DAT600: Algorithm Theory

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Assignment – 1

Algorithm	Worst case	Average case / Expected case
Insertion sort	$\Theta(n^2)$	$\Theta(n^2)$
Merge sort	$\Theta(nlgn)$	$\Theta(nlgn)$
Heapsort	$O(nlgn)$	-
Quicksort	$\Theta(n^2)$	$\Theta(nlgn)$ (expected)

1. Programming

[code is here]

Step counting

Following the hint from the lecture, the insertion sort algorithm was modified to count the number of steps taken. For

Insertion sort

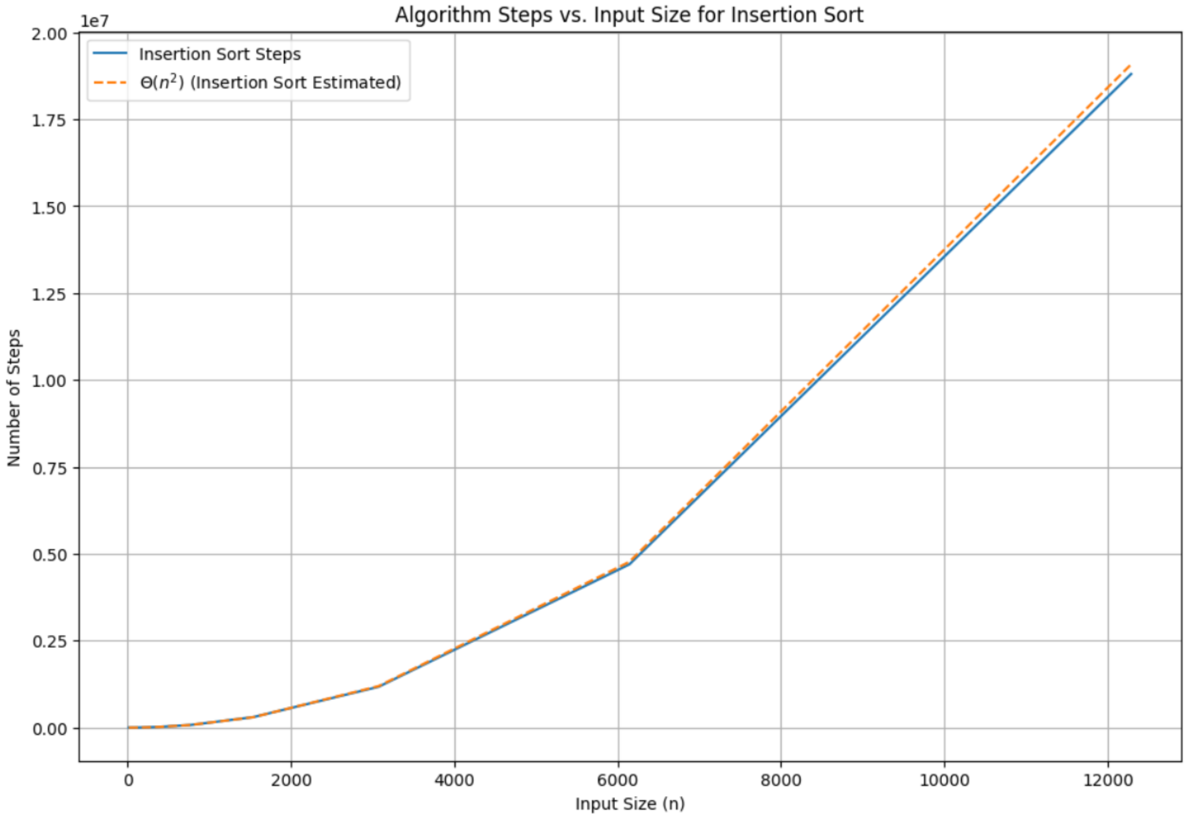
The plots below with increasing input size shows that the number of steps taken is quadratic, as expected.
The expected time complexity is $\Theta(n^2)$. So the calculated number of steps taken for the plot is

