^{\gamma_1 t}+C_2e^{\gamma_2 t}$. The transient term $e^{\gamma_2 t}$ decays at faster rate than $e^{\gamma_1 t}$. As time $t \to \infty$, $x_1 \to C_1e^{\gamma_1 t} \to 0$; $x_2 = \gamma_1 e^{\gamma_1 t} \to 0$. **The singular point node obtained in this case (0, 0) is stable in $(y, \dot{y})$ plane.**

ii)
$$\ddot{y}-8\dot{y}+17y = 34$$

$$\ddot{y}-8\dot{y}+17y-34 = 0$$

The above equation can be rewritten as,

$$\frac{d^2y}{dt^2}-8\frac{dy}{dt}+17y-34 = 0$$

$$\frac{d^2(y-2)}{dt^2}-8\frac{d}{dt}(y-2)+17(y-2) = 0$$

The characteristic equation is given by,

$$(s-2)^2 - 8(s-2) + 17(s-2) = 0$$

$$(s^2 - 4s + 4) - 8s + 16 + 17s - 34 = 0$$

$$s^2 + 5s - 14 = 0$$

$$(s+7)(s-2) = 0$$

$$\therefore \quad \gamma_1 = -7; \; \gamma_2 = 2$$

There are two roots one positive and one negative root. The singular point obtained in this case is called **saddle point** which is shifted from origin by 2 units. The slopes of the two straight line trajectories with slopes dependent on root values. The nature of phase trajectory is shown in the Fig. 8.26.

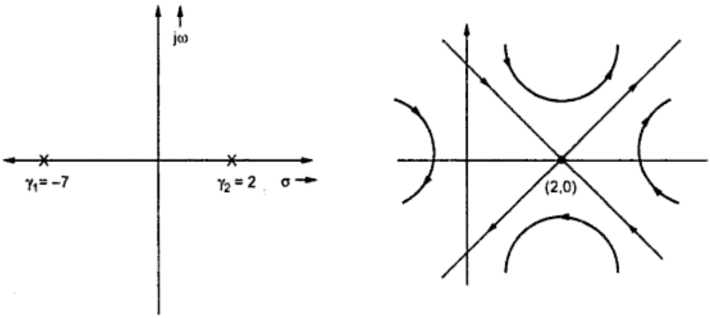


Fig. 8.26

8.10 Singular Points of a Nonlinear System

Singular point is a point on the phase plane where the slope of the trajectory is indeterminate. In the state model of a given system, when time variable is eliminated then we get,

$$\frac{dx_2}{dx_1} = \frac{f(x_1, x_2)}{x_2} = \frac{0}{0}$$

By equating the numerator and denominator equations to zero i.e. solutions of the equations $f(x_1, x_2) = 0$ and $x_2 = 0$ gives the locations of the singular points along with its number.

It is also important to know the behaviour of the trajectories in the vicinity of a singular point of a nonlinear system. The nonlinear equations are linearized at the singular point using say Taylor series expansion and then the nature of phase trajectory around the singular point is determined by linear system analysis which is described earlier.

8.11 Delta Method

In isocline method, entire phase plane must be filled with line segments of different possible slopes. If only a single solution curve is needed, only a few of these lines are put to use, thus wasting a lot of effort. A technique of construction known as the delta method leads more directly to the solution. This method applies to the solution of equations of the type,

$$\ddot{x}+f(\dot{x}, x, t) = 0 \qquad ...(1)$$

Where f must be continuous and single valued but may be nonlinear and time dependent.

The equation (1) is written as

$$\ddot{x}+\omega_0^2 x+f(\dot{x}, x, t)-\omega_0^2 x = 0 \qquad ...(2)$$

The constant ω_0^2 may be determined from the equation (1) itself or may have to chosen from other information. Once again it is convenient to introduce the definitions $\tau = \omega_0 t$ and $v = \frac{dx}{d\tau}$ so as to give,

$$\omega_0^2 v\frac{dv}{dx}+\omega_0^2 x+[f(v, x, \tau)-\omega_0^2 x] = 0 \qquad ...(3)$$

or

$$\frac{dv}{dx} = -\frac{[x+\delta(v, x, \tau)]}{v} \qquad ...(4)$$

where

$$\delta(v, x, \tau) = \frac{1}{\omega_0^2}f(v, x, t)-x \qquad ...(5)$$

Equation (4) is similar to that one used for the case of isocline method. Because of the way, in which it is set up, the S method is most immediately applicable to the equations with oscillatory solutions, although it is not limited to this class of equation. Function δ of equation (5) depends upon v, x and τ but it is assumed constant for small changes in these variables. Under this assumption, the variables of equation (4) can be separated and integrated to give,

$$v^2+(x+\delta)^2 = \text{constant} = R^2 \text{ (say)} \qquad ...(6)$$

Equation (6) represents a circle of radius R and centre at $(-\delta, 0)$. Thus for a suitable increment, the solution curve is the arc of a circle having these properties. The construction is shown in Fig. 8.21 where [x(0), v(0)] represents a point on the solution curve at the time $\tau(0)$. These values used in equation (5) allow the calculation of δ. This values of δ determines the centre of the circular arc located on the x-axis. The radius R is automatically fixed. A short circular arc represents a portion of solution curve. Actually it is more accurate to use the average values of v, x, τ for that increment. Again, the allowable length of the arc is a compromise.

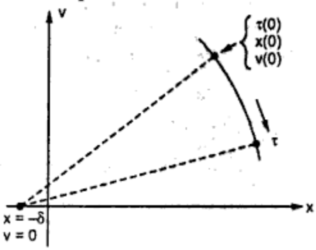


Fig. 8.27

Examples with Solutions

Example 8.2 : *A linear second order servo is described by the equation $\ddot{C}+2\xi\omega_n\dot{C}+\omega_n^2C = 0$ where $\xi = 0.15$, $\omega_n = 1$ rad/sec, C(0) = 1.5 and $\dot{C} = 0$. Determine the singular point. Construct the phase trajectory, using the method of isoclines*

(VTU: July/Aug.-2006)

Solution : Consider the equation

$$\ddot{C}+2\xi\omega_n\dot{C}+\omega_n^2C = 0$$

Given $\xi = 0.15,\ \omega_n = 1,\ C(0) = 1.5,\ \dot{C} = 0$

$$\ddot{C}+2(0.15)(1)\dot{C}+1^2C = 0$$

$$\ddot{C}+0.3\dot{C}+C = 0$$

Taking Laplace transform,

$$s^2C(s)+0.35\,C(s)+C(s) = 0$$

$$\left[s^2+0.35+1\right]C(s) = 0$$

Consider $$s^2+0.35+1 = 0$$

$$s_{1,2} = \frac{-0.3\pm\sqrt{(0.3)^2-4(1)(1)}}{2} = \frac{-0.3\pm j[1\cdot 9773]}{2}$$

$$= -0.15\pm j0\cdot 9886$$

The roots are complex conjugate and located in left half of s plane. In time domain the response will be underdamped consisting of oscillations and the singular point is thus **stabel focus.**

Consider again the differential equation

$$\frac{d^2C(t)}{dt^2}+0.3\frac{dC(t)}{dt}+C(t) = 0$$

Let $\quad x_1 = C(t), \quad x_2 = \dot{C}(t)$

$$\therefore \dot{x}_1 = \dot{C}(t) = x_2, \quad \dot{x}_2 = \ddot{C}(t) = - \ C(t) - 0.3\ \dot{C}(t) = -x_1 - 0.3x_2$$

$$m = \frac{dx_2}{dx_1}1 = \frac{dx_2/dt}{dx_1/dt} = \frac{-x_1-0.3x_2}{x_2}$$

$$\therefore \quad mx_2 = -x_1 - 0.3x_2$$

$$(m+0.3)x_2 = -x_1$$

$$\therefore \quad x_2 = -\left[\frac{1}{m+0.3}\right]x_1$$

The above equation is an equation of isocline.

At initial point (1.5,0)

$$0 = -\frac{1}{m+0.3}(1-5)$$

$$m = -0.3-\infty = -\infty$$

Due to this trajectory moves downwards

Using the method of isocline, the phase portrait can be drawn as shown in the Fig. 8.28

Considering various values of m

m	Isocline equation	$\theta = \tan^{-1}m$
1	$x_2 = -0.7692\ x_1$	45°
5	$x_2 = -0.1886\ x_1$	78.69°
10	$x_2 = -0.0970\ x_1$	84.28°
∞	$x_2 = 0$	90°
−1	$x_2 = 1.4285\ x_1$	− 45°
−5	$x_2 = 0.2127\ x_1$	−78.69°
−10	$x_2 = 0.1030\ x_1$	− 84.28°
$-\infty$	$x_2 = 0$	−90°

The type of singular point obtained is stable focus.

ii) Consider, $\ddot{y} + 3\dot{y} + 2y = 0$

Let the state variables be $x_1 = y$ and $x_2 = \dot{y}$

The corresponding state model is

$\dot{x}_1 = \dot{y} = x_2;\ \dot{x}_2 = \ddot{y} = -3x_2 - 2x_1$

Eliminating the time variable,

$$\frac{dx_2}{dx_1} = \frac{-3x_2 - 2x_1}{x_2}$$

The characteristic equation is given by,

$$s^2 + 3s + 2 = 0$$

$$\therefore \quad (s+2)(s+1) = 0$$

$$s = -2;\ s = -1$$

We have, $\gamma_1 = -1$ and $\gamma_2 = -2$

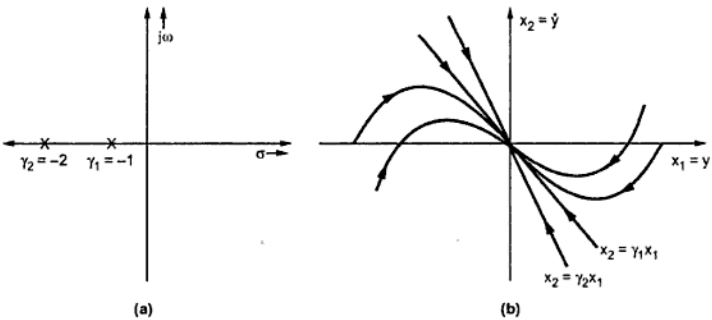


Fig. 8.30

The nature of phase trajectory is as shown in the Fig. 8.30 and the type of singular point obtained is stable node.

The response in this case is given by $y(t) = C_1 e^{\gamma_1 t} + C_2 e^{\gamma_2 t}$. The transient term $e^{\gamma_2 t}$ decays at faster rate than $e^{\gamma_1 t}$. As time $t \to \infty$, $x_1 \to C_1 e^{\gamma_1 t} \to 0$; $x_2 = \gamma_1 e^{\gamma_1 t} \to 0$. The singular point node obtained in this case (0, 0) is stable in $(y, \dot{y})$ plane.

iii) Consider, $\ddot{y} + 3\dot{y} - 10 = 0$

Let the state variables be $x_1 = y$; $x_2 = \dot{y}$

The corresponding state model is,

$$\dot{x}_1 = \dot{y} = x_2$$

$$\dot{x}_2 = \ddot{y} = -3\dot{y} + 10 = -3x_2 + 10$$

Eliminating the time variable

$$\frac{dx_2}{dx_1} = \frac{-3x_2 + 10}{x_2}$$

The characteristic equation is given by,

$$s^2 + 3s - 10 = 0$$

$$\therefore \quad s^2 + 5s - 2s - 10 = 0$$

$$\therefore \quad s(s+5) - 2(s+5) = 0$$

$$\therefore \quad (s+5)(s-2) = 0$$

$$\therefore \quad Y_1 = -5 \; ; \; Y_2 = 2$$

There are two roots one positive and one negative. The singular point obtained in this case is called saddle point. The slopes of the two straight line trajectories with slopes dependent on root values. The nature of phase trajectory is shown in the Fig. 8.31.

