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Inverse problem of non-homogeneous residual stress identification in thin plates

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ABSTRACT

Vibration of solid bodies with residual stresses has been attracting attention of researchers from different countries for a long time. Problems of residual stress analysis have its applications in fields of building, mechanical engineering, aircraft construction, biomechanics, manufacturing of composite and functionally-gradient materials. The most common model of residual stresses (or prestresses) is the homogeneous prestress state model; however, in fact the prestress state is often non-homogeneous under natural conditions. One of the most powerful nondestructive methods of reconstruction of non-homogeneous prestress state is the acoustical method.

In the present paper the direct problem formulations for 3D bodies and thin plates with non-homogeneous prestress fields are described. A formulation of the inverse problem on a reconstruction of non-homogeneous prestress state is given on the basis of acoustical method. The problem is reduced to the iterative process; at each step of the latter the direct problem and the integral Fredholm equation of the first kind are solved. Two ways of obtaining operator equations of the inverse problem are presented for two oscillation regimes – in-plane and out-of-plane plate vibration modes. Numerical results of solving the inverse problem on a reconstruction of the uniaxial prestress function in case of out-of-plane vibration for a thin rectangular plate are presented. Features and characteristics of the solutions obtained are revealed; the most auspicious conditions for a better quality of the identification procedure are pointed out.

1. Introduction

Stresses that exist in a solid without an application of any external (force or thermal) actions are termed residual stresses (or prestresses). Such stresses are usually the results of welding, heat treatment, rolling-and-pressing procedures and other manufacturing processes. Also the residual stresses can arise from rigid connection of differing materials in contact area (Birger, 1963; Chernishov et al., 1996).

At present the development and perfection of prestress identification techniques is important task of solid mechanics. This is due to the fact that the presence of concentrators or considerable heterogeneities of residual stresses in a construction may cause significant loss of durability or even a destruction of the construction. A distinguishing feature of the residual stress is that its presence is not evident until a failure or a breakage occurs. On the other hand, in special cases some types of residual stress state in the construction may enhance its reliability and durability during its exploitation. For this reason, sometimes the construction is advisedly exposed to the residual stresses at a fabrication stage (for example, prestressed concrete).

Investigations on prestress identification using semi-destructive methods have been actively conducted until now. Most of such methods are based on physical intrusion of an indentor inside a body or even on its partial destruction. Among these methods there are multifarious variants of the hole-drilling method, the crack compliance method and others (Prime, 1999; Lee et al., 2004; Švaříček and Vlk, 2007; Moharami and Sattari-Far, 2008). In addition to the disadvantage of destructive peculiarity, one more shortcoming of such methods is physical limitation of setting sensors, strain indicators or indentors nearby induced deformations.

However, over the last two decades a tendency of the prestress identification techniques to shift away from semi-destructive methods towards promising non-destructive methods has been revealing itself. The most widespread non-destructive methods are the X-ray diffraction method (Larsson and Odén, 2004; Shiro et al., 2008), the acoustical method (Vatulyan, 2007b; Tovstik, 2009; He and Kobayashi, 2001; Guz, 2005; Sathish et al., 2005) and the ultrasound method (Karabutov et al., 2008; Vangi, 2001; Sanderson and Shen, 2010; Uzun and Bilge, 2011), the magneto-acoustical emission (the magnetic Barkhausen noise technique) (O'Sullivan et al., 2004; Yelbay et al., 2010), and the holographic technique (Chernishov et al., 1996). Also it is worth noting the eigenstrain method (Cao et al., 2002; Jun and Korsunsky, 2010; Korsunsky et al., 2006) and different mixed methods including

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several techniques of residual stress identification. The new method of nano-indenting has been developed quite recently to investigate residual stresses in near-surface layers of a body (Zhu et al., 2010; Dean et al., 2011). Besides experimental identification techniques, the analytical approaches of residual stress identification have been also developed (Schajer and Prime, 2006; Farrahi et al., 2009; Isakov et al., 2003; Xiong et al., 2009).

The X-ray diffraction methods are generally recognized as the most precise methods of measuring stresses near to the surface. These methods are commonly used for checking other techniques. But X-ray methods also have a number of disadvantages (Prime, 1999; Walker, 2001). One of the most powerful nondestructive methods of a reconstruction of non-homogeneous prestress state is the acoustical method. Its main advantages over the others methods are its inexpensiveness (for instance, in comparison with X-ray methods), the operations efficiency and its applicability to various materials.

It should be noted that mostly the homogeneous prestress state model is used (Tovstik, 2009; Guz, 2005). However, the prestress state is often non-homogeneous in reality, and components of the prestress tensor depend on coordinates, especially near stress concentrators such as cavities, cracks, insertions, weld seams and so on. At present there are many studies devoted to the prestress effect in areas of cracks and welds and to the methods on a prevention of defect growth (Lammi and Lados, 2011; Ihara et al., 2011; Lee and Chang, 2011). One more important area of research closely related to the previous one is an effect of shot peening process and laser shock peening in defect regions on prestress field in surface layers of a sample (Liu et al., 2011; Mylonas and Labeas, 2011; Brockman et al., 2012).

In the present paper the theoretical research on opportunities of non-homogeneous prestress field reconstruction in thin elastic isotropic plates oscillating in in-plane and out-of-plane regimes is conducted using the acoustical method. The prestress field is characterized by the Piola stress tensor according to the linearized model described in (Guz, 2002). Weak and strong formulations of direct problems for both vibration regimes are given. On the basis of linearization method the inverse problem is reduced to the iterative process (Vatulyan, 2010a). At every step of this process the direct problem and the Fredholm integral equation of the first kind with continuous kernel are solved.

The numerical calculations of solving the direct and inverse problems were made using the Finite Element Method in the package "FreeFem++" (Hecht et al., 2009) and the Tikhonov regularization procedure in the programming language "Fortran". The series of numerical experiments on an identification of smooth laws of uniaxial prestress state in rectangular plate is conducted. The most auspicious loading regimes and frequency ranges for the identification procedure are given.