$$\frac{steps}{n^2} \text{ for each element in the list}$$

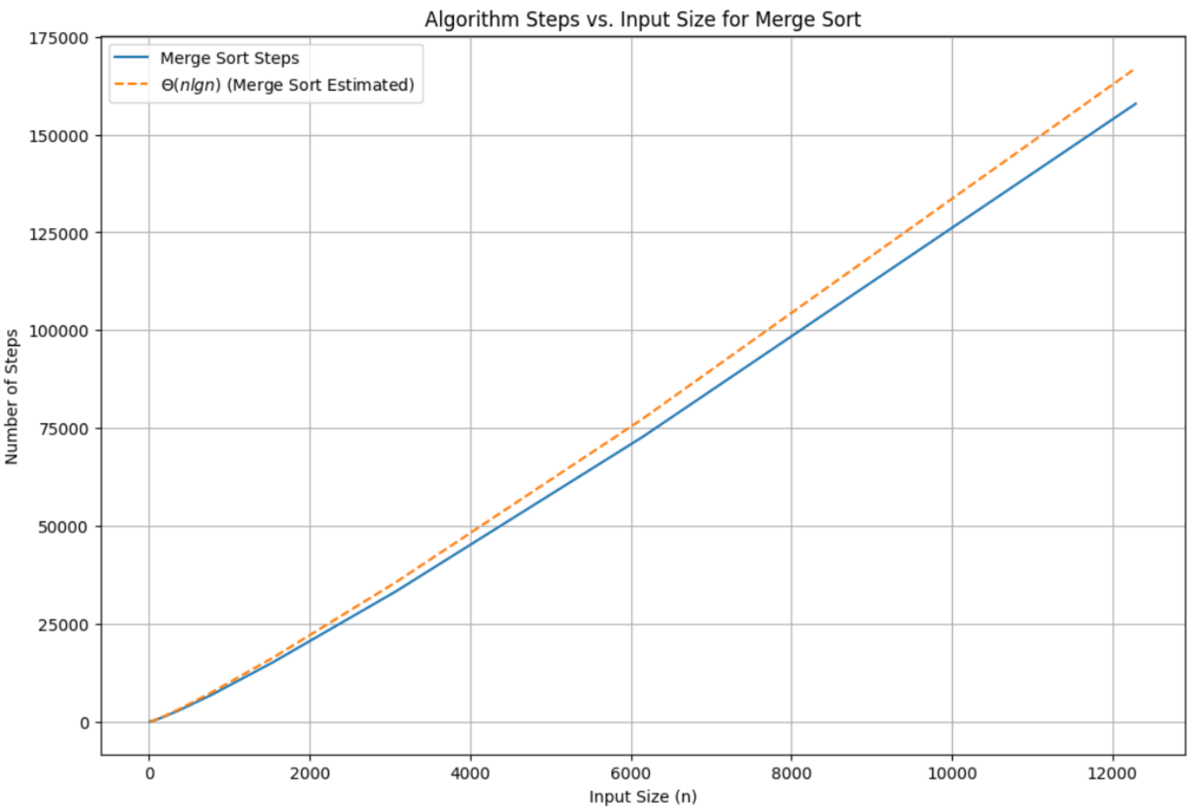
For all the other algorithms, the number of steps taken is logarithmic. The expected time complexity is $\Theta(nlgn)$. So the calculated number of steps taken for the plot is

