Research Statement

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Abstract. Modular forms and modular curves are indispensable tools in the study of elliptic curves and abelian varieties, and their relation to Galois representations has long been a central topic in number theory. In order to explicitly describe the image of the Galois representations attached to elliptic curves defined over $\mathbb{Q}$, Zywina [Zyw22] introduced agreeable subgroups of $\mathrm{GL}_2(\hat{\mathbb{Z}})$. Let $G \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$ be an open subgroup such that $\det(G) = \hat{\mathbb{Z}}^\times$ and $-I \in G$. Let X_G be the modular curve associated to G, parametrizing elliptic curves with G-structure. My doctoral research utilizes agreeable subgroups to study modular curves defined over $\mathbb{Q}$ in families of abelian twists. In particular, I have given **a complete classification of modular curves of genus at most 24.** More precisely, fixing a non-negative integer g, there are finitely many families $\{\mathscr{F}_i\}_{i\in I}$ of abelian twists of modular curves such that any modular curve X_G of genus at most g lies in one of $\mathscr{F}_i$. Furthermore, this classification has been used to give an efficient and implemented algorithm for computing projective models, modular forms, (singular) plane models, and gonality bounds for modular curves of genus ≤ 12 defined over $\mathbb{Q}$. This constitutes progress towards Mazur's Program B and is of great interest to number theorists. In the summer of 2025, our implementation was adopted by the L-functions and Modular Forms Database (LMFDB) to compute models for one million modular curves and to update their database.

In the last decades, number theorists have extensively studied various properties of modular curves. Many of these results can be investigated in the special case of a family of twists of modular curves, which are easier to treat and well-behaved. A major part of my research program going forward is to utilize my doctoral thesis to study the vast literature on modular curves in finitely many families of abelian twists.

In my postdoctoral research, I will

(1) investigate the isogeny decomposition of Jacobians of modular curves in families of twists,

(2) apply various methods in the literature to study the existence of local points in families of twists,

(3) extend the result of my thesis to modular curves over number fields by removing the assumption of surjective determinant,

(4) investigate $\mathbb{Q}$-gonality of geometrically trigonal modular curves,

(5) explore the relationship between descent obstructions and the birational section property for modular curves [Koe05, Sti15], in the spirit of Grothendieck's anabelian geometry,

(6) study the obstructions to the existence of rational points on modular curves, by comparing the Brauer–Manin obstruction and finite descent obstructions in the sense of [Sto07],

(7) conduct a systematic study of geometrically bielliptic modular curves, investigating whether their bielliptic structures descend to $\mathbb{Q}$.

1 Background and past work

In algebraic and arithmetic geometry, modular curves are important objects of study. Let $G \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$ be an open subgroup where $\hat{\mathbb{Z}}$ denotes the profinite completion of $\mathbb{Z}$. Unless stated otherwise, we assume that $\det(G) = \hat{\mathbb{Z}}^\times$ and $-I \in G$. The group G is uniquely determined by its image in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, where N is an integer divisible by the level of G. To systematically investigate the elliptic curves, the modular curves were defined, most generally by Deligne and Rapoport [DR73]. In this context, the modular curve X_G is the generic fiber of the coarse space for the moduli stack over $\mathbb{Z}[1/N]$ that parametrizes elliptic curves with G-level structure. When $\det(G) = \hat{\mathbb{Z}}^\times$, X_G is a smooth, projective, and geometrically irreducible curve defined on $\mathbb{Q}$.

Modular curves do not come with equations; instead, they are defined by the moduli problem they represent. In general, it is a hard problem to find an explicit model for an arbitrary modular curve X_G of genus g. The algorithm I created reduces this problem to the computation of finitely many modular curves and a collection of linear algebra problems that are easily handled. The result underlying this algorithm, namely, the classification of modular curves into finitely many families of twists, generates new questions regarding the behavior of many properties of modular curves under twisting.