2. Direct problem

2.1. Formulation of steady-state vibration problem for 3D prestressed body

Consider steady-state vibration of elastic isotropic prestressed body with the boundary $S = S_u \cup S_\sigma$ which is clamped at the boundary part S_u and acted by the loading oscillating with the frequency ω at the boundary part S_σ. The linearized formulation of the boundary problem is given by

$$\begin{cases} T_{ij,j} + \rho\omega^2 u_i = 0, \\ T_{ij} = \sigma_{ij} + u_{i,m}\sigma^0_{mj}, \\ \sigma_{ij} = C_{ijkl}u_{k,l}, \\ u_i|_{S_u} = 0, \quad T_{ij}n_j|_{S_\sigma} = P_i, \end{cases} \tag{1}$$

where u_i – components of displacement vector, T_{ij} – components of the non-symmetric Piola stress tensor,[1] σ^0_{ij} – components of symmetric prestress tensor, C_{ijkl} – components of elasticity tensor, ε_{ij} – components of linear deformation tensor, $i,j = 1,2,3$. The components of the residual stress tensor satisfy equilibrium equations, i.e. $\sigma^0_{ij,j} = 0$.

2.2. Formulation of steady-state vibration problem for thin prestressed plate

Let us view steady-state vibration of thin elastic isotropic plate of volume V, boundary ∂V and thickness $2h$, with arbitrary section outline. Denote a plane section in the middle level by S. Regarding $\partial S = l = l_u \cup l_\sigma$, the plate is clamped on the lateral surface part $l_u \times [-h,h]$ and it is acted by the loading oscillating with the frequency ω at $l_\sigma \times [-h,h]$ (Fig. 2).

The plate contains non-homogeneous 2D prestress field, i.e. only three components of the symmetric prestress tensor are nonzero:

$$\sigma^0_{\alpha\beta}(x_1,x_2) \neq 0, \quad \alpha,\beta = 1,2. \tag{2}$$

2.2.1. In-plane vibration

The problem of in-plane vibration of prestressed thin plate is given by the set which is similar to the formulation of the problem for 3D body (1), with the only difference that the indexes i,j take on values just 1 or 2; instead of the third relation of the set the following relation takes place:

$$\sigma_{ij} = \lambda^* \delta_{ij}u_{k,k} + 2\mu\varepsilon_{ij}, \tag{3}$$

at that we consider plane stress state, i.e. $\lambda^* = \frac{2\lambda\mu}{\lambda+2\mu}$ (next, the "star" will be omitted).

The weak formulation of the problem (1) is given by

$$\int_{l_\sigma} P_i v_i dl_\sigma - \int_S (\lambda u_{i,i} v_{j,j} + 2\mu\varepsilon_{ij}(u_k)\varepsilon_{ij}(v_k) + u_{i,m}\sigma^0_{mj} v_{i,j} - \omega^2\rho u_i v_i) dS = 0, \tag{4}$$

where v_i are test functions satisfying the same essential conditions as the functions u_i do (Vatulyan, 2008).

2.2.2. Out-of-plane vibration

Let us take on the Kirchhoff–Love hypotheses for thin plates; we neglect the stress component σ_{33}, and consider that all the components of the displacement vector depend on one function $w = w(x_1,x_2)$ (Timoshenko and Woinowsky-Krieger, 1959):

$$u_1 = -x_3 w_{,1}; \quad u_2 = -x_3 w_{,2}; \quad u_3 = w. \tag{5}$$

In the paper (Nedin and Vatulyan, 2011b) the proper boundary problem formulation is derived using the hypotheses (5) and (2).

The stationary vibration of the plate is caused by periodic forces with intensity $qe^{i\omega t}$, directed transversely to the plate plane and applied to the boundary part l_σ. The problem formulation then takes form of the following boundary problem:

$$\begin{cases} D\Delta^2 w + \frac{2}{3}h^3(\sigma^0_{m\beta}w_{,\alpha m})_{,\alpha\beta} - 2h\sigma^0_{\alpha\beta}w_{,\alpha\beta} \\ \quad -2h\rho\omega^2 w = 0, \\ w|_{l_u} = 0, \quad \frac{\partial w}{\partial n}\Big|_{l_u} = 0, \quad G_t|_{l_\sigma} = 0, \\ \left[\frac{\partial}{\partial \tau}(H_t) - N_t\right]\Big|_{l_\sigma} = q. \end{cases} \tag{6}$$

[1] This formula is based on the model of residual stress introduced by Guz (2002). It is worth noting another models commonly used by researchers (see Truesdell, 1972; Hoger, 1986; Robertson, 1998).

where $D = \frac{2Eh^3}{3(1-\nu^2)}$ – cylindrical rigidity of plate, $G_t = \frac{2}{3}h^3\tilde{T}_{\alpha\beta}n_\alpha n_\beta$ – bending moment at the boundary l, $H_t = \frac{2}{3}h^3(\tilde{T}_{2\beta}n_\beta n_1 - \tilde{T}_{1\beta}n_\beta n_2)$ – twisting moment at l, $N_t = -D\frac{\partial(\Delta w)}{\partial n} - \frac{2}{3}h^3(\sigma^0_{m\beta}w_{,\alpha m})_{,\beta}n_\alpha + 2h\sigma^0_{\alpha\beta}n_\alpha w_{,\beta}$ – shear force at l, $\widetilde{T}_{\alpha\beta} = \lambda^*\delta_{\alpha\beta}w_{,kk} + 2\mu w_{,\alpha\beta} + w_{,\alpha m}\sigma^0_{m\beta}$ $(\alpha,\beta,k,m = 1,2)$, n_1 and n_2 – components of the vector of normal to the boundary l, $\frac{\partial}{\partial n}$ and $\frac{\partial}{\partial \tau}$ – normal and tangentional derivatives to l relatively. Here we neglect the inertia of rotation in the equation and in the last boundary condition.