$$\frac{steps}{n * log_2 n} \text{ for each element in the list}$$

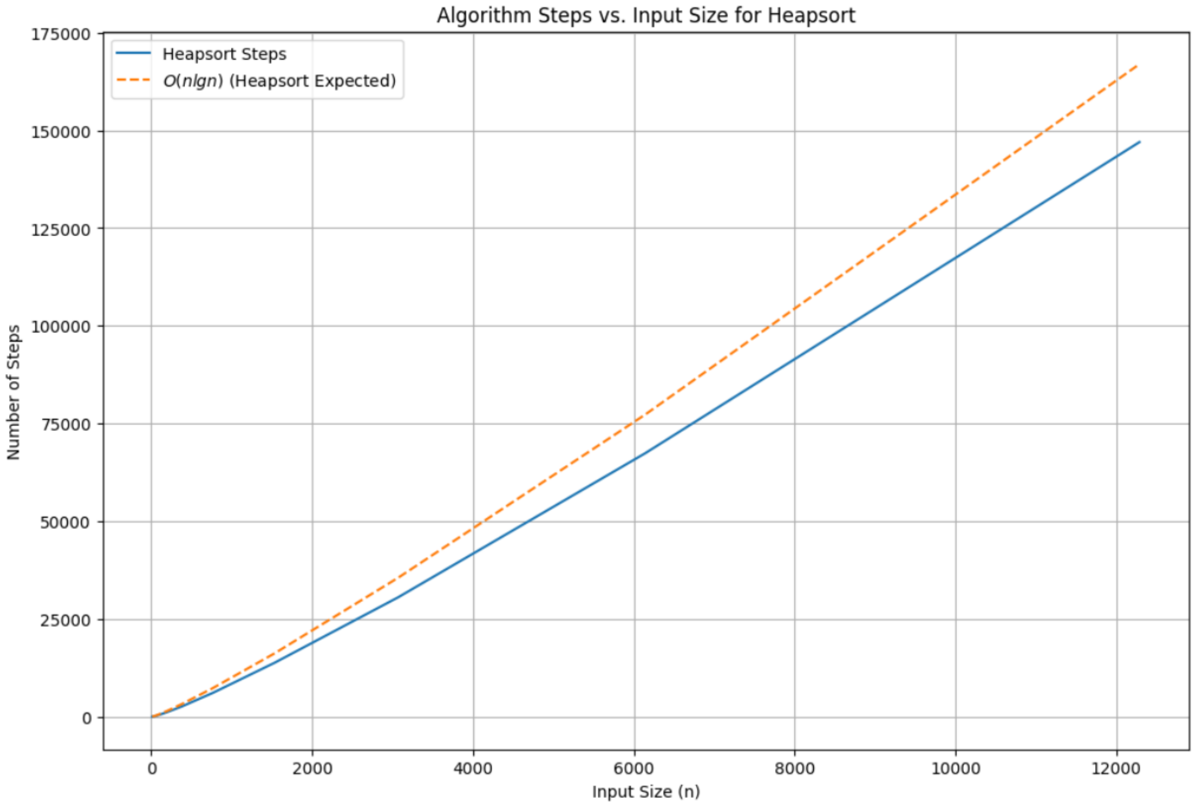
This is how the estimated number of steps points are calculated in the plots.


Algorithm Steps vs. Input Size for Insertion Sort
1e7
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
Number of Steps
Insertion Sort Steps
Θ(n^2) (Insertion Sort Estimated)
0
2000
4000
6000
8000
10000
12000
Input Size (n)

