We snortin' fart bubbles

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Lecture 25 (09.04.2024)

Theorem 1: Bolzano-Weierstrass Theorem for Sequences of Points in $\mathbb{R}^2$

Theorem 1. *Every bounded sequence of points in $\mathbb{R}^2$ has at least one limit point.*

Proof. Let $\{P_n = (x_n, y_n)\}$ be a bounded sequence of points. Then there is a constant M such that for any n

$$\sqrt{x_n^2 + y_n^2} < M.$$

So, $|x_n| < M$ and $|y_n| < M$ for any n, and by the Bolzano-Weierstrass theorem for sequences of real numbers, there exists a subsequence $\{x_{n_k}\}$ of sequence $\{x_n\}$ that is convergent to some number a. Subsequence $\{y_{n_k}\}$ is also bounded by M. So, applying the Bolzano-Weierstrass theorem again we can conclude that there is a subsequence $\{y_{n_{k'}}\}$ of $\{y_{n_k}\}$ that is convergent to some number b. So, sequence of points $\{(x_{n_{k'}}, y_{n_{k'}})\}$ converges to (a, b). The point (a, b) is a limit point.

Theorem 2: Bolzano-Weierstrass Theorem for Sets in $\mathbb{R}^2$

Theorem 2. *Every bounded infinite set of points in $\mathbb{R}^2$ has at least one limit point.*

Proof. Let G be a bounded infinite set of points in $\mathbb{R}^2$. Then there exists a sequence $P_n \in G$ such that $P_n \neq P_m$ for $m \neq n$. According to theorem 1, $\{P_n\}$ has a limit point Q that is a limit point of G.

Lecture 26 (16.04.2024)

Theorem 1

Theorem 3. *If a function $f(x, y)$ is differentiable at the point (x_0, y_0) then it is continuous at this point.*

Proof. Indeed, by definition 2, at U function $f(x,y)$ can be presented in the form

$$f(x,y) = f(x_0,y_0) + f'_x(x_0,y_0)\Delta x + f'_y(x_0,y_0)\Delta y + \varepsilon \cdot \rho.$$

As $f'_x(x_0,y_0)$ and $f'_y(x_0,y_0)$ are finite numbers and Δx, Δy, ε and ρ tend to 0 as $(x,y) \to (x_0,y_0)$,

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

So, theorem is proved.

Theorem 2

Theorem 4. *Let $f(x,y)$ be defined in an open set G and partial derivatives $f'_x(x,y)$ and $f'_y(x,y)$ exist and are continuous in G. Then function $f(x,y)$ is differentiable at each point of G.*

Proof. Let (x_0,y_0) be any point of G. We can rewrite Δf in the form

$$\Delta f = f(x,y) - f(x_0,y_0) = [f(x,y) - f(x,y_0)] + [f(x,y_0) - f(x_0,y_0)].$$

Considering expressions in square brackets as functions of one variables we can apply Lagrange formula

$$f(x,y) - f(x,y_0) = f'_y(x,b)\Delta y,$$

$$f(x,y_0) - f(x_0,y_0) = f'_x(a,y_0)\Delta x$$

for some b between y and y_0 and a between x and x_0.

We get

$$\begin{aligned}\Delta f &= f'_x(a,y_0)\Delta x + f'_y(x,b)\Delta y \\ &= (f'_x(x_0,y_0) + (f'_x(a,y_0) - f'_x(x_0,y_0)))\,\Delta x \\ &\quad + \big(f'_y(x_0,y_0) + (f'_y(x,b) - f'_y(x_0,y_0))\big)\,\Delta y.\end{aligned}$$

The last two terms can be rewritten in the form

$$\alpha\Delta x + \beta\Delta y = \left(\alpha\frac{\Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} + \beta\frac{\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}\right)\rho.$$

Applying the continuity of partial derivatives we obtain that α, $\beta \to 0$ as $\Delta x, \Delta y \to 0$. So, $\varepsilon \to 0$ and $f(x,y)$ is differentiable at (x_0,y_0).