Modular curves can be defined in alternative ways, e.g., by specifying its function field or by using the graded ring of modular forms with respect to the congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ arising from G. It is clear, using any of these definitions, that the inclusion $G \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$ induces a j-map $X_G \to \mathbb{P}^1_{\mathbb{Q}}$, where $\mathbb{P}^1_{\mathbb{Q}} \cong X_{\mathrm{GL}_2(\hat{\mathbb{Z}})}$ is called the j-line. Over $\mathbb{C}$, the j-line corresponds to isomorphism classes of elliptic curves.

Zywina introduced in [Zyw22] an efficient and implemented algorithm to compute the image of the Galois representation ρ_E, associated to an elliptic curve E defined over $\mathbb{Q}$. To this end, he introduced agreeable subgroups of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ and used a special subset of them to compute these images. Along with agreeable subgroups, he also introduced the family of groups attached to a pair $(\mathcal{G}, B)$.

We say that a subgroup $\mathcal{G}$ of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ is agreeable if it is open in $\mathrm{GL}_2(\hat{\mathbb{Z}})$, satisfies $\det(\mathcal{G}) = \hat{\mathbb{Z}}^\times$, contains all the scalar matrices, and if the levels of $\mathcal{G} \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$ in $\mathrm{GL}_2(\hat{\mathbb{Z}})$ and $\mathcal{G} \cap \mathrm{SL}_2(\hat{\mathbb{Z}}) \subseteq \mathrm{SL}_2(\hat{\mathbb{Z}})$ in $\mathrm{SL}_2(\hat{\mathbb{Z}})$ have the same odd prime divisors. Let $G \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$ be an open subgroup. There is a unique minimal agreeable subgroup satisfying $G \subseteq \mathcal{G}$, which is called the agreeable closure of G. The group G is normal in $\mathcal{G}$ and $\mathcal{G}/G$ is finite and abelian. In particular, commutator subgroups of $\mathcal{G}$ and G agree with each other.

Let $\mathcal{G}$ be an agreeable subgroup of $\mathrm{GL}_2(\hat{\mathbb{Z}})$. Fix a closed subgroup B of $\mathcal{G}$ satisfying $[\mathcal{G}, \mathcal{G}] \subseteq B \subseteq \mathrm{SL}_2(\hat{\mathbb{Z}})$.

Definition 1. The family of groups associated to the pair $(\mathcal{G}, B)$ is the set $\mathscr{F}(\mathcal{G}, B)$ of open subgroups H of $\mathcal{G}$ that satisfy $\det(H) = \hat{\mathbb{Z}}^\times$ and $H \cap \mathrm{SL}_2(\hat{\mathbb{Z}}) = B$.

Assume that $\mathscr{F}(\mathcal{G}, B)$ is nonempty. Pick a group $G \in \mathscr{F}(\mathcal{G}, B)$. We know that G is an open subgroup of $\mathcal{G}$ and since $[\mathcal{G}, \mathcal{G}] \subset G$, we have that G is a normal subgroup of $\mathcal{G}$ and the quotient $\mathcal{G}/G$ is finite and abelian. Consider any homomorphism $\gamma\colon \hat{\mathbb{Z}}^\times \to \mathcal{G}/G$. It gives rise to the group

$$G_\gamma := \{g \in \mathcal{G} : g \cdot G = \gamma(\det g)\}.$$

Lemma 2. *With notation as above, the set $\mathscr{F}(\mathcal{G}, B)$ consists of the groups G_γ with $\gamma\colon \hat{\mathbb{Z}}^\times \to \mathcal{G}/G$ a homomorphism.*

Remark 3. Since the genus of an open subgroup of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ is determined by its intersection with $\mathrm{SL}_2(\hat{\mathbb{Z}})$, all the groups in $\mathscr{F}(\mathcal{G}, B)$ have the same genus.

