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Technical Notes

Modified *p-k* Method for Flutter Solution with Damping Iteration

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Nomenclature

B = generalized damping matrix
g = real part of the nondimensional Laplace parameter p
K = generalized stiffness matrix
k = reduced frequency
L = reference length
l = number of iteration steps
M = generalized mass matrix
n = number of natural modes used in the aeroelastic model
p = nondimensional Laplace parameter
Q = generalized aerodynamic influence coefficient matrix
q = generalized coordinates vector
s = Laplace parameter
V = true airspeed
γ = transient decay-rate coefficient
ρ = air density

Introduction

FLUTTER solution techniques have been well developed in aeronautical engineering and may continue to be an interesting topic for modern aeroelasticians. Many efforts are devoted to getting a "true damping" solution of the flutter equation. For example, the *p-k* method [1] is widely used in routine flutter analysis for giving approximate true damping results, which are useful in flight flutter test programs. The promising g method for reliable damping prediction proposed by Chen [2] includes a first order of damping term in the flutter equation that is rigorously derived from the Laplace-domain aerodynamics. Recently, the g method was improved by Qiu and Sun [3] with an additional second order of damping term included in the flutter equation. Borglund [4] applied approximated p-domain aerodynamics up to first order of damping term by Chen [2] to compute the nominal eigenvalues using a modified version of the *p-k* solver in [5].

In this Note, the g method equation is reformulated and solved by a modified *p-k* method with damping iteration. Implications of the second order of damping term are also drawn. The imaginary unit number i is replaced with $(p-g)/k$, so the modified *p-k* equation does not have any extra term such as the added aerodynamic damping matrix in the standard *p-k* equation [6]. Because the modified *p-k* equation is real rather than complex as in the g method, it may reduce the computational time when compared with that of the g method. The present algorithm closely resembles that of the standard *p-k* method; thus, one only needs to make minor alterations to existing codes in practice.

p-k Method and *g* Method Flutter Equation

Assuming linear unsteady aerodynamics computed by panel method in the frequency domain or by linearized frequency-domain CFD results, the *p-k* method flutter equation presented by Hassig [1] reads

$$[(V^2/L^2)Mp^2 + (V/L)Bp + K - \tfrac{1}{2}\rho V^2 Q(ik)]\{q\} = 0 \tag{1}$$

where $p=(V/L)s$ stands for nondimensional Laplace parameter, the reduced frequency is $k=\omega L/V$, and p can be expressed as

$$p = g + ik \tag{2}$$

with $g=\gamma k$.

Equation (1) was modified by Rodden and Johnson [6] as

$$[(V^2/L^2)Mp^2 + (V/L)Bp + K - \tfrac{1}{2}\rho V^2 (Q^I(k)/k)p - \tfrac{1}{2}\rho V^2 Q^R(k)]\{q\} = 0 \tag{3}$$

where $Q^R(k)$ and $Q^I(k)$ are the real part and imaginary part of $Q(ik)$, i.e.,

$$Q(ik) = Q^R(k) + iQ^I(k) \tag{4}$$

Equation (3) can be rewritten in a state-space form:

$$[A(k) - pI_{2n}]\{u\} = 0 \tag{5}$$

where $[A(k)]$ is the real matrix,

$$[A(k)] = \begin{bmatrix} 0_n & I_n \\ C_0 & D_0 \end{bmatrix} \tag{6}$$

with

$$C_0 = -(L^2/V^2)M^{-1}[K - \tfrac{1}{2}\rho V^2 Q^R(k)]$$
$$D_0 = -(L^2/V^2)M^{-1}[(V/L)B - \tfrac{1}{2}\rho V^2 (Q^I(k)/k)]$$

and $\{u\} = \{\{q\}, p\{q\}\}^T$.

The *p-k* method is then used to solve this nonlinear eigenvalue problem of Eq. (5).

However, by substituting Eq. (2) into the fourth term of Eq. (3), this equation can be rewritten as

$$[(V^2/L^2)Mp^2 + (V/L)Bp + K - \tfrac{1}{2}\rho V^2 (Q^I(k)/k)g - \tfrac{1}{2}\rho V^2 Q(ik)]\{q\} = 0 \tag{7}$$

By comparing Eq. (7) with Equation (1), it is found that there is an extra term, called the added aerodynamic damping matrix:

$$-\tfrac{1}{2}\rho V^2 (Q^I(k)/k)g$$

which is demonstrated to be valid only at small k or $g=0$ or for linearly varying $Q(ik)$ [2].

