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Asymptotics of eigenvalues of large symmetric Toeplitz matrices with smooth simple-loop symbols

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This paper is devoted to the asymptotic behavior of all eigenvalues of the increasing finite principal sections of an infinite symmetric (in general non-Hermitian) Toeplitz matrix. The symbol of the infinite matrix is supposed to be moderately smooth and to trace out a simple loop in the complex plane. The main result describes the asymptotic structure of all eigenvalues. The asymptotic expansions constructed are uniform with respect to the location of the eigenvalues in the bulk of the spectrum.

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1. Introduction

Given a function $a(t)$ in L^1 on the complex unit circle $\mathbb{T}$ we denote by a_l the l-th Fourier coefficient

$$a_l = \frac{1}{2\pi} \int_0^{2\pi} a(e^{ix}) e^{-ilx} dx, \tag{1.1}$$

$l \in \mathbb{Z}$, and by $T_n(a)$ the $n \times n$ Toeplitz matrix $(a_{j-k})_{j,k=1}^n$.

The object of our study is the behavior of the spectral characteristics (eigenvalues and singular numbers, eigenvectors, determinants, condition numbers, etc.) of Toeplitz matrices in the case when the dimension of the matrices tends to infinity. It has been intensively studied for a century (see [1], [2], [3], [4], and literature cited there). This problem is important for statistical mechanics and other applications ([1], [5], [6], [7], [8]). First of all, we mention the numerous versions of the Szegö theorem on the asymptotic distribution of eigenvalues and theorems of Abram-Parter type on the asymptotic distribution of singular numbers ([9], [10], [11], [12]). There is a rich literature devoted to the asymptotics of the determinants of Toeplitz matrices (see monographs [2], [3], papers [13], [14], [15], [5] and literature cited there). Much attention has been paid to the asymptotics of the largest and smallest eigenvalues ([16], [17], [18]).

We note that articles on the individual asymptotics of all eigenvalues have appeared quite recently. In particular, the cases of real-valued symbols (self-adjoint Toeplitz operators) satisfying the so-called SL (Simple-Loop) condition were studied in the articles [19], [5], [20]. In these articles the authors successively considered polynomial symbols, infinitely smooth symbols, and finally symbols having four continuous derivatives.

Finally, a symbol which has a continuous first derivative and satisfies certain additional conditions at the minimum and maximum points is considered in [21]. In [22], the case of a symbol that has a 4th order zero and thus does not satisfy the conditions of a simple loop is studied.

The asymptotic expansions of all eigenvalues are constructed in the case of essentially complex-valued symbols having singularities of the Fisher-Hartwig type in the articles [23], [24], [25], [26]. We note that of the complex-valued (non-selfadjoint) case is more complicated than the real-valued one, because finding the location of the limit set of eigenvalues of Toeplitz matrices with n tending to infinity is a nontrivial question. This question is resolved in [13] for the case of the Fisher-Hartwig singularities considered in the above-mentioned papers. In this paper it is shown that the limit set coincides with the image of the symbol in the complex plane.

There is another well-known case when the limit set also coincides with the symbol image. It is the case when this image is a “curve without an interior”. Using this fact, in [27] we solved the problem of the asymptotic behavior of the eigenvalues of Toeplitz symmetric matrices with a polynomial symbol satisfying the following condition. Namely, the symbol which passes its own curve-image exactly two times when the variable t makes one turn on the unit circle.

Note that the asymptotic structure of the eigenvectors in the case of $n \to \infty$ is considered in the papers [28], [29], [30].

In this paper we generalize the results of the article [27], extending the class of symbols from polynomials to the class of smooth functions that are merely required to have two continuous derivatives. For this purpose the method used in [27] needed a significant change. The main obstacle here is that the considered symbol is defined only on the unit circle and does not allow, in general, unlike the case of the polynomial symbol of [27], an analytic continuation to the neighborhood of the unit circle $\mathbb{T}$. At the same time, the eigenvalues are not located on the image-curve of the symbol $a(t)$, but they are located in some of its neighborhoods. Consequently, there arises the question about the continuation of $a(t)$ to the complex plane. In this regard, we replace the symbol with a polynomial approximation (first terms of the Laurent series) $a_n(t)$ of degree $n-1$ (see (2.4)) and note that the operator corresponding to the Toeplitz matrix does not change. The function $a_n(t)$ is considered in an annulus with the width of order $1/n$, containing $\mathbb{T}$. We show that all eigenvalues lie in the image of the mapping $w = a_n(z)$ of this annulus. On the one hand, it is necessary to transfer the methods and results of [20], [21] from the real segment to a region in the complex plane. On the other hand, we ensure that all constructions are uniform with respect to the parameters of the family of functions $\{a_n\}$, $n \in \mathbb{N}$.

The paper is organized as follows. Section 2 contains the main results of the work. In Section 3 we consider an example with numerical calculations of all eigenvalues for different values of n. The presented figures bring up several questions about the location the eigenvalues. The main results that are formulated in Section 2 allows us to answer these questions. In particular we give the asymptotics of the eigenvalues that are located near the points $z = a(1)$ and $z = a(-1)$ (see Lemma 3.2), where the derivative of the symbol vanishes. This result is a generalization to the complex case of the well-known results about the asymptotics of the smallest and largest eigenvalues of large Toeplitz matrices with real-valued symbols (see [9], [18]).

Section 4 presents the results on the smoothness properties of the functions $a(t)$ and $a_n(t)$ that we need and the functions $b(t,\lambda)$, $b_n(t,\lambda)$ are constructed on the basis of $a(t)$, $a_n(t)$ (see (2.10), (2.12)). In Section 5, a nonlinear equation is introduced for determining the eigenvalues and then its asymptotic properties are investigated. Section 6 is devoted to the analysis of the solvability of the above mentioned nonlinear equation in the complex domain surrounding the image curve of the symbol $a(t)$. The main results are proved in Section 7.