The weak formulation of the problem (6) is given by

$$\int_S \Big[Dw_{,\alpha\beta}v_{,\alpha\beta} - \frac{2}{3}h^3\rho\omega^2\nabla w\cdot\nabla v + \frac{2}{3}h^3\sigma^0_{m\beta}w_{,\alpha m}v_{,\alpha\beta} + 2h\sigma^0_{\alpha\beta}w_{,\alpha}v_{,\beta} - 2h\rho\omega^2 wv\Big]dS - \int_{l_\sigma} qv\,dl_\sigma = 0, \tag{7}$$

where v is the test function satisfying the same essential conditions as the function w does.

2.3. Direct problem solving

Solving, testing and analysis of the direct problems of in-plane and out-of-plane vibrations of thin prestressed plate are given in Nedin and Vatulyan (2011a,b) in detail.

In both vibration cases the numerical solutions were obtained using FEM in the package "FreeFem++". To estimate the accuracy of FEM solutions, a comparison of the problem solution for a band plate (section shape is oblong rectangular with relation of sides $a/b > 10$) with the analytical solution of the problem in case of homogeneous characteristics (the material parameters, the density, the prestress tensor components) is made. The comparison of the numerical and theoretical solutions showed high accuracy of the FEM results obtained.

Also an analysis of the prestress level influence on the frequency response function was made for rectangular plate for the first and the second frequency ranges. A difference of frequency response functions turned out to be sufficient enough to employ the prestress reconstruction procedure; the discrepancy was most significant near the resonant frequencies.

3. Inverse problem

3.1. Formulation of inverse problem on non-homogeneous prestress identification in 3D body

Consider a solid body with characteristics which are the same as those given in Section 2.1 and containing non-homogeneous prestress field described by the components $\sigma^0_{ij}(x_1,x_2,x_3)$ of the prestress tensor.

The inverse problem on an identification of non-homogeneous prestresses in the body is given by the set (1). Now the unknown functions are $\sigma^0_{ij}(x_1,x_2,x_3)$; the known functions are the material parameters and the loading.

Also assume that some additional information on the displacement field is given. Note, that the most popular ways of its assignment: setting the displacement field u_i inside the area for a fixed frequency, or setting the displacement field u_i at the boundary part for a limited number of frequencies $\omega_k \in [\omega_-,\omega_+]$. The inverse problem is linear in the first case, and it's nonlinear in the second one (Vatulyan, 2010b, 2007b). Let us consider the second way of the additional information assignment:

$$\{u_i|_{S_\sigma}\}_k, \omega_k \in [\omega_-,\omega_+], \quad k = \overline{1,m}.$$

The problem is to identify the functions σ^0_{ij} on the basis of the information given.

3.2. Derivation of inverse problem equation

3.2.1. In-plane vibration

Let us view the inverse problem on prestress identification for a plate in case of in-plane vibration. In this section we use one of the methods of derivation of the operator equation for inverse problem solving based on weak formulation of direct problem.

For that let us take the weak formulation (4) of the direct problem (1):

$$A(\underline{\underline{\sigma}}^0,\underline{u},\underline{v}) - B(\underline{v}) = 0,$$

where the corresponding bilinear and linear forms are

$$A(\underline{\underline{\sigma}}^0,\underline{u},\underline{v}) = \int_S (\lambda u_{i,i}v_{j,j} + 2\mu\varepsilon_{ij}(u_k)\varepsilon_{ij}(v_k) + u_{i,m}\sigma^0_{mj}v_{i,j} - \omega^2\rho u_i v_i)dS,$$

$$B(\underline{v}) = \int_{l_\sigma} P_i v_i dl_\sigma.$$

Let us fix two states (conditions) corresponding to the following sets of characteristics: (1) $u_i^{(1)}$, $\sigma_{ij}^{0(1)}$; (2) $u_i^{(2)}$, $\sigma_{ij}^{0(2)}$.

Consider the equality

$$A(\underline{\underline{\sigma}}^{0(1)},\underline{u}^{(1)},\underline{u}^{(2)}) - A(\underline{\underline{\sigma}}^{0(2)},\underline{u}^{(2)},\underline{u}^{(1)}) = B(\underline{u}^{(2)}) - B(\underline{u}^{(1)}). \tag{8}$$

After reorganizing the left-hand side of (8) we have:

$$\begin{aligned}
&A(\underline{\underline{\sigma}}^{0(1)},\underline{u}^{(1)},\underline{u}^{(2)}) - A(\underline{\underline{\sigma}}^{0(2)},\underline{u}^{(2)},\underline{u}^{(1)})\\
&= \int_S (\lambda u^{(1)}_{i,i}u^{(2)}_{j,j} + 2\mu\varepsilon_{ij}(u^{(1)}_k)\varepsilon_{ij}(u^{(2)}_k) + u^{(1)}_{i,m}\sigma^{0(1)}_{mj}u^{(2)}_{i,j} - \omega^2\rho u^{(1)}_i u^{(2)}_i)dS\\
&\quad - \int_S (\lambda u^{(2)}_{i,i}u^{(1)}_{j,j} + 2\mu\varepsilon_{ij}(u^{(2)}_k)\varepsilon_{ij}(u^{(1)}_k) + u^{(2)}_{i,m}\sigma^{0(2)}_{mj}u^{(1)}_{i,j}\\
&\quad - \omega^2\rho u^{(2)}_i u^{(1)}_i)dS\\
&= \int_S (u^{(1)}_{i,m}\sigma^{0(1)}_{mj}u^{(2)}_{i,j} - u^{(2)}_{i,m}\sigma^{0(2)}_{mj}u^{(1)}_{i,j})dS.
\end{aligned}$$

The right-hand side of (8) takes form:

$$B(\underline{u}^{(2)}) - B(\underline{u}^{(1)}) = \int_{l_\sigma} P_i(u^{(2)}_i - u^{(1)}_i)dl_\sigma.$$