Based on the analytic property of the complex function $Q(p) = Q(g+ik)$ and using the Cauchy–Riemann equations, Chen [2] expanded $Q(p)$ along the imaginary axis (i.e., $g=0$) using the damping perturbation method for small g:

$$Q(p) \approx Q(ik) + gQ'(ik) \tag{8}$$

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where $Q'(ik) = dQ(ik)/d(ik)$ can be calculated from $Q(ik)$ by a central differencing scheme, except at $k = 0$. At $k = 0$, a forward differencing scheme is suggested by Chen [2].

The g method equation is formulated through replacing $Q(ik)$ in Eq. (1) by $Q(p)$ of Eq. (8):

$$[(V^2/L^2)Mp^2 + (V/L)Bp + K - \tfrac{1}{2}\rho V^2 Q'(ik)g - \tfrac{1}{2}\rho V^2 Q(ik)]\{q\} = 0 \tag{9}$$

Substituting Eq. (2) into Eq. (9) will yield a second-order linear system in terms of g, then a reduced-frequency-sweep technique is introduced by the g method to search for the condition $\mathrm{Im}(g) = 0$ and solve for the eigenvalues g.

In [3], $Q(p)$ was further expanded up to second order of damping perturbation:

$$Q(p) \approx Q(ik) + gQ'(ik) + \tfrac{1}{2}g^2 Q''(ik) \tag{10}$$

Equation (9) provides a reliable aerodynamic damping matrix that is valid in the complete k domain, and this eigenvalue problem in terms of g can be solved successfully by the g method. Alternatively, it might be of interest to combine this novel damping perturbation approach with the standard p-k method in flutter solution. In the following section, a modified p-k method with damping perturbation is presented, and the flutter results of this method for a sample test case are validated by the g-method solution. Implications of the second order of damping term are also studied using the modified p-k method.

p-k Method Equation with Damping Perturbation

To facilitate the eigenvalue analysis of Eq. (9) by the p-k method, we start by eliminating the pure imaginary unit number i of $Q(ik)$ and $Q'(ik)$ in Eq. (9). These terms can be written in real part and imaginary part:

$$Q(ik) = Q^R(k) + iQ^I(k) \tag{11}$$

$$Q'(ik) = Q'^R(k) + iQ'^I(k) \tag{12}$$

Solving for the imaginary unit number i in Eq. (2) gives

$$i = (p - g)/k \tag{13}$$

It is noted that Eq. (13) is not valid at $k = 0$. Fortunately, there is no need to introduce Eq. (13) for $k = 0$, because imaginary part of matrix $Q(ik)$ will vanish as $k = 0$ (corresponding to stationary aerodynamics).

Substituting Eq. (13) into Eqs. (11) and (12) yields

$$Q(ik) = Q^R(k) + Q^I(k)(p - g)/k \tag{14}$$

$$Q'(ik) = Q'^R(k) + Q'^I(k)(p - g)/k \tag{15}$$

Replacing $Q(ik)$ and $Q'(ik)$ in Eq. (9) by Eqs. (14) and (15) yields the p-k method equation in terms of p,

$$[(V^2/L^2)Mp^2 + (V/L)Bp + K - \tfrac{1}{2}\rho V^2 (Q'^R(k) + Q'^I(k)(p-g)/k)g - \tfrac{1}{2}\rho V^2 (Q^R(k) + Q^I(k)(p-g)/k)]\{q\} = 0 \tag{16}$$

Equation (16) can be rewritten in a state-space form:

$$[A(g,k) - pI_{2n}]\{u\} = 0 \tag{17}$$

where $[A(g,k)]$ is the real matrix,

$$[A(g,k)] = \begin{bmatrix} 0_n & I_n \\ C & D \end{bmatrix} \tag{18}$$

with

$$C = -(L^2/V^2)M^{-1}[K - \tfrac{1}{2}\rho V^2 (Q^R(k) + Q'^R(k)g - Q^I(k)g/k - Q'^I(k)g^2/k)]$$

$$D = -(L^2/V^2)M^{-1}[(V/L)B - \tfrac{1}{2}\rho V^2 (Q^I(k)/k + Q'^I(k)g/k)]$$

and $\{u\} = \{\{q\}, p\{q\}\}^T$.

By comparing Eq. (17) to Eq. (5), it is clearly seen that the new state matrix A depends not only on the reduced frequency k (imaginary part of p), but also on the damping term g (real part of p). Thus, modifications to the standard p-k method should be made.

