Ch5,6 Review

Thursday, November 12, 2020 9:19 AM

Topics

- Coordinate systems
- Vectors
- Circles
- Ellipses
- Parabolas
- Completing the square

Coordinate Systems

Rectangular/Cartesian coordinates: (x,y)

Polar coordinates: (θ, r)

Rectangular $\mapsto$ Polar: Find θ, r from x, y

$r = \sqrt{x^2+y^2}$, $\tan\theta = \frac{y}{x}$

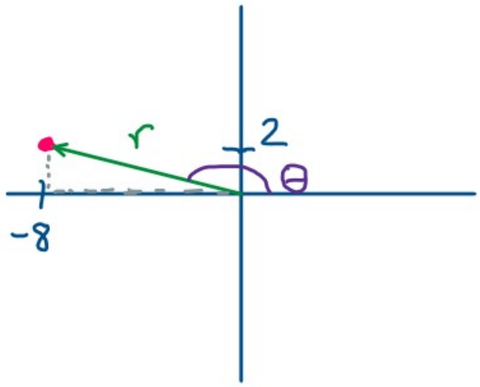
Note: $\tan^{-1}$ always gives an angle in QI or QIV. If your point in QII or QIII, find your θ by adding 180° to the angle.

Polar $\mapsto$ Rectangular: Find x, y from θ, r.

$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$

Ex.: Convert $(-8,2)$ to polar coordinates



$$r = \sqrt{(-8)^2 + (2)^2}$$

$$r = \sqrt{64 + 4}$$

$$r = \sqrt{68}$$

$$\boxed{r = 2\sqrt{17}}$$

$$\tan\theta = \frac{2}{-8} = -\frac{1}{4}$$

$$\Rightarrow \theta = \tan^{-1}\left(-\frac{1}{4}\right) \approx -14.04^\circ$$

However, our point is clearly in the 2nd quadrant.

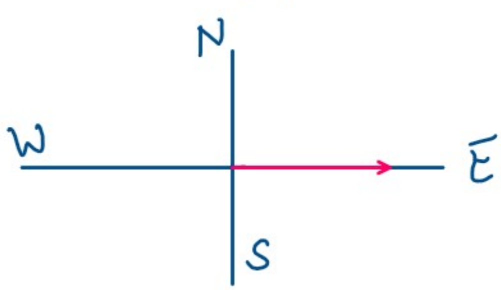
$$\Rightarrow \theta \approx 180^\circ + (-14.04^\circ) = \boxed{165.96^\circ}$$

Vectors

Vectors have magnitude and direction. They look like directed line segments on a graph.

Q: If someone is travelling 5 mph, can this be represented as a vector?

A: No. Need direction in addition to magnitude. 5 mph, east is a vector.



You may see vectors in any of these 3 forms:

i) Rectangular Form

$\vec{v} = \langle a, b \rangle$ where $a = \|\vec{v}\| \cos\theta$,
$b = \|\vec{v}\| \sin\theta$

ii) Polar Form

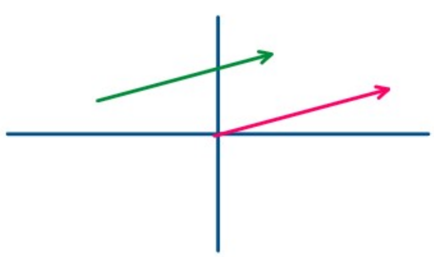
Magnitude/Norm/Length: $\|\vec{v}\| = \sqrt{a^2+b^2}$

$\tan\theta = \frac{y}{x}$

iii) Basis form

$\vec{v} = a\vec{i} + b\vec{j}$ where $\vec{i} = \langle 1, 0 \rangle$,
$\vec{j} = \langle 0, 1 \rangle$

Adding and Subtracting Vectors



$\vec{v}_1 = \langle 6, 2 \rangle$

$\vec{v}_2 = \langle 6, 2 \rangle$

$\boxed{\vec{v}_1 = \vec{v}_2}$

In other words, these are the same vector.

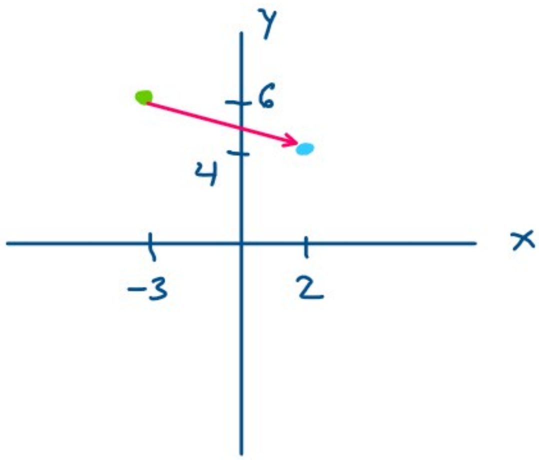
Takeaway: A vector is the journey, not the destination.