Therefore, the equality (8) becomes

$$\int_S (u^{(1)}_{i,m}\sigma^{0(1)}_{mj}u^{(2)}_{i,j} - u^{(2)}_{i,m}\sigma^{0(2)}_{mj}u^{(1)}_{i,j})dS = \int_{l_\sigma} P_i(u^{(2)}_i - u^{(1)}_i)dl_\sigma. \tag{9}$$

Let us use the linearization method in the following form:

$$\sigma^{0(2)}_{ij} = \sigma^{0(1)}_{ij} + \delta\sigma^0_{ij}, \quad u^{(2)}_{i,j} = u^{(1)}_{i,j} + \delta u_{i,j}, \tag{10}$$

where $\delta\sigma^0_{ij}$, $\delta u_{i,j}$ are corrections to the corresponding functions. Substituting the expressions (10) into the integrand of the left-hand side of Eq. (9) we obtain:

$$\begin{aligned}
u^{(1)}_{i,m}\sigma^{0(1)}_{mj}u^{(2)}_{i,j} - u^{(2)}_{i,m}\sigma^{0(2)}_{mj}u^{(1)}_{i,j} &= u^{(1)}_{i,m}\sigma^{0(1)}_{mj}(u^{(1)}_{i,j} + \delta u_{i,j}) - (u^{(1)}_{i,m}\\
&\quad + \delta u_{i,m})(\sigma^{0(1)}_{mj} + \delta\sigma^0_{mj})u^{(1)}_{i,j}\\
&= u^{(1)}_{i,m}\sigma^{0(1)}_{mj}u^{(1)}_{i,j} + u^{(1)}_{i,m}\sigma^{0(1)}_{mj}\delta u_{i,j}\\
&\quad - u^{(1)}_{i,m}\sigma^{0(1)}_{mj}u^{(1)}_{i,j} - u^{(1)}_{i,m}\delta\sigma^0_{mj}u^{(1)}_{i,j}\\
&\quad - \delta u_{i,m}\sigma^{0(1)}_{mj}u^{(1)}_{i,j} - \delta u_{i,m}\delta\sigma^0_{mj}u^{(1)}_{i,j}.
\end{aligned} \tag{11}$$

By simplifying (11) and neglecting nonlinear items we have

$$u^{(1)}_{i,m}\sigma^{0(1)}_{mj}u^{(2)}_{i,j} - u^{(2)}_{i,m}\sigma^{0(2)}_{mj}u^{(1)}_{i,j} = -u^{(1)}_{i,m}\delta\sigma^0_{mj}u^{(1)}_{i,j}.$$

Hence, the equality (8) becomes

$$\int_S u_{i,m}^{(1)} \delta\sigma_{mj}^0 u_{i,j}^{(1)} dS = \int_{l_\sigma} P_i (u_i^{(1)} - u_i^{(2)}) dl_\sigma, \tag{12}$$

at that $i, j, m = 1, 2$.

Remark. Such method of derivation the equations for solving various inverse problems is general because it implies considering two physically-different states of the problem. In paper (Dudarev and Vatulyan, 2011) the equality (12) was obtained without fixing two states.

Denote $u_i^{(2)} =: f_i$ at l_σ. By expanding the sum in the left-hand side of (12) we finally have

$$\begin{aligned}
&\int_S (\delta\sigma_{11}^0 [(u_{1,1}^{(1)})^2 + (u_{2,1}^{(1)})^2] + 2\delta\sigma_{12}^0 [u_{1,1}^{(1)} u_{1,2}^{(1)} + u_{2,1}^{(1)} u_{2,2}^{(1)}] \\
&\quad + \delta\sigma_{22}^0 [(u_{1,2}^{(1)})^2 + (u_{2,2}^{(1)})^2]) dS \\
&\quad = \int_{l_\sigma} P_i (u_i^{(1)} - f_i) dl_\sigma.
\end{aligned} \tag{13}$$

3.2.2. Out-of-plane vibration

Let us consider the inverse problem on prestress identification for a plate oscillating in out-of-plane regime. In Section 3.2.1 the operator equation of the inverse problem was derived in in-plane case on the basis of weak formulation of the problem. We should mention that this is not the only way of derivation of operator equations. In this section the operator equation for out-of-plane vibration case will be derived from the corollary of the generalized reciprocity relation constructed in the paper (Vatulyan, 2007a):

$$\int_V \delta\sigma_{mj}^0 u_{i,m}^{(1)} u_{i,j}^{(1)} dV + \int_{S_\sigma} P_i (f_i - u_i^{(1)}) dS_\sigma = 0, \tag{14}$$

where $\delta\sigma_{mj}^0$ are corrections to the unknown prestress functions, $f_i = u_i^{(2)}$ is known displacement field on the boundary part S_σ (the superscript (1) describing the state will be omitted below), $i, j, m = 1, 2, 3$.

Remark. This relation is given for the case of 3D body considered in Section 2.1 (see Fig. 1). Also note that it has the same form as (12).

The relation (14) binds together the known and unknown problem parameters.

Now let us use the hypotheses for thin plate (5) and consider the case of 2D prestress state (2). Therefore the first item of the relation (14) takes the following form:

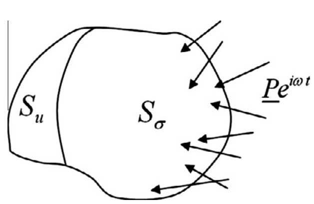

Fig. 1. 3-Dimensional solid body with the boundary $S = S_u \cup S_\sigma$. The periodic forces $\underline{P}e^{i\omega t}$ are applied at S_σ; the body is clamped at S_u.