Each group in $G_\gamma \in \mathscr{F}(\mathcal{G}, B)$ corresponds to a modular curve X_{G_γ}.The group $\mathcal{G}$ acts on X_G where the restricted action of G is trivial. Hence, there is an action of $\mathcal{G}/G$ on X_G. We have $\mathrm{Aut}(X_G/X_{\mathcal{G}}) = \mathcal{G}/G$ where $\mathrm{Aut}(X_G/X_{\mathcal{G}})$ is the group of automorphisms f of the curve X_G that satisfy $\pi_G \circ f = \pi_G$. Precomposing γ with the cyclotomic character, we get a homomorphism $\xi\colon\ \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \longrightarrow \mathcal{G}/G$ which can be viewed as a 1-cocycle of X_G. Twisting with the cocycle ξ, we obtain a new curve $(X_G)_\xi$ and we have $X_{G_\gamma} = (X_G)_\xi$. Hence, the family $\mathscr{F}(\mathcal{G}, B)$ of groups corresponds to a family of twists of modular curves. For the rest of this statement, a family of curves, family of twists, and family of groups all refer to a family $\mathscr{F}(\mathcal{G}, B)$ as in Definition 1.

Agreeable subgroups play a significant role in the study of modular curves. Zywina's work shows that, in a certain sense, they provide complete information about how the image of the Galois representation ρ_E varies in $\mathrm{GL}_2(\hat{\mathbb{Z}})$. In a similar vein, we establish that, up to a bounded genus, there exist only finitely many agreeable subgroups, and moreover, they divide modular curves into finitely many well-behaved collections.

Theorem 4. [Kar25a] *Fix a non-negative integer g. Let G be an open subgroup of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ with full determinant and $-I \in G$ of genus g.*

1. *There are only finitely many families of modular curves of genus g. These families are effectively computable.*

2. *There is an effective and implemented algorithm that takes as input the group G and outputs a projective curve $C_G \subseteq \mathbb{P}^r_{\mathbb{Q}}$ for some $r > 0$ such that C_G is isomorphic to X_G.*

Let g be a non-negative integer. Theorem 4 gives rise to an algorithm that computes projective models for modular curves of genus at most g. Let $\mathscr{F} := \mathscr{F}(\mathcal{G}, B)$ be a non-empty family of curves of genus g. Fix a representative curve $X_G \in \mathscr{F}$. We can compute a projective model for X_G using a subspace of modular forms

$V \subseteq M_{k,G}$, where V is acted on by $\mathcal{G}$ [Kar25a, Zyw22]. An explicit Hilbert 90 calculation allows us to twist the projective model of X_G and the space of modular forms V with respect to the induced action of $\mathcal{G}/G$ to get a projective model for X_{G_γ}. Precisely, our algorithm has two parts: a one-time precomputation that takes a significant amount of time and the implemented algorithm that computes the projective model of X_G where the input $G \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$ is an open subgroup of genus at most g. This algorithm has been implemented for $g \leq 12$.

Genus	0	1	2	3	4	5	≤ 6	≤ 12	≤ 24
Families	638	1753	1209	3865	1573	6181	15943	48819	166141
Agreeable Groups	418	1078	885	2244	1151	3659	9998	30233	95981

Table 1: Number of families and agreeable groups up to conjugacy in $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ for small genus

Algorithm 1 (Computing projective models of modular curves of genus at most g)**.**
For precomputation, we

(1) compute the finitely many agreeable subgroups and the families $\mathscr{F}(\mathcal{G}, B)$ of genus at most g,
(2) find a representative H from each family $\mathscr{F}(\mathcal{G}, B)$,
(3) compute a model for X_H,
(4) compute the j-map $\pi : X_H \to \mathbb{P}^1_{\mathbb{Q}}$.

Given an open subgroup G of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ with $-I \in G$ and $\det(G) = \hat{\mathbb{Z}}^\times$, we

(1) find the family $\mathscr{F}(\mathcal{G}, B)$ in our list that contains G,
(2) compute the cocycle ξ such that $X_G = (X_H)_\xi$,
(3) twist the modular curve X_H and the map $\pi : X_H \to \mathbb{P}^1_{\mathbb{Q}}$ via effective Hilbert 90.

For the implementation, I have created a `Magma` package which also computes modular forms on X_H and arbitrary relative j-maps between modular curves [Kar25b]. In the case where X_H is geometrically hyperelliptic and has genus greater than 2, [Kar25b] can be used to compute the $\mathbb{Q}$-gonality of X_H. In the summer of 2025, this `Magma` package was used by LMFDB to compute models for more than one million modular curves in their database. I continue my involvement with the LMFDB to improve the gonality and rational points data for modular curves.