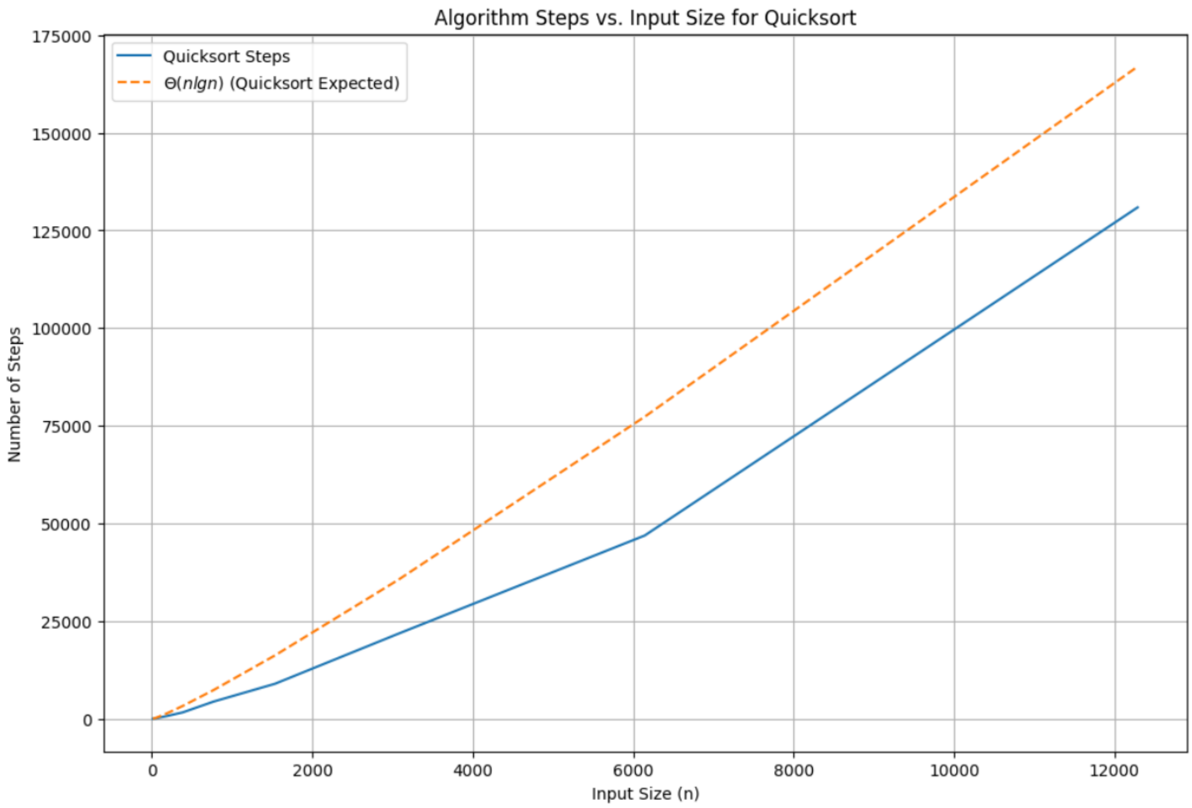
Merge sort


Algorithm Steps vs. Input Size for Merge Sort
175000
150000
125000
100000
75000
50000
25000
0
Number of Steps
Merge Sort Steps
Θ($nlgn$) (Merge Sort Estimated)
0
2000
4000
6000
8000
10000
12000
Input Size (n)

Heapsort



Quicksort



2. Compare true execution time

After implementing the algorithms in Python and Go, we ran the algorithms on a list of 10 000 random integers. The results are shown in the table below. As expected, the Python implementations are a lot slower than the Go implementations. The results are also as expected in terms of the algorithms. Insertion sort is the slowest, followed by heapsort, quicksort, and merge sort.

<!--
py assignment1.py
Function insertion_sort, time elapsed: 1231119.752 us.
Function heapsort, time elapsed: 52052.732 us.
Function merge_sort, time elapsed: 20185.461 us.
Function quicksort, time elapsed: 37180.236 us.

go run ./go/assignment1.go
Function: insertionSort, Time elapsed: 11233 us.
Function: mergeSort, Time elapsed: 911 us.
Function: quickSort, Time elapsed: 614 us.
Function: heapSort, Time elapsed: 642 us.
-->

Algorithm	Time elapsed Python	Time elapsed Go
Insertion sort	1231119.752 µs	11233 µs
Merge sort	20185.461 µs	911 µs
Heapsort	52052.732 µs	642 µs
Quicksort	37180.236 µs	614 µs