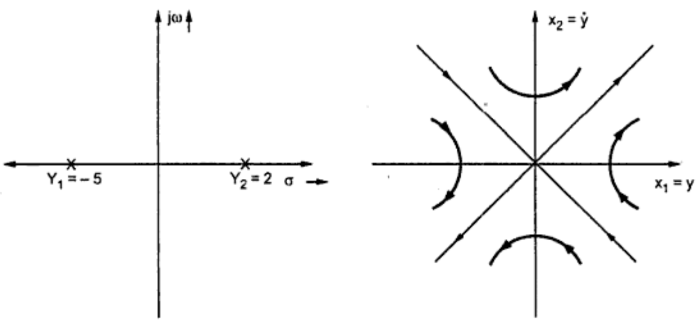


Fig. 8.31

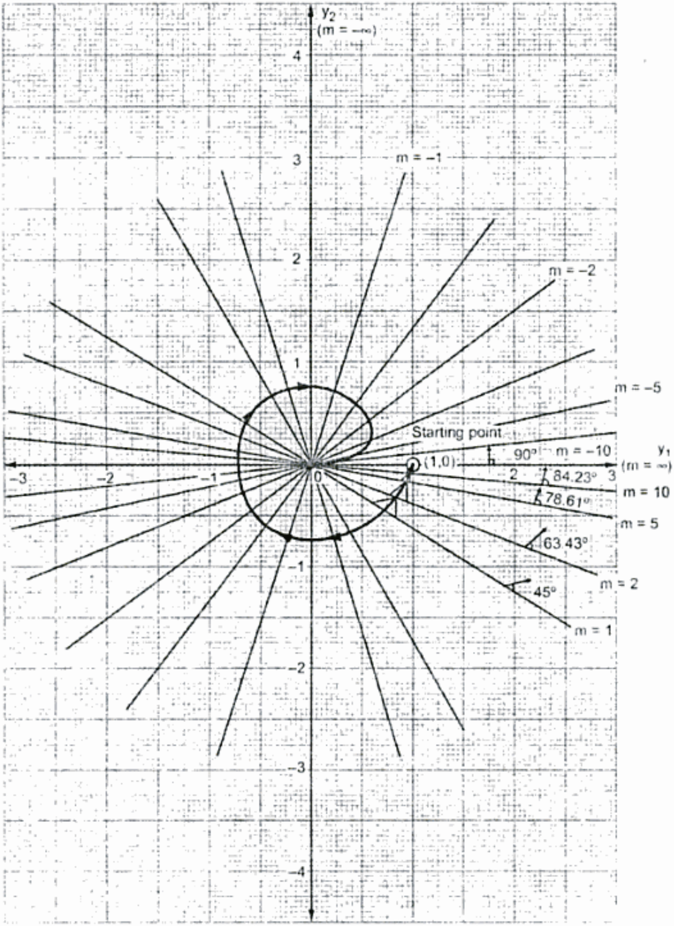

y_2
(m = −∞)
m = −1
m = −2
m = −5
Starting point
(1,0)
90°
m = −10
y_1
(m = ∞)
84.23°
m = 10
78.61°
m = 5
63.43°
m = 2
45°
m = 1

Fig 8.32

m	Isocline equation	$\theta = \tan^{-1} m$
1	$y_2 = -0.625y_1$	45°
2	$y_2 = -0.3846y_1$	63.43°
5	$y_2 = -0.1785\ y_1$	78.69°
10	$y_2 = -0.0943y_1$	84.28°
∞	$y_2 = 0$	90°
−1	$y_2 = 2.5y_1$	−45°
−2	$y_2 = 0.7142y_1$	−63.43°
−5	$y_2 = 0.2272y_1$	−78.69°
−10	$y_2 = 0.1063y_1$	−84.28°
−∞	$y_2 = 0$	−90°

The phase-plane plot is shown in the Fig. 8.32.(See Fig. 8.32. on previous page)

Example 8.5 : *Draw the phase-plane trajectory for the following equation using isocline method :* $\ddot{x}+2\xi\omega\dot{x}+\omega^2 x=0$ *Given,* $\xi = 0.5$, $\omega = 1$, *Initial point (0, 6)*

(VTU: Jan./ Feb.-2007)

Solution: Consider the equation

$$\ddot{x}+2\xi\omega_n \dot{x}+\omega_n^2 x = 0$$

Given that $\xi = 0.5$, $\omega_n = 1$, $x = 6$, $\frac{dx}{dt} = 0$

$$\ddot{x}+2(0{\cdot}5)(1)\dot{x}+1^2 x = 0$$

$$\ddot{x}+\dot{x}+x = 0$$

Let $y_1 = x$

$$y_2 = \dot{x}$$

$$\dot{y}_1 = \dot{x} = y_2 \ ; \dot{y}_2 = \ddot{x} = -\dot{x} - x = -y_2 - y_1$$

$$m = \frac{dy_2}{dy_1} = \frac{dy_2/dt}{dy_1/dt} = \frac{-y_1-y_2}{y_2}$$

$$my_2 = -y_1 - y_2$$

$$(m+1)y_2 = -y_1$$

$$y_2 = -\left[\frac{1}{m+1}\right]y_1$$

The above equation is an equation of isocline.

$$m+1 = -\frac{y_1}{y_2}$$

$$m = -\frac{y_1}{y_2}-1$$

Δ t initial point (6, 0).

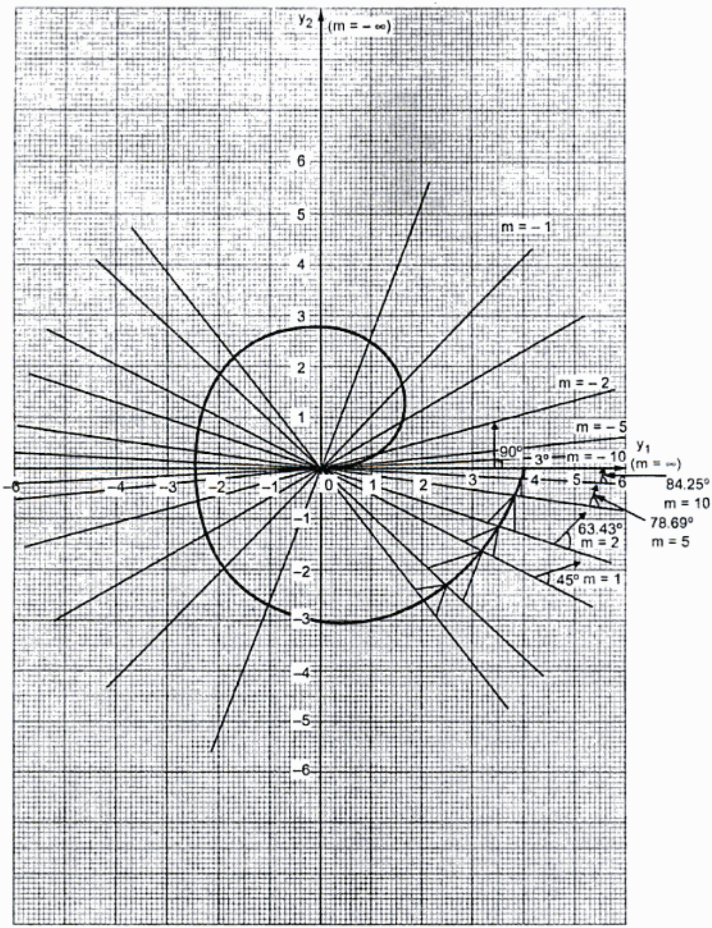
$$m = -\frac{6}{0}-1=-\infty$$

∴ The trajectory moves downwards

Using the method of isocline, the phase portrait can be drawn considering various values of m.

m	Isocline equation	$\theta = \tan^{-1}m$
1	$y_2 = -0.5\ y_1$	45°
2	$y_2 = -0.33\ y_1$	63.43°
5	$y_2 = -0.166\ y_1$	78.69°
10	$y_2 = -0.090\ y_1$	84.28°
∞	$y_2 = 0$	90°
–2	$y_2 = 1y_1$	– 45°
–5	$y_2 = 0.25\ y_1$	– 63.43°
–10	$y_2 = 0.1111\ y_1$	–78.69°
–20	$y_2 = 0.0526\ y_1$	–84.28°
–∞	$y_2 = 0$	–90°

The phase-plane plot is as shown in the Fig. 8.33.



y_2
(m = – ∞)
6
5
4
3
2
1
–1
–2
–3
–4
–5
–6
m = – 1
m = – 2
m = – 5
90°
3°
m = – 10
y_1
(m = ∞)
–6
–4
–3
–2
–1
0
1
2
3
4
5
6
84.25°
m = 10
78.69°
m = 5
63.43°
m = 2
45° m = 1

Fig. 8.33

9

Liapunov's Stability Analysis

9.1 Introduction

The state equation for a general time invariant system has the form $\dot{x} = f(x, u)$. If the input u is constant then the equation will have form $\dot{x} = F(x)$. For this system, the points, at which derivatives of all state variables are zero, are the singular points. These singular points are nothing but equilibrium points where the system stays if it is undisturbed when the system is placed at these points.

The stability of such a system is defined in two different ways. If the input to the system is zero with arbitrary initial conditions, the resulting trajectory in phase-plane, discussed in earlier chapter, tends towards the equilibrium state.

If the input to the system is provided then the stability is defined as for bounded input, the system output is also bounded.

For linear systems with non-zero eigen values, there is only one equilibrium state and the behaviour of such systems about this equilibrium state totally determines the qualitative behaviour in the entire state space.

In case of nonlinear systems, the behaviour for small deviations about the equilibrium point is different from that for large deviations. Hence local stability for such systems does not indicate the overall stability in the state space. Also the non-linear systems having multiple equilibrium states, the trajectories move from one equilibrium point and tend to other with time. Thus stability in case of non-linear system is always referred to equilibrium state instead of global term stability which is the total stability of the system.

In case of linear control systems, many of the stability criteria such as Routh's stability test, Nyquist stability criterion etc. are available. But these cannot be applied for non-linear systems.

The second method of Liapunov which is also called direct method of Liapunov is the most common method for obtaining the stability of non-linear systems. This method is equally applicable to time varying systems, stability analysis of linear, time in variant systems and for solving quadratic optimal control problem.

9.2 Stability in the Sense of Liapunov

Consider a system defined by the state equation $\dot{x} = f(x, t)$. Let us assume that this system has a unique solution starting at the given initial condition. Let us consider this solution as $F(t : x_0, t_0)$ where $x = x_0$ at $t = t_0$ and t is the observed time.

$$\therefore \qquad F(t_0 : x_0, t_0) = x_0$$

If we consider a state x_e for system with equation $\dot{x} = f(x, t)$ in such a way that $f(x_e, t) = 0$ for all t then this x_e is called equilibrium state. For linear, time invariant systems having A non-singular, there is only one equilibrium state while there are one or more equilibrium states if A is singular.

In case of non-linear systems as we have seen previously there are more than one equilibrium states. The isolated equilibrium states that is isolated from each other can be shifted to origin i.e. $f(0, t) = 0$ by properly shifting the coordinates. These equilibrium states can be obtained from the solution of equation $f(x_e, t) = 0$.

Now we will consider the stability analysis of equilibrium states at the origin. We will consider a spherical region of radius R about an equilibrium state x_e such that

$$\| x - x_e \| \leq R$$

$\| x - x_e \|$ is called Euclidean norm and is defined as

$$\| x - x_e \| = [x_1 - x_{1e})^2 + (x_2 - x_{2e})^2 + \dots (x_n - x_{ne})^2]^2$$

Let $S(\delta)$ consists of all points such that $\| x_0 - x_e \| \leq \delta$ and let $S(\varepsilon)$ consists of all points such that $\| F(t : x_0, t_0) - x_e \| \leq \varepsilon$ for all $t \geq t_0$.

Any equilibrium state x_e of the system $\dot{x} = f(x, t)$ is said to be stable in the sense of Liapunov if corresponding to each $S(\varepsilon)$ there is $S(\delta)$ such that trajectories starting in $S(\delta)$ do not leave $S(\varepsilon)$ as time t increases indefinitely. The real number δ depends on ε and in general also depends on t_0. If δ does not depend on t_0, the equilibrium state is said to be uniformly stable.

The region $S(\varepsilon)$ must be selected first and for each $S(\varepsilon)$, there must be a region $S(\delta)$ in such a way that the trajectories starting within $S(\delta)$ do not leave $S(\varepsilon)$ as time t progresses.

There are many types of stability definitions such as asymptotic stability, asymptotic stability in large. We will also see the definition of instability along with definitions of these types of stability.

9.3 Asymptotic Stability

An equilibrium state x_e is said to be asymptotically stable if it is stable in the sense of Liapunov and every solution starting within $S(\delta)$ converges without leaving $S(\varepsilon)$ to x_e as t increases indefinitely.

The asymptotic stability is more important than mere stability. The asymptotic stability is a local stability. Hence establishing asymptotic stability does not indicate the proper operation of the system. The size of the largest region of the asymptotic stability is required. This region is called domain of attraction which is part of state space where asymptotically stable trajectories originate.

9.4 Asymptotic Stability in the Large

If the asymptotic stability holds for all states that is all points in state space, from which trajectories originate, the equilibrium state is said to be asymptotically stable in the large. Alternatively we can say that the system is asymptotically stable in the large if it is asymptotically stable for every initial state regardless of how near or how far it is from the origin.

The equilibrium state x_e of the system is said to be asymptotically stable in the large if it is stable and if every solution converges to x_e as t increases indefinitely. The necessary condition for asymptotic stability is to be determined which is normally different. Practically it stability in large is that there must be one equilibrium state in the whole state space.

In control system problems, asymptotic stability in large is required in the absence of which, the largest region of asymptotic stability is to be determined which is normally difficult. Practically, it is sufficient to get a region of asymptotic stability large enough so that no disturbance will exceed it.