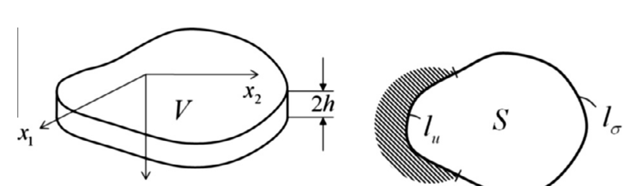

Fig. 2. On the left: the picture of 3D plate of volume V, boundary ∂V and thickness $2h$. On the right: the plate section of area S and boundary $\partial S = l = l_u \cup l_\sigma$.

$$\begin{aligned}
\int_V \delta\sigma_{mj}^0 u_{i,m} u_{i,j} dV &= \int_V [\delta\sigma_{11}^0 (u_{1,1}^2 + u_{2,1}^2 + u_{3,1}^2) + 2\delta\sigma_{12}^0 (u_{1,1} u_{1,2} \\
&\quad + u_{2,1} u_{2,2} + u_{3,1} u_{3,2}) + \delta\sigma_{22}^0 (u_{1,2}^2 + u_{2,2}^2 + u_{3,2}^2)] dV \\
&= \int_V [\delta\sigma_{11}^0 (x^2 w_{,11}^2 + x^2 w_{,21}^2 + w_{,1}^2) \\
&\quad + 2\delta\sigma_{12}^0 (x^2 w_{,11} w_{,12} + x^2 w_{,21} w_{,22} + w_{,31} w_{,32}) \\
&\quad + \delta\sigma_{22}^0 (x^2 w_{,12}^2 + x^2 w_{,22}^2 + w_{,32}^2)] dV \\
&= \int_S \left\{ \delta\sigma_{11}^0 \left[\frac{2}{3} h^3 (w_{,11}^2 + w_{,12}^2) + 2h w_{,1}^2 \right] \right. \\
&\quad + 2\delta\sigma_{12}^0 \left[\frac{2}{3} h^3 w_{,12} \Delta w + 2h w_{,1} w_{,2} \right] \\
&\quad \left. + \delta\sigma_{22}^0 \left[\frac{2}{3} h^3 (w_{,12}^2 + w_{,22}^2) + 2h w_{,2}^2 \right] \right\} dS.
\end{aligned}$$

Taking into account the way of plate loading, finally the operator relation (14) for the out-of-plane vibration regime of thin prestressed plate takes form

$$\begin{aligned}
&\int_S \left\{ \delta\sigma_{11}^0 \left[\frac{2}{3} h^3 (w_{,11}^2 + w_{,12}^2) + 2h w_{,1}^2 \right] \right. \\
&+ 2\delta\sigma_{12}^0 \left[\frac{2}{3} h^3 w_{,12} \Delta w + 2h w_{,1} w_{,2} \right] \\
&\left. + \delta\sigma_{22}^0 \left[\frac{2}{3} h^3 (w_{,12}^2 + w_{,22}^2) + 2h w_{,2}^2 \right] \right\} dS = \int_{l_\sigma} q(w - f) dS.
\end{aligned} \tag{15}$$

Let us view the case of the uniaxial pre-tension along the axis x_1, when the only one nonzero component of the prestress tensor is σ_{11}^0. As a result of the fact that the prestress tensor satisfies the equilibrium equations, the function σ_{11}^0 depends on only the coordinate x_2. In that case the relation (15) becomes

$$\int_S \delta\sigma_{11}^0 \left[\frac{2}{3} h^3 (w_{,11}^2 + w_{,12}^2) + 2h w_{,1}^2 \right] dS = \int_{l_\sigma} q(w - f) dS. \tag{16}$$

3.3. Numerical experiments

3.3.1. Description

Consider inverse problem on uniaxial non-homogeneous prestress identification for rectangular thin plate in out-of-plate oscillation regime (Fig. 3).

The problem takes form (6); at that the known functions are plate dimensions a, b and h, material parameters – cylindrical rigidity D and density ρ, and load intensity q. Let us set a problem on an identification of the law of variation $\sigma_{11}^0(x_2)$. As additional information we use the data on the displacement field w at the boundary part l_σ in a set of frequencies $\omega_k \in [\omega_-, \omega_+]$.

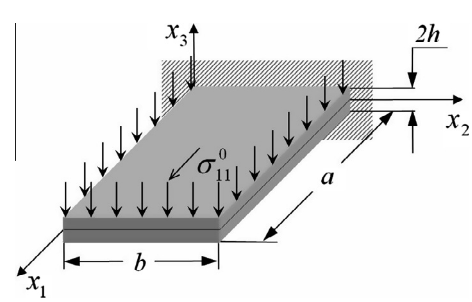

Fig. 3. The rectangular thin plate with the uniaxial non-homogeneous prestress σ_{11}^0. The uniformly distributed periodic load at the boundary part produces the out-of-plate oscillation regime.

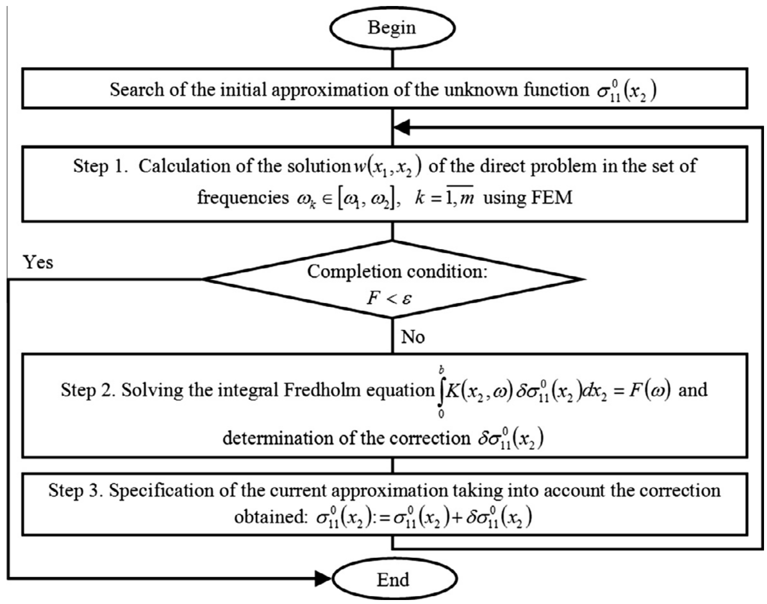

Fig. 4. Flow block of the iterative process of solving the inverse problem.

