Comparison of Global Solution Methods to a Zero Lower Bound Model

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Abstract

During the Great Recession, the U.S. Federal Reserve lowered policy rates to zero, creating a kink in its policy rule, and calling into question traditional solution methods. Recent work has explored fully nonlinear models to represent the ZLB, but not much work has compared the performance of different solution methods. Two suggested solution methods are linear interpolation This paper examines the effect of making the policy functions conditional on whether the ZLB binds. We consider a fully nonlinear New Keynesian model with capital, and a solution algorithm with policy function iteration and linear interpolation. We find no evidence that the regime-indexed policy functions are smoother. In addition, the solution algorithms perform similarly with respect to accuracy, but the solution containing the regime-indexed policy functions is much slower, all else equal.

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1 Introduction

As a result of the global financial crisis of 2007-9 and the subsequent recession, central banks lowered their policy rate to its zero lower bound (ZLB). Although the ZLB constraint has always existed, for most of the data prior to the recession, policy rates were well above zero. In the wake of this unprecedented lowering of policy rates, the ZLB holds for a considerable portion of historic data for three major economies — the US, Japan, and the Euro Area. The ZLB creates a kink in the policy rule and calls into question linear estimation methods.

Researchers have responded to this inherent nonlinearity in different ways. Some have failed to incorporate the ZLB period data. Others have chosen to estimate linear models on the entire data set; however, this could lead to inaccurate estimates of model parameters and model predictions. Some have estimated a piecewise linear version of the nonlinear model (e.g., Guerrieri and Iacoviello, 2017). A few have estimated fully nonlinear models that treats the ZLB as an occasionally binding constraint (e.g., Gust et al., 2017; Plante et al., 2018; Richter and Throckmorton, 2016). These papers solve the nonlinear model using projection methods, which is explored in several other papers (e.g. Aruoba et al., 2018; Fernańdez-Villaverde et al., 2015). Solving a nonlinear model using projection methods provides the most comprehensive treatment of the ZLB but is computationally intensive.

As it is important to incorporate all historical data and yield accurate parameter values and predictions, the path forward seems to be in fully nonlinear models. Among those that use projection methods to solve a nonlinear model, some use policy function iteration with linear interpolation (e.g., Plante et al., 2018; Richter and Throckmorton, 2016). In terms of the approximating function, others estimate alternative functions like regime-indexed policy functions (Gust et al., 2017) or piecewise smooth policy functions (Aruoba et al. 2018) to avoid the kink introduced by the ZLB. With respect to the grid construction, some use the Smolyak collocation method developed by Judd et al. (2010) to reduce the burden of dimensionality (e.g., Gust et al., 2017; Fernańdez-Villaverde et al. 2015; Aruoba et al., 2018). Not much work has been done in comparing the accuracy and speed of different global nonlinear solution methods. We will examine the performance of Atkinson et al. (2019) and Gust et al. (2017) in terms of accuracy and speed as examples of different ways to solve nonlinear models.

The Atkinson et al. (2019) solution method is policy function iteration with linear interpolation to approximate future variables. The state space features evenly-spaced nodes and exogenous state variables approximated with the Markov chain from Rouwenhorst (1995). The Rouwenhorst approximation is useful because it only requires interpolation along the dimensions of the endogenous state variables. This yields a faster and more accurate solution than quadrature methods.They update the policy functions with linear interpolation to accurately capture the kink experienced at the ZLB.

The Gust et al. (2017) solution method approximates policy functions underlying the model's decision rule using an antistropic Smolyak method with Chebyshev polynomials. (See Judd et al. (2014) for a discussion of antistropic Smolyak methods). They approximate the exogenous state variables using Gauss-Hermite quadrature. Instead of directly computing the policy functions, they estimate functions at and away from the ZLB, which builds on Christiano and Fisher (2000). The idea is that the policy functions feature a kink or non-differentiability at the ZLB and regime-indexing the policy functions will yield smoother functions.

In this paper, we consider the effect of making the policy functions conditional on whether the ZLB binds. To isolate the effect of splitting the policy functions, we use policy function iteration on a fixed point with linear interpolation following Atkinson et al. (2019). This solution method also provides the motivation for a single policy function, and the Gust et al. (2017) solution provides the motivation for a regime-indexed policy function. Henceforth, the solution method based on the Atkinson et al. (2019) solution algorithm will be denoted ART and the solution algorithm that estimates smoother functions based on Gust et al. (2017) will be denoted GHLS. It is important to note that we do not consider the effect of a Smolyak basis with Chebyshev polynomials, so this is a modified Gust et al. (2017) solution method. The key point in this paper is to explore how splitting up the policy functions conditional on the ZLB impacts the speed and accuracy of the solution of a nonlinear model.

First, we construct the state space with evenly-spaced nodes and approximate the exogenous state variables with Rouwenhorst (1995). After obtaining initial conjectures for a set of policy functions from the log-linear solution, we iterate on the policy functions, linearly interpolating and numerically integrating the policy functions each step, using a fixed point iteration scheme. The algorithm converges once the maximum distance between successive guesses of policy functions falls below a convergence criterion. For the ART portion, we update a single policy function. For the GHLS portion, we update regime-indexed policy functions.

The standard representation of the Euler equation is

$$u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta],$$

where $u(\cdot)$ is the instantaneuous utility function, $0 < \beta < 1$ is the discount factor, c_t is consumption at period t, k_t is the capital stock at the beginning of the period, $0 < \delta \leq 1$ is the rate of depreciation, and $f(\cdot)$ is the production function. $u(\cdot)$ and $f(\cdot)$ are differentated according to the choice variables. The Euler equation is the fundamental dynamic equation in intertemporal optimization problems that include dynamic constraints. It states that the marginal cost of forgoing consumption today must equal the discounted marginal benefit of investing in capital for one period.