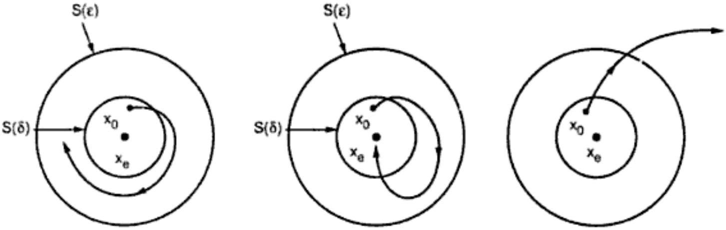
9.5 Instability

An equilibrium state x_e is said to be unstable if for some real number $\varepsilon > 0$ and any real number $\delta > 0$, irrespective of how small it is, there is always a state x_0 in $S(\delta)$ such that the trajectory starting at this states leaves $S(\varepsilon)$.

9.6 Graphical Representation

The graphical representation of stability, asymptotic stability and instability is shown in the Fig. 9.1 (a), (b) and (c) respectively.

The region $S(\delta)$ is binding the initial state x_0 and the region $S(\varepsilon)$ corresponds to the boundary for the trajectory starting at x_0. The correct region of allowable initial conditions are not specified for the definitions of stability that we have seen previously. Thus these definitions apply to the neighbourhood of the equilibrium state only if S(ε) is not corresponding to the entire state plane.



(a) Stability (b) Asymptotic stability (c) Instability

Fig. 9.1

In Fig. 9.1 (c), the trajectory is leaving S(ε) and shows that the equilibrium state is unstable. It can not be stated that the trajectory will go to infinity as it may approach a limit cycle outside the region S(ε). In case of linear, time invariant system if it is unstable trajectories starting near the unstable equilibrium state will go to infinity which is not necessarily true in case of non linear systems.

The definitions of stability that we have seen so far are not the only ones defining the stability of equilibrium state. There are other ways of defining stability also. In classical control theory, the systems which are asymptotically stable are called stable systems and those which are stable in the sense of Liapunov but are not asymptotically stable are called unstable.

9.7 Some Important Definitions

In this section we will consider some important definitions which are useful in understanding Liapunov's stability criterion.

9.7.1 Positive Definiteness

A scalar function F(x) is said to be positive definite in a particular region which includes the origin of state space if F(x) > 0 for all non-zero states x in that region and F(0) = 0.

e.g. $$F(x) = x_1^2 + 2x_2^2$$

9.7.2 Negative Definiteness

A scalar function F(x) is said to be negative definite if – F(x) is positive definite.

e.g. $$F(x) = -x_1^2 - (3x_1 + 2x_2)^2$$

9.7.3 Positive Semidefiniteness

A scalar function F(x) is said to be positive semidefinite if it is positive at all states in the particular region except at the origin and at certain other states where it is zero.

e.g. $$F(x) = (x_1 + x_2)^2$$

9.7.4 Negative Semidefinite

A scalar function F(x) is said to be negative semidefinite if − F(x) is positive semidefinite.

9.7.5 Indefiniteness

A scalar function F(x) is said to be indefinite in the particular region if it assumes both positive and negative values irrespective how small the region is

e.g. $$F(x) = x_1 x_2 + x_2^2$$

9.8 Quadratic Form

A class of scalar functions which plays important role in the stability analysis based on Liapunov's second method is the quadratic form.

e.g. $$F(x) = X^T Px = [x_1 \; x_2 \; \ldots \; x_n] \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{12} & P_{22} & \cdots & P_{2n} \\ \vdots & & & \\ P_{1n} & \cdots & \cdots & P_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

P is real symmetric matrix and x is a real vector.

9.9 Hermitian Form

If x is a complex n vector and P is a Hermitian matrix then the complex quadratic form is called Hermitian form.

e.g. $$F(x) = x^* Px = [\overline{x_1} \; \overline{x_2} \cdots \overline{x_n}] \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ \overline{P}_{12} & P_{22} & \cdots & P_{2n} \\ \vdots & & & \\ \overline{P}_{1n} & \overline{P}_{2n} & \cdots & P_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The stability analysis in state space the Hermitian form is commonly used than the quadratic from as it is more general. The positive definiteness of the quadratic form or Hermitian form F(x) can be determined by Sylvester's criterion which states the necessary and sufficient conditions for quadratic or Hermitian form to be positive definite is

$$P_{11} > 0 \begin{vmatrix} P_{11} & P_{12} \\ \bar{P}_{12} & P_{22} \end{vmatrix} > 0 \ldots\ldots \begin{bmatrix} P_{11} & \cdots & \cdots & P_{1n} \\ \bar{P}_{12} & \cdots & \cdots & P_{2n} \\ \vdots & & & \\ \bar{P}_{1n} & \cdots & \cdots & P_{nn} \end{bmatrix} > 0$$

$F(x) = x^* Px$ is positive semidefinite if P is singular and all the principle minors are non-negative.

F(x) is negative definite if – F(x) is positive definite. Similarly F(x) is negative semidefinite if – F(x) is positive semidefinite.

9.10 Liapunov's Second Method

A system which is vibrating is stable if its total energy is continuously decreasing. This indicates that the time derivative of the total energy must be negative. The energy is decreased till an equilibrium state is reached. The total energy is a positive definite function.

This fact obtained from classical mechanics theory is generalised in Liapunov's second method. If the system has an asymptotically stable equilibrium state then the stored energy decays with increase in time till it attains minimum value at the equilibrium state.

But there is no simple way for defining an energy function. For purely mathematical system. This difficulty was overcome as Liapunov introduced Liapunov function method which is fictitious energy function.

Liapunov functions depend on $x_1, x_2, \ldots x_n$ and t. It is given as $F(x_1, x_2, \ldots x_n, t)$ or as F(x, t). In Liapunov's second method, the sign behaviour of F(x, t) and its time derivative $\dot{F}(x, t) = dF(x, t)/dt$ gives us information about stability, asymptotic stability or instability of an equilibrium state without requiring to solve the equations directly to get the solution.

9.11 Liapunov's Stability Theorem

Consider a scalar function V(x), where x is n vector and is positive definite, then the states x that satisfy V(x) = C, where C is a positive constant, lie on a closed hyper surface in n dimensional state space at least in the neighbourhood of origin. This is shown in the Fig. 9.2.

If V(x) is a positive definite function obtained for a given system such that its time derivative taken along the trajectory is always negative then V(x) becomes smaller and smaller in terms of C and finally reduced to zero as x reduces to zero. This indicates asymptotic stability of the origin. Liapunov's main stability theorem is based on this and gives a sufficient condition for asymptotic stability.

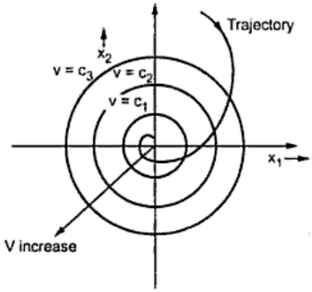


Fig. 9.2

Liapunov's stability theorem is as given below.

Consider a system described by equation $\dot{x} = f(x, t)$ where $f(o, t) = 0$ for all t. If there exists a scalar function V(x, t) having continuous first partial derivatives and satisfying the conditions such as V(x, t) is positive definite and V(x, t) is negative definite then the equilibrium state at the origin is uniformly asymptotically stable. If $V(x, t) \to \infty$ as $\| x \| \to \infty$ where $\| x \|$ is norm of x, then the equilibrium state at the origin is uniformly asymptotically stable in the large.

The above theorem is a basic theorem of second method. The theorem 2 is stated below.

Consider the system described by $\dot{x} = f(x, t)$ where $f(0, t) = 0$ for all $t \geq t_0$. If there exists a scalar function V(x, t) having continuous first partial derivatives and V(x, t) is positive definite, $\dot{V}(x, t)$ is negative semidefinite $\dot{V}(\phi(t : x_0, t_0), t)$ does not vanish identically in $t \geq t_0$ for any t_0 and any $x_0 \neq 0$ where $\phi(t : x_0, t_0)$ denotes or indicates the solution starting from x_0 at t_0 then the equilibrium state at origin of the system is uniformly asymptotically stable in the large.

The equilibrium state at origin is unstable when there exists a scalar function U(x, t) having continuous, first partial derivatives and satisfying the conditions U (x, t) is positive definite in some region about the origin and $\dot{U}(x, t)$ is positive definite in the same region.

9.12 Stability of Linear and Nonlinear Systems

If the equilibrium state in case of linear, time invariant system is asymptotically stable locally then it is asymptotically stable in the large. But in case of a nonlinear system, the equilibrium state has to be asymptotically stable in the large for the state to be locally asymptotically stable. Hence the asymptotic stability of the equilibrium state of linear, time invariant systems and those of nonlinear systems is different.

If it is required to test the asymptotic stability of any equilibrium state for a nonlinear system then the stability analysis of linearised models of non-linear systems is totally insufficient. The nonlinear systems are to be tested without making them linearized. There are various method based on Liapunov's second method such as Krasovskii's method which can be used to test sufficient conditions for asymptotic stability of nonlinear systems.

We shall study in detail Krasovskii's method for asymptotic stability test.

9.13 Construction of Liapunov's Functions for Nonlinear Systems by Krasovskii's Method

Liapunov's direct method is a most powerful tool available for doing the stability analysis of nonlinear systems. The stability analysis by Liapunov's method includes determination of a positive definite function V(x) called Liapunov's function. But unfortunately there is no universal technique or method for selecting Liapunov's function which will be unique for a special problem. Some of the Liapunov's function will give better results than other functions. There are various techniques available for construction of Liapunov's functions.

Similarly for a given function V(x), there is no general method which allows us to confirm that the function is positive definite. But if V(x) is in the quadratic form in x_i's then Sylvester's theorem can be applied to test the positive definiteness of the function.

If it is not possible to obtain the Liapunov's function of required type then it does not indicate that the system is unstable. The theorems that we have seen so far just provides sufficient conditions for stability. The indication from the fact that Liapunov's function of required type is not obtained is that the attempt to establish the stability of the system is failed.

Krasovskii's method gives a sufficient condition for the equilibrium state to be asymptotically stable.

Consider the system described by following equation. $\dot{x} = f(x)$; $f(0) = 0$. The origin is a singular point.

Let us consider Liapunov's function as,

$$V = f^T P f$$

Here P is a symmetric positive definite matrix.

Now,
$$\dot{V} = \dot{f}^T P f + f^T P \dot{f}$$

$$\dot{f} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} = Jf$$

J is called Jacobian Matrix given by

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Substituting $\dot{f}$ in expression for $\dot{V}$ we have,

$$\dot{V} = f^T J^T pf + f^T PJf$$

$$= f^T (J^T P + PJ) f$$

Let $$Q = J^T P + PJ$$

As V is positive definite, for the system to be asymptotically stable, Q should be negative definite. If in addition V(x) is tending to infinity as norm of x tends to infinity ($\|x\| \to \infty$) the system is asymptotically stable in the large.

Alternatively if Liapunov function is taken as

$$V(x) = f^T(x) \,.\, f(x)$$

J(x) is the Jacobian Matrix given earlier

$$J(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

We have $$\dot{f}(x) = \frac{\partial f(x)}{\partial x} \cdot \frac{\partial x}{\partial t} = J(x)\,\dot{x} = J(x) \,.\, f(x)$$

Now we have, $\dot{V}(x) = \dot{f}^T(x) \cdot f(x) + f^T(x) \cdot \dot{f}(x) = [J(x) \cdot f(x)]^T f(x) + f^T(x) \cdot J(x) \cdot f(x)$

$$= f^T(x) \cdot J^T(x) \cdot f(x) + f^T(x) \cdot J(x) \cdot f(x)$$

$$= f^T(x) \,[J^T(x) + J(x)]\, f(x)$$

Let $\hat{J}(x) = J^T(x) + J(x)$. If $\hat{J}(x)$ is negative definite then $\dot{V}(x)$ will also be negative definite. Therefore V(x) is Liapunov function and the origin is asymptotically stable.

If V(x) is tending to infinity as $\| x \| \to \infty$ then the equilibrium state is asymptotically stable in large.

The Krasovskii's theorem is different from the normal linearization technique. It is not limited to small departures from the equilibrium state. The Liapunov function V(x) and $\dot{V}(x)$ are expressed interms of f(x) or $\dot{x}$ rather than in terms of x. Krasovskii method gives sufficient condition for asymptotic stability of non-linear systems and necessary and sufficient condition in case of linear systems.

An equilibrium state of a non-linear system may be stable even if the conditions stated above are not satisfied. So comments on stability of non-linear systems cannot be vigoursly made by applying this method.