1.1 Universal elliptic curves

Let $\mathscr{F}(\mathcal{G}, B)$ be a family where $-I \notin B$. We call $\mathscr{F}(\mathcal{G}, B)$ a fine family of modular curves since the elements of $\mathscr{F}(\mathcal{G}, B)$ are fine moduli spaces. The table above exhibits the numbers for coarse families, but there are also finitely many fine families of curves of genus at most g.

In joint work with Rakvi and Steve Huang, we use the classification of fine families of modular curves to give a classification of universal elliptic curves associated to $G \subset \mathrm{GL}_2(\hat{\mathbb{Z}})$ where $\det(G) = \hat{\mathbb{Z}}^\times$ and $-I \notin G$. Following the same ideas as above, we have implemented an algorithm that computes models for universal elliptic curves defined over $\mathbb{Q}$. We hope to use our `Magma` package to update the LMFDB with universal elliptic curves data.

1.2 Progress on Mazur's Program B

In [Maz77a], Mazur classifies the $\mathbb{Q}$-torsion of a non-CM elliptic curve E and describes the following program:

Mazur's Program B. *Given a number field K and a subgroup H of $\mathrm{GL}_2(\hat{\mathbb{Z}}) = \prod_p \mathrm{GL}_2(\mathbb{Z}_p)$ classify all elliptic curves E/K whose associated Galois representation on torsion points maps $\mathrm{Gal}(\overline{K}/K)$ into $H \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$.*

Let G_1 and G_2 be the groups $\pm\rho^*_{E_1}(\mathrm{Gal}_{\mathbb{Q}})$ and $\pm\rho^*_{E_2}(\mathrm{Gal}_{\mathbb{Q}})$, where E_1 and E_2 are elliptic curves with j-invariants $-7 \cdot 11^3$ and $-7 \cdot 137^3 \cdot 2083^3$, respectively. Note that these groups are well-defined up to conjugacy in $\mathrm{GL}_2(\hat{\mathbb{Z}})$.

In [Zyw22], it is conjectured that if $G \subset \mathrm{GL}_2(\hat{\mathbb{Z}})$ is an open subgroup with surjective determinant containing $-I$, and if X_G has genus at least 54 and G is not conjugate to G_1 or G_2 in $\mathrm{GL}_2(\hat{\mathbb{Z}})$, then X_G contains no non-CM rational points over $\mathbb{Q}$. Hence conjecturally, explicitly computing the families of modular curves up to genus 53, along with extending our algorithm to such families, allows us to compute a projective model for any modular curve over rationals that contains a non-CM rational point (except X_{G_1} and X_{G_2} whose points are well understood).

Following the conjectures of Zywina, we suggest the following challenging program:

Program 5. *Following the conjectures of Zywina, we suggest the following steps to resolve Mazur's Program B:*

1. *Prove Serre's uniformity problem (or a stronger version of it: Conjecture 1.2 in* [Zyw22]*).*
2. *Classify all the rational points on a finite number of special modular curves as described in Section 14 of* [Zyw22]*.*
3. *Classify all congruence subgroups of* $\mathrm{SL}_2(\mathbb{Z})$ *(in the sense of* [CP03]*) up to genus 53 (or genus β as in Lemma 14.7 in* [Zyw22]*).*
4. *Compute all families of modular curves up to the genus mentioned above, in the sense of Definition 1.*
5. *Investigate the behavior of rational points in a family of twists.*

A major part of my future research is the study of various properties of curves in families of modular curves, with the aim of developing a deeper understanding of the rational structure of spaces of modular forms and the distribution of rational points on modular curves. I believe that partitioning modular curves into well-structured collections and examining them individually provide a uniform framework for studying modular curves over $\mathbb{Q}$.

2 Ongoing Work

In this section, I will describe the research projects that I currently work on.

2.1 Isogeny decomposition of modular Jacobians & Sato–Tate groups

(with Zachary Couvillion, in preparation) Fix a curve $X_G \in \mathscr{F}(\mathcal{G}, B)$ and consider the modular curves X_{G_γ} in $\mathscr{F}(\mathcal{G}, B)$. It is natural to investigate how $J_{G_\gamma} := \mathrm{Jac(X_{G_\gamma})}$ decomposes into simple isogeny factors as γ varies.