Remark. In Nedin and Vatulyan (2011b) special case was considered when additional information contained the data on displacement field w in the hole area S. In that case, the inverse problem is linear and it may be reduced to a single solving of the integral Fredholm equation of the first kind. In present paper we deal with nonlinear inverse problem on the uniaxial non-homogeneous prestress identification just for the out-of-plate vibration regime. The case of in-plane vibration was considered in Nedin and Vatulyan (2011a).

3.3.2. Reduction of the inverse problem to the iterative process

The inverse problem formulated belongs to the class of ill-posed problems. One of the most efficient methods of its solving is the method of iterative processes (Vatulyan, 2010a).

Let us use the relation (16) obtained and represent it in the following form:

$$\int_0^b K^{(n-1)}(x_2,\omega)\delta\sigma_{11}^{0(n)}(x_2)dx_2 = F^{(n-1)}(\omega), \quad \omega \in [\omega_-, \omega_+], \tag{17}$$

where

$$K^{(n-1)}(x_2,\omega) = \int_0^a \left[\frac{2}{3}h^3((w_{,11}^{(n-1)})^2 + (w_{,12}^{(n-1)})^2) + 2h(w_{,1}^{(n-1)})^2\right]dx_1, \tag{18}$$

$$F^{(n-1)}(\omega) = \int_{l_\sigma} q(w^{(n-1)} - f)dS, \tag{19}$$

$l_\sigma = \{x_1 \in [0,a]; x_2 = 0\} \cup \{x_1 \in [0,a]; x_2 = b\} \cup \{x_1 = a; x_2 \in [0,b]\}$.

Here, in round brackets the iteration number is indicated. For a given value n the relation (16) can be considered as the integral Fredholm equation of the first kind depending on the correction $\delta\sigma_{11}^{0(n-1)}(x_2)$ to the unknown uniaxial prestress function. On the basis of the formulated equation (17) one can construct the iterative process of solving the inverse problem. The process is represented in the form of flow block in Fig. 4. Its brief description is given below.

First of all, the automatic search of initial approximation of the unknown uniaxial prestress function $\sigma_{11}^0(x_2)$ is fulfilled (for instance, in class of constants or linear functions) (see next section).

After that the iteration process consisting of three steps occurs as follows.

Step 1. The direct problem (6) is solved for $\sigma_{11}^{0(n-1)}(x_2)$ (that means the $(n-1)$-approximation of the function $\sigma_{11}^0(x_2)$), using the FEM in the package FreeFem++. As a result the displacement fields $w_k^{(n-1)}(x_1,x_2)$ are calculated for m frequency values ω_k, $k=\overline{1,m}$. At the first iteration ($n=1$) the direct problem is solved for the initial approximation $\sigma_{11}^{0(0)}(x_2)$ selected automatically for the function to identify.

Step 2. The integral Fredholm equation of the first kind (17) is solved, and the correction $\delta\sigma_{11}^{0(n)}(x_2)$ to the current approximation of $\sigma_{11}^0(x_2)$ is calculated in the set of points $\{x_2\}_i \in [0,b]$. At that the additional information for solving the inverse problem is contained in the function $f(\omega_k) = w|_{l_\sigma}(\omega_k)$, which is given point by point for several frequency values ω_k, $k=\overline{1,n}$ (this function characterizes the displacements measured under the load in evenly distributed set of points at the boundary part l_σ). The solving of the equation (17) is implemented using the Tikhonov regularization procedure[2] which is realized by means of programming language Fortran in the module linked to the package FreeFem++.

Step 3. The current approximation of the function to identify is redefined by formula $\sigma_{11}^{0(n)} = \sigma_{11}^{0(n-1)} + \delta\sigma_{11}^{0(n)}$, where $\delta\sigma_{11}^{0(n)}$ is the correction calculated at the Step 2.

[2] Generally, the solving of the integral Fredholm equation of the first kind

$$\int_a^b K(x,s)u(s)ds = f(x), \quad x \in [c,d]$$

(where $K(x,s)$ is a continuous kernel, $f(x)$ is a given function, and $u(s)$ is an unknown function) is well known to be an ill-posed problem, therefore one cannot find its solution correctly by ordinary methods. Tikhonov proposed to understand by its solution the function $u(s)$ which minimize the following functional

$$M^\alpha[u] = \int_c^d \left[\int_a^b K(x,s)u(s)ds - f(x)\right]^2 dx + \alpha\int_a^b u^2(s)ds, \quad \alpha > 0,$$

if we only know that $u(s)$ is continuous on $[a,b]$. Tikhonov proved that such problem is solvable and it has unique solution. The value α called regularization parameter (in most cases this value is small: $\alpha = 10^{-4} \div 10^{-8}$). For more detail see Tikhonov and Arsenin (1979).

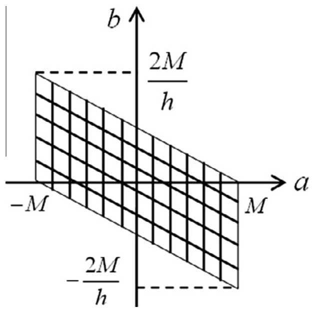

Fig. 5. Parallelepiped domain for the coefficients α and β.

Then the steps described above are performed again with redefined correction $\sigma_{11}^{0(n)}$, and so on until the completion condition is satisfied.

As a completion condition one can choose a minimization of the discrepancy functional

$$F = \|F^{(n)}(\omega)\| = \sqrt{\sum_{i=1}^{m} F^{(n)}(\omega_i)^2} < \varepsilon,$$

where $F^{(n)}(\omega)$ was defined by (19) previously, ε is the accuracy prescribed, m is the number of frequencies (Fig. 4).