Let us have an illustration of Krasovskii's method consider a second order non-linear system described by following equations

$$\dot{x}_1 = g_1(x_1) + g_2(x_2)$$

$$\dot{x}_2 = x_1 + bx_2$$

Let us consider that $g_1(0) = g_2(0) = 0$ and $g_1(x_1)$ and $g_2(x_2)$ are real and differentiable. We will also assume that

$[g_1(x_1) + g_2(x_2)]^2 + [x_1 + bx_2]^2 \to \infty$ as $\| x \| \to \infty$

Now we will obtain the sufficient conditions for asymptotic stability of the equilibrium state x = 0 $f_1 = \dot{x}_1 = f(x_1) = g_1(x_1) + g_2(x_2)$

$$f_2 = \dot{x}_2 = f(x_2) = x_1 + bx_2$$

Let J(x) be the Jacobian matrix given as

$$J(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1}[g_1(x_1)+g_2(x_2)] & \frac{\partial}{\partial x_2}[g_1(x_1)+g_2(x_2)] \\ \frac{\partial}{\partial x_1}[x_1+bx_2] & \frac{\partial}{\partial x_2}[x_1+bx_2] \end{bmatrix}$$

$$= \begin{bmatrix} g_1'(x_1) & g_2'(x_2) \\ 1 & b \end{bmatrix}$$

Here $\quad g_1'(x_1) = \frac{\partial g_1(x_1)}{\partial x_1}$ and $g_2'(x_2) = \frac{\partial g_2(x_2)}{\partial x_2}$

Now we have, $\hat{J}(x) = J^T(x) + J(x)$

$$= \begin{bmatrix} g_1'(x_1) & 1 \\ g_2'(x_2) & b \end{bmatrix} + \begin{bmatrix} g_1'(x_1) & g_2'(x_2) \\ 1 & b \end{bmatrix}$$

$$= \begin{bmatrix} g_1'(x_1)+g_1'(x_1) & 1+g_2'(x_2) \\ g_2'(x_2)+1 & b+b \end{bmatrix}$$

$$= \begin{bmatrix} 2\,g_1'(x_1) & 1+g_2'(x_2) \\ 1+g_2'(x_2) & 2b \end{bmatrix}$$

By Krasovskii's theorem if $\hat{J}(x)$ is negative definite then the equilibrium state x = 0 of the system considered is asymptotically stable in the large.

Hence if $g_1'(x_1) < 0$ for all $x_1 \neq 0$ and $4b\, g_1'(x_1) - [1 + g_2'(x_2)]^2 > 0$ for all $x_1 \neq 0$, $x_2 \neq 0$ then the equilibrium state at x = 0 is asymptotically stable in the large. This two conditions are sufficient for asymptotic stability.

9.14 The Direct Method of Liapunov and the Linear System

For linear systems, Liapunov's direct method proves to be a simple method for stability analysis. Use of Liapunov's method for linear systems is helpful in extending the thinking towards nonlinear systems.

Consider linear system described by state equation

$$\dot{x} = Ax$$

The linear system described by above equation is asymptotically stable in the large at the origin if and only if for any symmetric, positive definite matrix Q, there exists a symmetric positive definite matrix P which is the unique solution $A^T P + PA = -Q$.

The proof of above theorem can be given. For this we will assume the symmetric positive definite matrix P exists which is the unique solution of the equation $V(x) = x^T Px$.

Consider the scalar function, $V(x) = x^T Px$

Here $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$

We have, $\quad V(x) = \dot{x}^T P x + x^T P \dot{x}$

$$\therefore \quad \dot{V}(x) = A(x)^T Px + x^T PA\, x = x^T A^T px + 2^T PA\, x$$

$$= x^T (A^T P + PA)\, x$$

$$= -x^T Qx$$

As Q is positive definite, $\dot{V}(x)$ is negative definite

Let norm of x define as

$$\| x \| = (x^T Px)^{1/2}$$

$$\therefore \quad V(x) = \| x \|^2$$

$$\therefore \quad V(x) \to \infty \text{ as } \| x \| \to \infty$$

The system is therefore asymptotically stable in the large at the origin. The result is also necessary. To prove this, assume that the system is asymptotically stable and P is negative definite.

$$\therefore \quad V(x) = x^T Px$$

$$\dot{V}(x) = -[\dot{x}^T P x + x^T p \dot{x}]$$

$$= -[-x^T Qx]$$

$$= x^T Qx > 0$$

This is the contradiction as $V(x) = x^T Px$ satisfies instability theorem. Hence the conditions for the positive definiteness of P are necessary and sufficient for asymptotic stability of the system.

The Liapunov's direct method applied to linear time invariant systems is same as the Hurwitz stability criterion.

Examples with Solutions

Example 9.1 : *Show that the following quadratic form is positive definite*

$$V(x) = 8x_1^2 + x_2^2 + 4x_3^2 + 2x_1x_2 - 4x_1x_3 - 2x_2x_3$$

Solution : The above given V(s) can be written as

$$V(x) = x^T P x = [x_1 \; x_2 \; x_3] \begin{bmatrix} 8 & 1 & -2 \\ 1 & 1 & -1 \\ -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Applying Sylvester's criterion we have,

$$8 > 0 ; \begin{vmatrix} 8 & 1 \\ 1 & 1 \end{vmatrix} = 7 > 0; \begin{vmatrix} 8 & 1 & -2 \\ 1 & 1 & -1 \\ -2 & -1 & 4 \end{vmatrix} = 20 > 0$$

As all the successive principal minors of the matrix P are positive, V(x) is positive definite.

Example 9.2 : *Determine the stability of a non-linear system governed by the equations*

$$\dot{x}_1 = -x_1 + 2x_1^2 x_2$$

$$\dot{x}_2 = -x_2$$

Solution : Let us select the Liapunov's function to be $V = x_1^2 + x_2^2$

We have, $V = x_1^2 + x_2^2$

$$\therefore \quad \frac{dV}{dt} = \frac{\partial V}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial V}{\partial x_2} \cdot \frac{dx_2}{dt}$$

Now, $\frac{\partial V}{\partial x_1} = 2x_1; \; \frac{\partial V}{\partial x_2} = 2x_2$

$$\frac{dx_1}{dt} = \dot{x}_1 = -x_1 + 2x_1^2 x_2$$

$$\frac{dx_2}{dt} = \dot{x}_2 = -x_2$$

Substituting,

$$\frac{dV}{dt} = 2x_1(-x_1 + 2x_1^2 x_2) + (2x_2)(-x_2)$$

$$= -2x_1^2(1 - 2x_1 x_2) - 2x_2^2$$

Thus $\frac{dV}{dt}$ is negative definite if $1 - 2x_1 x_2 > 0$. Thus Liapunov's criterion is satisfied and hence the origin of the system is asymptotically stable.

Example 9.3 : *Determine the stability of the system described by the following equation*

$$\dot{x} = Ax \; ; \; A = \begin{bmatrix} -1 & 1 \\ -2 & -4 \end{bmatrix}$$

Solution : To check the stability of the system described by equation $\dot{x} = Ax$ we will solve the equation $A^T P + PA = -Q$ where for any symmetric, positive definite matrix Q, there exists a symmetric positive definite matrix P.

We can find out matrix P for any arbitrary choice of positive, definite real symmetric matrix Q. Let us select Q = I, where I is identity matrix

$$\therefore \quad A^T P + PA = -Q$$

Let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$

$$\therefore \quad \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -2 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -p_{11}-2p_{21} & -p_{12}-4p_{22} \\ p_{11}-4p_{21} & p_{12}-4p_{22} \end{bmatrix} + \begin{bmatrix} -p_{11}-2p_{12} & p_{11}-2p_{22} \\ -p_{21}-2p_{22} & p_{21}-4p_{22} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2p_{11}-2p_{21}-2p_{12} & p_{11}-5p_{12}-2p_{22} \\ p_{11}-5p_{21}-2p_{22} & p_{12}-5p_{22}+p_{21} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

But P is symmetric matrix $\therefore p_{12} = p_{21}$

$$\therefore \begin{bmatrix} -2p_{11}-4p_{12} & p_{11}-5p_{12}-2p_{22} \\ p_{11}-5p_{12}-2p_{22} & 2p_{12}-5p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$-2\,p_{11} - 4\,p_{12} = -1$$

$$p_{11} - 5\,p_{12} - 2\,p_{22} = 0$$

$$2p_{12} - 5\,p_{22} = -1$$

Solving the three equations using Cramer's rule,

$$\Delta = \begin{vmatrix} -2 & -4 & 0 \\ 1 & -5 & -2 \\ 0 & 2 & -5 \end{vmatrix} \begin{aligned} &= -2(25+4)+4(-5) \\ &= -2(29)-20 = -78 \end{aligned}$$

$$\Delta_{p11} = \begin{vmatrix} -1 & -4 & 0 \\ 0 & -5 & -2 \\ -1 & 2 & -5 \end{vmatrix} \begin{aligned} &= -1(25+4)+4(0-2) \\ &= -29-8 \\ &= -37 \end{aligned}$$

$$\Delta_{p12} = \begin{vmatrix} -2 & -1 & 0 \\ 1 & 0 & -2 \\ 0 & -1 & -5 \end{vmatrix} \begin{aligned} &= -2(0-2)+1(-5) \\ &= 4-5 = -1 \end{aligned}$$

$$\Delta_{p22} = \begin{vmatrix} -2 & -4 & -1 \\ 1 & -5 & 0 \\ 0 & 2 & -1 \end{vmatrix} \begin{aligned} &= -2(5)+4-1(2) \\ &= -10-4-2 \\ &= -16 \end{aligned}$$

$$p_{11} = \frac{\Delta_{p11}}{\Delta} = \frac{-37}{-78} = \frac{37}{78}$$

$$p_{12} = p_{21} = \frac{\Delta_{p12}}{\Delta} = \frac{-1}{-78} = \frac{1}{78}$$

$$p_{22} = \frac{\Delta_{p22}}{\Delta} = \frac{-16}{-78} = \frac{16}{78}$$

$$\therefore \qquad P = \begin{bmatrix} \frac{37}{78} & \frac{1}{78} \\ \frac{1}{78} & \frac{16}{78} \end{bmatrix}$$

Using Sylvester's criterion it can be seen that P is positive definite. Therefore origin of the sytem under consideration is asymptotically stable in the large.

Example 9. 4 : *Use Krasovskii's theorem to show that the equilibrium state $x = 0$ of the system described by*

$$\dot{x}_1 = -3x_1 + x_2$$

$$\dot{x}_2 = x_1 - x_2 - x_2^3$$

is asymptotically stable in the large. **(VTU: Jan./Feb.-2005)**

Solution : We have, $\dot{x}_1 = f_1(x) = -3x_1 + x_2$

$$\dot{x}_2 = f_2(x) = x_1 - x_2 - x_2^3$$

The Jacobian matrix is given by

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \qquad \frac{\partial f_1}{\partial x_1} = -3 \qquad \frac{\partial f_1}{\partial x_2} = 1$$

$$\frac{\partial f_2}{\partial x_1} = 1 \quad ; \quad \frac{\partial f_2}{\partial x_2} = -1 - 3x_2^2$$

$$\therefore \qquad J = \begin{bmatrix} -3 & 1 \\ 1 & -1-3x_2^2 \end{bmatrix}$$

Let $\qquad Q = J^T P + PJ$

Let $\qquad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be identity matrix

$$Q = \begin{bmatrix} -3 & 1 \\ 1 & -1-3x_2^2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 1 & -1-3x_2^2 \end{bmatrix}$$

$$Q = \begin{bmatrix} -3 & 1 \\ 1 & -1-3x_2^2 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ 1 & -1-3x_2^2 \end{bmatrix} = \begin{bmatrix} -6 & 2 \\ 2 & -2-6x_2^2 \end{bmatrix}$$

For system to be asymptotically stable Q must be negative definite i.e. – Q must be positive definite

$$-Q = -\begin{bmatrix} -6 & 2 \\ 2 & -2-6x_2^2 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & 2+6x_2^2 \end{bmatrix}$$

Using Sylvester's criterion, $6 > 0$ and $12 + 36\,x_2^2 - 4 > 0$ i.e. $12\,(1 + 3x_2^2) > 4$ then the equilibrium state $x = 0$ is asymptotically stable in the large.

Example 9.5 : *Investigate the stability of the following non-linerar system using direct method of Liaupnov.*

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - x_1^2\,x_2$$

(VTU: July /Aug.- 2005,Jan./Feb.-2005)

Solution : Given that $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 - x_1^2\,x_2$

Let the Liapunov function be, $V = x_1^2 + x_2^2$

$$\begin{aligned} \therefore \qquad \dot{V} &= 2\,x_1\,\dot{x}_1 + 2\,x_2\,\dot{x}_2 \\ &= 2\,x_1\,x_2 + 2\,x_2\,(-x_1 - x_1^2\,x_2) \\ &= 2x_1\,x_2 - 2x_1\,x_2 - 2\,x_1^2\,x_2^2 \\ &= -2\,x_1^2\,x_2^2 \end{aligned}$$

It can be seen that $\dot{V} < 0$ for all non-zero values of x_1 and x_2. Hence the function is negative definite. Therefore the origin of the system is asymptotically stable in large.