It is known that J_G decomposes as $\prod_f A_f^{e_f}$ into simple isogeny factors where f is a cusp form for $\Gamma_1(N^2)$ and N is the level of G [RSZB22, Appendix A]. Current algorithms for computing this decomposition involve the computation of all new forms of $\Gamma_1(N^2)$, which become inefficient as N increases. We are working to address two questions:

(1) Can the decomposition of J_{G_γ} be deduced from the decomposition of J_G and the continuous homomorphism γ?

(2) Can the newforms associated with the simple isogeny factors of J_{G_γ} be obtained by twisting the newforms associated with the simple isogeny factors of J_G?

We have already obtained several positive results in this direction. For instance, when $\mathscr{F}(\mathcal{G}, B)$ is a family of genus-2 curves, or when $[\mathcal{G} : G] = 2$, one can determine the decomposition of J_{G_γ} into simple isogeny factors by analyzing J_G and γ. This case already accounts for roughly one third of all families up to genus 24. In general, however, the existence of self-twists of modular forms complicates matters. One might expect the twist J_{G_γ} to decompose as $\prod_f A_{f\otimes\xi}^{e_{f\otimes\xi}}$, with multiplicities varying depending on the cocycle (where ξ is a product of Dirichlet characters). Yet the situation is more subtle. Under certain conditions, individual isogeny factors A_f may split into lower-dimensional abelian varieties, or conversely, they may "come together" to form higher-dimensional ones upon twisting. Influential works [Rib04, Pyl04, Rib90, Rib80, RSZB22] show that a simple isogeny factor of J_H is a GL_2-type abelian variety, and over $\overline{\mathbb{Q}}$ it can only further decompose into special abelian varieties known as *building blocks*. The structure of these building blocks and their fields of definition has been extensively studied [Que09, Pyl04, Gui12a, Gui12b, GQ14, Rib80].

Our primary goal is to classify twists arising from continuous homomorphisms $\gamma : \widehat{\mathbb{Z}}^\times \to \mathcal{G}/G$ that alter the dimensions appearing in the decomposition of J_{G_γ}. This includes understanding the relationship between the spaces $S_{2,G}$ and the Jacobian J_G (i.e., defining the Jacobian through cusp forms, as in the case of Iwahori-level groups) and describing the action of Hecke operators on $S_2(\Gamma_G)$ and $S_{2,G}$ [Ass21, Kat04]. In [Ass21], the author presents an algorithm for computing the Hecke decomposition of $S_2(\Gamma_G)$ using modular symbols. We aim to extend this algorithm by investigating the $\mathbb{Q}$-structures of $S_2(\Gamma_G)$ corresponding to twists in $\mathscr{F}$.

The endomorphism algebra $\mathrm{End}^0(A_f)$ of a simple isogeny factor A_f determines the field of definition of the building blocks associated with A_f [Que09]. Since the Albert classification type of modular Jacobians is related to the self-twists of the associated modular form f [Rib80], one can study the existence of inner twists of modular forms to understand the decomposition of Jacobians in families of modular curves. As expected, this investigation includes the study of the relationship between building blocks and Brauer groups over number fields [Que09, Pyl04]. In the specific case of genus 2 modular curves, the endomorphism algebras of dimension 2 abelian varieties have been extensively analyzed [BFGR06, FKRS12, FG20, FFG22]. We plan to use the cited results to classify endomorphism algebras arising from genus 2 modular curves. .

Completely determining the isogeny decomposition of Jacobians within families has numerous applications. Since there exists a map $X_{G_\gamma} \to X_{\mathcal{G}}$, the Jacobian of the agreeable closure can be identified within the decomposition of J_{G_γ} for each γ. It is possible to explicitly enumerate some families in which a constant rank 0 elliptic curve E appears in the Jacobian decompositions, and to employ the methods of [MR25] to provably compute all rational points in those families of modular curves. This approach is particularly useful for families of modular curves where J_G is completely decomposable into elliptic curves over $\mathbb{Q}$. Following an analysis of the ranks of the elliptic curves appearing in the decomposition, one can apply the techniques of [MR25] effectively.