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2. The main results

Let $\alpha \geq 0$. We denote by W^α the weighted Wiener algebra of all complex-valued functions $a : \mathbb{T} \to \mathbb{C}$ whose Fourier coefficients satisfy

$$\|a\|_\alpha := \sum_{j=-\infty}^{\infty} |a_j|(|j|+1)^\alpha < \infty. \tag{2.1}$$

Let m be the integer part of α. It is readily seen that if $a \in W^\alpha$ then the function g defined by $g(\varphi) := a(e^{i\varphi})$ is a 2π-periodic C^m function on $\mathbb{R}$. In what follows we consider complex-valued symmetric simple-loop functions in W^α. To be more precise, for $\alpha \geq 2$, we let CSL^α be the set of all $a \in W^\alpha$ such that $g(\varphi) = a(e^{\mathrm{i}\varphi})$ has the following properties.

(1) The function $g(\varphi)$ is symmetric in the sense that:

$$g(\varphi) = g(2\pi - \varphi), \qquad \varphi \in [0, 2\pi]. \tag{2.2}$$

(This is equivalent to the condition $a_j = a_{-j}$, $j = 1, 2, \ldots$.)

(2) $\mathrm{Im}(g)$ is a simple (without self-intersections) arc with non-coincident endpoints M_0, M_1: $g(0) = g(2\pi) = M_0$, $g(\pi) = M_1$, $M_0 \neq M_1$, so that $g'(\varphi) \neq 0$ for $\varphi \in (0, \pi)$ and $g''(0) = g''(2\pi) \neq 0$, $g''(\pi) \neq 0$.

It should be noted that if we have (2.2) then

$$g'(0) = g'(2\pi) = g'(\pi) = 0. \tag{2.3}$$

We introduce the following notation. Let $f(t) = \sum_{j=-\infty}^{\infty} f_j t^j$ $(t \in \mathbb{T})$ be a function from the space $L_2(\mathbb{T})$, so that $\sum_{j=-\infty}^{\infty} |f_j|^2 < \infty$. We consider the projectors

$$[P_n f](t) := \sum_{j=0}^{n-1} f_j t^j, \qquad n = 1, 2, \ldots.$$

We will also denote the image of the operator P_n by $L_2^{(n)}$. Note that for a symbol $a \in L_\infty(\mathbb{T})$ the Toeplitz matrix $T_n(a)$ can be identified with the operator

$$T_n(a) : L_2^{(n)} \to L_2^{(n)}, \text{ defined by } T_n(a)f = P_n(af).$$

We introduce then the functions

$$a_n(t) = \sum_{j=-(n-1)}^{n-1} a_j t^j \tag{2.4}$$

and note that

$$T_n(a) = T_n(a_n). \tag{2.5}$$

Therefore, we will use the function $a_n(t)$ instead of the symbol $a(t)$, when it will be convenient, and respectively the function $g_n(\varphi) := a_n(e^{\mathrm{i}\varphi})$ instead of $g(\varphi)$.

Note that the functions $a_n(t)$ and $g_n(\varphi)$ satisfy all conditions of the definition of CSL^α for a sufficiently large n. Besides, if $a(t) \in W^\alpha$ then

$$\sup_{t\in\mathbb{T}} |a(t) - a_n(t)| = o\left(1/n^\alpha\right), \quad n \to \infty \tag{2.6}$$

(see Lemma 4.1, i) below).

Introduce the sets

$$\mathcal{R}(a) := \{g(\varphi) : \ \varphi \in (0, \pi)\}; \tag{2.7}$$

$$\Pi_n(a) = \left\{\psi = \varphi + \mathrm{i}\delta \,|\, \varphi \in [cn^{-1}, \pi - cn^{-1}], \ \delta \in [-Cn^{-1}, Cn^{-1}]\right\}, \tag{2.8}$$

where c small enough and C large enough are fixed positive numbers. Let us denote

$$\mathcal{R}_n(a) := \{g_n(\psi) : \ \psi \in \Pi_n(a)\}. \tag{2.9}$$

It is well known (see for example [2]) that the limit spectrum of the operator family $\{T_n(a)\}_{n=1}^\infty$ coincides with the curve $\mathcal{R}(a)$. Thus, for sufficiently large n the spectrum of $T_n(a)$ is located in a neighborhood of $\mathcal{R}(a)$. Moreover, we will show that $\mathrm{Sp}T_n(a) \subset \mathcal{R}_n(a)$.

According to the conditions (1)-(2), for each $\lambda \in \mathcal{R}(a)$ there exists exactly one $\varphi_1(\lambda) \in (0, \pi)$ such that $g(\varphi_1(\lambda)) = \lambda$. The symmetry implies that the function $\varphi_2(\lambda) := (2\pi - \varphi_1(\lambda)) \in (\pi, 2\pi)$ also has this property: $g(\varphi_2(\lambda)) = \lambda$.

For all $\lambda_0 \in \mathcal{R}(a)$ consider the function

$$\hat{b}(t, \lambda_0) = \frac{(a(t) - \lambda)e^{\mathrm{i}\varphi_1(\lambda_0)}}{(t - e^{\mathrm{i}\varphi_1(\lambda_0)})(t^{-1} - e^{\mathrm{i}\varphi_1(\lambda_0)})}. \tag{2.10}$$

(Note that $e^{-\mathrm{i}\varphi_2(\lambda)} = e^{\mathrm{i}\varphi_1(\lambda)}$, therefore the function $t^{-1} - e^{\mathrm{i}\varphi_1(\lambda)}$ goes to zero at the single point $t_2 = e^{\mathrm{i}\varphi_2(\lambda)}$.) In a similar manner, for all $\lambda \in \mathcal{R}_n(a)$ there exist $\varphi_{1,n}(\lambda) \in \Pi_n$, such that

$$g_n(\varphi_{1,n}(\lambda)) = g_n(2\pi - \varphi_{1,n}(\lambda)) = \lambda. \tag{2.11}$$

It will be shown below that the point $\varphi_{1,n}(\lambda)$ satisfying condition (2.11) (see Lemma 4.3) is unique.