Example 9.6 : *A second order system is represented by*

$$\dot{x} = Ax \text{ where } A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

Assuming matrix Q to be identity matrix, solve for matrix P in the equation $A^TP + PA = -Q$. Use Liapunov theorem and determine the stability of the origin of the system. Write the Liapunov function V(x). **(VTU: July/Aug.-2005, Jan./Feb.-2007)**

Solution : $\dot{x} = Ax$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \; ; \; A^T = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

Let

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix},\; Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Consider $$A^TP + PA = -Q$$

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = -\begin{bmatrix} +1 & 0 \\ 0 & +1 \end{bmatrix}$$

$$\begin{bmatrix} -p_{12} - p_{22} \\ p_{11} - p_{12} p_{12} - p_{22} \end{bmatrix} + \begin{bmatrix} -p_{12} & p_{11} - p_{12} \\ -p_{22} & p_{12} - p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2p_{12} & p_{11} - p_{12} - p_{22} \\ p_{11} - p_{12} - p_{22} & 2(p_{12} - p_{22}) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$-2\,p_{12} = -1 \qquad \therefore\ p_{12} = +1/2$$

$$p_{11} - p_{12} - p_{22} = 0 \qquad \therefore\ p_{11} - p_{22} = p_{12} = \frac{1}{2}$$

$$2\,(p_{12} - p_{22}) = -1 \qquad \therefore\ p_{12} - p_{22} = -\frac{1}{2}$$

$$\therefore \qquad p_{22} = p_{12} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

$$p_{11} = p_{12} + p_{22} = \frac{1}{2} + 1 = \frac{3}{2}$$

$$P = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

Using Sylvester's criterion, P can be tested for positive definiteness

$$\frac{3}{2} > 0 \quad \begin{vmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{3}{2} - \frac{1}{4} = \frac{5}{4} > 0$$

$\therefore$ P is positive definite. Hence the equilibrium state at origin is asymptotically stable in the large. The Liapunov's function is

$$V(x) = x^T P x = [x_2 \quad x_2] \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2}x_1 + \frac{1}{2}x_2 & \frac{1}{2}x_1 + x_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\therefore \qquad V(x) = \frac{3}{2}x_1^2 + \frac{1}{2}x_1x_2 + \frac{1}{2}x_1x_2 + x_2^2$$

$$= \frac{1}{2}(3x_1^2 + 2x_1x_2 + 2x_2^2)$$

Example 9.7 : *A system is described by the following equation :* **(VTU: Jan/Feb.- 2008)**

$\dot{x} = Ax$ *Where* $A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}$.

Assuming matrix Q to be the identify matrix, solve for matrix P and comment on the stability of the system using the equation $A^T P + PA = -Q$.

Solution : To check the stability of the system described by equation $\dot{x} = Ax$, We will solve the equation, $A^T P + PA = -Q$ where for any symmetric, positive definite matrix Q, there exists a symmetric, positive definite matrix Q, there exists a symmetric positive definite matrix P.

We can find out matrix P for any arbitrary choice of positive, definite, real symmetric matrix Q. Let us select Q = I, where I is Identity matrix.

$$A^T P + PA = -Q.$$

Let $$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}, \quad A^T = \begin{bmatrix} -1 & 1 \\ -2 & -4 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -p_{11} + p_{21} & -p_{12} + p_{22} \\ -p_{11} - 4p_{21} & -2p_{12} - 4p_{22} \end{bmatrix} + \begin{bmatrix} -p_{11} + p_{12} & -2p_{11} - 4p_{12} \\ -p_{21} + p_{22} & -2p_{21} - 4p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2p_{11} + p_{12} + p_{21} & -5p_{12} - 2p_{11} + p_{22} \\ -2p_{11} - 5p_{21} + p_{22} & -2p_{12} - 2p_{21} - 8p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

But P is symmetric matrix $\therefore p_{12} = p_{21}$

$$\begin{bmatrix} -2p_{11} + p_{12} + p_{21} & -2p_{11} - 5p_{12} + p_{22} \\ -2p_{11} - 5p_{12} + p_{22} & -2p_{12} - 2p_{21} - 8p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$-2p_{11} + p_{12} + p_{21} = -1$$

$$-2p_{11} - 5p_{12} + p_{22} = 0$$

$$-2p_{12} - 2p_{21} - 8p_{22} = -1$$

Solving these three equations using Cramer's rule

$$\Delta = \begin{bmatrix} -2 & 1 & 1 \\ -2 & -5 & 1 \\ -2 & -2 & -8 \end{bmatrix} \begin{array}{l} = -2(40+2)-1(16+2)+1(4-10) \\ = -2(42)-18-6 \\ = -84-18-6 = -108 \end{array}$$

$$\Delta p_{11} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -5 & 1 \\ -1 & -2 & -8 \end{bmatrix} \begin{array}{l} = -1(40+2)-1(1)+1(-5) \\ = -80-1-5 = -86 \end{array}$$

$$\Delta p_{12} = \begin{bmatrix} -2 & -1 & 1 \\ -2 & 0 & 1 \\ -2 & -1 & -8 \end{bmatrix} \begin{array}{l} = -2(0+1)+1(16+2)+1(2) \\ = -2+18+2 \\ = 18 \end{array}$$

$$\Delta p_{22} = \begin{bmatrix} -2 & 1 & -1 \\ -2 & -5 & 0 \\ -2 & -2 & -1 \end{bmatrix} \begin{array}{l} = -2(5)-1(2)-1(4-10) \\ = -10-2+6 = -6 \end{array}$$

$$p_{11} = \frac{\Delta p_{11}}{\Delta} = \frac{-86}{-108} = \frac{86}{108}$$

$$p_{12} = p_{21} = \frac{\Delta p_{12}}{\Delta} = \frac{18}{-108}$$

$$p_{22} = \frac{\Delta p_{22}}{\Delta} = \frac{-6}{-108} = \frac{6}{108}$$

$$P = \begin{bmatrix} \frac{86}{108} & \frac{-18}{108} \\ \frac{-18}{108} & \frac{6}{108} \end{bmatrix}$$

Using Sylvester's criterion, it can be seen that P is positive definite. Therefore origin of the system under consideration is asymptotically stable in the large.

Example 9.8 : *Consider the system with differential equation* $\ddot{e}+k\dot{e}+k_1\dot{e}^3+e=0$. *Examine the stability by Liapunov's method, given that* $k > 0$ *and* $k_1 > 0$.

(VTU: July/Aug.-2007)

Solution: Consider the given equation,

$$\ddot{e}+k\dot{e}+k_1\dot{e}^3+e=0, k > 0, k_1 > 0$$

Let

$$x_1 = e$$

$$\dot{x}_1 = \dot{e} = x_2$$

$$\dot{x}_2 = \ddot{e} = -k\dot{e}-k_1\dot{e}^3-e = -kx_2-k_1x_2^3-x_1$$

As there is no specific procedure for selecting Liapunov's function but it has to be selected from experience. Let us select $f = x_1^2 + x_2^2$ as Liapunov's function

$$\therefore \quad \frac{df}{dt} = 2x_1 \dot{x}_1 + 2x_2 \dot{x}_2$$

$$= 2x_1(x_2) + 2x_2\left(-kx_2 - k_1x_1^3 - x_1\right)$$

$$= 2x_1x_2 - 2kx_2^2 - 2k_1x_2^4 - 2x_1x_2$$

$$= -2kx_2^2 - 2k_1x_2^4 = -2kx_2^2 + k_1x_2^4$$

The above equation is negative semidefinite and hence the system is stable.

Example 9.9 : *Examine the stability of the system described by the following equation by Krasovaskii's theorem.*

$\dot{x}_1 = -x_1$

$\dot{x}_2 = x_1 - x_2 - x_2^3$ **(VTU: July/Aug.-2007)**

Solution :

We have,
$$\dot{x}_1 = f_1(x) = -x_1$$

$$\dot{x}_2 = f_2(x) = x_1 - x_2 - x_2^3$$

The Jacobian matrix is given by,

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \quad \begin{matrix} \frac{\partial f_1}{\partial x_1} = -1 & ; \frac{\partial f_2}{\partial x_1} = 1 \\ \frac{\partial f_1}{\partial x_2} = 0 & ; \frac{\partial f_2}{\partial x_2} = -1 - 3x_2^2 \end{matrix}$$

$$J = \begin{bmatrix} -1 & 0 \\ 1 & -1-3x_2^2 \end{bmatrix}$$

Let
$$Q = J^T P + PJ$$

Let
$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ be identity matrix}$$

$$Q = \begin{bmatrix} -1 & 1 \\ 0 & -1-3x_2^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1-3x_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 0 & -1-3x_2^2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & -1-3x_2^2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2-6x_2^2 \end{bmatrix}$$

$$Q = \begin{bmatrix} -2 & 1 \\ 1 & -2\left(1+3x_2^2\right) \end{bmatrix}$$

For system to be asymptotically stable Q must be negative definite i.e. – Q must be positive definite.

$$-Q = -\begin{bmatrix} -2 & 1 \\ 1 & -2\left(1+3x_2^2\right) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2\left(1+3x_2^2\right) \end{bmatrix}$$

Using Sylvester's criterion, 2>0 and $4+12x_2^2-1>0$ i.e. $4+12x_2^2>1$ i.e. $4\left(1+3x_2^2\right)>1$ then equilibrium state x = 0 is asymptotically stable in the large.

Example 9.10 : *Deteremine whether or not following quadratic form is positive definite :* **(VTU: Jan./Feb.-2007)**

$$Q\left(x_{1,}x_2\right) = 10x_1^2+4x_2^2+x_3^2+2x_1x_2-2x_2x_3-4x_1x_3$$

Solution : Consider the given equation

$$Q\left(x_{1,}x_2\right) = 10x_1^2+4x_2^2+x_3^2+2x_1x_2-2x_2x_3-4x_1x_3$$

Consider,

$$Q = \begin{bmatrix} x_1 \, x_2 \, x_3 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} p_{11}x_1+p_{12}x_2+p_{13}x_3 & p_{12}x_1+p_{22}x_2+p_{23}x_3 & p_{13}x_1+p_{23}x_2+p_{33}x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= p_{11}x_1^2+p_{22}x_2^2+p_{33}x_3^2+2p_{12}x_1x_2+2p_{23}x_2x_3+2p_{13}x_1x_3$$

Equating coefficients of above equation with those from given equation

$p_{11}=10$; $p_{22}=4$; $p_{33}=1$

$2p_{12}=2$; $p_{12}=1$

$2p_{23}=-2$; $p_{23}=-1$

$2p_{13}=-4$; $p_{13}=-2$

The given equation can be rewritten as,

$$Q = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 10 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Applying Sylvester's criterion we have,

$$10 > 0 \; ; \begin{vmatrix} 10 & 1 \\ 1 & 4 \end{vmatrix} = 39 > 0 \; ; \begin{bmatrix} 10 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{bmatrix}$$

$$= 10(4-1)-1(1-2)-2(-1+8)$$

$$= 30+1-14 = 17 > 0$$

As all the successive principal minors of the matrix P are positive, $Q(x_1, x_2)$ is positive definite.

Example 9.11 : *Using Lyapunov's direct method, find the range of K to guarantee stability of the system shown in the Fig. 9.3.* **(VTU: Jan./Feb.- 2006)**

Solution :

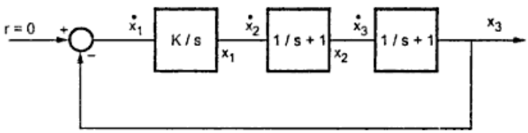


Fig. 9.3

The above diagram can be redrawn as,

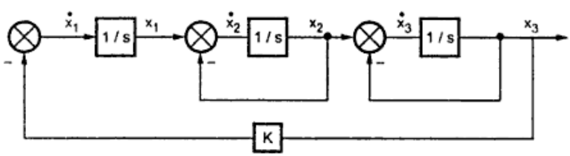


Fig. 9.4

The state equations are,

$$\dot{x}_1 = -Kx_3$$

$$\dot{x}_2 = x_1 - x_2$$

$$\dot{x}_3 = x_2 - x_3$$

In matrix from above equations are written as,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -K \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Here $$A = \begin{bmatrix} 0 & 0 & -K \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad \therefore A^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ -K & 0 & -1 \end{bmatrix}$$

To find the range of values of K for stability, we will solve the equation $A^T P + PA = -Q$.