In a related direction, [FLGS18, FKRS12, FS14, FG18, FKS21] investigate the Sato–Tate groups of abelian varieties of dimensions 2 and 3 and describe how these groups behave under twisting. Building on these results, for families of twists of modular curves of genus at most 3, I plan to classify all Sato–Tate groups that can arise from low-genus modular curves.

2.2 Local points for genus 0

A projective genus 0 curve is isomorphic to a plane curve, and if it possesses a rational point, i.e., $X(\mathbb{Q}) \neq \emptyset$, then X is isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$. Let $X_G \in \mathscr{F}(\mathcal{G}, B)$ be a modular curve of genus 0, and assume that X_G has a rational point. We aim to investigate the conditions under which X_{G_γ} also possesses a rational point. Since genus 0 curves satisfy the Hasse principle, it suffices to examine $X_{G_\gamma}(\mathbb{Q}_p)$ for rational primes p. Existing data on modular curves and previous work in the literature [Ozm12, Rak24, Cla09, Cla07, Cla08] suggest connections to quadratic reciprocity, and we anticipate a result analogous to the main theorem of [Ozm12].

Furthermore, working with the cohomological data arising from twists in the Brauer group of the base field allows one to relate this question to the solutions of finitely many norm equations, as in [Pre13]. This investigation would enable us to:
(1) completely characterize which genus 0 modular curves X_G are isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$ using cohomological information in families of twists, and
(2) classify geometrically hyperelliptic modular curves of genus greater than 2 with $\mathbb{Q}$-gonality 2 by analyzing the canonical models of such curves in families of twists.

3 Future work

In this section, I will briefly outline research projects I plan to work on during my postdoctoral position, some of which are in very early stages of development.

3.1 Geometric and rational gonality 3

Let $X := X_G$ be as above, of genus g. Given a geometrically hyperelliptic curve of genus greater than 2, our algorithm determines whether X_G has $\mathbb{Q}$-gonality 2 or 4. It is also possible to investigate the situation in which the $\overline{\mathbb{Q}}$-gonality of X_G is 3.

(1) If X has genus ≥ 5 and $X(\mathbb{Q}) \neq \emptyset$, then X has $\mathbb{Q}$-gonality 3.

(2) If X does not possess a rational point, or if X has genus 4, then the canonical model of X is cut out by quadratic and cubic forms in $\mathbb{P}^{g-1}$. The quadratic forms define a ruled surface, whose rulings can be used to construct degree 3 maps to $\mathbb{P}^1_{\mathbb{Q}}$. Currently, I am developing `Magma` packages capable of handling these ruled surfaces and analyzing how the rulings vary under twists of the canonical model of X.

3.2 Extending the classification to modular curves defined over number fields

We have classified modular curves associated with open subgroups $G \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ satisfying $\det(G) = \widehat{\mathbb{Z}}^\times$, which means that the curve X_G is defined over $\mathbb{Q}$. When this assumption is removed, the associated modular curve X_G is defined over a finite abelian extension $K_G/\mathbb{Q}$. I plan to generalize my previous work to classify modular curves corresponding to arbitrary open subgroups $G \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ containing $-I$. This project will involve using the class field theory of number fields (which is simpler in the case $K_G = \mathbb{Q}$) to define suitable families of modular curves and to study the K-rational spaces of modular forms. The algorithm and implementation described in previous sections can be extended to compute projective models of modular curves defined over arbitrary number fields.

Once this project is complete, the ongoing work described above can also be extended to modular curves over number fields.

3.3 Finite abelian descent

The isogeny decomposition of modular Jacobians is also connected to Stoll's work [Sto07] on finite descent obstructions and rational points on curves. Stoll proves that if a curve maps nontrivially into an Abelian variety A/k such that $A(k)$ is finite and $\text{Ш}(k, A)$ has no nontrivial divisible elements, then the information coming from the finite abelian descent cuts out the rational points on C. Stoll also proves that Brauer–Manin obstruction is equal to finite abelian descent on curves.