Together with (2.10) we consider the function

$$\hat{b}_n(t, \lambda) = \frac{(a_n(t) - \lambda)e^{\mathrm{i}\varphi_{1,n}(\lambda)}}{(t - e^{\mathrm{i}\varphi_{1,n}(\lambda)})(t^{-1} - e^{\mathrm{i}\varphi_{1,n}(\lambda)})}, \qquad \lambda \in \mathcal{R}_n(a), \tag{2.12}$$

which is a polynomial of powers of t and t^{-1} of finite degree and does not vanish in the domain $t \in \mathbb{T}$, $\lambda \in \mathcal{R}_n(a)$. We show that this function allows a Wiener-Hopf factorization of the form

$$\hat{b}_n(t,\lambda) = \hat{b}_{n,+}(t,\lambda)\hat{b}_{n,+}(t^{-1},\lambda),$$

where $\hat{b}_{n,+}(t,\lambda)$ is a polynomial of degree $n-2$ of the variable t. We introduce the function

$$\theta_n(\lambda) = \log \frac{\hat{b}_{n,+}(e^{\mathrm{i}\varphi_{1,n}(\lambda)},\lambda)}{\hat{b}_{n,+}(e^{-\mathrm{i}\varphi_{1,n}(\lambda)},\lambda)}, \qquad \lambda \in \mathcal{R}_n(a). \tag{2.13}$$

When $\lambda \in \mathcal{R}(a)$, which is equivalent to $e^{\mathrm{i}\varphi_1(\lambda)} \in \mathbb{T}$, the function $\theta_n(\lambda)$ can be represented as

$$\theta_n(\lambda) = \frac{1}{2\pi\mathrm{i}} \int\limits_{\mathbb{T}} \frac{\log \hat{b}_n(\tau,\lambda)}{\tau - e^{\mathrm{i}\varphi_{1,n}(\lambda)}} d\tau - \frac{1}{2\pi\mathrm{i}} \int\limits_{\mathbb{T}} \frac{\log \hat{b}_n(\tau,\lambda)}{\tau - e^{-\mathrm{i}\varphi_{1,n}(\lambda)}} d\tau. \tag{2.14}$$

We also introduce the function

$$\theta(\lambda) = \frac{1}{2\pi\mathrm{i}} \int\limits_{\mathbb{T}} \frac{\log \hat{b}(\tau,\lambda)}{\tau - e^{\mathrm{i}\varphi_1(\lambda)}} d\tau - \frac{1}{2\pi\mathrm{i}} \int\limits_{\mathbb{T}} \frac{\log \hat{b}(\tau,\lambda)}{\tau - e^{-\mathrm{i}\varphi_1(\lambda)}} d\tau, \quad \lambda \in \mathcal{R}(a), \tag{2.15}$$

where the integrals in (2.14) and (2.15) are understood in the sense of the principal value. It is more convenient for us to consider the introduced functions as functions of the parameter $s := \varphi_{1,n}(\lambda)$ $(\lambda = g_n(s))$:

$$\eta(s) := \theta(g(s)), \qquad s \in (0,\pi) \tag{2.16}$$

and

$$\eta_n(s) := \theta_n(g_n(s)), \qquad s \in \Pi_n(s) \tag{2.17}$$

Introduce the values

$$d_{j,n} = \frac{\pi j}{n+1}, \qquad j = 1,2,\ldots,n, \tag{2.18}$$

$$e_{j,n} = d_{j,n} - \frac{\eta_n(d_{j,n})}{n+1}, \qquad j = 1,2,\ldots,n. \tag{2.19}$$

We will also need the following small areas:

$$\Pi_{j,n}(a) := \left\{ s \in \Pi_n(a),\ |s - e_{j,n}| \le \frac{c_n}{n+1} \right\}, \quad j = 1,2,\ldots,n, \tag{2.20}$$

where the constants c_n do not depend on j and decrease to 0 with $n \to \infty$ (see (6.5)). Now we are ready to formulate the main results of this work. Let $\lambda_j^{(n)}$, $j = 1, 2, \dots, n$ be a numeration of the eigenvalues of the operator $T_n(a)$.

Theorem 1. *Let a be a symbol such that $a \in \mathsf{CSL}^\alpha$, $\alpha \geq 2$. Then, for every sufficiently large natural number n, the following statements hold:*

i) all eigenvalues of $T_n(a)$ are different, and $\lambda_j^{(n)} \in g(\Pi_{j,n})$ for $j = 1, 2, \dots, n$,
ii) the values $s_{j,n}$ such that $\lambda_j^{(n)} = g_n(s_{j,n})$ satisfy the equation

$$(n+1)s + \eta_n(s) = \pi j + \Delta_n(s), \qquad j = 1, 2, \dots, n, \tag{2.21}$$

with $|\Delta_n(s)| = o(1/n^{\alpha-2})$ as $n \to \infty$ uniformly respect to $s \in \Pi_n(a)$,
iii) equation (2.21) has a unique solution in the domain $\Pi_{j,n}$.