Let us select $$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This selection of Q is valid as this does not make $x^T Q x$ identically equal to zero except at the origin.

$$-[x_1 \; x_2 \; x_3] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -x_3^2$$

Consider $A^T P + PA = -Q$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ -K & 0 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \begin{bmatrix} 0 & 0 & -K \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Solving the equation we get

$$P = \begin{bmatrix} \frac{1}{K(2-K)} & 0 & \frac{-1}{2(2-K)} \\ 0 & \frac{1}{2(2-K)} & \frac{1}{2(2-K)} \\ \frac{-1}{2(2-K)} & \frac{1}{2(2-K)} & \frac{1}{(2-K)} \end{bmatrix}$$

For P to be positive definite it is necessary and sufficient that

$$K(2-K)>0$$

$$K > 0 \; ; \; 2 - K > 0 \; , \; 2 > K$$

Thus for $0 < K < 2$, the system is asymptotically stable.

Review Questions

1. *Write a note on stability in the sense of Liapunov.*
2. *Define i) stability ii) Asymptotic stability iii) Asymptotic stability in the large.*
3. *Show the graphical representation of stability, asymptotic stability and instability.*
4. *Define i) positive definiteness ii) negative definiteness iii) positive semidefiniteness iv) negative semidefiniteness v) indefiniteness*
5. *Discuss Quadratic form and Hermitian form.*
6. *Explain Liapunov's second method and Liapunov's stability theorem.*
7. *Write a note on stability of linear and nonlinear systems.*
8. *Write a short note on Krasovskii's method of constructing Liapunov's functions for nonlinear systems.*
9. *Examine the stability of the origin of the following system*

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -6x_1 - 5x_2$$

10. *Write a Liapunov function for the system*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Determine the stability of the origin of the system.

11. *Determine the stability of the equilibrium state of the following system.*

$$\dot{x}_1 = -x_1 - 2x_2 + 2$$

$$\dot{x}_2 = x_1 - 4x_2 - 1$$

12. *State and explain Liapunov's theorems on*

i) Asymptotic stability ii) Global asymptotic stability and iii) Instability.

13. *Use Krasovskii's theorem to show that the equilibrium state $x = 0$ of system described by*

$$\dot{x}_1 = -3x_1 + x_2$$

$$x_2 = x_1 - x_2 - x_2^3$$

14. *Investigate the stability of following non-linear system using direct method Liapunov.*

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - x_1^2 x_2$$

15. *A second order system is represented by*

$$\dot{x} = Ax \text{ where } A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

Assuming matrix Q to be identity matrix, solve for matrix p in the equation $A^T P + PA = -Q$. Use Liapunov theorem and determine the stability of the origin of system. Write the Liapunov function V(x).

❑❑❑

Modern Control Theory

✎ **Chapterwise University Questions with Answer**

Jan./Feb. - 2005

July/Aug. - 2005

Jan./Feb. - 2006

July/Aug. - 2006

Jan./Feb. - 2007

July/Aug. - 2007

Jan./Feb. - 2008

1 State Variable Analysis and Design

Q.1 *Mention the disadvantages of conventional control theory and explain how these are overcome in modern control theory with particular reference to (i) Non-linear systems (ii) Time varying system (iii) Analysis (iv) Design and (v) Computer applications.*

(Jan./Feb.-2005, 10 Marks)

Ans. : Refer section 1.1.

Q.2 *Explain the concepts of state variable, state and state model of a linear system.*

(Jan./Feb.-2006, 6 Marks)

Ans. : Refer section 1.2.

Q.3 *Define the concept of i) State ii) State variables iii) State space iv) State vector.*

(July/Aug.-2006, Jan./Feb.-2008, 6 Marks; Jan./Feb.-2007, 10 Marks)

Ans. : Refer section 1.2.

Q.4 *Compare classical control theory against modern control theory.*

(July/Aug.-2006, 4 Marks)

Ans. : Refer section 1.1.

Q.5 *List advantages of modern control theory over conventional control theory.*

(July/Aug.- 2007, 4 Marks)

Ans. : Refer section 1.1.2.

Q.6 *Derive the equation of the vector model differential state equation.*

(July/Aug.- 2007, 2 Marks)

Ans. : Refer section 1.3.

Q.7 *Mention the differences between state space techniques and classical approach.*

(Jan./Feb.-2008, 6 Marks)

Ans. : Refer section 1.1.

❑❑❑

Q.6 *Linearize the following equation in the neighbourhood of the origin.*

$$\frac{d^2\theta}{dt^2} = tan^{-1}2\theta - 3\sin\theta + 2ue^{u/2} + u^3$$

Obtain the approximate response θ(t) for u = 0.02, with the system initially at equilibrium. **(Jan./Feb.-2006, 6 Marks)**

Ans. :

$$\frac{d^2\theta}{dt^2} = \tan^{-1} 2\theta - 3\sin\theta + 2ue^{u/2} + u^3$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\ldots \approx \theta \qquad \text{.... in the neighbourhood of origin}$$

$$\tan^{-1}2\theta = 2\theta + \frac{(2\theta)^3}{3} + \frac{(2\theta)^5}{5} + \ldots\ldots \approx 2\theta \qquad \text{.... in the neighbourhood of origin}$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = 2\theta - 3\theta + 2ue^{u/2} + u^3$$

$$e^{u/2} = 1 + \frac{u}{2} + \frac{(u/2)^2}{2!} + \frac{(u/2)^3}{3!} + \ldots\ldots$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = -\theta + 2u\left[1 + \frac{u}{2} + \frac{u^2}{8} + \frac{u^3}{48} + \ldots\right] + u^3$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = -\theta + 2u \qquad \text{... neglecting higher order terms}$$

$$\therefore \quad \frac{d^2\theta}{dt^2} + \theta = 2u \qquad \text{... linearized equation}$$

Select $X_1(t) = \theta$, $X_2(t) = \dot{X}_1(t)$

$$\therefore \quad \dot{X}_2(t) = -\theta + 2u = -X_1(t) + 2u$$

Thus the state model is,

$$\begin{bmatrix} \dot{X}_1(t) \\ \dot{X}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

$$\therefore A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$[sI - A] = s\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} = \begin{bmatrix}s & -1\\1 & s\end{bmatrix}$$

$$\text{Adj }[sI - A] = \begin{bmatrix}s & -1\\+1 & s\end{bmatrix}^T = \begin{bmatrix}s & 1\\-1 & s\end{bmatrix}$$

$$\therefore \quad [sI - A]^{-1} = \frac{\text{Adj }[sI-A]}{|sI-A|} = \frac{\begin{bmatrix}s & 1\\-1 & s\end{bmatrix}}{s^2+1} = \begin{bmatrix}\frac{s}{s^2+1} & \frac{1}{s^2+1}\\ \frac{-1}{s^2+1} & \frac{s}{s^2+1}\end{bmatrix}$$

$$ZSR = L^{-1}\{\phi(s)\,B\,U(s)\} \text{ where } U(s) = \frac{0.02}{s}$$

$$= L^{-1}\left\{\begin{bmatrix}\frac{s}{s^2+1} & \frac{1}{s^2+1}\\ \frac{-1}{s^2+1} & \frac{s}{s^2+1}\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}\left[\frac{0.02}{s}\right]\right\} = L^{-1}\left\{\begin{bmatrix}\frac{0.04}{s(s^2+1)}\\ \frac{0.04}{s^2+1}\end{bmatrix}\right\}$$

$$L^{-1}\left\{\frac{0.04}{s(s^2+1)}\right\} = L^{-1}\left\{\frac{A}{s} + \frac{Bs+C}{s^2+1}\right\}$$

$$\therefore \quad As^2 + A + Bs^2 + sC = 0.04 \qquad \text{i.e. } A + B = 0,\ C = 0,\ A = 0.04,\ B = -0.04.$$

$$\therefore L^{-1}\left\{\frac{0.04}{s} - \frac{0.04s}{s^2+1}\right\} = 0.04 - 0.04\cos t$$

$$\theta(t) = X_1(t) = \mathbf{0.04 - 0.04\cos t} \qquad \ldots u = 0.02$$

Q.7 *Choosing appropriate physical variables as state variables, obtain the state model for the electric circuit shown in Fig. 2.* **(Jan./Feb.-2006, 8 Marks)**

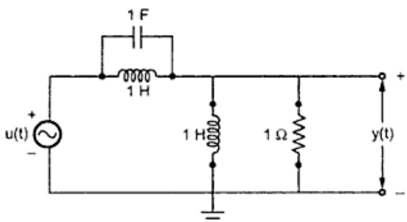


Fig. 2

Ans. : Refer example 2.29.

Q.8 *For the transfer function* $\dfrac{Y(s)}{R(s)} = \dfrac{s(s+2)(s+3)}{(s+1)^2 (s+4)}$

Obtain the state model in

i) Phase variable canonical form

ii) Jordan canonical form **(Jan./Feb.-2006, 8 Marks)**

Ans. : Refer example 2.30.

Q.9 *Fig. 3 shows the block diagram of a speed control system with state variable feedback. The drive motor is an armature controlled dc motor with armature resistance R_a, armature inductance L_a, motor torque constant K_T, inertia referred to motor shaft J, viscous friction coefficient referred to the motor shaft B, back emf constant K_b and tachometer K_t. The applied armature voltage is controlled by a three phase full-converter. e_c is control voltage, e_a is armature voltage, e_r is the reference voltage corresponding to the desired speed. Taking X_1 = ω (speed) and $X_2 = i_a$ (armature current) as the state variables, $u = e_r$ as the input, and y = ω as the output, derive a state variable mode for the feedback system.* **(July/Aug.-2006, 6 Marks)**

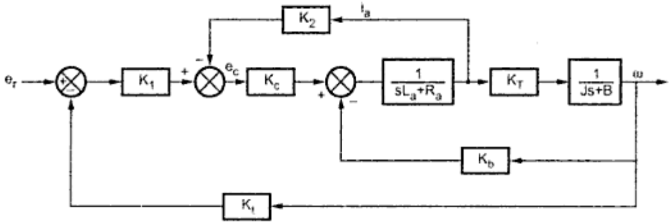


Fig. 3

Ans. : Refer example 2.31.

Q.10 *For a RLC network shown in Fig. 4, write the state model in matrix notation choosing $X_1(t) = i(t)$ and $X_2(t) = v_C(t)$ where $X_1(t)$ and $X_2(t)$ are state variables, $v_C(t)$ is output, $v(t)$ is input.* **(July/Aug.-2006, 8 Marks)**

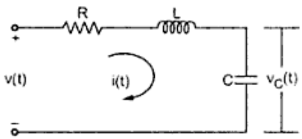


Fig. 4

Ans. : Refer example 2.1.

Q.11 *For a transfer function given by* $G(s) = \dfrac{2}{s^2+3s+2}$ *write the state model in the i) Phase variable form ii) Diagonal form.* **(July/Aug.-2006, 8 Marks)**

Ans. : Refer example 2.32.

Q.12 *State the properties of Jordan matrix.* **(July/Aug.-2006, 6 Marks)**

Ans. : Refer section 2.3.1.

Q.13 *Obtain the state space representation model for the following electrical circuit in Fig. 5. Given R = 1 MΩ and C = 1 μF.* **(Jan./Feb.-2007, 10 Marks)**

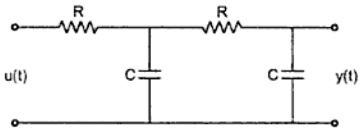


Fig. 5

Ans. : Refer example 2.22.

Q.14 *Obtain the state space representation of the following system and draw its phase variable diagram :* $\dddot{Y} + 6\ddot{Y} + 11\dot{Y} + 6Y = 6u$

(Jan./Feb.- 2007, 10 Marks; July/Aug.-2007, 6 Marks)

Ans. : Taking Laplace transform of both sides and neglecting initial conditions we get, $\dfrac{Y(s)}{U(s)} = \dfrac{6}{s^3+6s^2+11s+6}$.

Then refer example 2.15.

Q.15 *Obtain the state model of the electrical network shown in Fig. 6 selecting 'V', 'i_1' and 'i_2' as state variables and voltage across R_2 and current I_2 through R_2 are the output variables Y_1 and Y_2.* **(July/Aug.-2007, 8 Marks)**

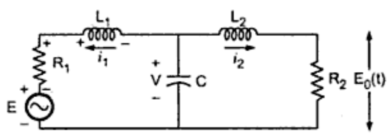


Fig. 6

Ans. : Refer example 2.19 for the procedure. The direction of i_1 is opposite in this example. Thus the matrices A and B are,

$$A = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & +\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & +\frac{1}{L_2} \\ -\frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix}$$

There are two outputs,

$$Y_1 = E_0(t) = i_2 R_2 = R_2 X_2$$

$$Y_2 = i_2 = X_2$$

$$\therefore \quad C = \begin{bmatrix} 0 & R_2 \\ 0 & 1 \end{bmatrix}$$

Q.16 *For the network shown in Fig. 7, choosing $i_1(t) = X_1(t)$ and $i_2(t) = X_2(t)$ as state variables, obtain the state equation and output equation in vector matrix form.*
(Jan./Feb.-2008, 8 Marks)

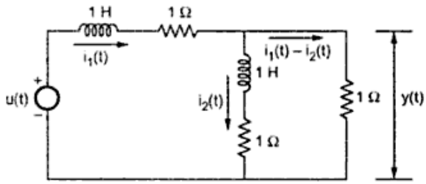


Fig. 7

Ans. : Refer example 2.33.