Most importantly for us, this is the case when A is modular of analytic rank 0 [Kol88, KL89, Wan96]. We plan to complete our project on the decomposition of Jacobians in families and to investigate the ranks and Shafarevich–Tate groups of the abelian varieties appearing in these decompositions. In addition, we aim to perform explicit computations to identify certain families of modular curves for which the Brauer–Manin obstruction is the only obstruction to the existence of rational points. Such computations for genus 2 and 3 families are already underway.

Additionally, the families of modular curves we study consist of coverings of the associated agreeable closure. Let $\mathscr{F}(\mathcal{G}, B)$ be a family and $X_G \in \mathscr{F}(\mathcal{G}, B)$ a fixed curve. Every curve in $\mathscr{F}(\mathcal{G}, B)$ comes equipped with a morphism $X_{G_\gamma} \to X_{\mathcal{G}}$, where the covering maps are abelian and étale over an open subset of $X_{\mathcal{G}}$. By descent theory, a certain finite subset of the maps $X_{G_\gamma} \to X_{\mathcal{G}}$ suffices to cover the rational points of $X_{\mathcal{G}}$, assuming $X_{\mathcal{G}}$ satisfies appropriate hypotheses [RZB15, Sko01]. In cases where the rational points of X_{G_γ} are more accessible, this approach can provide information about the rational points of $X_{\mathcal{G}}$. Note that the finitely many modular curves mentioned in (2) of Program 5 are all associated to agreeable subgroups.

3.4 $\mathbb{Q}$−gonality

Let X_G and $\mathscr{F}(\mathcal{G}, B)$ be as above. Beyond studying families with geometric gonality 2 and 3, it is natural to ask how twisting the map $X_G \to X_{\mathcal{G}}$ affects the $\mathbb{Q}$-gonality of X_G. Bounds for the gonality of algebraic curves have been widely investigated in the literature [RX18, Poo07, NO24, DvH14]. In collaboration with the LMFDB group, I am working to incorporate refined bounds into their database, and I plan to carry out a systematic study of this phenomenon for modular curves of arbitrary genus and geometric gonality.

A particularly promising direction for this project is the study of local gonalities over finite fields $\mathbb{F}_q$. In characteristic p, one encounters other modular curves, known as Igusa curves, which encode information about the level structure of elliptic curves when the level is divisible by p [Poo07]. Results on the gonality of Igusa curves, and the methods used to obtain them [HS91, SN96, Bak99, Ogg74], have applications in the characteristic 0 case. I intend to explore two problems:

(1) proving lower bounds for the gonality of Igusa curves that are linear in the index of the associated subgroups [Poo07], and

(2) extending the techniques of Ogg and Harris–Silverman to regular, smooth, and integral models of other modular curves.

These directions tie directly to questions about the existence of local points on modular curves, where the study of such models has already yielded some results (see 3.5 below). I plan to pursue both problems in parallel.

3.5 Local points of modular curves of positive genus

Curves of genus 0 satisfy the Hasse principle, making it natural to ask how the existence of local points changes under twisting. After completing the genus 0 case, I plan to extend this analysis to modular curves of positive genus.

Previous work [Ozm12] shows that regular models of modular curves X_G, and a study of their special fibers provide important tools for studying this question. Such models are well understood for $X_0(N)$ following the work of [DR73], and they have also been constructed for the split and non-split Cartan subgroups and their normalizers [EP24]. Semistable models of modular curves have been studied as well [Maz77b, DR73, Wei16, EP21, Edi90]. My goal is to analyze the geometry of these models and their special fibers under twisting, in order to obtain finer information about the local behavior of the associated modular curves.

3.6 Section property

As Stoll observes [Sto07, Remark 8.9], there is a connection between descent obstructions and Grothendieck's section conjecture. A smooth projective $\mathbb{Q}$-variety is said to be excellent with respect to all coverings if $X(\mathbb{Q}) = X(\mathbb{A}_{\mathbb{Q}})_{\bullet}^{\text{f-cov}}$ [Sto07], i.e., if the descent obstructions coming from all coverings of X cut out the rational points of X. It is known that the curves $X_0(N)$, $X_1(N)$, and $X(N)$ are excellent with respect to all coverings, which implies that they possess the birational section property in the sense of [Koe05]. I plan to extend these results to determine which classes of modular curves are excellent with respect to finite or finite–abelian covers.