The interest rate appears in the household's budget constraint in models where the household has the choice of owning a nomimal bond. The household must choose a combination of consumption, labor, and bond holdings to maximize expected lifetime utility. Thus, the household's choice of consumption depends on the interest rate and whether the economy is at the ZLB. To approximate regime-indexed consumption policy functions c_t^{ZLB} and c_t^{normal}, we provide initial guesses based on the log-linear solution. We refine the policy function guesses as follows: For each node in the state space, we solve for the time t variables with either the c_t^{ZLB} or c_t^{normal} according to i_t, the interest rate at time t. With the time t variables, we linearly interpolate both policy functions. Next, we calculate the interest rate on each node in the state space, construct an aggregate consumption policy function based on whether the economy is at the ZLB on each node, and calculate time $t+1$ variables acordingly. We then update c_t^{ZLB} with an interest rate of 1, and c_t^{normal} with i_t.

In terms of accuracy, both the ART and GHLS yield similar Euler equation errors, with ART being slightly more accurate for the model with capital. The ART method converges much more quickly than the GHLS method, particularly when capital is introduced to the model. There is not a clear advantage of the regime-indexed policy function approximation when using an evenly spaced grid with linear interpolation. All else equal, using regime-indexed policy functions increases the computing time. However, introducing Chebyshev polynomials and sparse grid methods as in Gust et al. (2017) may reduce solution time and allow us to solve larger models with more state variables. Further research could explore how much accuracy is gained or lost relative to using non regime-indexed policy functions approximated with Chebyshev polynomials.

The remainder of the paper is as follows: Section 2 outlines our setup for the model with capital. Section 3 describes regime-indexed polynomials and outlines the solution algorithm. Section 4 discusses the results in terms of solution time, curvature of policy functions, and Euler equation errors. Section 5 concludes.

2 Model

This is a widely-employed New Keynesian model which is similar to the model in Atkinson et al. (2019), but includes capital accumulation as in Gust et al. (2017). Thus, it is an appropriate model to benchmark the accuracy and speed of the two solution methods.

2.1 Firms The production sector consists of a continuum of monopolistically competitive intermediate goods firms and a final goods firm. Intermediate firm $f \in [0,1]$ produces a differentiated good, $y(f)$, according to $y_t(f) = (k_{t-1}(f))^{\alpha}(z_t n_t(f))^{1-\alpha}$, where $n(f)$ is the labor hired by firm f and $k(f)$ is the capital rented by firm f. $z_t = g_t z_{t-1}$ is technology, which is common across firms.

Deviations from the steady-state growth rate, $\bar{g}$, follow

$$g_t = \bar{g} + \sigma_g \varepsilon_{g,t},\ \varepsilon_g \sim \mathbb{N}(0,1). \tag{1}$$

The final goods firm purchases output from each intermediate firm to produce the final good, $y_t \equiv [\int_0^1 y_t(f)^{(\theta-1)/\theta} df]^{\theta/(\theta-1)}$, where $\theta > 1$ is the elasticity of substitution. Dividend maximization determines the demand for intermediate good f, $y_t(f) = (p_t(f)/p_t)^{-\theta} y_t$, where $p_t = [\int_0^1 p_t(f)^{1-\theta} df]^{1/(1-\theta)}$ is the price level. Following Rotemberg (1982), intermediate firms pay a price adjustment cost, $adj_t^p(f) \equiv \varphi(p_t(f)/(\bar{\pi} p_{t-1}(f)) - 1)^2) y_t/2$, where $\varphi > 0$ scales the cost and $\bar{\pi}$ is the steady-state gross inflation rate. Given this cost, firm f chooses $n_t(f)$, $k_{t-1}(f)$, and $p_t(f)$ to maximize the expected discounted present value of future dividends, $E_t \sum_{k=t}^{\infty} q_{t,k} d_k(f)$, subject to its production function and the demand for its product, where $q_{t,t} \equiv 1$, $q_{t,t+1} \equiv \beta(\lambda_t/\lambda_{t+1})$ is the pricing kernel between periods t and $t+1$, $q_{t,k} \equiv \prod_{j=t+1}^{k>t} q_{j-1,j}$, and $d_t(f) = p_t(f) y_t(f)/p_t - w_t n_t(f) - adj_t^p(f)$. In symmetric equilibrium, the optimality conditions reduce to

$$y_t = (k_{t-1})^{\alpha} (z_t n_t)^{1-\alpha}, \tag{2}$$

$$w_t = (1-\alpha) mc_t y_t / n_t, \tag{3}$$

$$r_t^k = \alpha mc_t y_t / k_{t-1}, \tag{4}$$

$$\varphi(\pi_t^{gap} - 1)\pi_t^{gap} = 1 - \theta + \theta mc_t + \beta \varphi E_t[(\lambda_t/\lambda_{t+1})(\pi_{t+1}^{gap} - 1)\pi_{t+1}^{gap}(y_{t+1}/y_t)], \tag{5}$$

where $\pi_t^{gap} = \pi_t/\bar{\pi}_t$ and $\pi_t = p_t/p_{t-1}$ is the gross inflation rate. If $\varphi = 0$, the real marginal cost of producing a unit of output (mc_t) equals $(\theta-1)/\theta$, which is the inverse of the markup of price over marginal cost.

2.2 HOUSEHOLDS The households choose $\{c_t, n_t, b_t, x_t, k_t\}_{t=0}^{\infty}$ to maximize expected lifetime utility given by $E_0 \sum_{t=0}^{\infty} \beta[\log(c_t - hc_{t-1}^a) - \chi n_t^{1+\eta}/(1+\eta)]$, where β is the discount factor, $\chi > 0$ determines steady-state labor, $1/\eta$ is the Frisch elasticity of labor supply, c is consumption, c^a is aggregate consumption, h is the degree of external habit persistence, b is the real value of a privately-issued 1-period nominal bond, x is investment, and E_0 is an expectation operator conditional on information available in period 0. The household's budget constraint is given by

$$c_t + x_t + b_t/(i_t s_t) = w_t n_t + r_t^k k_{t-1} + b_{t-1}/\pi_t + d_t,$$

where i is the gross nominal interest rate, r^k is the capital rental rate, and d is a real dividend from ownership of intermediate firms. The nominal bond, b is subject to a risk premium, s, that follows

$$s_t = (1-\rho_s)\bar{s} + \rho_s s_{t-1} + \sigma_s \varepsilon_{s,t},\ 0 \le \rho_s < 1,\ \varepsilon_s \sim \mathbb{N}(0,1), \tag{6}$$

where $\bar{s}$ is the steady-state value. An increase in s_t boosts saving, which lowers period-t demand.