Q.17 *Obtain the state model in phase variable form and write the block diagram for the system represented by,* **(Jan./Feb.-2008, 6 Marks)**

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 10y = 3u(t).$$

Ans. : Refer example 2.16.

Q.18 *For the following transfer function obtain the state model in canonical form:*
(Jan./Feb.-2008, 8 Marks)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{6}{s^3 + 6s^2 + 11s + 6}.$$

Ans. : Refer example 2.15.

3 Matrix Algebra and Derivation of Transfer Function

Q.1 *Determine the transfer matrix for the system* **(Jan./Feb.-2005, 10 Marks)**

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ -5 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} ; \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 8 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Ans. : Refer example 3.20.

Q.2 *Find the transformation matrix P that transforms the matrix A into diagonal or Jordan form where* $A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}$. **(Jan./Feb.-2005, 10 Marks)**

Ans . : Refer example 3.15.

Q.3 *What are genralized eigen vectors? How are they determined?* **(July/Aug.-2005, 5 Marks)**

Ans. : Refer section 3.10.

Q.4 *Convert the following state model into canonical form* **(July./Aug.-2005, 8 Marks)**

$$\dot{X} = \begin{bmatrix} 1 & -4 \\ 3 & -6 \end{bmatrix} X + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$

$$Y = [1 \quad 0]X$$

Ans. : Refer example 3.21.

Q.5 *Convert the following square matrix A into Jordan canonical form using a suitable non-singular transformation matrix P.* **(July./Aug.-2005, 7 Marks)**

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -9 & -6 \end{bmatrix}$$

Ans. : Refer example 3.22.

Q.6 *Consider the matrix*

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

i) Find the eigen values and eigen vectors of A

ii) Write the modal matrix

iii) Show that the modal matrix indeed diagonalizes A. **(Jan./Feb.-2006, 12 Marks)**

Ans. : Refer example 3.23.

Q.7 *Given the state model* $\dot{X} = AX - Bu,\ y = CX$

$$\text{where } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } C = [1\ \ 0\ \ 0]$$

i) Simulate and find the transfer function $\dfrac{Y(s)}{U(s)}$ *using Mason's gain formula.*

ii) Determine the transfer function from the state model formulation. **(Jan./Feb.-2006, 7 Marks)**

Ans. : Refer example 3.24.

Q.8 *The vector* $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ *is an eigen vector of* $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. *Find the eigen value of A corresponding to the vector given.* **(July/Aug.-2006, 5 Marks)**

Ans. : Refer example 3.25.

Q.9 *Show that the characteristic equation and eigen values of a sytem matrix are unvariant under linear transformation.* **(July/Aug.-2006, 8 Marks)**

Ans. : Refer section 3.9.

Q.10 *Determine the transfer function of the given state vector differential equation below :*

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & -5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} u$$

(July/Aug.-2007, 8 Marks)

$$= \begin{bmatrix} \frac{1}{s} & \frac{s+5}{s(s+4)(s+1)} & \frac{1}{s(s+4)(s+1)} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$$

$$= \frac{1}{s} + \frac{s+5}{s(s+4)(s+1)}(-1) + \frac{5}{s(s+4)(s+1)}$$

$$= \frac{1}{s} - \frac{s+5}{s(s+4)(s+1)} + \frac{5}{s(s+4)(s+1)}$$

$$= \frac{1}{s} + \frac{-s-5+5}{s(s+4)(s+1)} = \frac{1}{s} - \frac{s}{s(s+4)(s+1)}$$

$$= \frac{1}{s} - \frac{1}{(s+4)(s+1)} = \frac{(s+4)(s+1) - s}{s(s+4)(s+1)}$$

$$\therefore \quad \text{T.F.} = \frac{s^2+5s+4-s}{s(s+4)(s+1)} = \frac{s^2+4s+4}{s(s+4)(s+1)}$$

Q.11 *Obtain eigen values, eigen vectors and state model in canonical form for a system described by the following state model :* **(July/Aug.-2007, 10 Marks)**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \textit{ uand } y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Ans. : Refer example 3.14 for obtaining M. Once M is obtained find M^{-1} and the new matrices $\tilde{B} = M^{-1}B$ and $\tilde{C} = CM$.

$$M = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -4 & -3 \\ -1 & 1 & 3 \end{bmatrix}, \; M^{-1} = \begin{bmatrix} 4.5 & 2.5 & 1 \\ -3 & -2 & -1 \\ 2.5 & 1.5 & 1 \end{bmatrix}, \; \tilde{B} = \begin{bmatrix} 6.5 \\ -5 \\ 4.5 \end{bmatrix}, \tilde{C} = [1 \quad 2 \quad 1].$$

Q.12 *Obtain the model matrix for the matrix given below :* **(Jan./Feb.-2008, 4 Marks)**

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix}$$

Ans. : Refer example 3.14.

4 Solution of State Equations

Q.1 *For a system represented by $\dot{X} = AX$ the response to one set of initial conditions is $X(t) = \begin{bmatrix} 2e^{-4t} \\ e^{-4t} \end{bmatrix}$ and another set of initial condition is $X(t) = \begin{bmatrix} 4e^{-2t} \\ e^{-2t} \end{bmatrix}$. Determine matix A and the state transition matrix $\phi(t)$.* **(Jan./Feb.-2005, 10 Marks)**

Ans. : Refer example 4.19.

Q.2 *Mention the conditions for complete controllability and complete observability of continous time systems. Using these, explain the principle of duality between controllability and observability.*

(Jan./Feb.-2005,Jan./Feb,-2008 10 Marks,;July/Aug.-2007 6 Marks)

Ans. : Refer sections 4.12 and 4.13.

Q.3 *Use controllability and observability matrices to determine whether the system represented by the flow graph shown in Fig. 1 is completely controllable and completely observable.*

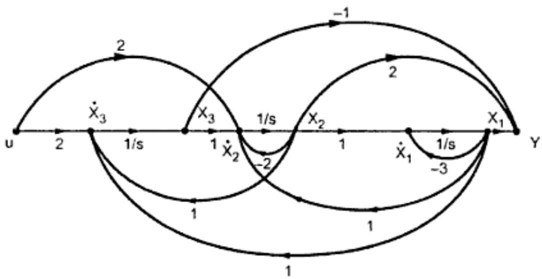


Fig. 1

(Jan./Feb.-2005, 10 Marks)

Ans. : Refer example 4.33.

Q.4 *Given the time invariant system :*

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ 0 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \;:\; y \; [1 \quad 0] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

and that $u(t) = e^{-1}$ and $y(t) = 2-\alpha te^{-1}$, find $X_1(t)$ and $X_2(t)$.
Find also $X_1(0)$ and $X_2(0)$. What happens if $\alpha = 0$? **(Jan./Feb.-2005,10 Marks)**

Ans. : Refer example 4.34.

Q.5 *What is a state transition matrix ? List the properties of state transition matrix.*
(July/Aug.-2005, July/Aug.-2007,Jan.Feb.-2008,6 Marks; July/Aug.-2006, 5 Marks, Jan./Feb.-2007, 10 Marks)

Ans. : Refer sections 4.3 and 4.4.

Q.6 *Given the state model of a system :*

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$Y = [1 \quad 0] \, X$$

$$\text{with initial conditions } X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Determine :

i) *The state transition matrix.*

ii) *The state transition equation X(t) and output Y(t) for an unit step input.*

iii) *Inverse state transition matrix.* **(July/Aug.-2005, 14 Marks)**

Ans. : Refer example 4.35.

Q.7 *Explain the concept of controllability and observability.*
(July/Aug.-2005, July/Aug.-2006, 6 Marks)

Ans. : Refer section 4.11.

Q.8 *Determine the controllability and observability of the following state model.*

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$Y = [10 \quad 5 \quad 1] X$$

(July/Aug.-2005, 8 Marks)

Ans. : Refer example 4.36.

Q.9 *A system represented by following state model is controllable but not observable. Show that the non-observability is due to a pole-zero cancellation in $C[sI-A]^{-1}B$.*

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$Y = [1 \quad 1 \quad 0] X$$

(July/Aug.-2005, 6 Marks)

Ans. : Refer example 4.37.

Q.10 *List atleast three important properties of the state transition matrix.*

(Jan./Feb.-2006, 3 Marks)

Ans. : Refer section 4.4.

Q.11 *Consider the homogeneous equation $\dot{X} = AX$, where A is a 3 × 3 matrix. The following three solution for three different initial conditions are available,*

$$\begin{bmatrix} e^{-t} \\ -e^{-t} \\ 2e^{-t} \end{bmatrix}, \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \\ 0 \end{bmatrix}, \begin{bmatrix} 2e^{-3t} \\ -6e^{-3t} \\ 0 \end{bmatrix}$$

i) Identify the initial conditions

ii) Find the state transition matrix

iii) Hence or otherwise find the system matrix A. **(Jan./Feb.-2006, 10 Marks)**

Ans. : Refer example 4.38.

Q.12 *Obtain the time response y(t) of the system given below by first transforming the state model into a 'Canonical model'.*

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u, \; Y = [1 \quad 0 \quad 0] \; X$$

u is a unit step function and $X^T(0) = [0 \quad 0 \quad 2]$ **(Jan./Feb.-2006, 12 Marks)**

Ans. : Refer example 4.39.

Q.17 *Obtain the state transition matrix using :*

i) *Laplace transformation method and*

ii) *Cayley-Hamilton method.*

For the system describe by,

$$X(t) = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} X(0)$$

(Jan./Feb.-2007, 10 Marks)

Ans. : Refer example 4.43.

Q.18 *State the conditions for completely controllability and complete observability. Determine the state controllability and observability of the system described by,*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [u]$$

$$Y = [4 \quad 5 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(Jan./Feb.-2007, 10 Marks)

Ans. : Refer section 4.11 and example, 4.36 for the procedure. The system is completely controllable and completely observable.

Q.19 *Obtain the observable phase variable state model of*

$T.F. = T(s) = \dfrac{Y(s)}{U(s)} = \dfrac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$. *Draw the signal flow graph of T(s).*

(July/Aug.-2007, 8 Marks)

Ans. : Refer example 4.46.

Q.20 *Obtain the controllable phase variable form of the transfer function*

$$T.F. = T(s) = \frac{Y(s)}{0(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}.$$

(July/Aug.-2007, 6 Marks)

Ans. : Refer example 4.47.

Q.21 *Compute e^{At} for the given matrix :*

$$A_1 = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}; A_2 = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}; A = \begin{bmatrix} 6 & \omega \\ -\omega & 6 \end{bmatrix}$$

(July/Aug.-2007, 6 Marks)

Ans. : Refer examples 4.27 and 4.28.

Q.26 *Determine the complete time response of the system given by :*

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} X(t), \text{ where } X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } Y(t) = [1 \quad -1]\, X(t).$$

(Jan./Feb.-2008, 12 Marks)

Ans. : Refer example 4.7.

Q.27 *Determine the controllability and observability using Kalman's test for the system described by,*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = [10 \quad 0 \quad 0]$$

(Jan./Feb.-2008, 10 Marks)

Ans. : Refer example 4.36 for the procedure and the given system **is completely controllable and observable.**

❑❑❑

Q.11 *Consider the system defined by* $\dot{x} = Ax + Bu$, *where* $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. *By using state feedback control* $u = -Kx$, *it is desired to have the closed loop poles at* $s = -2 \pm j4$ *and* $s = -10$. *Determine the state feedback gain matrix "K" by any one method.* **(Jan./Feb.-2008, 8 Marks)**

Ans. : Refer example 5.10

Q.12 *Consider the system* $\dot{x} = Ax + Bu$ *and* $Y = Cx$, *where* $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ *and* $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. *Determine the observer gain matrix by the use of :*

i) The direct substitution method.

ii) Ackermann's formula.

Assume that the desired Eigen values of the observer gain matrix are $\mu_1 = -2 + j3.4641$ *and* $\mu_2 = -2 - j3.4641$, $\mu_3 = -5$. **(Jan./Feb.-2008, 8 Marks)**

Ans. : Refer example 5.11.

□□□

6 Controllers

Q.1 *Explain effects of a PI controller on the static and dynamic response of a system.* **(July/Aug.-2005, 5 Marks)**

Ans. : Refer section 6.9.

Q.2 *Consider a typical second order, type one system with unity feedback, being controlled by a PD controller and show that i) Damping increase with PD control.* **(July/Aug.-2005, 4 Marks)**

Ans. : Refer section 6.14.