Theorem 2. *Under the conditions of the Theorem 1,*

$$s_{j,n} = d_{j,n} + \sum_{k=1}^{[\alpha]-1} \frac{p_k(d_{j,n})}{(n+1)^k} + \Delta_2^{(n)}(j)$$

where

$$\Delta_2^{(n)}(j) = \begin{cases} o(1/n), & \alpha = 2 \\ O(1/n^{\alpha-1}), & \alpha > 2 \end{cases}$$

as $n \to \infty$ uniformly in j. The functions p_k can be calculated explicitly; in particular

$$p_1(s) = -\eta(s), \quad p_2(s) = \eta(s)\eta'(s).$$

Theorem 3. *Under the conditions of Theorem 1,*

$$\lambda_j^{(n)} = g(d_{j,n}) + \sum_{k=1}^{[\alpha]-1} \frac{r_k(d_{j,n})}{(n+1)^k} + \Delta_3^{(n)}(j) \tag{2.22}$$

where

$$\Delta_3^{(n)}(j) = \begin{cases} o\left(d_{j,n}(\pi - d_{j,n})/n\right), & \alpha = 2, \\ O\left(d_{j,n}(\pi - d_{j,n})/n^{\alpha-1}\right), & \alpha > 2, \end{cases}$$

as $n \to \infty$ uniformly in j. The coefficients r_k can be calculated explicitly; in particular

$$r_1(\varphi) = -g'(\varphi)\eta(\varphi) \quad \text{and} \quad r_2(\varphi) = \frac{1}{2}g''(\varphi)\eta^2(\varphi) + g'(\varphi)\eta(\varphi)\eta'(\varphi).$$

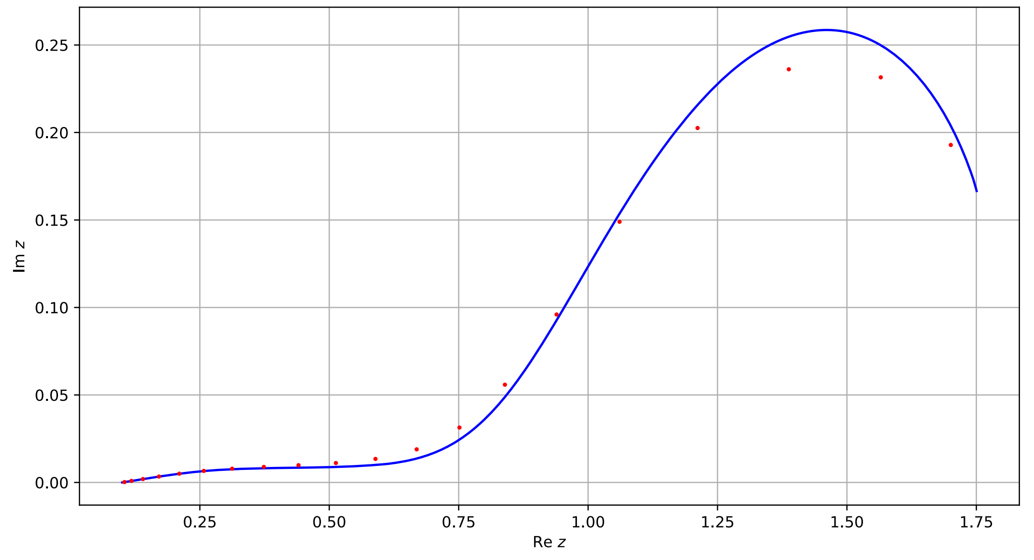

Fig. 3.1. Image of the symbol $a_1(t)$ and eigenvalues of the matrix $T_{20}(a_1)$.

3. Numerical example and the consequences of the main results

Define a symbol $a_1(e^{\imath\varphi}) = g_1(\varphi)$ by

$$g_1(\varphi) = c_1 \sin(c_0\varphi^2) + c_2((1+\varphi)^{5/2} + (1-\varphi)^{5/2}), \qquad \varphi \in [-\pi;\pi],$$

(it is more convenient for us to consider the symbol in this section on the segment $[-\pi;\pi]$) where

$$c_0 = \frac{1}{5} - \frac{1}{6}\mathrm{i}, \qquad c_1 = \frac{(1-\pi)^{3/2} - (\pi+1)^{3/2}}{16\pi c_0 \cos(\pi^2 c_0)}, \qquad c_2 = \frac{1}{20}.$$

The expression for the constants $c_{1,2}$ are derived from the conditions $g'(-\pi) = g'(\pi)$. (The equalities $g(-\pi) = g(\pi)$ and $g''(-\pi) = g''(\pi)$ are a consequence of the symmetry of function $g(\varphi)$.) It can be verified that the constructed function satisfies the conditions (1) and (2) at the begining of Section 2. (The condition $g'(\varphi) \neq 0$ for $\varphi \in (-\pi;\pi)$ can be verified numerically.)

We can see that the third derivative of the symbol $a_1(t)$ has singularities at the points $t = e^{\mathrm{i}}$ and $t = e^{-\mathrm{i}}$. It is easy to see that $a_1(t) \in \mathsf{CSL}^{2.5-\delta}$ for arbitrary small $\delta > 0$.