Theorem 3

Theorem 5. *Let $f(x, y)$ be differentiable at (x_0, y_0), and $x = x(t)$, $y = y(t)$ are differentiable functions at $t = t_0$, then $F(t) = f(x(t), y(t))$ is differentiable at $t = t_0$ and*

$$F'(t_0) = f'_x(x_0, y_0)x'(t_0) + f'_y(x_0, y_0)y'(t_0).$$

Proof. As $f(x, y)$ is differential at (x_0, y_0)

$$f(x, y) - f(x_0, y_0) = f'_x(x_0, y_0)\Delta x + f'_y(x_0, y_0)\Delta y + \varepsilon \cdot \rho,$$

where $\varepsilon = \varepsilon(\Delta x, \Delta y) \to 0$ as $\Delta x, \Delta y \to 0$ and $\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$.

The last formula can be rewritten in the form

$$\begin{aligned} f(x(t), y(t)) - f(x(t_0), y(t_0)) = f'_x(x_0, y_0)(x(t) - x(t_0)) \\ + f'_y(x_0, y_0)(y(t) - y(t_0)) + \varepsilon \cdot \rho, \end{aligned} \tag{1}$$

$$F(t) - F(t_0) = f'_x(x_0, y_0)(x(t) - x(t_0)) + f'_y(x_0, y_0)(y(t) - y(t_0)) + \varepsilon \cdot \rho.$$

Dividing by $\Delta t = t - t_0$, we have

$$\begin{aligned} \frac{F(t) - F(t_0)}{\Delta t} &= f'_x(x_0, y_0)\frac{x(t) - x(t_0)}{\Delta t} \\ &\quad + f'_y(x_0, y_0)\frac{y(t) - y(t_0)}{\Delta t} + \varepsilon \cdot \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}. \end{aligned} \tag{2}$$

Taking the limit $\Delta t \to 0$, as $\Delta x, \Delta y \to 0$ while $\Delta t \to 0$, we get

$$F'(t_0) = f'_x(x_0, y_0)x'(t_0) + f'_y(x_0, y_0)y'(t_0).$$

Lecture 27 (23.04.2024)

Theorem 1

Theorem 6. *Let $f(x)$ be a function having continuous first derivatives in a neighborhood of a point a and $v \in \mathbb{R}^n$, $|v| = 1$. Then $\frac{\partial f}{\partial v}(a)$ exists and*

$$\frac{\partial f}{\partial v}(a) = v_1\frac{\partial f}{\partial x_1}(a) + v_2\frac{\partial f}{\partial x_2}(a) + \cdots + v_n\frac{\partial f}{\partial x_n}(a) = \sum_{i=1}^{n} v_i\frac{\partial f}{\partial x_i}(a).$$

Proof. Let us consider the function $F(t) = f(a + vt) = f(a_1 + v_1t, a_2 + v_2t, \ldots, a_n + v_nt)$. By definition,

$$F'(0) = \lim_{t\to 0}\frac{F(t) - F(0)}{t} = \lim_{t\to 0}\frac{f(a + vt) - f(a)}{t} = \frac{\partial f}{\partial v}(a).$$

On the other hand, by derivative of the composition,

$$F'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a+vt)\frac{dx_i}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a+vt)v_i.$$

and

$$F'(0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)v_i.$$

Comparing these, we get the proof of the theorem.

Theorem 2

Theorem 7. *The gradient vector ∇F at $P \subset S$ is perpendicular to the tangent vector to any curve γ on S that passes through P. In other words, the gradient vector ∇F at $P \subset S$ is orthogonal to the surface $F(x, y, z) = 0$ at point P.*

Proof. Suppose that S is a surface with equation $F(x, y, z) = 0$, that is, it is a level surface of a function F of three variables, and let $P \in S$ be a point on S. Let γ be any curve that lies on the surface S and passes through the point P. The curve γ is described by a continuous vector function $r(t) = (x(t), y(t), z(t))$. Let t be the fixed parameter value corresponding to P, that is, $r(t) = P$. Since $\gamma \subset S$, any point of $r(t)$ must satisfy the equation of S, that is,

$$F(x(t), y(t), z(t)) = 0,$$

so that $\frac{d}{dt}F(x(t), y(t), z(t)) = 0$. On the other hand, from the derivative of a composition we have that

$$\frac{\partial F}{\partial x}x'(t) + \frac{\partial F}{\partial y}y'(t) + \frac{\partial F}{\partial z}z'(t) = 0.$$