3. Basic proofs

a) Show that for any real constants a and b, where $b > 0$, $(n+a)^b = \Theta(n^b)$

If $(n+a)^b = \Theta(n^b)$, then constants $c_1, c_2,$ and n_0 exists such that $0 \leq c_1 n^b \leq (n+a)^b \leq c_2 n^b$ for all $n < n_0$

$$\frac{c_1 * n^b}{n^b} \leq \frac{(n+a)^b}{n^b} \leq \frac{c_2 * n^b}{n^b} \text{ for all } n > n_0$$
$$c_1 \leq \frac{(n+a)^b}{n^b} \leq c_2$$

For high values of n, the constant a is negligible, thus $\lim_{x\to\infty} \frac{(n+a)^b}{n^b} = \frac{n^b}{n^b} = 1$. Since we know $\frac{(n+a)^b}{n^b}$ trends towards 1, we know there exists c_1, c_2 and n_0 that fulfills $\Theta(n^b) = (n+a)^b$

b) Show that $\frac{n^2}{lgn} = o(n^2)$

If $\frac{n^2}{lgn} = o(n^2)$, constants n_0 exist such that for any constant $c > 0, 0 \leq \frac{n^2}{lgn} \leq cn^2$ for all $n \geq n_0$

In other terms, it holds true if $\lim_{n\to\infty} \frac{\frac{n^2}{lgn}}{n^2} = 0$

$$\lim_{n\to\infty} \frac{\frac{n^2}{lgn} * \frac{lgn}{n^2}}{n^2 * \frac{lgn}{n^2}} = \lim_{n\to\infty} \frac{1}{lgn} = 0$$

c) Show that $n^2 \neq o(n^2)$

If $n^2 \neq o(n^2)$, it implies $n^2 = \Omega(n^2)$

$0 \leq n^2 \leq cn^2$ with positive constant $c < 0$ and $n_0 < n$.

We can already see that $c = 1$ fulfills for all n

Because $0 \leq n^2 \leq n^2$

4. Divide and conquer

$$T(n) = 3T(\frac{n}{2}) + \Theta(n)$$

Master theorem method

$$T(n) = 3T(\frac{n}{2}) + \Theta(n)$$

Master method
The different cases are:

1. If $f(n) = O(n^{\log_b a-\epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
    - Here, a, b are constants in the recurrence $T(n) = aT(\frac{n}{b}) + f(n)$, and ϵ is a positive constant.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.
    - This case applies when $f(n)$ grows at the same rate as $n^{\log_b a}$.
3. If $f(n) = \Omega(n^{\log_b a+\epsilon})$ for some constant $\epsilon > 0$, and if $af(\frac{n}{b}) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.
    - This case is used when $f(n)$ grows faster than $n^{\log_b a}$.

For our case:
$a = 3, b = 2, f(n) = \Theta(n)$
We have the following

- $n^{log_b a} = n^{log_2 3} \approx n^{1.58}$ since $log_2 3 \approx 1.58$
- $f(n) = \Theta(n^1)$

Since $1.58 - 1 > 0$ we can say that there exists a constant $\epsilon > 0$ such that $f(n) = O(n^{log_b a-\epsilon})$

