MA 202: Numerical Methods

Semester–II, Academic Year 2022-23

Tutorial set - 5

Instructions

- **Submission deadline:** April 30, 2023.
- Any request for extension will not be entertained. Late submission (even by 1 second) will not be accepted.
- Questions 3 and 6 (if time permits) will be solved during the tutorial. The students have to solve the rest of the problems on their own and submit the answers to all problems.
- You have to write computer programs for each problem separately. You may use MATLAB or Python to write the programs.
- Submission will be through google classroom.
- Please put all the programs in a single 'zip' file and upload on google classroom.
- Each program should be named as follows: Tut<Insert-Number>Prob<Insert-Number>.m/py.
- The 'zip' file should be named as: Tut<Insert-Number><Your-Name><Roll-Number>.zip /rar/.... If additional information is required to run your program, please submit that as a PDF file contained within the same 'zip' file. Handwritten solutions, wherever appropriate must also be included in this same 'zip' file.
- Please make sure that your programs do not contain any error. In case there is a run time error, not credit for that particular problem will be given.
- Please do not submit screenshots of your programs. You must submit executable files.

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(1) Suppose that the value of a function $f(x)$ is known at the following set of discrete grid points: x_i, $i = 1,\ 2,\ ...,\ N$. Assume that any two neighboring grid points have the same separation, $\Delta x = x_i - x_{i-1}$, $i = 2, 3, ..., N$. Answer the following:

(a) Derive a one sided finite difference approximation for $f''(x_i)$ using the value of the function at the points, x_i, x_{i+1} and x_{i+2}. Quantify the truncation error.

(b) Derive a finite difference approximation for $f^{(4)}(x_i)$ (i.e., the 4th derivative) using the value of the function at the points: x_{i-2}, x_{i-1}, x_i, x_{i+1} and x_{i+2}. Also report the truncation error.

(c) Derive a finite difference approximation for $f'''(x_i)$ using the value of the function at the points: x_{i-2}, x_{i-1}, x_{i+1} and x_{i+2}. Also report the truncation error.

(d) Can you derive an expression for $f''(x_i)$ using the function value at the grid points x_i, x_{i-1} and x_{i+1} when the grid points are unequally spaced? Assume $\Delta x_{i-1} = x_i - x_{i-1}$ and $\Delta x_i = x_{i+1} - x_i$.

(2) Develop a computer program to solve the Lorenz equations:

$$\begin{aligned}\dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= rx - y - xz \\ \dot{z} &= -bz + xy\end{aligned}$$

using the fourth-order RK method for $\sigma = 10$, $b = 2.67$, $r = 28$, and initial conditions of $x(0) = y(0) = z(0) = 5$. In all cases use a step size of $\Delta t = 0.01$ and simulate from $t = 0$ to 20. Plot: (i) $x(t)$ vs t, (ii) $y(t)$ vs t, (iii) $z(t)$ vs t, (iv) A 3D phase plot of $x(t)$ vs $y(t)$ vs $z(t)$.

(3) Develop a computer program to solve the equation for the Van der Pol oscillator:

$$\ddot{y} - \mu(1 - y^2)\dot{y} + y = 0$$

Subject to $y(0) = 4$ and $\dot{y}(0) = 0$. Take: $\mu = 0.1$, $\mu = 1$ and $\mu = 4.5$ and solve till $t \geq 50$. Try various other initial conditions and other values of μ and observe what is happening to the solution. Generate the following plots: (i) $y(t)$ vs t and (ii) a phase plot showing $\dot{y}$ vs y.

(4) Develop a computer code to solve the equation of motion of a harmonic oscillator given by: $\ddot{\theta} + \omega^2 \sin\theta = 0$, subject to $\theta(0) = -\pi/2$ and $\dot{\theta}(0) = 0$. Take, $\omega = 1,\ 2,\ 4$ and 8. Use the RK-4th order method. Plot $\theta(t)$ vs t and $\dot{\theta}$ vs t.

(5) The temperature distribution in a tapered canonical cooling fin is described by the following differential equation, which has been nondimensionallized

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \left(\frac{2}{x}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}x} - pu\right) = 0,$$

where $u =$ temperature $(0 \leq u \leq 1)$, $x =$ axial distance $(0 \leq x \leq 1)$, and p is a nondimensional parameter that describes the heat transfer and geometry

$$p = \frac{hL}{k}\sqrt{1 + \frac{4}{2m^2}},$$

where $h =$ a heat transfer coefficient, $k =$ thermal conductivity, $L =$ the length or height of the cone, and $m =$ slope of the cone wall. The equation has the boundary conditions

$$u(x = 0) = 0, \quad u(x = 1) = 1.$$

Develop a computer code to solve this equation for the temperature distribution using finite difference methods. Use second-order accurate finite difference analogues for the derivatives. Plot T vs x for the following values of $p = 10, 20, 50,$ and 100.

(6) It can be shown that the potential (ψ) around a spherical object carrying a weak surface charge may be written as (dimensionless form):

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = \lambda^2\psi, \quad 1 \leq r \leq 5,$$

subject to: $\psi(r = 1) = 1$ and $d\psi/dr = 0$ at $r = 5$. Develop a computer program to solve the above equation to deduce ψ, using the finite difference method. Plot ψ vs r for $\lambda = 3,\ 4,\ 6$ and 10.