Households also face an investment adjustment cost, so the law of motion for capital is given by

$$k_t = (1-\delta)k_{t-1} + x_t(1-\nu(x_t^g - 1)^2/2),\ 0 \leq \delta \leq 1, \tag{7}$$

where $x_t^g = x_t/(\bar{g}x_{t-1})$ is investment growth relative to its steady-state and $\nu \geq 0$ scales the cost.

The first order conditions to each household's constrained optimization problem are given by

$$\lambda_t = c_t - hc_{t-1}^a, \tag{8}$$

$$w_t = \chi n_t^\eta \lambda_t, \tag{9}$$

$$1 = \beta E_t[(\lambda_t/\lambda_{t+1})(s_t i_t/(\bar{\pi}\pi_{t+1}^{gap}))], \tag{10}$$

$$q_t = \beta E_t[(\lambda_t/\lambda_{t+1})(r_{t+1}^k + (1-\delta)q_{t+1})], \tag{11}$$

$$1 = q_t[1 - \nu(x_t^g - 1)^2/2 - \nu(x_t^g - 1)x_t^g] + \nu\beta\bar{g}E_t[q_{t+1}(\lambda_t/\lambda_{t+1})(x_{t+1}^g)^2(x_{t+1}^g - 1)], \tag{12}$$

$$\varphi(\pi_t^{gap} - 1)\pi_t^{gap} = 1 - \theta + \theta mc_t + \beta\varphi E_t[(\lambda_t/\lambda_{t+1})(\pi_{t+1}^{gap} - 1)\pi_{t+1}^{gap}(y_{t+1}/y_t)], \tag{13}$$

where $1/\lambda$ is the marginal utility of consumption and q is Tobin's q.

Monetary Policy The central bank sets the gross nominal interest rate, i, according to

$$i_t = \max\{1, i_t^n\}, \tag{14}$$

$$i_t^n = (i_{t-1}^n)^{\rho_i}(\bar{\imath}(\pi_t^{gap})^{\phi_\pi}(y_t^g)^{\phi_y})^{1-\rho_i}\exp(\sigma_i\varepsilon_{i,t}), 0 \leq \rho_i < 1, \varepsilon_i \sim \mathbb{N}(0,1), \tag{15}$$

where y^{gdp} is real GDP (i.e., output, y, minus the resources lost due to adjustment costs, adj^p), i^n is the gross notional interest rate, $\bar{\imath}$ and $\bar{\pi}$ are the target values of the inflation and nominal interest rates, and ϕ_π and ϕ_y are the responses to the inflation and output growth gaps. A more negative net notional rate indicates that the central bank is more constrained.

Competitive Equilibrium The aggregate resource constraint and real GDP definition are given by

$$c_t + x_t = y_t^{gdp} \tag{16}$$

$$y_t^{gdp} = [1 - \varphi(\pi_t^{gap} - 1)^2/2]y_t \tag{17}$$

The model does not have a steady-state due to the unit root in technology, z_t. Therefore, we define the variables with a trend in terms of technology (i.e., $\tilde{x}_t \equiv x_t/z_t$). The detrended equilibrium system is provided in Appendix A. A competitive equilibrium consists of sequences of quantities, $\{\tilde{c}_t, \tilde{y}_t, \tilde{y}_t^{gdp}, x_t^g, y_t^g, n_t, \tilde{k}_t, \tilde{x}_t\}_{t=0}^{\infty}$, prices, $\{\tilde{w}_t, i_t, i_t^n, \pi_t, \tilde{\lambda}_t, q_t, r_t^k, mc_t\}_{t=0}^{\infty}$, and exogenous variables, $\{s_t, g_t\}_{t=0}^{\infty}$, that satisfy the detrended equilibrium system, given the initial conditions, $\{\tilde{c}_{-1}, i_{-1}^n, \tilde{k}_{-1}, \tilde{x}_{-1}, \tilde{w}_{-1}, s_0, g_0, \varepsilon_{i,0}\}$, and three sequences of shocks, $\{\varepsilon_{g,t}, \varepsilon_{s,t}, \varepsilon_{i,t}\}_{t=1}^{\infty}$.

Subjective Discount Factor	β	0.9949	Rotemberg Price Adjustment Cost	φ	100
Frisch Labor Supply Elasticity	$1/\eta$	3	Inflation Gap Response	ϕ_π	2.0
Price Elasticity of Substitution	θ	6	Output Growth Gap Response	ϕ_y	0.5
Steady-State Labor Hours	$\bar{n}$	$1/3$	Habit Persistence	h	0.80
Steady-State Risk Premium	$\bar{s}$	1.0058	Risk Premium Persistence	ρ_s	0.80
Steady-State Growth Rate	$\bar{g}$	1.0034	Notional Rate Persistence	ρ_i	0.80
Steady-State Inflation Rate	$\bar{\pi}$	1.0053	Technology Growth Shock SD	σ_g	0.005
Capital Share of Income	α	0.35	Risk Premium Shock SD	σ_s	0.005
Capital Depreciation Rate	δ	0.025	Notional Interest Rate Shock SD	σ_i	0.0035
Investment Adjustment Cost	ν	4			

Table 1: Parameter values

2.3 Parameter Values

Table 1 shows the model parameters. Parameters are from Atkinson et al. (2019), and are chosen to be characteristic of U.S. data. The subjective discount factor, β is set to 0.9949, the time average of the values implied by the steady-state consumption Euler equation and the federal funds rate. The Frisch labor supply elasticity $1/\eta$ is set to 3 to match the macro estimate in Peterman (2016). The price elasticity of substitution θ is set to 6, matching GHLS and corresponding to to a 20% average markup. Steady-state labor hours $\bar{n}$ is set to 1/3 of available time. The steady-state risk premium, growth rate, and inflation rate are set to the time averages of per capital real DGP growth, the percent GDP implicit price deflator, and the Baa corporate bond yield relative to the yield on the 10-Year Treasury. The capital share of income α is equal to the Fernald (2012) utilization-adjusted quarterly-TFP estimates of the capital share of income from 1988-2017Q4. The other parameters are set to values consistent with posterior estimates from similar models in the literature. For the model without capital, all parameters are the same except σ_i which is set to 0.002. (For the model with capital, σ_i was originally set to 0.002 to match Atkinson et al. (2019), but the model failed to hit the ZLB during simulation, so we increased the parameter to 0.0035.)