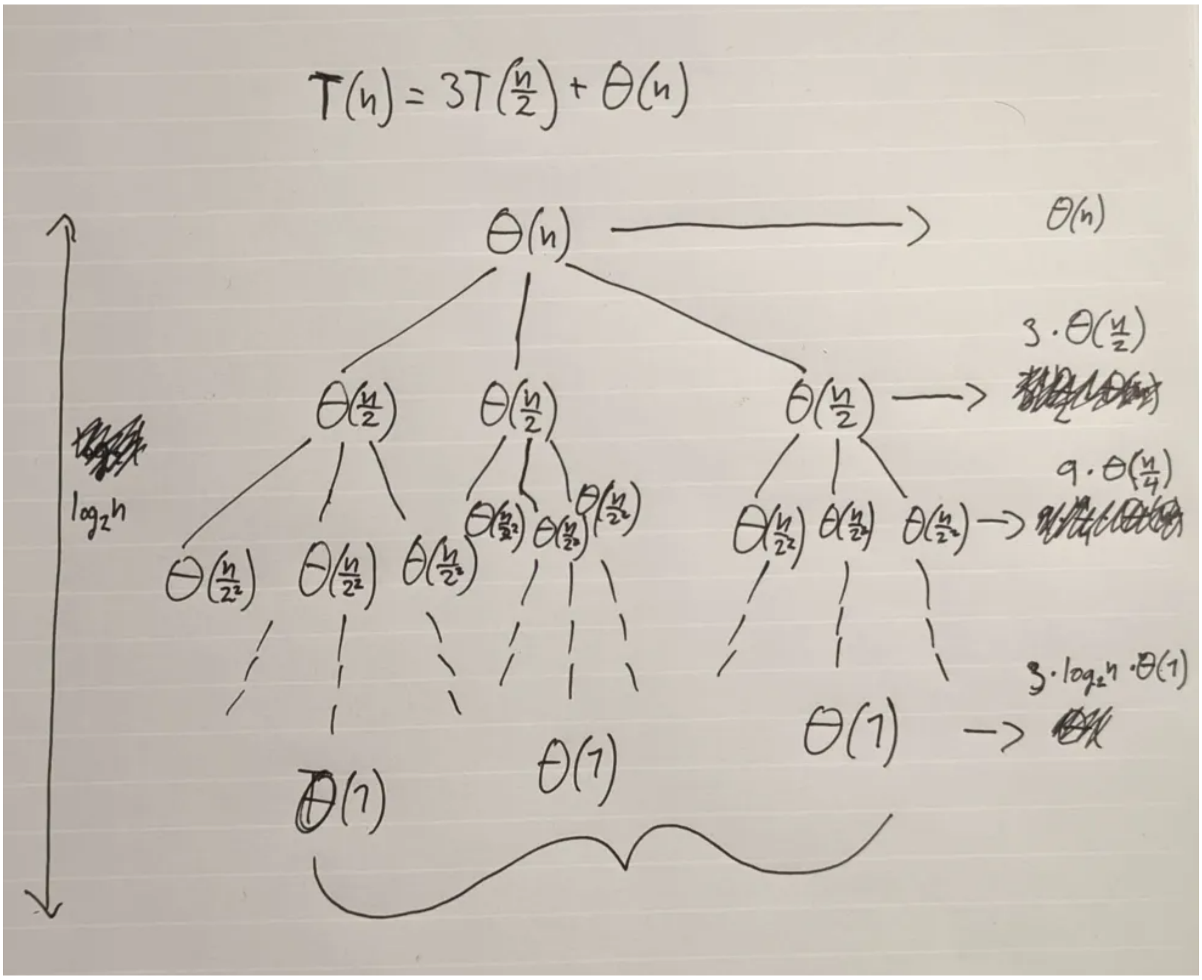
Therefore we are in case 1.

This gives us:

$T(n) = \Theta(n^{log_2 3})$

Recursion tree method

To use the visual method of the recursion tree we split the algorithm into a constant part and a recursive part. The constant part is $\Theta(n)$ which we set to cn (where c is a positive constant), and the recursive part is $3T(\frac{n}{2})$. For each recursive call, tree branches out into three branches, each with a size of $\frac{n}{2^{level}}$ assuming the root node is level 0. The depth of the tree is $\log_2 n$ since the size of the tree is halved for each level. The tree is shown [here](#).



The totals calculated to the right of the tree are the sum of the nodes per level.