The image of the symbol $a_1(t)$ and the eigenvalues of the matrices $T_n(a_1)$ for $n = 20$ and $n = 80$ are shown in the Fig. 3.1 and Fig. 3.2 correspondingly. If we look at these figures then we can make the following observations:

1. The limit set of the eigenvalues for $T_n(a_1)$ if $n \to \infty$ really coincides with $\mathcal{R}(a_1)$.

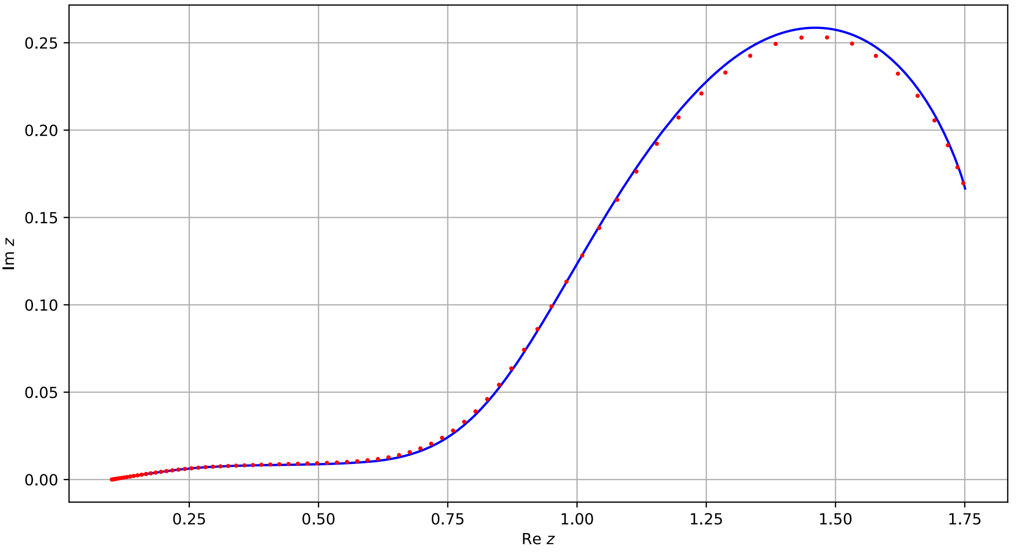

Fig. 3.2. Image of the symbol $a_1(t)$ and eigenvalues of the matrix $T_{80}(a_1)$.

2. The points of concentration for the eigenvalues of $T_n(a_1)$ are $z_1 = a_1(1)$ and $z_2 = a_1(-1)$. The distance between consecutive eigenvalues in neighborhoods of z_1 and z_2 is much less than outside of these neighborhoods.
3. Some eigenvalues are located below the curve $\mathcal{R}(a_1)$ and others above the curve.

We are going to show that Theorem 3 allows to explain and clarify these observations.

Designate the spectrum of $T_n(a)$ by $\mathrm{Sp}T_n(a) := \left\{\lambda_j^{(n)}\right\}_{j=1}^{n}$. Then the limit set of the eigenvalues of the sequence $\{T_n(a)\}_{n=1}^{\infty}$ is

$$\Lambda(a) := \limsup_{n\to\infty} \mathrm{Sp}(T_n(a)).$$

The next lemma is a direct consequence of Theorem 3.

Lemma 3.1. *Under the conditions of Theorem 1,*

$$\Lambda(a) = \mathcal{R}(a).$$

In addition

$$\sup_{1\le j\le n} \rho\left(\lambda_j^{(n)}, \mathcal{R}(a)\right) \le \frac{\mathrm{const}}{n}.$$

(Here $\rho(z, \mathcal{R}(a))$ is the distance between the point z and the curve $\mathcal{R}(a)$ in the complex plane.)

Consider now Observation 2. The situation in the neighborhoods of the points z_1 and z_2 is clarified by the following statement.

Lemma 3.2. *Let the conditions of the Theorem 1 be fulfilled. Then:*

i) if $\frac{j}{n+1} \to 0$, *then the following asymptotic formula is true:*

$$\lambda_j^{(n)} = g(0) + \frac{\pi^2 g''(0)}{2} \frac{j^2}{(n+1)^2} + \Delta_4^{(n)}(j),$$

where

$$\Delta_4^{(n)}(j) = o\left(j^2/n^2\right); \tag{3.1}$$

ii) if $\frac{n+1-j}{n+1} \to 0$, *then the following asymptotic formula is true:*

$$\lambda_j^{(n)} = g(\pi) + \frac{\pi^2 g''(\pi)}{2} \frac{(n+1-j)^2}{(n+1)^2} + \Delta_5^{(n)}(j),$$

where

$$\Delta_5^{(n)}(j) = o\left((n+1-j)^2/n^2\right). \tag{3.2}$$

Proof. Since $a(t) \in W^2$ then, according to (2.22), we have

$$\lambda_j^{(n)} = g(d_{j,n}) - \frac{g'(d_{j,n})\eta(d_{j,n})}{n+1} + o\left(\frac{d_{j,n}(\pi - d_{j,n})}{n}\right).$$

Consider i). We have in this case that

$$o\left(\frac{d_{j,n}(\pi - d_{j,n})}{n}\right) = o\left(\frac{j}{n^2}\right).$$

Taking into account that

$$g'(d_{j,n}) = g(0) + \frac{g''(0)}{2}(d_{j,n})^2 + o(d_{j,n})^2,$$

$g'(d_{j,n}) = O(d_{j,n})$, and $\eta(d_{j,n}) = O(d_{j,n})$, one has

$$\frac{|g'(d_{j,n})\eta(d_{j,n})|}{n+1} = o\left(\frac{j^2}{n^3}\right).$$

Thus we get i). Case ii) is proved analogously. □

Thus Lemma 3.2 shows that the distance between consecutive eigenvalues located in a neighborhood of the point $z_1 = g(0)$ ($j/n \to 0$) is $o(j/n^2)$. So if j is bounded ("the first eigenvalues") then this distance is of order

$$o\left(1/n^2\right). \tag{3.3}$$

The same result is true for neighborhoods of the point $z_2 = g(\pi)$ ("the last eigenvalues"). On the other hand for $\varepsilon < d_{j,n} < \pi - \varepsilon$, where $\varepsilon > 0$ is fixed small enough (inner eigenvalues) Theorem 3 gives that

$$|\lambda_{j+1}^{(n)} - \lambda_j^{(n)}| = o\left(1/n\right). \tag{3.4}$$

Thus (3.3) and (3.4) confirm and explain of the Observation 2 above.