If we're given the starting and ending points of a vector, we can write the vector in Rectangular form.

$$\left.\begin{array}{l} \text{Start}: (x_1, y_1) \\ \text{End}: (x_2, y_2) \end{array}\right\} \Rightarrow \boxed{\vec{v} = \langle x_2 - x_1, y_2 - y_1 \rangle}$$

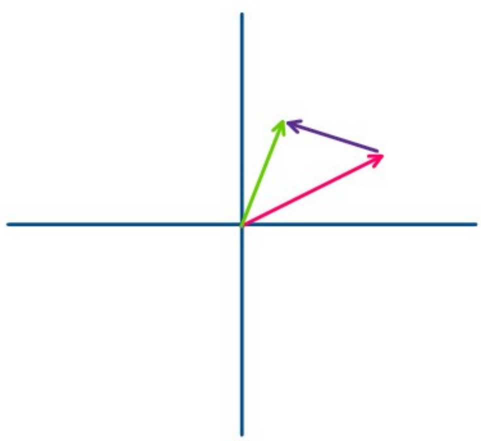
Ex.: $\vec{v}$ starts at $(-3, 6)$, ends at $(2, 4)$

$$\vec{v} = \langle 2-(-3), 4-6 \rangle = \langle 5, -2 \rangle$$



Adding Vectors

$\vec{v}_1 + \vec{v}_2$ geometrically can be seen as



$\vec{v}$ is the resulting vector from adding $\vec{v}_1$ and $\vec{v}_2$. Essentially, start the next vector where the previous one ends.

Q: Does order matter when adding vectors?

A: No. Try an example to see for yourself!

Subtracting Vectors

$\vec{v}_1 - \vec{v}_2$ is equivalent to $\vec{v}_1 + (-\vec{v}_2)$

The opposite (or negative) of a vector keeps the magnitude but flips the direction.

Ex.: $\vec{u} = \langle u_1, u_2 \rangle$, $\vec{v} = \langle v_1, v_2 \rangle$

$$\boxed{\begin{aligned} \vec{u} + \vec{v} &= \langle u_1 + v_1,\ u_2 + v_2 \rangle \\ \vec{u} - \vec{v} &= \langle u_1 - v_1,\ u_2 - v_2 \rangle \end{aligned}}$$

Scaling Vectors

We can increase and decrease the size of a vector by multiplying each coordinate by a scalar.

$\vec{v} = \langle v_1, v_2 \rangle$, $\boxed{k\vec{v} = \langle kv_1, kv_2 \rangle}$

Applications of Vectors

In this class, we will apply vectors to problems related to forces and navigation but vectors appear in many other problems in math and science.

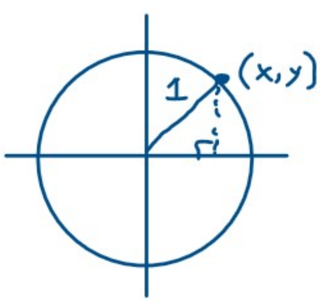
In our problems, you'll typically be given vectors in polar form and asked to add or subtract them.

Steps:
1) Convert to rectangular coordinates
2) Add or subtract
3) Convert back to polar coordinates

Activity: A boat travels 30 km in the direction of 54°, then turns and travels 13 km in the direction of 27°. How far is the boat from its initial location and at what bearing (angle)?

Circles

Like many topics in this course, this relates to the Unit Circle.



Any point on the Unit Circle can be written as

$x^2 + y^2 = 1^2$ by Pythag. Thrm.

Now, generalize this to circles of a different size. We could say

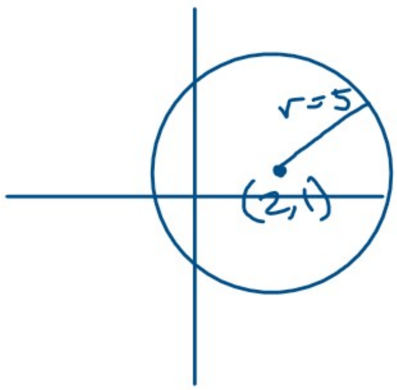
$x^2 + y^2 = r^2$ where r is the radius of the circle (or hypotenuse of any triangle from the center of the circle to a point on the circle)

Lastly, what if the center is not at the origin?
Calculate each leg as distance travelled in x, y direction.

A circle with center at (h,k) and radius r has the equation

$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$

<u>Ex.</u>: Radius 5, center at $(2,1)$



$$(x-2)^2 + (y-1)^2 = 25$$

where (x,y) are any point on the circle

Ellipses

Similar to circles but stretched/compressed unequally.