3.3.3. Initial approximation search

In this section one way of initial approximation search is given for the event that the initial approximation of the unknown prestress function is searched in the form of linear function $\sigma_{11}^{0(0)}(x_2) = \alpha + \beta x_2$. Assuming the priori information about the boundedness of the prestress function is known

$$-M \leqslant \sigma_{11}^{0(0)}(x_2) \leqslant M,$$

we have

$$\begin{cases} -M \leqslant \alpha + \beta x_2 \leqslant M, \\ 0 \leqslant x_2 \leqslant h. \end{cases} \tag{20}$$

This set forms the domain for the coefficients α and β on the compact set – the parallelepiped with vertexes $\{(-M, \frac{2M}{h}), (M, 0), (M, -\frac{2M}{h}), (-M, 0)\}$ (Fig. 5).

After minimizing the functional

$$F \rightarrow \min,$$

the selection of proper pair α and β is made on the grid of the domain, to construct the initial approximation.

3.3.4. Numerical results

The series of numerical experiments on an identification of the uniaxial non-homogeneous prestress function $\sigma_{11}^{0}(x_2)$ in a rectangular plate was conducted (Fig. 3). The various classes of functions to identify were considered – monotonous and non-monotone (linear, polynomial, exponential, trigonometrical and more complicated analytical dependences). At that the vibration frequencies ω of load applied were selected from the three frequency ranges: below the first resonant frequency (the first range), between the first and the second resonants (the second range), and between the second and the third resonant (the third range). In all the experiments the prestress levels $\max|\sigma_{11}^{0}(x_2)|$ were chosen in such a way that the relation $\max\frac{\sigma_{11}^{0}}{E}$ (E – Young modulus) changed in the range $10^{-5} \div 10^{-3}$. The unknown prestress function was identified in the form of its values in a set of points evenly distributed in the domain $x_2 \in [0, b]$. In the experiments below the number of points is 12. As the additional information of the inverse problem we used values of the function $f(\omega)$ from the Eq. (19) in set of points distributed at the loaded part l_σ of the plate boundary. The best reconstruction results were achieved when considering 10 points at each of three loaded plate sides forming l_σ (see computational results below). Mostly, when increasing or decreasing this number of points, the descrapancy of the reconstruction obtained increased.

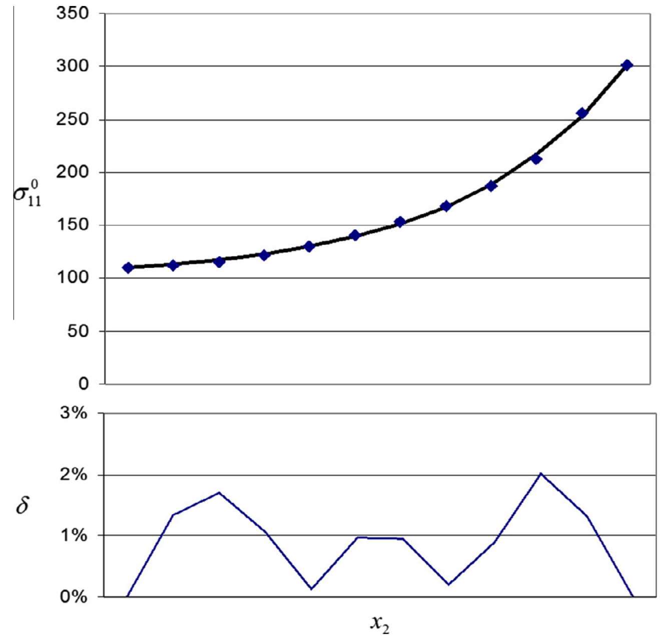

Fig. 6. Plate dimensions: $a = 2$ m, $b = 1$ m, $h = 0.1$ m, exact prestress function $\sigma_{11}^{0}(x_2) = -800(\sin \pi x_2/50 - 1.2)$. Number of iterations – 10.

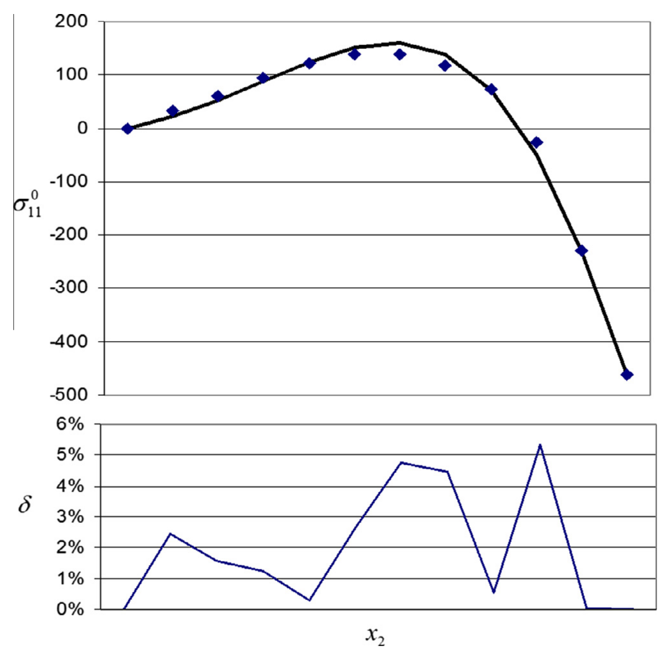

Fig. 7. Plate dimensions: $a = 2$ m, $b = 0.5$ m, $h = 0.1$ m, exact prestress function $\sigma_{11}^{0}(x_2) = -800(\sin \pi x_2/50 - 1.2)$. Number of iterations – 35.

In the problem of an identification of the uniaxial non-homogeneous prestresses in a plate in case of out-of-plane vibration regime, one can see the same tendencies as in case of in-plane vibrations given in Nedin and Vatulyan (2011a).