3 Solution Methods

3.1 Regime-indexed policy functions

Instead of directly approximating the policy functions, some papers elect to approximate alternative functions to avoid the kink experienced at the ZLB (e.g., Gust et al., 2017; Aruoba et al., 2018). Gust et al. (2017) uses policy functions that are indexed by the interest-rate regime. Because these functions do not depend on the current indicator function,Gust et al. (2017) claims that they are more likely to be smooth. The regime-specific functions still depend on a secondary effect that the non-differentiability in the nominal rate in the following period has on expectations of future variables. However, following Christiano and Fisher (2000) this effect should be small due to the presence of the expectations operators which act to smooth out regime-specific functions. The idea for regime-indexed policy functions is as follows:

$$\mathbf{pf}_t(d) = \mathbf{pf}_{t,1}(d)\mathbb{I}_t(d) + \mathbf{pf}_{t,2}(d)(1 - \mathbb{I}_t(d)), \tag{18}$$

where $\mathbb{I}_t(d)$ is defined by

$$\mathbb{I}_t(d) = \begin{cases} 1 & \text{if } i_t > 1 \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

The variable $i_t = \max\{1, i_t^n\}$ represents the value of the notional rate derived from evaluating the functions $\mathbf{pf}_{t,j}(d)$ for $j \in \{1, 2\}$ and using equation (15). (For each variable, $j = 1$ denotes the function associated with the regime with a positive nominal rate and $j = 2$ denotes the function associated with the ZLB regime.) The functions $\mathbf{pf}_{t,j}$ satisfy the residual functions $R_{t,l,j}$ for and $l \in \{1, 2, 3, 4\}$ which correspond to the household FOC bond, FOC capital, FOC investment, and the price Phillips curve, respectively, and $j = 1, 2$.

$$R_{t,1,1} = 1 - s_t i_t \beta E_t[(\lambda_t/\lambda_{t+1})(1/(\bar{\pi}\pi_{t+1}^{gap}))], \tag{20}$$

$$R_{t,1,2} = 1 - s_t \beta E_t[(\lambda_t/\lambda_{t+1})(1/(\bar{\pi}\pi_{t+1}^{gap}))], \tag{21}$$

$$R_{t,2,j} = q_t - \beta E_t[(\lambda_t/\lambda_{t+1})(r_{t+1}^k + (1-\delta)q_{t+1})], \tag{22}$$

$$R_{t,3,j} = 1 - q_t[1 - \nu(x_t^g - 1)^2/2 - \nu(x_t^g - 1)x_t^g] - \nu\beta\bar{g}E_t[q_{t+1}(\lambda_t/\lambda_{t+1})(x_{t+1}^g)^2(x_{t+1}^g - 1)], \tag{23}$$

$$R_{t,4,j} = \varphi(\pi_t^{gap} - 1)\pi_t^{gap} - (1-\theta) - \theta mc_t - \beta\varphi E_t[(\lambda_t/\lambda_{t+1})(\pi_{t+1}^{gap} - 1)\pi_{t+1}^{gap}(y_{t+1}/y_t)]. \tag{24}$$

3.2 Solution Algorithm

Our solution algorithm is as follows: Construct an evenly-spaced grid by discretizing the endogenous state variables and using Rouwenhorst (1995) method to approximate the exogenous processes for s_t, g_t, and $\varepsilon_{i,t}$. We use the Rouwenhorst method so that we do not have to interpolate along the dimensions of the exogenous state variables, providing a faster and more accurate than quadrature methods. The vector of policy functions is denoted $\mathbf{pf}_t$ and the realization on node d is denoted $\mathbf{pf}_t(d)$. Denote the vector of states $\mathbf{z}_t$.

1. To construct the initial policy functions $\mathbf{pf}_0$, we solve the log-linear analog of our model with the ZLB not impos ed using Sims's (2002) `gensys` algorithm. (For GHLS, we initialize the regime-indexed policy functions by setting them to the log-linear solution.)

2. For each node in the state space $d \in \{1, \ldots, D\}$:

   (a) Solve for the variables dated at time t given $\mathbf{pf}_t$ and $\mathbf{z}_t$. (For GHLS, we calculate time t variables with the non-ZLB policy functions if the interest rate $i_t > 1$, and the ZLB policy functions if the interest rate $i_t \leq 1$).

   (b) Linearly interpolate the policy functions at the updated state variables $\mathbf{z}_{t+1}$ to obtain $\mathbf{pf}_{t+1}(m)$ on every integration node $m \in \{1, \ldots, M\}$. (For GHLS, we interpolate the non-ZLB and ZLB policy functions separately.)

   (c) Given the interpolated policy functions $\{\mathbf{pf}(m)\}_{m=1}^{M}$, and integration nodes and weights provided by the Rouwenhorst method, approximate the expectation operators for the model. (For GHLS, use the interest rate array calculated from the interpolated policy functions to determine whether the model is at the ZLB at each node. We construct the aggregate policy functions accordingly, as in (18) and (19).)

   (d) Solve for time $t+1$ variables from the expectation operators. Back out the policy functions $\mathbf{pf}_d$ from the expectation operators and time $t+1$ variables. (For GHLS, we solve for the regime-indexed policy functions according to (20)-(24)).

3. Repeat step 2 until the maximum distance between successive approximations of the policy functions is below 10^{-6}.

3.3 Euler Equation Errors We use Euler equation errors to measure the accuracy of our solutions. To measure errors between nodes, we use Gauss-Hermite quadrature instead of the Rouwenhorst method following ART to allow exogenous variables to have realizations between grid points. The Euler equation errors are represented in absolute value of the errors in base 10 logarithms. An Euler equation error of -3 means the household makes an error equivalent to one per 1,000 consumption goods; i.e., an error of 0.1% on average.