<u>Circle at origin:</u> $x^2 + y^2 = r^2$

$$\Rightarrow \frac{x^2 + y^2}{r^2} = 1$$

$$\Rightarrow \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1 \quad (*)$$

Recall from transformations that multiplying/dividing inside parentheses compresses/widens

<u>Ex.</u>: $\sin(5x)$ reduces period from $2\pi \mapsto \underline{\underline{2\pi}}$

$\underline{Ex.}$: $\sin(5x)$ reduces period from $2\pi \mapsto \frac{2\pi}{5}$

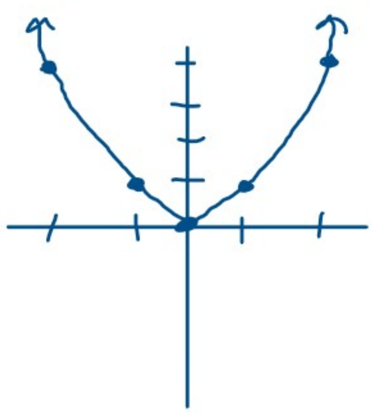
Therefore ⊛ is taking a unit circle and stretching it wider (x) and taller (y) by a factor of r.

Standard form of ellipse

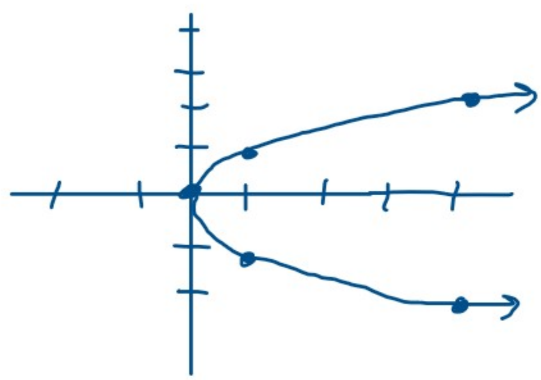
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Parabolas

Standard Parabola : $y = x^2$



$x = y^2$



Recall our transformations :

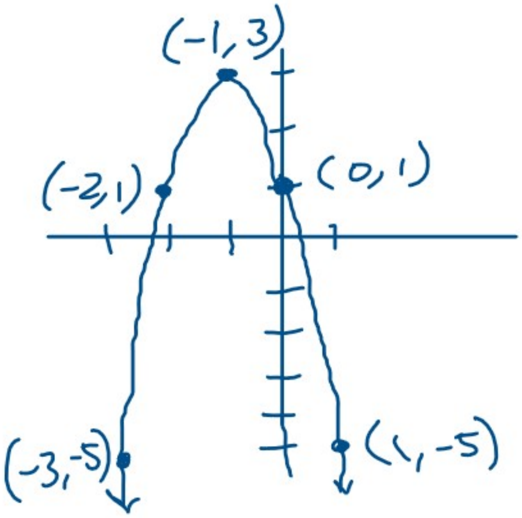
$$f(x) \longmapsto Af(Bx+C)+D$$

Activity: In your own words, how do A, B, C, D change $f(x)$?

General form of a parabola:

$$y - k = A(x-h)^2 \text{ where vertex is } (h,k)$$

Ex.: Graph $y = -2(x+1)^2 + 3$



Completing the Square

Standard forms of circles, ellipses and parabolas have no extra linear terms outside the parentheses.
Sometimes, given the equation in a messier form.
Completing the square helps us get to this "nice" form from messy form.

Steps:

i) Rearrange terms

ii) Factor out coefficients on x^2, y^2

iii) Determine perfect squares

iv) Supply the missing constant to both sides

v) Write in standard form

Ex.: Write the parabola below in standard form

$$y + 2x^2 = 10x - 22$$

Goal: $y - k = A(x-h)^2$

i) $y + 22 = -2x^2 + 10x$

ii) $y + 22 = -2(x^2 - 5x)$

iii) $y + 22 + c = -2(x - 5/2)(x - 5/2)$

iv) $y + 22 + \left(-\frac{25}{2}\right) = -2(x - 5/2)^2$

v) $y + \frac{19}{2} = -2(x - 5/2)^2$

Vertex: $(5/2, -19/2)$

Problems

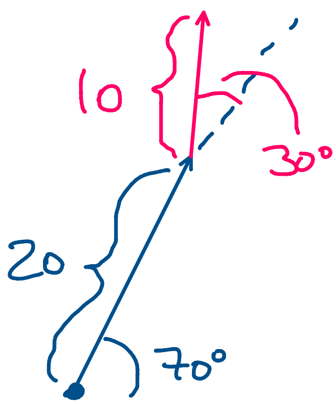
Problems

1) A vector, $\vec{v} = \langle v_1, v_2 \rangle$, has magnitude $\|\vec{v}\|$ and direction θ_v. For $k > 0$, what is the magnitude and direction of $k\vec{v}$?

2) Write vector $\vec{u} = \langle -2, -4 \rangle$ in polar, rectangular, and basis form.

3) Find c so that $3x^2 - 18x + c$ is a perfect square which will factor to $a(x+p)^2$. What are a, c, p?

4) A boat travels 20 miles in the direction of $70°$. It then turns counter-clockwise $30°$ and travels for another 10 miles. How far is the boat from its starting point and at what direction (angle)?



Credit to Kayla for the great question. See Kayla's review notes for the solution.