The last row of the tree will have $3^{log_2 n} = n^{log_2 3}$ subproblems of size $T(1)$, which yields $\Theta(n^{log_2 3})$

The sums of each level are defined as $(3/2)^{\hat{\imath}} * cn$

Taking the sum of each row gives the function:

$$T(n) = \sum_{i=0}^{\log_2 n-1} (3/2)^i * cn + \Theta(n^{log_2 3})$$

The function tells us that $\Theta(n^{log_2 3})$ overpowers the other part of the function as it is of higher order ($log_2 3 > 1$), so $T(n) = O(n^{log_2 3})$ seems likely.

Testing $O(n^{log_2 3})$:
We must show that $T(n) \leq dn^{log_2 3}$, d being a positive constant

$$\begin{aligned}
T(n) &\leq 3T(n/2)^{log_2 3} + cn \\
T(n) &\leq 3d(n/2)^{log_2 3} + cn \\
T(n) &\leq \frac{3dn^{log_2 3}}{2^{log_2 3}} + cn \\
T(n) &\leq dn^{log_2 3} + cn
\end{aligned}$$

As shown above there is still cn, but due to $dn^{log_2 3}$ being of higher order, cn will be overpowered for large enough n as $dn^{log_2 3} + cn = O(n^{lo_g 23})$

This is the same result as the master theorem method.

Code

Following the algorithms from *Introduction to Algorithms* we implemented the following algorithms in Python (and Go). The go code is mostly a 1:1 translation from the python code, with the exception of some list slicing.

Python

Insertion sort

Derived from **INSERTION-SORT** pseudocode in *Introduction to Algorithms* p.18.

```python
def insertion_sort(items: list[int]) -> list[int]:
    """Simple insertion sort algorithm

    Args:
        items: list of ints to be sorted.

    Returns:
        items: list of ints sorted in ascending order.
    """
    for i in range(1, len(items)):
        val = items[i]
        j = i - 1
        while j >= 0 and items[j] > val:
            items[j + 1] = items[j]
            j -= 1
        items[j + 1] = val
    return items
```

Merge sort

Derived from **MERGE-SORT** p.34 and **MERGE** p.31 pseudocode in *Introduction to Algorithms*.

```python

def merge_sort(items: list[int], left: int, right: int) -> list[int]:
    """Merge sort algorithm.

    Args:
        items: List of items to sort.
        left: Left index of items.
        right: Right index of items.

    Returns:
        List of sorted items.
    """

    def merge(items: list[int], left: int, middle: int, right: int) -> list[int]:
        """Merges two sorted lists into one sorted list.

        Args:
            items: List of items to sort.
            left: Index of leftmost item.
            middle: Index of middle item.
```

```
            right: Index of rightmost item.

        Returns:
            Merged sorted list of items.
        """
        n1 = middle - left + 1
        n2 = right - middle
        L = [items[left + i] for i in range(n1)]
        R = [items[middle + i + 1] for i in range(n2)]
        i, j, k = 0, 0, left
        while i < n1 and j < n2:
            if L[i] <= R[j]:
                items[k] = L[i]
                i += 1
            else:
                items[k] = R[j]
                j += 1
            k += 1
        rest = L[i:] if i < n1 else R[j:]
        items[k : k + len(rest)] = rest

    if left < right:
        middle = (left + right) // 2

        merge_sort(items, left, middle)
        merge_sort(items, middle + 1, right)
        merge(items, left, middle, right)
```

Heapsort

Derived from **HEAPSORT** p.160, **MAX-HEAPIFY** p.154, and **BUILD-MAX-HEAP** p.157 pseudocode in *Introduction to Algorithms*.

```
def heapsort(items: list[int]) -> list[int]:

    def max_heapify(items: list[int], heapsize: int, i: int) -> None:
        left = lambda: 2 * i + 1
        right = lambda: 2 * i + 2
        largest = i
        if left() < heapsize and items[left()] > items[i]:
            largest = left()
        else:
            largest = i

        if right() < heapsize and items[right()] > items[largest]:
            largest = right()

        if largest != i:
            items[i], items[largest] = items[largest], items[i]
            max_heapify(items, heapsize, largest)

    def build_max_heap(items: list[int], heapsize: int) -> None:
        for i in range(heapsize // 2 - 1, -1, -1):
            max_heapify(items, heapsize, i)
```

```
    build_max_heap(items, len(items))

    for i in range(len(items) - 1, 0, -1):
        items[i], items[0] = items[0], items[i]
        max_heapify(items, i, 0)
    return items
```