We simulate 10,000 periods of the model using the nonlinear solution with random shocks. For each period, we follow one iteration of step 2 of the algorithm outlined in Appendix B and then transform the residuals to Euler equation errors.

We considered adding a second measure of accuracy, a statistical measure from Den Haan and Marcet (1994) based on a transformation of the Euler equation residuals to a chi-square distribution. However, it proved uninformative, as several runs of the test statistic were all in the extreme tails of the distribution.

4 Results

	Model without capital			Model with capital		
	Total nodes	Iterations	Total Time	Total Nodes	Iterations	Total time
ART	2401	67	17.9s	78125	65	0h 15m 37.1s
GHLS	2401	66	46s	78125	1170	5h 32m 48.6s

Table 2: Solution times

4.1 Solution Times

Table 2 reports the solution times for the ART and GHLS solution methods for the models with and without capital. The solution times were computed with multi-core processing using the Parallel Computing Toolbox. With MATLAB executable functions (MEX) provided by the companion toolbox to Richter et al. (2014), we implemented the interpolation steps of the algorithm in Fortran. The solution times represent one run of each algorithm. The machine used to compute the solutions has 32 total cores (two CPUs at 2.30GHz, each with 16 cores).

The GHLS method is slower for both the model with and without capital. The speed differential blows up for the model with capital. The performance is somewhat contingent upon the parameterization. The GHLS method was faster with $\sigma_v = 0.002$, but we selected $\sigma_v = 0.0035$ so that the ZLB constraint was better represented in the simulation. The GHLS solution algorithm handles a labor policy (consumption policy for model without capital) at and away from the ZLB. As a result, the solution algorithm requires two interpolation steps. In addition, the GHLS solution algorithm must determine at which nodes the state space is at the ZLB, and solve for time t and $t + 1$ variables accordingly.

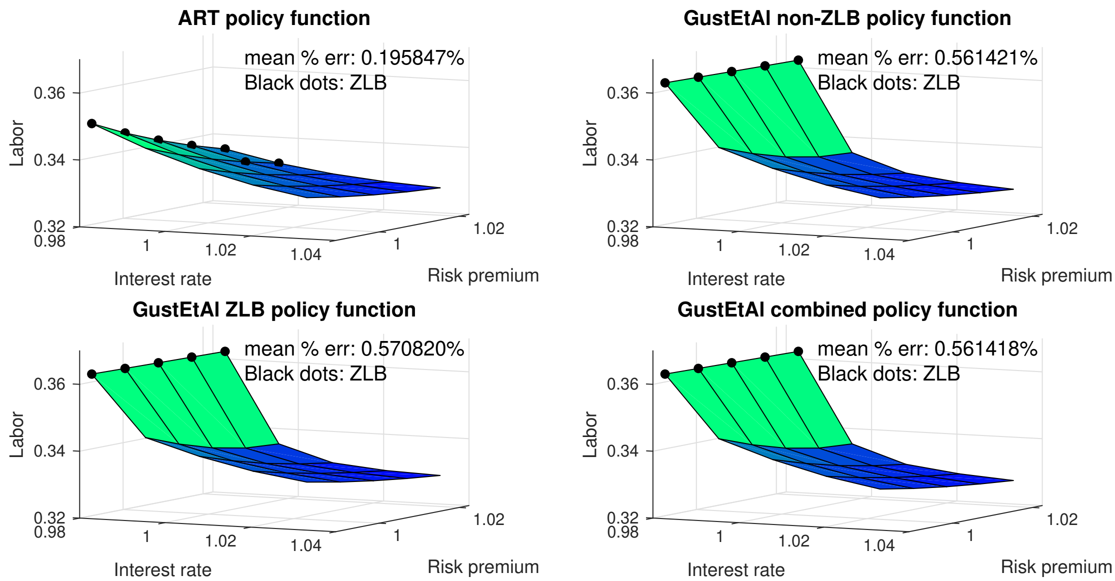

Figure 1: Labor policy function for model with capital

4.2 Policy Functions Figure 1 shows the cross-section of the labor policy functions with the interest rate and risk premium for the model with capital. The black dots on the nodes of the graph represent labor values corresponding to the ZLB. The GHLS labor policy function is computed at and away from the ZLB, and combined later, so all three are shown in the figure. The GHLS combined policy function is very similar to the policy function away from the ZLB.

Visually, the ART labor policy function is smoother. Labor hours smoothly increase as the interest rate falls and hits the ZLB. The GHLS labor policy function features a kink at the ZLB where labor rises dramatically.

	Model without capital		Model with capital	
	Mean % Error	RMSE	Mean % Error	RMSE
ART policy	0.46545%	0.0020137 c units	0.19585%	.0029035 l units
GHLS policy	0.45556%	0.0019792 c units	0.56142%	0.0076665 l units

Table 3: Smoothness measures for labor policy functions (c for model with capital and n for model with capital). GHLS combined policy functions are reported.

Table 3 reports the RMSE (root mean square error) and mean percent error from linear policy functions. The mean percent error is calculated as $1/M \sum_{i=1}^{N} [\mathbf{pf.n}(i) - \hat{\mathbf{n}}(i)]/\bar{n}$ where N is the number of periods, **pf.n** is the labor policy function, $\hat{\mathbf{n}}$ is the linear regression model of **pf.n** fit to the state variables, and $\bar{n}$ is steady state labor. The corresponding results for the model without capital is reported in Appendix C.

The RMSE and the mean percent error indicate that the ART labor policy function is smoother and better fits a linear plane for the model with capital. For the model without capital, both functions are smooth, but the GHLS policy function slightly better fits a linear plane than the ART policy function.