We pass to Observation 3. Consider the "inner eigenvalues", that is, $\varepsilon < d_{j,n} < \pi - \varepsilon$. Suppose that $a(t) \in W^3$. Then the formula (2.22) gives us that

$$\lambda_j^{(n)} = g(d_{j,n}) - \frac{g'(d_{j,n})\eta(d_{j,n})}{n+1} + O\left(\frac{1}{n^2}\right).$$

Let $\tilde{e}_{j,n} := d_{j,n} - \dfrac{\operatorname{Re}\eta(d_{j,n})}{n+1}$. Then

$$\lambda_j^{(n)} = g(\tilde{e}_{j,n}) - \mathrm{i}\frac{g'(\tilde{e}_{j,n})\operatorname{Im}\eta(d_{j,n})}{n+1} + O\left(\frac{1}{n^2}\right). \tag{3.5}$$

Thus $\lambda_j^{(n)}$ is located on the normal to the curve $\mathcal{R}(a)$ at the point $z = g(\tilde{e}_{j,n})$ with exactitude $O\left(1/n^2\right)$.

In addition we can see that the point $\lambda_j^{(n)}$ is located "above" or "below" of the curve $\mathcal{R}(a)$ depending the sign of the real number $\operatorname{Im}\eta(d_{j,n})$. Moreover, formula (3.5) gives us that $\dfrac{|g'(\tilde{e}_{j,n})\operatorname{Im}\eta(d_{j,n})|}{n+1} + O\left(1/n^2\right)$ is the distance between $\lambda_j^{(n)}$ and $\mathcal{R}(a)$.

It should be noted that Theorem 3 has not only a qualitative sense but also a quantitative (numerical) one. If one has numerical values for the function $\eta(d_{j,n})$ we can calculate all eigenvalues $\lambda_j^{(n)}$, $j = 1, 2, \ldots, n$ very rapidly for different values of n. This idea was applied in the article [31] (in the case of real-valued symbols) where the function $\eta(d_{j,n})$ was calculated with the help of the eigenvalues $\lambda_j^{(n)}$ for not very large n.

In the rest of this section we illustrate the accuracy of our asymptotic formulas. We introduce the following notation for the approximated eigenvalues from (2.22):

$$\lambda_{1,j}^{(n)} = g(d_{j,n}) - \frac{g'(\varphi)\eta(\varphi)}{n+1},$$

$$\lambda_{2,j}^{(n)} = g(d_{j,n}) - \frac{g'(\varphi)\eta(\varphi)}{n+1} + \frac{\frac{1}{2}g''(\varphi)\eta^2(\varphi) + g'(\varphi)\eta(\varphi)\eta'(\varphi)}{(n+1)^2}.$$

Table 1
Comparative accuracy of the calculation of the spectrum of $T_n(a_1)$.

n	20	40	80	160	320
$\Delta_1^{(n)}$	3.2e-03	8.8e-04	2.3e-04	5.9e-05	1.5e-05
$\Delta_2^{(n)}$	3.9e-04	5.6e-05	7.2e-06	9.2e-07	1.2e-07

The relative approximation error will be characterized by the values

$$\Delta_1^{(n)} = \max_{j=\overline{1,n}} \left| \frac{\lambda_{1,j}^{(n)} - \lambda_j^{(n)}}{\lambda_j^{(n)}} \right|, \qquad \Delta_2^{(n)} = \max_{j=\overline{1,n}} \left| \frac{\lambda_{2,j}^{(n)} - \lambda_j^{(n)}}{\lambda_j^{(n)}} \right|.$$

The resulting accuracy of the spectrum of $T_n(a_1)$ is shown in Table 1.
We can see that for even $n = 20$ the accuracy is good enough for both approximations.

4. Preliminary results

Let the function f be $f(t) = \sum_{j=-\infty}^{\infty} f_j t^j \in L_2(\mathbb{T})$. We introduce the operator Q_n by the formula

$$(Q_n f)(t) := \sum_{j=n}^{\infty} f_j t^j. \tag{4.1}$$

The following lemma holds.

Lemma 4.1.

i) If $f \in W^\alpha(\mathbb{T})$, $\alpha \ge 0$, then

$$\sup_{t\in\mathbb{T}} |(Q_n f)(t)| \le \frac{\|Q_n f\|_\alpha}{n^\alpha}.$$

ii) For a natural number $k \le \alpha$, we have

$$f^{(k)}(t) \in W^{\alpha-k}(\mathbb{T}) \quad \textit{and} \quad \sup_{t\in\mathbb{T}} \left| \left(Q_n f^{(k)}\right)(t) \right| \le \frac{\|Q_n f\|_\alpha}{n^{\alpha-k}}.$$

iii) For a real number $k > \alpha$, we have

$$\sum_{j=-(n-1)}^{n-1} |f_j||j|^k \le 2n^{k-\alpha}\|f\|_\alpha.$$

Proof. Items i) and ii) were proved in [20], [21]. Let us prove iii):

$$\sum_{j=-(n-1)}^{n-1} |f_j||j|^k = \sum_{j=-(n-1)}^{n-1} (|f_j|(1+|j|)^\alpha) \frac{|j|^k}{(1+|j|)^\alpha}$$
$$\leq (1+n)^{k-\alpha} \sum_{j=-n}^{n} |f_j|(1+|j|)^\alpha \leq 2n^{k-\alpha}\|f\|_\alpha. \quad \square$$

Let $a \in W^\alpha$. Consider the functions

$$g(\varphi) = a(e^{\mathrm{i}\varphi}) = \sum_{j=-\infty}^{\infty} a_j e^{\mathrm{i}j\varphi} \quad \text{and} \quad g_n(\varphi) = a_n(e^{\mathrm{i}j\varphi}) = \sum_{j=-(n-1)}^{(n-1)} a_j e^{\mathrm{i}j\varphi}.$$

The following lemma gives an asymptotic representation when $n \to \infty$ of the function $g_n(\varphi)$ in the complex domain $\Pi_n(a)$ using the function $g(\varphi)$ (and its derivatives) defined on $[0, \pi]$.