We have that

$$(\nabla F(P), r'(t)) = 0.$$

Lecture 28 (27.04.2024)

Theorem 1

Theorem 8. *If partial derivatives $f'_x(x, y)$, $f'_y(x, y)$, $f''_{xy}(x, y)$ and $f''_{yx}(x, y)$ are defined at a neighborhood of (x_0, y_0) and continuous at (x_0, y_0) then*

$$f''_{xy}(x_0, y_0) = f''_{yx}(x_0, y_0).$$

Proof. Let us consider a function

$$\Delta = [f(x, y) - f(x, y_0)] - [f(x_0, y) - f(x_0, y_0)]$$

and $\varphi(x) = f(x, y) - f(x, y_0)$. Then

$$\Delta = \varphi(x) - \varphi(x_0)$$

and, applying Lagrange theorem to $\varphi(x)$ we have that for some point ξ from the interval between points x and x_0

$$\Delta = \varphi'(\xi)\Delta x, \quad \Delta x = x - x_0.$$

So,

$$\Delta = [f'_x(\xi, y) - f'_x(\xi, y_0)]\Delta x.$$

Applying the Lagrange theorem to function of y (in square brackets), we get for some point η from the interval between point y and y_0 with $\Delta y = y - y_0$ the following presentation

$$\Delta = f''_{xy}(\xi, \eta)\Delta x \Delta y. \quad (1)$$

Let us present now the function Δ in the form

$$\Delta = [f(x, y) - f(x_0, y)] - [f(x, y_0) - f(x_0, y_0)]$$

and introduce the function $\psi(y) = f(x, y) - f(x_0, y)$. So,

$$\Delta = \psi(y) - \psi(y_0).$$

Applying the Lagrange theorem to $\psi(y)$ we have that for some point β from the interval between points y and y_0

$$\Delta = \psi'(\beta)\Delta y, \quad \Delta y = y - y_0$$

and

$$\Delta = [f'_y(x, \beta) - f'_y(x_0, \beta)]\Delta y.$$

Applying the Lagrange theorem again to the function $f'_y(x, \beta)$ we obtain

$$\Delta = f''_{yx}(\alpha, \beta)\Delta x \Delta y, \quad (2)$$

where α belongs to the interval between x and x_0. Comparing (1) and (2), we obtain

$$f''_{xy}(\xi, \eta) = f''_{yx}(\alpha, \beta).$$

Theorem follows from the continuity of partial derivatives at (x_0, y_0) as (ξ, η) and (α, β) tend to (x_0, y_0) if $\Delta x, \Delta y \to 0$.

Lecture 29 (30.04.2024)

Theorem 1

Theorem 9. *(Taylor formula for a function of two variables). If a point $M_1(x+\Delta x, y+\Delta y) \in B(x, y)$ then the increment $\Delta f = f(M_1) - f(M)$ can be presented* *in the form*

$$\begin{aligned}\Delta f &= df(x,y) + \frac{d^2 f(x,y)}{2!} + \cdots + \frac{d^n f(x,y)}{n!} + \frac{d^{n+1} f(x+\theta\Delta x, y+\theta\Delta y)}{(n+1)!}\\ &= \sum_{k=1}^{n} \frac{d^k f(x,y)}{k!} + \frac{d^{n+1} f(x+\theta\Delta x, y+\theta\Delta y)}{(n+1)!}, \quad \theta \in [0,1].\end{aligned}$$

Proof. Let us introduce the function $F(t) = f(x+t\Delta x, y+t\Delta y)$, $t \in [0,1]$ and find its derivatives up to the $(n+1)$-th order

$$\begin{aligned}F'(t) &= f_x(x+t\Delta x, y+t\Delta y)\Delta x + f_y(x+t\Delta x, y+t\Delta y)\Delta y\\ &= \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right) f(x+t\Delta x, y+t\Delta y).\end{aligned}$$