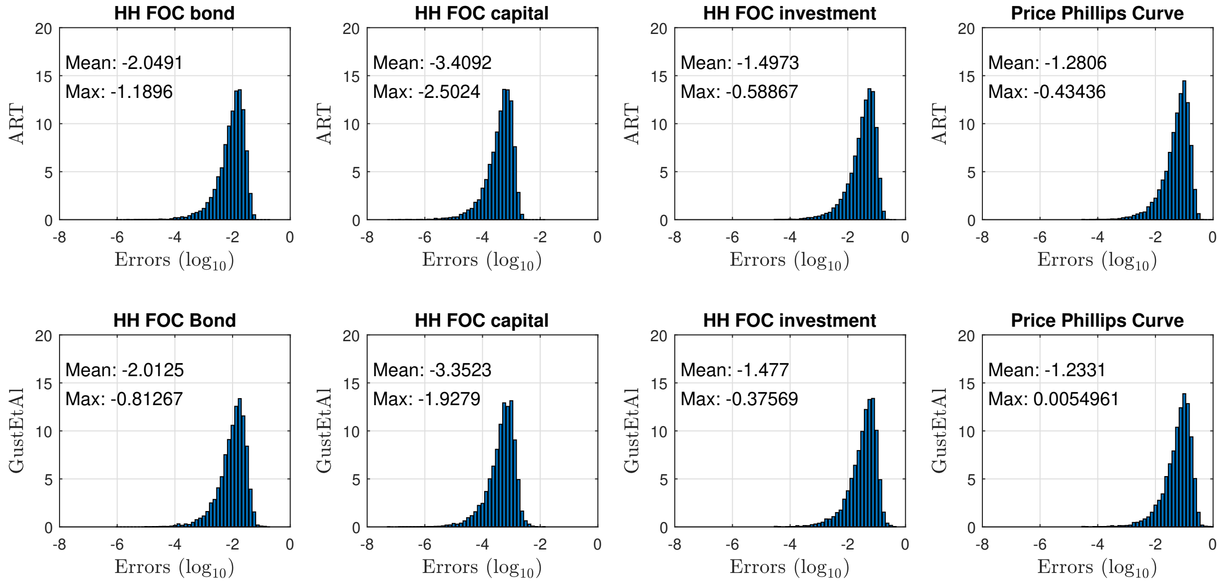

Figure 2: Euler equation errors for model with capital

4.3 Euler Equation Errors

Figure 2 shows the distribution of the absolute value of the Euler equation errors in base 10 logarithms for the household FOC bond, FOC capital, FOC investment, and price Phillips curve in the model with capital. We also report the mean and maximum error. The corresponding results for the model without capital is reported in Appendix C.

The Euler equation errors are very similar between solution algorithm, though the ART Euler equation errors are slightly more negative, and thus correspond to a smaller household error. The mean errors are similar, though the maximum error is considerably higher for the GHLS Euler equation error. The shapes of the errors are comparable.

5 Conclusion

During the Great Recession, the U.S. Federal Reserve lowered policy rates to zero, creating a kink in its policy rule, and calling into question traditional solution methods. Recent work has explored fully nonlinear models to represent the ZLB, but not much work has compared the performance of different solution methods. This paper analyzes the effect that different ways of representing the policy functions can have on a nonlinear solution algorithm. We use a method based on Atkinson et al. (2019) to motivate directly approximating the policy functions and a method based on Gust et al. (2017) to motivate regime-indexing the policy functions. Although the regime-indexed policy functions were meant to be smoother and easier to compute, the GHLS method was slower and featured more of a non-linearity at the ZLB then the ART representation in the model with caiptal.

The regime-indexed policy functions also yielded a solution that was less accurate in terms of Euler equation errors in the model with capital.

Future work includes solving the GHLS solution method using Smolyak discretization methods with Chebyshev polynomials. We would expect this to speed up the solution algorithm and improve the accuracy of the solution. If this addition made the algorithm faster, we could also discretize the grid more finely to improve the accuracy of the solution and compare this to the ART method.

A Detrended Equilibrium System

Medium-Scale Model The detrended system includes (1), (6), (14), (15) and

$$\tilde{y}_t = (\tilde{k}_{t-1}/g_t)^{\alpha} n_t^{1-\alpha}, \tag{25}$$

$$r_t^k = \alpha mc_t g_t \tilde{y}_t/\tilde{k}_{t-1}, \tag{26}$$

$$\tilde{w}_t = (1-\alpha) mc_t \tilde{y}_t/n_t, \tag{27}$$

$$\tilde{y}_t^{gdp} = [1-\varphi(\pi_t^{gap}-1)^2/2]\tilde{y}_t, \tag{28}$$

$$y_t^g = g_t \tilde{y}_t^{gdp}/(\bar{g}\tilde{y}_{t-1}^{gdp}), \tag{29}$$

$$\tilde{\lambda}_t = \tilde{c}_t - h\tilde{c}_{t-1}/g_t, \tag{30}$$

$$\tilde{w}_t = \chi n_t^{\eta} \tilde{\lambda}_t, \tag{31}$$

$$\tilde{c}_t + \tilde{x}_t = \tilde{y}_t^{gdp}, \tag{32}$$

$$x_t^g = g_t \tilde{x}_t/(\bar{g}\tilde{x}_{t-1}), \tag{33}$$

$$\tilde{k}_t = (1-\delta)(\tilde{k}_{t-1}/g_t) + \tilde{x}_t(1-\nu(x_t^g-1)^2/2), \tag{34}$$

$$1 = \beta E_t[(\tilde{\lambda}_t/\tilde{\lambda}_{t+1})(s_t i_t/(\bar{\pi}\pi_{t+1}^{gap} g_{t+1}))], \tag{35}$$

$$q_t = \beta E_t[(\tilde{\lambda}_t/\tilde{\lambda}_{t+1})(r_{t+1}^k + (1-\delta)q_{t+1})/g_{t+1}], \tag{36}$$

$$1 = q_t[1-\nu(x_t^g-1)^2/2 - \nu(x_t^g-1)x_t^g] + \nu\beta\bar{g}E_t[q_{t+1}(\tilde{\lambda}_t/\tilde{\lambda}_{t+1})(x_{t+1}^g)^2(x_{t+1}^g-1)/g_{t+1}], \tag{37}$$

$$\varphi(\pi_t^{gap}-1)\pi_t^{gap} = 1-\theta+\theta mc_t + \beta\varphi E_t[(\tilde{\lambda}_t/\tilde{\lambda}_{t+1})(\pi_{t+1}^{gap}-1)\pi_{t+1}^{gap}(\tilde{y}_{t+1}/\tilde{y}_t)]. \tag{38}$$

The variables are $\tilde{c}, \tilde{n}, \tilde{x}, \tilde{k}, \tilde{y^{gdp}}, \tilde{y}, x^g, y^g, \tilde{w}, r^k, \pi, i, i^n, q, mc, \tilde{\lambda}, g$, and s.