Lemma 4.2. *Let the point $\psi = \varphi + \mathrm{i}\delta$ be in $\Pi_n(a)$. If $a(t) \in W^\alpha$, $\alpha \geq 0$, $m = [\alpha]$ then*

$$g_n(\psi) = g(\varphi) + \sum_{k=1}^{m} g^{(k)}(\varphi)(\mathrm{i}\delta)^k + \sum_{k=0}^{m+1} \alpha_{n,k}(\varphi)(\mathrm{i}\delta)^k,$$

where $\alpha_{n,k}(\varphi) \in W^0$,

$$\|\alpha_{n,k}\|_0 = o(n^{k-\alpha}), \qquad k = 0, 1, \ldots, m$$

and

$$\|\alpha_{n,m+1}\|_0 = O(n^{m+1-\alpha}).$$

Proof. We can represent $g_n(\psi)$ in the form

$$g_n(\psi) = \sum_{j=-n-1}^{n-1} a_j e^{\mathrm{i}j(\varphi+\mathrm{i}\delta)}.$$

As $|j\delta| \leq C$, we can use the asymptotics

$$e^{\mathrm{i}j(\mathrm{i}\delta)} = 1 + \sum_{k=1}^{m} \frac{(-\delta j)^k}{k!} + O(\delta j)^{m+1}.$$

So we obtain

$$\begin{aligned} g_n(\psi) &= \sum_{j=-(n-1)}^{n-1} a_j e^{\mathrm{i}j\varphi}\left(1+\sum_{k=1}^{m}\frac{(-\delta j)^k}{k!}+O(\delta j)^{m+1}\right) \\ &= g(\varphi)+\alpha_{n,0}(\varphi)+\sum_{k=1}^{m}(\mathrm{i}\delta)^k \sum_{j=-(n+1)}^{(n-1)} a_j e^{\mathrm{i}j\varphi}(ij)^k+\sum_{j=-(n+1)}^{n+1} a_j\, O(\delta j)^{m+1} \\ &= g(\varphi)+\sum_{k=1}^{m} g^{(k)}(\varphi)(\mathrm{i}\delta)^k+\alpha_{n,0}(\varphi)+\sum_{k=1}^{m}(\mathrm{i}\delta)^k\alpha_{n,k}(\varphi)+\delta^{m+1}\alpha_{n,m+1}(\varphi). \end{aligned}$$

It's easy to see, due to Lemma 4.1, that

$$\|\alpha_{n,0}(\varphi)\|_0 = \sum_{|j|>(n-1)} |a_j| \le (1+n)^{-\alpha} \sum_{|j|>(n-1)} |a_j|(1+|j|)^{\alpha} = o(n^{-\alpha}).$$

Similarly, for $k = 1, 2, \ldots, m$,

$$\|\alpha_{n,k}(\varphi)\|_0 = \sum_{|j|>(n-1)} |a_j||j|^k \le (1+n)^{k-\alpha} \sum_{|j|>(n-1)} |a_j|(1+|j|)^{\alpha} = o(n^{k-\alpha}).$$

Finally,

$$\begin{aligned} \|\alpha_{n,m+1}(\varphi)\|_0 &\le \mathrm{const} \sum_{j=-(n-1)}^{n-1} |a_j|(1+|j|)^{m+1} \\ &\le \mathrm{const}\, n^{m+1-\alpha} \sum_{j=-(n-1)}^{n-1} |a_j|(1+|j|)^{\alpha} = O(n^{m+1-\alpha}). \quad \square \end{aligned}$$

Now we can prove the correctness of the introduction of the value $\varphi_{1,n}(\lambda)$ (see (2.11))

Lemma 4.3. *Let $a(t) \in W^{\alpha}$, $\alpha \ge 2$. Then the mapping*

$$g_n:\ \Pi_n(a) \ \to\ \mathcal{R}_n(a)$$

is a bijection for every sufficiently large natural number n.

Proof. The surjectivity of this mapping follows from the definition of the set $\mathcal{R}_n(a)$.

We will prove injectivity by contradiction. Suppose that for each n there exists a couple of different points $\psi_{1,n},\ \psi_{2,n} \in \Pi_n(a)$ and a point $\lambda_n \in \mathcal{R}_n(a)$ such that

$$g_n(\psi_{1,n}) = g_n(\psi_{2,n}) = \lambda_n. \tag{4.2}$$

Without loss of generality, we can assume that the sequences $\{\psi_{1,n}\}_{n=1}^{\infty}$, $\{\psi_{2,n}\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$ have limits $\varphi_1,\ \varphi_2 \in [0,\pi]$ respectively, and $\lambda_0 \in \mathcal{R}(a)$. It is obvious that

$\varphi_1 = \varphi_2 := \varphi_0$ because of $g(\varphi_1) = g(\varphi_1) = \lambda_0$, and the mapping $g: \ [0, \pi] \to \mathcal{R}(a)$ is injective by condition (2). Suppose that $\varphi_0 \notin \{0, \pi\}$. Then by (4.2) we have

$$g_n(\psi_{2,n}) - g_n(\psi_{1,n}) = 0. \tag{4.3}$$

Applying Taylor's formula, we obtain

$$g_n'(\psi_{1,n})(\psi_{2,n} - \psi_{1,n}) + o\left(|\psi_{2,n} - \psi_{1,n}|\right) = 0.$$