Modifications to the Solution Algorithm of the Standard *p-k* Method

To adapt the standard p-k method to the dependence of matrix A on damping as stated above, it is suggested to include the damping iteration $g_{l+1} = p_l^R$ in addition to the reduced-frequency iteration $k_{l+1} = p_l^I$ in the solution algorithm of the standard p-k method. The damping convergence threshold should also be included:

$$|g_{l+1} - g_l| < \varepsilon$$

where ε is a user input with a default value of 0.001. With these modifications, the conventional reduced-frequency "lining-up" process is still performed in the modified algorithm. It is implied that the present p-k method will give the same number of roots as the number of structural modes used in the flutter analysis.

Sample Test Case and Discussion

The sample test case is selected from those of the ZONA/ZAERO Applications Manual [7]. The generalized mass, stiffness, and aerodynamic matrices are obtained from ZAERO output file. The extracted data is further adopted in the following flutter solutions by the modified p-k method with damping iteration. Note that the flutter equation is identical for the modified p-k method and the g method, so the slight difference between the flutter results calculated by both methods is mainly caused by the differences in their solution algorithms of these methods.

Fifteen-Degree Sweptback Wing at $M = 0.45$

This test case is denoted as HA145E in [5]. Four natural modes are used in flutter analysis with assumed equal structural damping coefficients $g = 0.02$ in these modes. The ZONA6 [7] method is employed to calculate subsonic unsteady aerodynamics. The comparison of the flutter characteristics between the modified p-k method and the g method is shown in Table 1. Figure 1 shows the damping-vs-velocity diagram (V–g diagram) and the flutter-frequency-vs-velocity diagram (V–f diagram) calculated by both methods.

Table 1 shows that the results of the modified p-k method agree well with those obtained by the g method. Four branches can be observed in Fig. 1, corresponding to the four aeroelastic modes for this test case. Good agreement between these methods is obtained in terms of the overall V–g and V–f diagram comparisons, as shown in Fig. 1. This result is expected, because both approaches solve the same flutter equation, i.e., Eq. (9).

Implication of Second Order of Damping Term

In the same way, second order of damping term could be added into the state matrix A straightforwardly by expressing $Q''(ik)$ as

$$Q''(ik) = Q''^R(k) + Q''^I(k)(p - g)/k \tag{19}$$

Table 1 Flutter characteristics of the 15 deg sweptback wing

Method	V_f, in./s	ω_f, Hz
Modified p-k method	6349.2	107.79
g method	6332.9	108.33

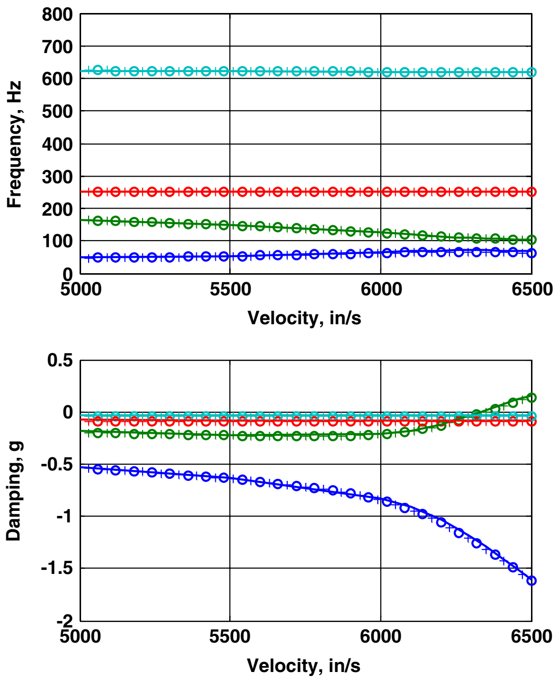

Fig. 1 *V–g* and *V–f* diagrams of the 15 deg sweptback wing at $M = 0.45$ by modified *p-k* method (first order of damping perturbation \circ , second order of damping perturbation +) and *g* method (solid line).

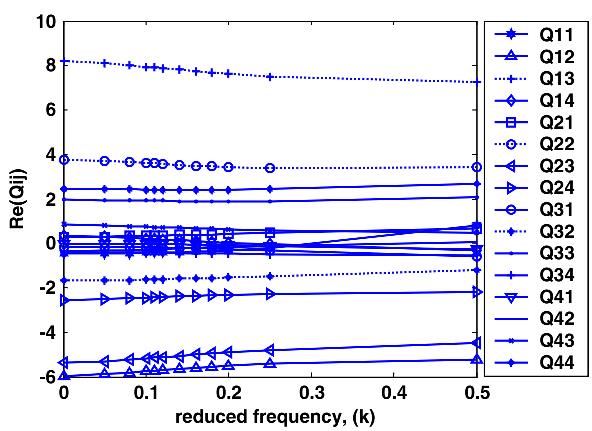

Fig. 2 Generalized aerodynamic coefficients vs reduced frequency of the 15 deg sweptback wing at $M = 0.45$.