Small-Scale Model The detrended system includes (1), (6), (14), (15), (31), (35), (38), and

$$\tilde{\lambda}_t = \tilde{c}_t, \tag{39}$$

$$\tilde{c}_t = [1-\varphi(\pi_t^{gap}-1)^2/2]\tilde{y}_t, \tag{40}$$

$$\tilde{y}_t = n_t. \tag{41}$$

The variables are $\tilde{c}, i^n, i, \tilde{\lambda}, \tilde{w}, \pi^{gap}, \tilde{y}, n, g$, and s.

B NONLINEAR SOLUTION METHOD

The following discussion is based on Atkinson et al. (2019) and Richter and Throckmorton (2014). We express the detrended nonlinear system compactly as

$$E[f(\mathbf{s}_{t+1}, \mathbf{s}_t, \varepsilon_{t+1})|\mathbf{z}_t, \vartheta] = 0,$$

where f is a vector-valued function, $\mathbf{s}_t$ is a vector of variables, $\varepsilon_t \equiv [\varepsilon_{s,t}, \varepsilon_{g,t}, \varepsilon_{i,t}]'$ is a vector of shocks, $\mathbf{z}_t$ is a vector of states ($\mathbf{z}_t \equiv [\tilde{c}_{-1}, i^n_{t-1}, \tilde{k}_{t-1}, \tilde{x}_{t-1}, s_t, g_t, \varepsilon_{i,t}]'$ for the model with capital and $\mathbf{z}_t \equiv [\tilde{c}_{t-1}, i^n_{t-1}, s_t, g_t, \varepsilon_{i,t}]'$ for the model without capital), and ϑ is a vector of parameters.

We use the Markov chain method in Rouwenhorst (1995) to discretize the endogenous state variables, s_t, g_t, and $\varepsilon_{i,t}$ following ART. The bounds on $\tilde{c}_{t-1}$, i^n_{t-1}, $\tilde{k}_{t-1}$, and $\tilde{x}_{t-1}$ are set to $\pm 2.5\%$, $\pm 6\%$, $\pm 8\%$, and $\pm 15\%$, respectively, following Atkinson et al. (2019). We discretize the states into 5 evenly-spaced points for the model with capital and 7 evenly-spaced points for the model without capital. The product of the points in each dimension, D, represents the total nodes in the state space ($D = 78125$ for the model with capital and $D = 2401$ for the model without capital). The realization of $\mathbf{z}_t$ on node d is denoted $\mathbf{z}_t(d)$. The Rouwenhorst method provides integration nodes, $[s_{t+1}(m), g_{t+1}(m), \varepsilon_{i,t+1}(m)]$, with weights, $\phi(m)$, for $m \in \{1, \dots, M\}$.

The vector of policy functions is denoted $\mathbf{pf}_t$ and the realization on node d is denoted $\mathbf{pf}_t(d)$ ($\mathbf{pf}_t(d) \equiv [\tilde{\pi}^{gap}_t(\mathbf{z}_t), n_t(\mathbf{z}_t), q_t(\mathbf{z}_t), mc_t(\mathbf{z}_t)]$ for the model with capital and $\mathbf{pf}_t(d) \equiv [\tilde{\pi}^{gap}_t(\mathbf{z}_t), \tilde{c}_t(\mathbf{z}_t)]$ for the model without capital. For the GHRT policy functions, any variables directly dependent on the interest rate (n_t for the model with capital and $\tilde{c}_t$ for the model without capital) are the result of computing regime-indexed policy functions using (18) and (19). The policy functions are selected so that solving for other variables in the nonlinear system is straightforward.

The following steps outline the global policy function iteration algorithm:

1. Use Sims's (2002) `gensys` algorithm to solve the log-linear model without the ZLB constraint and obtain conjectures for the policy functions $\mathbf{pf}_0$. (For GHLS, the initial guesses for n^{normal}_t and n^{ZLB}_t are set to the log-linear solution for n_t from `gensys`).

2. On each node $d \in \{1, \dots, D\}$:

   (a) ART: Solve for all variables dated at time t, given $\mathbf{pf}_t(d)$ and $\mathbf{z}_t(d)$.
   GHLS: Solve for i_t. If $i_t > 1$, solve for all time t variables with n^{normal}_t. Otherwise, solve for all time t variables with n^{ZLB}_t.

(b) ART: Linearly interpolate the policy functions $\mathbf{pf}_{j-1}$, at the updated state variables $\mathbf{z}_{t+1}(m)$, to obtain $\mathbf{pf}_{t+1}(m)$ on every integration node, $m \in \{1, \ldots, M\}$.
GHLS: Linearly interpolate n_t^{normal} and n_t^{ZLB} in separate steps with the remainder of the policy functions in $\mathbf{pf}_{j-1}$.

(c) ART: Given $\{\mathbf{pf}_{t+1}(m)\}_{m=1}^{M}$, solve for the other elements of $\mathbf{s}_{t+1}(m)$ and approximate the expectation operators

$$E[f(\mathbf{s}_{t+1}, \mathbf{s}_t(d), \varepsilon_{t+1})|\mathbf{z}_t(d), \vartheta] \approx \sum_{m=1}^{M} \phi(m) f(\mathbf{s}_{t+1}(m), \mathbf{s}_t(d), \varepsilon_{t+1}(m)).$$

GHLS: Solve for $\{i_t(m)\}_{m=1}^{M}$ using the interpolated policy functions and time t variables. If the value of $\{i_t(m)\}_{m=1}^{M}$ on node d is greater than 1, assign $\{n_t(m)\}_{m=1}^{M}$ on node d to the corresponding value in $\{n_t^{normal}(m)\}_{m=1}^{M}$, otherwise assign $\{n_t(m)\}_{m=1}^{M}$ on node d to the corresponding value in $\{n_t^{ZLB}(m)\}_{m=1}^{M}$. Solve for the other elements in $\mathbf{s}_{t+1}$ and approximate the expectation operators.