I also plan to utilize the results of [Sti15] and [HS12] to study the properties of sections of $\pi_1(X_{G_\gamma}/\mathbb{Q})$ as X_{G_γ} varies within a family $\mathscr{F}(\mathcal{G}, B)$. The existence of certain types of sections of $\pi_1(X_{G_\gamma}/\mathbb{Q})$ (such as birationally liftable or birationally adelic sections) provides information about the existence of adelic and local points on X_{G_γ}. By combining these with the methods described in sections 3.4 and 3.5, one can obtain substantial information regarding the local points of modular curves.

3.7 Bielliptic modular curves over $\mathbb{Q}$

In [Zyw25], beyond classifying modular curves of geometric gonality 2 and 3, the author also classifies modular curves that are geometrically bielliptic. These correspond to finitely many infinite families in our framework, each consisting of geometrically bielliptic curves defined over $\mathbb{Q}$. Let $X_G \in \mathscr{F}(\mathcal{G}, B)$ be one such curve. I propose to investigate whether X_{G_γ} is rationally bielliptic, i.e., whether there exists a degree-2 morphism $X_{G_\gamma} \to E$ defined over $\mathbb{Q}$ to an elliptic curve E, as $\gamma : \widehat{\mathbb{Z}}^\times \to \mathcal{G}/G$ varies.

The works of [JKS20, BKS23, BGK20, BG20, Bar99, BKX13] give a general understanding of bielliptic modular curves and their geometrically bielliptic structure. The works of [FK91, Kuh88] give some information about rationally bielliptic curves of genus 2. In [BP24], the authors compute the rational points of bielliptic genus 2 modular curves over $\mathbb{Q}$, whose Jacobians have rank 2, using quadratic Chabauty methods. The specific case of genus 2 modular curves is easier to approach, as the Jacobian of such a curve is necessarily geometrically split.

I aim to undertake a systematic study of all geometrically bielliptic families of genus-2 modular curves, with the goal of identifying those twists that are rationally bielliptic. At the very least, I expect to isolate families containing a positive density of rationally bielliptic modular curves.

3.8 Future interest: explicit computations of certain automorphic forms

Explicit computations of classical (elliptic) modular forms have revealed striking phenomena, leading to the formulation of numerous conjectures, several of which have since been established as theorems. Extending such computational efforts to the broader theory of automorphic forms would be highly valuable.

During my involvement with the LMFDB project, my primary focus was on classical (elliptic) modular forms and modular curves. At the same time, I had the opportunity to interact with number theorists whose research revolved around Maass forms, Hilbert modular forms, and Siegel modular forms, through which I became familiar with contemporary methods for studying these objects [DV13, LD25a, RRST12]. In this light, an immediate area of interest for me is to investigate Hilbert and Siegel modular forms and to explore possible avenues of generalization. Two directions present themselves immediately:

(1) One direction is to focus on classical Hilbert modular forms and to construct analogous families of twists, particularly in the case of Hilbert modular forms of parallel weight 2 over a totally real field of strict class number 1. This special case exhibits many similarities with elliptic modular forms. The Eichler–Shimura and Shimura–Taniyama conjectures admit analogous formulations, and many of the techniques initiated by Wiles [Wil95] have been successfully adapted to totally real fields [DV13]. Moreover, there exist explicit computational methods for Hilbert modular forms, which generalize established techniques for elliptic modular forms, such as modular symbols and Brandt matrices. Another important tool is the lifting of elliptic modular forms via base change, which plays a central role in computations with Hilbert modular forms. In the future, I hope to work on the implementation of these methods, and investigate how lifting elliptic modular forms behave in families of twists.

(2) An elliptic modular form may be viewed as a Siegel modular form of degree 1. Consequently, a natural direction for generalizing known methods is the study of degree 2 Siegel modular forms. Distinct computational approaches are available, including Eisenstein series, Ikeda and Maass lifts, and theta constants and series [RRST12]. I am particularly interested in methods for lifting elliptic modular forms to degree 2 Siegel modular forms and in examining their behavior under twisting.

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