Submatrices of $A(g,k)$ up to second order of damping perturbation are as follows:

$$C = -(L^2/V^2)M^{-1}[K - \tfrac{1}{2}\rho V^2(Q^R(k) + Q'^R(k)g + \tfrac{1}{2}Q''^R(k)g^2 - Q^I(k)g/k - Q'^I(k)g^2/k - \tfrac{1}{2}Q''^I(k)g^3/k)]$$
$$D = -(L^2/V^2)M^{-1}[(V/L)B - \tfrac{1}{2}\rho V^2(Q^I(k)/k + Q'^I(k)g/k + \tfrac{1}{2}Q''^I(k)g^2/k)] \tag{20}$$

Now we would reexamine the test case selected above by updating state matrix A with Eq. (20) in the modified *p-k* method algorithm. The resulting $V–g$ and $V–f$ diagrams up to the second order of damping perturbation are also depicted in Fig. 1 compared with those obtained by the first order of damping perturbation. The $V–g$ and $V–f$ diagrams seem to be almost identical between both approaches. This could be explained by the linearly varying property of $Q^I(k)$, as already demonstrated by Fig. 1 of [2], while each element of matrix $Q^R(k)$ shown in Fig. 2 is also nearly linear, i.e., $Q'^R(k) \approx$ const and $Q'^I(k) \approx$ const, so $Q''^R(k) \approx Q''^I(k) \approx 0$. When $Q(ik)$ is linearly varying, the second order of damping term $\frac{1}{2}g^2Q''(ik)$ would become zero; therefore, the higher order of damping perturbation will reduce to the first order of damping perturbation and not affect the flutter solution.

Conclusions

A modified *p-k* method with damping iteration is presented by solving the *g* method equation in terms of nondimensional Laplace parameter p instead of damping g. The state matrix of the modified *p-k* method depends on both damping and reduced frequency, which needs certain accommodations in the standard *p-k* method to reflect such a change, i.e., including the iteration loop for damping g and setting its convergence threshold.

As a consequence of the conventional eigenvalue solution by reduced-frequency lining-up process performed in the *p-k* method, the modified *p-k* method gives the same number of roots as the number of structural modes used in the flutter analysis. The selected test case illustrates that the modified *p-k* method with damping perturbation leads to almost the same results as those obtained by the *g* method.

The effect of the higher order of damping term is studied by the modified *p-k* method in one particular case. It is found that the effect on the flutter results is moderate for the higher order of damping term when the generalized aerodynamic matrix is nearly linear in reduced frequency k.

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References

[1] Hassig, H. J., "An Approximate True Damping Solution of the Flutter Equation by Determinant Iteration," *Journal of Aircraft*, Vol. 8, No. 11, 1971, pp. 885–889.
doi:10.2514/3.44311

[2] Chen, P. C., "Damping Perturbation Method for Flutter Solution: The *g* Method," *AIAA Journal*, Vol. 38, No. 9, 2000, pp. 1519–1524.
doi:10.2514/2.1171

[3] Qiu, J., and Sun, Q., "New Improved g Method for Flutter Solution," *Journal of Aircraft*, Vol. 46, No. 6, 2009, pp. 2184–2186.
doi:10.2514/1.46328

[4] Borglund, D., "Robust Eigenvalue Analysis Using the Structured Singular Value: The μ-p Flutter Method," *AIAA Journal*, Vol. 46, No. 11, 2008, pp. 2806–2813.
doi:10.2514/1.35859

[5] Bäck, P., and Ringertz, U., "Convergence of Methods for Nonlinear Eigenvalue Problems," *AIAA Journal*, Vol. 35, No. 6, 1997, pp. 1084–1087.
doi:10.2514/2.200

[6] Rodden, W. P., and Johnson, E. H., *MSC/NASTRAN Aeroelastic Analysis User's Guide*, Ver. 68, MacNeal-Schwendler, Los Angeles, 1994.

[7] *ZAERO Applications Manual*, Ver. 7.3, ZONA Technology, Scottsdale, AZ, 2005.

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