(d) ART: Solve for $\mathbf{pf}_t(d)$ from the expectation operators and $\mathbf{s}_{t+1}(m)$. Set $\mathbf{pf}_j(d) = \mathbf{pf}_t(d)$.
GHLS: Using the expectation operators, solve for the $n_t(d)$ policy in the style of (20) and the n_t^{ZLB} policy in the style of (21). Solve for the rest of the policy functions in $\mathbf{pf}_t(d)$.

3. Repeat step 2 until $\text{maxdist}_j < 10^{-6}$, where $\text{maxdist}_j \equiv \max\{|\mathbf{pf}_j - \mathbf{pf}_{j-1}|\}$. When that criterion is satisfied, the algorithm has converged to an approximate nonlinear solution.

C Results for model without capital

The following reports the corresponding results for the model wtihout capital.

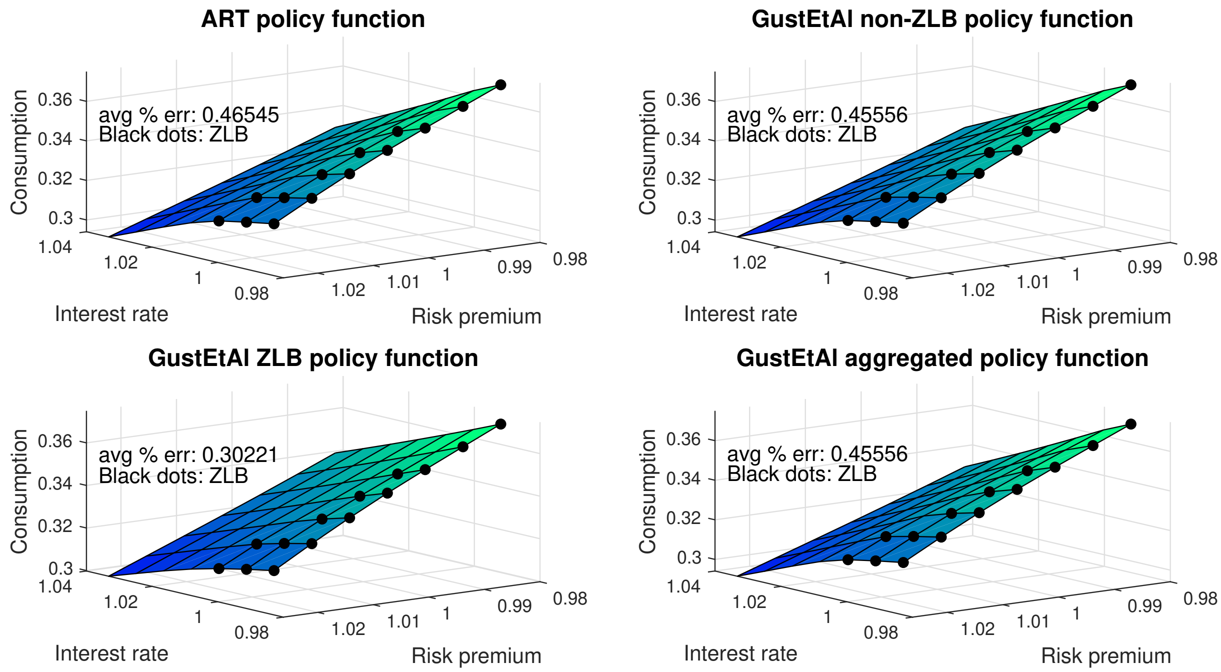

Figure 3: Consumption policy function for model without capital

Figure 3 shows the cross-section of the consumption policy functions with the interest rate and risk premium. The policy functions are all fairly smooth and only have a slight nonlinearity at the ZLB. The average percent error of the consumption policy functions from a linear plane is slightly lower for the GHLS policy functions.

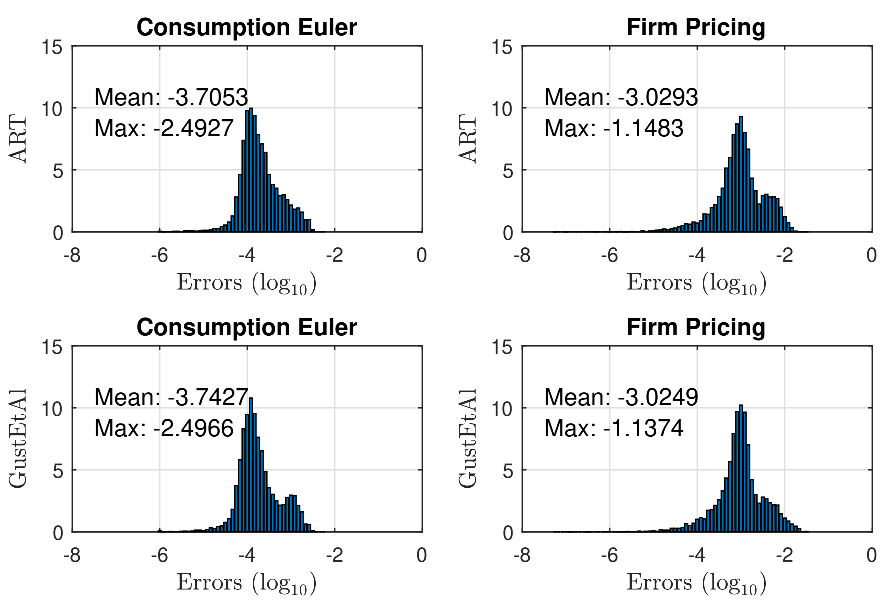

Figure 4: Euler equation errors for model without capital

Figure 4 shows the distribution of the absolute value of the Euler equation errors in base 10 logarithms for the consumption Euler equation and the price Phillips curve. The mean and maximum Euler equations are comparable between algorithms; the GHLS solution does slightly better for the consumption Euler equation and the ART solution does slightly better for the firm pricing equation.

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