Q.3 *A temperature control system has the block diagram given in Fig.1. The input signal is a voltage and represents the desired temperature θ_r. Find the steady-state error of the system when θ_r is a unit step function and i) D(s) = 1 ii) $D(s) = 1 + \frac{0.1}{s}$ iii) D(s) = 1 + 0.3 s. What is the effect of the integral term in the PI controller and the derivative term in PD controller on the steady state error ?* **(July/Aug.-2007, 8 Marks)**

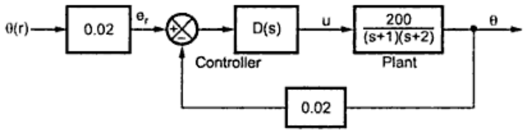


Fig. 1

Ans. : Refer example 6.16.

Q.4 *What is a controller ? Explain P, I, PI and PID controllers.* **(Jan./Feb.-2007, 10 Marks)**

Ans. : Refer sections 6.1, 6.5, 6.6, 6.9 and 6.11

Q.5 *Define controller. Explain properties of P, PI and PID controllers with the help of block diagram.* **(July/Aug.-2007, 8 Marks)**

Ans. : Refer sections 6.1, 6.5, 6.9 and 6.11

Q.6 *Define controller. Explain PD, PI and PID controllers. What are the advantages of PID controller ?* **(Jan./Feb.2008, 6 Marks)**

Ans. : Refer sections 6.1, 6.10, 6.9 and 6.11

7 Nonlinear Systems

Q.1 *Explain the following behaviour of non-linear systems :*
(i) Frequency amplitude dependence .
(ii) Multivalued responses and jump resonances. **(Jan/Feb.-2005, 10 Marks)**

Ans. : Refer section 7.2.

Q.2 *Discuss the basic features of the following non-linearities*

i) Non-linear friction

ii) On-off controllers

iii) Back lash **(Jan./Feb.-2006, 9 Marks)**

Ans. : Refer sections 7.4.5 and 7.4.6.

Q.3 *What are inherent nonlinearities ? Explain any three of them.* **(July/Aug.-2006, 6 Marks)**

Ans. : Refer section 7.4.

Q.4 *Sketch the following non-linearities :*

i) Ideal relay ii) Relay with dead zone iii) Relay with dead zone and hysterisis iv) Relay with hysterisis v) Dead zone. **(July/Aug.-2006, 4 Marks)**

Ans. :

Q.5 *Discuss the basic features of the Backlash non-linearities with suitable figures.* **(July/Aug.-2007, 4 Marks)**

Ans. : Refer section 7.4.6.

Q.6 *Explain any three non-linearities in control systems.* **(Jan./Feb.-2008, 6 Marks)**

Ans. : Refer section 7.4.

❑❑❑

8 Phase Plane Method

Q.1 *Determine the kind of singularity for each of the following differential equations.*

(i) $\ddot{y}+3\dot{y}+2y=0$

(ii) $\ddot{y}-8\dot{y}+17y=34$ **(Jan./Feb.-2005, 10 Marks)**

Ans. : Refer example 8.1.

Q.2 *What is a phase plane plot? Describe delta method or any other method of drawing phase plane trajectories.* **(Jan./Feb.-2005, 10 Marks)**

Ans. : Refer sections 8.1 and 8.5.

Q.3 *With reference to non-linear system explain :*

(i) Jump resonance

(ii) Limit cycles **(July/Aug.-2005, 6 Marks)**

Ans. : Refer sections 8.3 and 8.2.

Q.4 *What are singular points? Explain the classification of singular points based on the location of eigen values of the system.* **(July/Aug.-2005; Jan./Feb.-2008, 8 Marks)**

Ans. : Refer section 8.9.

Q.5 *Fig. 1 shows phase portraits for type - 0 systems. Classify them into the categories. Stable focus, stable node, saddle point and so-on. (See Fig. 1 on next page.)* **(Jan./Feb.-2006;July/Aug.-2007, 6 Marks)**

Ans. : Stable focus

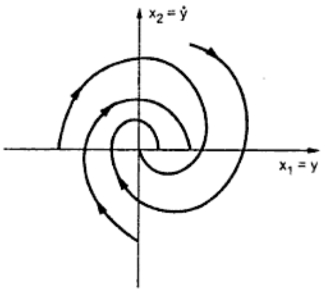


Fig. 2 Stable focus

Unstable node

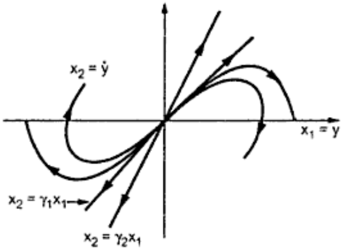


Fig. 6 Unstable node

Saddle point

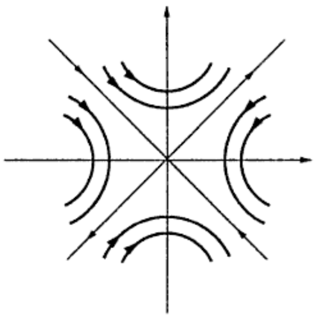


Fig. 7 Saddle point

Q.6 *Explain the concept of jump resonance with a suitable example.*

(Jan./Feb.-2006, 5 Marks)

Ans. : Refer section 8.3.

Q.7 *Explain the delta method of constructing phase trajectories.* **(Jan./Feb.-2006, 7 Marks)**

Ans. : Refer section 8.11.

Q.8 *Using isocline method, draw the phase trajectory for the system.*

$$\frac{d^2x}{dt^2} + 0.6\,\frac{dx}{dt} + x = 0$$

with $x = 1$ *and* $\frac{dx}{dt} = 0$ *as initial condition.*

(Jan./Feb.-2006, 8 Marks)

Ans. : Refer section 8.4.

Q.9 *A linear second order servo is described by the equation* $\ddot{C}+2\xi\omega_n\dot{C}+\omega_n^2 C = 0$ *where* $\xi = 0.15$, $\omega_n = 1$ *rad/sec,* $C(0) = 1.5$ *and* $\dot{C} = 0$. *Determine the singular point. Construct the phase trajectory, using the method of isoclines.* **(July/Aug.-2006, 10 Marks)**

Ans. : Refer example 8.2.

Q.10 *Define singular point on a phase plane. Explain different types of singular points* **(July/Aug.-2006, 10 Marks)**

Ans. : Refer section 8.9.

Q.11 *What are singular points? Explain different singular points adopted in non-linear control systems.* **(Jan./Feb.-2007, 8 Marks)**

Ans. : Refer section 8.9.

Q.12 *Find out singular points for the following systems :*

i) $\ddot{x}+0.5\dot{x}+2x=0$

ii) $\ddot{y}+3\dot{y}+2y=0$

iii) $\ddot{y}+3\dot{y}-10=0$ **(Jan./Feb.-2007, 12 Marks)**

Ans. : Refer example 8.3.

Q.13 *Draw the phase-plane trajectory for the following equation using Isocline method :*

$$\ddot{x}+2\xi\omega\dot{x}+\omega^2 x = 0$$

Given, $\xi=0.5$, $\omega=1$, *Initial point (0, 6).* **(Jan./Feb.-2007, 12 Marks)**

Ans. : Refer example 8.5.

Q.14 *Discuss the basic features of the jump resonance non-linearities with suitable figures : Jump resonance.* **(July/Aug.-2007, 8 Marks)**

Ans. : Refer section 8.3.

Q.15 *Explain the construction of a phase trajectory by Delta method.* **(Jan./Feb.-2008, 6 Marks)**

Ans. : Refer section 8.5.

❑❑❑

9 Liapunov's Stability Analysis

Q.1 *State and explain Liapunov's theorems on*
(i) Asymptotic stability
(ii) Global asymptotic stability and (iii) Instability.
(Jan./Feb.-2005; July/Aug.-2006; Jan./Feb.-2008, 10 Marks)

Ans. : Refer sections 9.3, 9.4 and 9.5.

Q.2 *Use Krasovskii's theorem to show that the equilibrium state $x = 0$ of the system described by*

$\dot{x}_1 = -3x_1 + x_2$

$\dot{x}_2 = x_1 - x_2 - x_2^3$ *is asymptotically state in the large.* **(Jan./Feb.-2005, 10 Marks)**

Ans. : Refer example 9.4.

Q.3 *Define :*
(i) Stability, (ii) Asymtotic stability (iii) Asymptotic stability in the large.
(July/Aug.-2005, 5 Marks, July/Aug.-2007, 6 Marks)

Ans. : Refer sections 9.29.3, 9.4 and 9.5.

Q.4 *Investigate the stability of the following non-linear system using direct method of Liapunov.*

$\dot{x}_1 = x_2$

$\dot{x}_2 = -x_1 - x_1^2 x_2$ **(July/Aug.-2005, 5 Marks)**

Ans. : Refer example 9.5.

Q.5 *A second order system is represented by,*

$\dot{x} = Ax$ *where* $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$

Assuming matrix Q to be identity matrix, solve for matrix P in the equation $A^T P + PA = -Q$. Use Liapunov theorem and determine the stability of the origin of the system. Write the Liapunov function V(x). **(July/Aug.-2005, 10 Marks)**

Ans. : Refer example 9.6.

Q.6 *Using Liapunovs direct method, find the range of K to guarantee stability of the system shown in Fig. 1.*

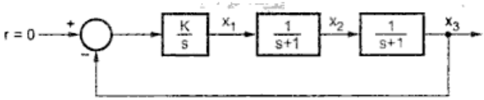


Fig. 1

(Jan./Feb.-2006, 14 Marks)

Ans. : Refer example 9.11.

Q.7 *Choose an appropriate Liapunov function and check the stability of the equilibrium state of the system described by*

$\dot{x} = x_2$

$\dot{x}_2 = -x_1 - x_1^2 x_2$ **(Jan./Feb.-2006, 6 Marks)**

Ans. : Refer example 9.5.

Q.8 *Determine whether or not following quadratic form is positive definite :*

$Q(x_1, x_2) = 10x_1^2 + 4x_2^2 + x_3^2 + 2x_1x_2 - 2x_2x_3 - 4x_1x_3$ **(Jan./Feb.-2007, 10 Marks)**

Ans. : Refer example 9.10.

Q.9 *Explain with an example - i) Liapunov main stability theorem ii) Liapunov second method and iii) Krasovskii's theorem.* **(Jan./Feb.-2007, 10 Marks)**

Ans. : Refer sections 9.10, 9.11 and 9.13.

Q.10 *Find the Liapunov function for the system :*

$$X(t) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} X.$$

(Jan./Feb.-2007, 8 Marks)

Ans. : Refer example 9.6.

Q.11 *Explain Liapunov's stability criterion.* **(July/Aug.-2007, 6 Marks)**

Ans. : Refer section 9.11.

Q.12 *Consider the system with differential equation.*

$\ddot{e}+k\dot{e}+k_1e^3+e=0$. *Examine the stability by Liapunov's method, given that* $K>0$ *and* $K_1>0$. **(July/Aug.-2007, 6 Marks)**

Ans. : Refer example 9.8.

Q.13 *Examine the stability of the system described by the following equation by Krasovdskii's.*

$\dot{x}_1=-x_1 \quad \dot{x}_2=x_1-x_2-x_2^3$ **(July/Aug.-2007, 8 Marks)**

Ans. : Refer example 9.9.

Q.14 *A system is described by the following equation :*

$\dot{x}=Ax$ Where $A=\begin{bmatrix}-1 & -2\\ 1 & -4\end{bmatrix}$.

Assuming matrix Q to be the identify matrix, solve for matrix P and comment on the stability of the system using the equation $A^TP+PA=-Q$.

(Jan/Feb.- 2008, 10 Marks)

Ans. : Refer example 9.7.

□□□

Contents

- **State variable analysis and design**
  Concept of state, State variables and state model, State modeling of linear systems, Linearization of state equations.
- State space representation using physical variables, Phase variables and canonical variables.
- Derivation of transfer function from state model, Digitalization, Eigen values, Eigen vectors, Generalized Eigen vectors.
- Solution of state equation, State transition matrix and its properties, Computation using Laplace transformation, Power series method, Cayley- Hamilton method, Concept of controllability and observability, Methods of determining the same.
- **Pole placement techniques**
  Stability improvements by state feedback, Necessary and sufficient conditions for arbitrary pole placement, State regulator design, And design of state observer, Controllers- P, PI, PID.
- **Non-linear systems**
  Behavior of non-linear system, Common physical non linearity-saturation, Friction, Backlash, Dead zone, Relay, Multivariable non-linearity.
- Phase plane method, Singular points, Stability of non-linear system, Limit cycles, Construction of phase trajectories.
- Liapunov stability criteria, Liapunov functions, Direct method of Liapunov and the linear system, Hurwitz criterion and Liapunov's direct method, Construction of Liapunov functions for non-linear system by Krasvskii's method.

First Edition : 2008

Price INR 225/-

ISBN 978-81-8431-506-6

Technical Publications Pune

1, Amit Residency, 412 Shaniwar Peth, Pune - 411030, M.S., India.
Telefax : +91 (020) 24495496/97, Email : technical@vtubooks.com
Visit us at : www.vtubooks.com