Quicksort

Derived from **QUICKSORT** and **PARTITION** pseudocode in *Introduction to Algorithms* p.171.

```
def quicksort(items: list[int], left: int, right: int) -> None:
    def partition(items: list[int], left: int, right: int) -> int:
        x = items[right]
        i = left - 1
        for j in range(left, right):
            if items[j] <= x:
                i += 1
                items[i], items[j] = items[j], items[i]
        items[i + 1], items[right] = items[right], items[i + 1]
        return i + 1

    if left < right:
        split = partition(items, left, right)
        quicksort(items, left, split - 1)
        quicksort(items, split + 1, right)
```

Go

```
func insertionSort(items []int) {
    for i := 1; i < len(items); i++ {
        key := items[i]
        j := i - 1
        for j >= 0 && items[j] > key {
            items[j+1] = items[j]
            j--
        }
        items[j+1] = key
    }

}

func merge(items []int, left int, middle int, right int) {
    n1 := middle - left + 1
    n2 := right - middle
    L := make([]int, n1)
    R := make([]int, n2)
    for i := 0; i < n1; i++ {
        L[i] = items[left+i]
    }
    for j := 0; j < n2; j++ {
        R[j] = items[middle+j+1]
    }
```

```go
    i, j, k := 0, 0, left
    for i < n1 && j < n2 {
        if L[i] <= R[j] {
            items[k] = L[i]
            i++
        } else {
            items[k] = R[j]
            j++
        }
        k++
    }
    if i < n1 {
        for i < n1 {
            items[k] = L[i]
            i++
            k++
        }
    } else {
        for j < n2 {
            items[k] = R[j]
            j++
            k++
        }
    }
}

func mergeSort(items []int, left int, right int) {
    if left < right {
        middle := (left + right) / 2
        mergeSort(items, left, middle)
        mergeSort(items, middle+1, right)
        merge(items, left, middle, right)
    }

}

func quickSort(items []int, left int, right int) {
    if left < right {
        split := partition(items, left, right)
        quickSort(items, left, split-1)
        quickSort(items, split+1, right)
    }
}
func partition(items []int, left int, right int) int {
    x := items[right]
    i := left - 1
    for j := left; j >= left && j < right; j++ {
        if items[j] <= x {
            i++
            items[i], items[j] = items[j], items[i]
        }
    }
    items[i+1], items[right] = items[right], items[i+1]
    return i + 1
}
```

```go
func maxHeapify(items []int, heapsize int, i int) {
    left := func(i int) int {
        return 2*i + 1
    }
    right := func(i int) int {
        return 2*i + 2
    }
    largest := i
    if left(i) < heapsize && items[left(i)] > items[i] {
        largest = left(i)
    }
    if right(i) < heapsize && items[right(i)] > items[largest] {
        largest = right(i)
    }
    if largest != i {
        items[i], items[largest] = items[largest], items[i]
        maxHeapify(items, heapsize, largest)
    }
}

func buildMaxHeap(items []int, heapsize int) {
    for i := heapsize / 2; i >= 0; i-- {
        maxHeapify(items, heapsize, i)
    }
}

func heapSort(items []int) {
    buildMaxHeap(items, len(items))
    for i := len(items) - 1; i >= 0; i-- {
        items[i], items[0] = items[0], items[i]
        maxHeapify(items, i, 0)
    }
}
```