First of all, in most cases of numerical experiments a small deterioration of the identification of the unknown function $\sigma_{11}^{0}(x_2)$ at

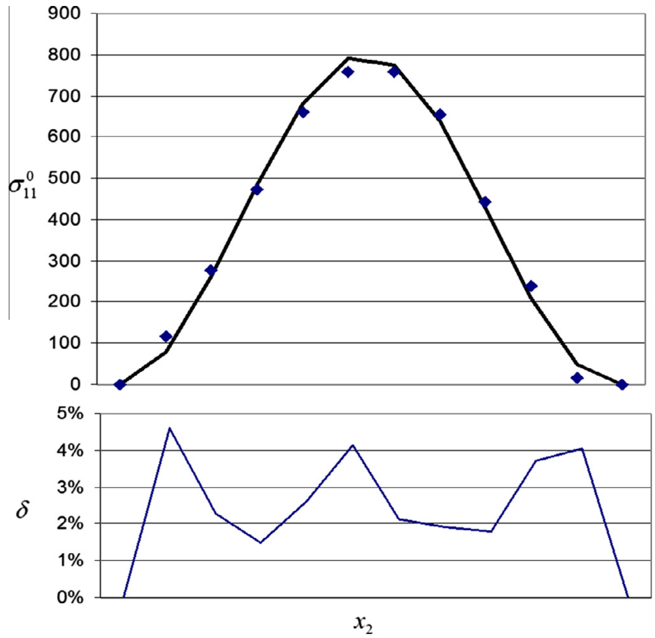

Fig. 8. Plate dimensions: $a = 2$ m, $b = 0.5$ m, $h = 0.1$ m, exact prestress function $\sigma_{11}^0(x_2) = -800(\sin \pi x_2/50 - 1.2)$. Number of iterations – 20.

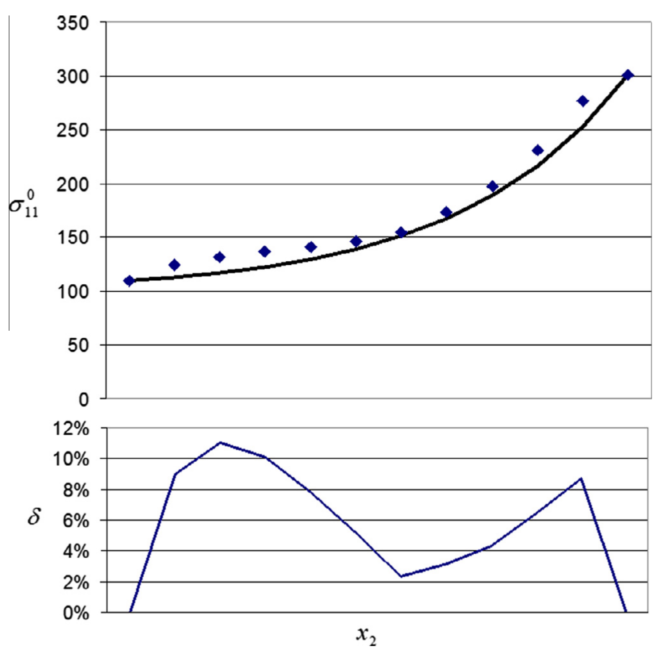

Fig. 9. Reconstruction result with 5%-noisy measurement data. Plate dimensions: $a = 2$ m, $b = 1$ m, $h = 0.1$ m, exact prestress function $\sigma_{11}^0(x_2) = -800(\sin \pi x_2/50 - 1.2)$. Number of iterations – 13.

the ends of the segment $x_2 \in [0, b]$ took place. In some cases the mistake at the ends exceeded 20%, while in the distance from the ends it changed within the limits 1–10%. Such deterioration of the inverse problem solution is principally connected with the fact that at every step of the iterative process the numerical calculations of the second partial derivatives of FEM displacement functions u_i being included in the kernel (18) of the integral equation (17) occur; such an operation is known to be an ill-posed problem. It is worth noting that the kernel itself does not tend to zero near the ends of the segment $x_2 \in [0, b]$. In the event that the values of the function to identify at the segment ends are considered to be given, the identification quality about the ends significantly increased.

Secondly, almost for all the examples viewed, the best reconstruction results were obtained while considering the second and the third frequency ranges, rather than the first one.

The numerical results for the reconstruction of the uniaxial stresses $\sigma_{11}^0(x_2)$ are given below (Figs. 6–8) for the following plate parameters: $E = 1.96 \cdot 10^{11}$ Pa, $\nu = 0.29$, $\rho = 7.8 \cdot 10^3$ kg/m^3. The identification results are presented for the second frequency range. The function values $\sigma_{11}^0(x_2)$ at the ends $x_2 \in [0, b]$ were accepted to be given. The firm line denotes exact function $\sigma_{11}^0(x_2)$, the small squares – identification result. The error value δ was calculated in percent by the formula $\frac{|\sigma_{11}^0(x_2) - \sigma_{11}^{0*}(x_2)|}{\max_{x_2 \in [0,b]}(|\sigma_{11}^{0*}(x_2)|)} 100\%$, where $\sigma_{11}^0(x_2)$ is the reconstructed law of variation of the prestress, $\sigma_{11}^{0*}(x_2)$ is the exact one.

Also the effect of measurement error on the identification accuracy was investigated. To do this, the perturbation of function $f(\omega)$ was induced as follows:

$$\tilde{f}(\omega) = f(\omega) + \gamma A R(\omega),$$

where γ is the parameter which denotes a noise degree, A is characteristic amplitude of the function $f(\omega)$, $R(\omega)$ is random function with maximum amplitude less or equaling one. As an example, in Fig. 9 the computation result for $\gamma = 0.05$ is given. The prestress function is selected in the same way as in Fig. 6. The maximum discrepancy of the reconstruction in this case does not exceed 11%.

4. Conclusion

The formulations of direct problems of steady-state vibration of thin elastic isotropic plates with non-homogeneous residual stresses are presented in cases of in-plane and out-of-plane vibration regimes. The operator equations of inverse problems for both vibration cases are derived. The iterative process of solving the inverse problem is described. The series of numerical experiments on an identification of uniaxial non-homogeneous residual stresses in rectangular plate oscillating in out-of-plane regime is conducted. The recommendations on a selection of the most effective probe regimes are given.

Acknowledgments

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