In the same way

$$\begin{aligned}F''(t) &= f_{xx}(x+t\Delta x, y+t\Delta y)(\Delta x)^2 + 2f_{xy}(x+t\Delta x, y+t\Delta y)\Delta x\Delta y\\ &\quad + f_{yy}(x+t\Delta x, y+t\Delta y)(\Delta y)^2\\ &= \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)^2 f(x+t\Delta x, y+t\Delta y)\end{aligned}$$

and so on. In particular,

$$F^{(k)}(t) = \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)^k f(x+t\Delta x, y+t\Delta y).$$

Considering the point $t = 0$ we have

$$\begin{aligned}F'(0) &= \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right) f(x,y) = df(x,y),\\ F''(0) &= \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)^2 f(x,y) = d^2 f(x,y),\\ &\vdots\\ F^{(k)}(0) &= \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)^k f(x,y) = d^k f(x,y).\end{aligned}$$

On the other hand, we can use the Maclaurin formula for the function $F(t)$ with $\Delta t = 1$

$$F(1) = F(0) + \frac{F'(0)}{1!} + \frac{F''(0)}{2!} + \cdots + \frac{F^{(n)}(0)}{n!} + \frac{F^{(n+1)}(\theta)}{(n+1)!}.$$

So, as $F(1) = f(M_1)$, $F(0) = f(M)$, substituting formulas for $F'(0)$, $F''(0)$, $\ldots$, $F^{(k)}(0)$ we have the formulation of the theorem.

Lecture 31 (25.05.2024)

Theorem 1

Theorem 10. *Let $f(x)$ have first derivatives in a domain $D \subset \mathbb{R}^n$. If $f(x)$ has a local extremum at a point $a \in D$, then*

$$\nabla f(a) = 0, \quad \text{or} \quad \frac{\partial f}{\partial x_i}(a) = 0, \quad i = 1, \ldots, n.$$

Proof. Let $a = (a_1, a_2, \ldots, a_n)$. Then functions of one variable

$$\varphi_i(t) = f(a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n), \quad i = 1, \ldots, n$$

have relative extremum at $t = a_i$. Hence $\varphi_i'(a_i) = 0$. That is,

$$\frac{\partial f}{\partial x_i}(a) = 0, \quad i = 1, \ldots, n.$$

Theorem follows.

Theorem 3

Theorem 11. *Let $f(x)$ have continuous first and second derivatives in a domain $D \subset \mathbb{R}^n$. Then:*

1. *If a is a local minimum (maximum) of $u = f(x)$, then the quadratic form $Q_f(a)[v]$ defined by*
$$Q_f(x)[v] = \big(\nabla^2 f(x)v, v\big)$$
is nonnegative (nonpositive) definite.

2. *(Sufficient condition) Let a be a critical point of f in D. Then:*
   - (a) *If $Q_f(a)[v]$ is positive definite, then f has a relative minimum at a.*
   - (b) *If $Q_f(a)[v]$ is negative definite, then f has a relative maximum at a.*
   - (c) *If $Q_f(a)[v]$ is indefinite, then f has neither relative minimum nor relative maximum at a.*

Proof. 1. Let a be a point of a local minimum of $f(x)$. Then, by definition, there exists a neighborhood $B_r(a)$, such that for any point $x \in B_r(a)$ we have that $f(x) \geq f(a)$. Also, for any $k \in \mathbb{R}$: $Q_f(x)[kv] = k^2 Q_f(x)[v]$. So, it is enough to prove the theorem for all $v \in \mathbb{R}^n$ such that $|v| < r$.

Consider the function of one variable $\varphi(t) = f(a + tv)$ where $t \in [-1, 1]$. By definition of local minimum, $\varphi(t) \geq \varphi(0) = f(a)$ for $t \in [-1, 1]$ and $t = 0$ is a point of a local minimum of φ at $[-1,1]$. So, $\varphi'(0)=0$ and $\varphi''(0)\geq 0$. By the derivative of composition,

$$\varphi'(t)=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(a+tv)v_i,$$

$$\varphi''(t)=\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i\partial x_j}(a+tv)v_iv_j=Q_f(a+tv)[v],$$

so, for $t=0$,

$$Q_f(a)[v]=\varphi''(0)\geq 0.$$

Part 1 of the theorem is proved.

2. Let a be a critical point of $u=f(x)$ and $Q_f(a)[v]$ is positive definite. Then there exists a neighborhood $B_r(a)$ such that for any $x\in B_r(a)$ the quadratic form $Q_f(x)[v]$ is also positive definite. Let us fix $x\in B_r(a)$ and consider $v=x-a$. If $\varphi(t)=f(a+tv)$, then $\varphi(1)=f(x)$, $\varphi(0)=f(a)$ and

$$\varphi'(0)=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(a)v_i=0,$$

as a is a critical point. We obtain

$$\begin{aligned}f(x)=&f(a+v)=\varphi(1)=\varphi(0)+\varphi(1)-\varphi(0)\\=&f(a)+\int_0^1\varphi'(t)\,dt=f(a)+\int_0^1(\varphi'(t)-\varphi'(0))\,dt\\=&f(a)+\int_0^1\left(\int_0^t\varphi''(\tau)\,d\tau\right)\,dt\geq f(a).\end{aligned}\tag{3}$$

Theorem follows.

Lecture 32 (01.06.2024)

Theorem 1

Theorem 12. *Let $P(x_0,y_0,z_0)$ be a solution of the constrained optimization problem*

$$f(x,y,z)\to\max(\min)\quad\textit{subject to}\quad g(x,y,z)=0,$$

where the functions f and g are continuously differentiable in a domain $D\subset\mathbb{R}^3$ and $\nabla g(P)\neq 0$. Then there exists a Lagrange multiplier λ such that P is a critical point of the Lagrange function

$$L(x,y,z,\lambda)=f(x,y,z)-\lambda g(x,y,z).$$

Proof. Suppose that the function $f(x, y, z)$ has an extreme value at a point $P(x_0, y_0, z_0) \in S$, where S is defined by the constraint $g(x, y, z) = 0$. Let $C \subset S$ be a curve with the vector function $r(t) = (x(t), y(t), z(t))$ that lies on S and passes through P. Assume that t_0 is the parameter value corresponding to the point $r(t_0) = P$. The composite function $F(t) = f(x(t), y(t), z(t))$ represents the values that f takes on the curve C. Since f has an extremum value at P, the function $F(t)$ has an extreme value at t_0, so $F'(t_0) = 0$. If f is differentiable, then

$$F'(t_0) = \frac{\partial f}{\partial x}(P)x'(t_0) + \frac{\partial f}{\partial y}(P)y'(t_0) + \frac{\partial f}{\partial z}(P)z'(t_0) = 0,$$

so that

$$(\nabla f(P), r'(t_0)) = 0.$$

This shows that the gradient vector $\nabla f(P)$ is orthogonal to the tangent vector $r'(t_0)$ to every such curve C. We also know that the gradient vector $\nabla g(P)$ is also orthogonal to every such curve. This means that the gradient vectors $\nabla f(P)$ and $\nabla g(P)$ must be parallel. Therefore, if $\nabla g(P) \neq 0$, there is a number λ such that

$$\nabla f(P) = \lambda \nabla g(P).$$

Let us now remark that the condition $\nabla f(P) = \lambda \nabla g(P)$ can be rewritten as

$$\frac{\partial f}{\partial x}(P) = \lambda \frac{\partial g}{\partial x}(P), \quad \frac{\partial f}{\partial y}(P) = \lambda \frac{\partial g}{\partial y}(P), \quad \frac{\partial f}{\partial z}(P) = \lambda \frac{\partial g}{\partial z}(P),$$

or,

$$\frac{\partial}{\partial x}(f - \lambda g)(P) = 0, \quad \frac{\partial}{\partial y}(f - \lambda g)(P) = 0, \quad \frac{\partial}{\partial z}(f - \lambda g)(P) = 0.$$

So, if we introduce the Lagrange function

$$L(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z),$$

where λ is called the Lagrange multiplier, the last set of relations can be written as a condition for P to be a critical point of L:

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial y} = 0, \quad \frac{\partial L}{\partial z} = 0,$$

together with the constraint

$$\frac{\partial L}{\partial \lambda} = -g(x, y, z) = 0.$$

Thus, we have proved the theorem. Kill me please.