The latter is impossible since

$$\lim_{n\to\infty} g_n'(\psi_{1,n}) = g'(\psi_0) \neq 0.$$

Thus, $\varphi_0 = 0$ or $\varphi_0 = \pi$. Let's suppose for definiteness that $\varphi_0 = 0$. Then for every sufficiently large natural number n we have $|\psi_{1,n}| \leq \sigma$ and $|\psi_{2,n}| \leq \sigma$ where $\sigma > 0$ is a small number. We will show that in this case the equality (4.3) is also impossible. Indeed

$$g_n(\psi_{2,n}) - g_n(\psi_{1,n}) = \int\limits_{I_n} g_n'(s)ds = \int\limits_{I_n} \left(\int\limits_{[0,s]} g_n''(u)du \right) ds, \tag{4.4}$$

where $I_n := [\psi_{1,n}, \psi_{2,n}]$ and $[0, s]$ are segments of the complex plane connecting these points. We note that identity (4.4) is true because $g_n'(0) = 0$. We rewrite (4.4) as

$$g_n(\psi_{2,n}) - g_n(\psi_{1,n}) = \frac{g_n''(\psi_{1,n})}{2}(\psi_{2,n}^2 - \psi_{1,n}^2) + \int\limits_{I_n} \left(\int\limits_{[0,s]} (g_n''(u) - g_n''(\psi_{1,n}))du \right) ds. \tag{4.5}$$

Let us estimate the integral term. Since $a(t) \in W^\alpha(\mathbb{T})$, $\alpha \geq 2$, we have $a''(t) \in W^{\alpha-2}(\mathbb{T})$. Therefore, the function $g_n''(\psi)$ is uniformly continuous in $\Pi_n(a)$ with respect to n. Thus, the following estimate holds for the difference in the integral expression (4.5):

$$\sup_{|u|\leq\sigma,\, |\psi_{1,n}|\leq\sigma} |g_n''(u) - g_n''(\psi_{1,n})| = o(1), \qquad \sigma \to 0,$$

which is uniform with respect to n. Replacing the variables in (4.5) by $u = e^{\mathrm{i}\arg s}v$ and $s = p\psi_{2,n} + (1-p)\psi_{1,n}$ we obtain

$$\left| \int\limits_{I_n} \left(\int\limits_{[0,s]} (g_n''(u) - g_n''(\psi_{1,n}))du \right) ds \right|$$
$$= |\psi_{2,n} - \psi_{1,n}| \left| \int\limits_0^1 e^{\mathrm{i}\arg s} \left(\int\limits_0^{|s|} \left(g_n''(e^{\mathrm{i}\arg s}v) - g_n''(\psi_{1,n})\right) dv \right) dp \right|$$

$$\leq |\psi_{2,n}-\psi_{1,n}|o(1)\int_0^1\left(\int_0^{|s|}dv\right)dp \leq o(|\psi_{2,n}-\psi_{1,n}|)\int_0^1 |p\psi_{2,n}+(1-p)\psi_{1,n}|dp$$

$$\leq o(|\psi_{2,n}-\psi_{1,n}|)\left(|\psi_{2,n}|+|\psi_{1,n}|\right).$$

Thus from (4.5) we get for a sufficiently large n the estimate

$$|g_n(\psi_{2,n})-g_n(\psi_{2,n})| \geq |\psi_{2,n}-\psi_{1,n}|\left(\frac{|g''(0)|}{4}|\psi_{2,n}+\psi_{1,n}|-o(|\psi_{2,n}|+|\psi_{1,n}|)\right).$$

Further, taking in account that

$$\psi_{1,n}=\varphi_{1,n}+\mathrm{i}\delta_{1,n}, \qquad \psi_{2,n}=\varphi_{2,n}+\mathrm{i}\delta_{2,n},$$

where $\varphi_{1,2,n}\geq\frac{c}{n}$ and $|\delta_{1,2,n}|\leq\frac{C}{n}$, we obtain

$$\begin{aligned}\frac{|g_n(\psi_{2,n})-g_n(\psi_{2,n})|}{|\psi_{2,n}-\psi_{1,n}|} &\geq \frac{|g''(0)|}{4}\sqrt{(\varphi_{1,n}+\varphi_{2,n})^2+(\delta_{1,n}+\delta_{2,n})^2} \\ &\quad - o\left(\sqrt{\varphi_{1,n}^2+\delta_{1,n}^2}+\sqrt{\varphi_{2,n}^2+\delta_{2,n}^2}\right) \\ &> \frac{|g''(0)|}{4}\left(\varphi_{2,n}+\varphi_{1,n}\right)-o\left(\varphi_{1,n}+|\delta_{1,n}|+\varphi_{2,n}+|\delta_{2,n}|\right) \\ &\geq \frac{|g''(0)|}{4}\cdot\frac{2c}{n}-o\left(\frac{1}{n}\right)>0.\end{aligned}$$

Consequently, $|g_n(\psi_{2,n})-g_n(\psi_{1,n})|>0$, which contradicts (4.3). The case $\varphi_0=\pi$ is treated similarly. $\square$

Remark 4.1. The function g_n is analytic and therefore it is a conformal mapping of the domain $\Pi_n(a)$ onto $\mathcal{R}_n(a)$.

Now consider the function $\hat{b}(t,\lambda_0)$ of the form (2.10), where $\lambda_0\in\mathcal{R}(a)$. It is convenient to consider it in two forms. As a function, the second argument of which is $s_0=\varphi_1(\lambda_0)$ $(=g^{-1}(\lambda_0))\ \in(0,\pi)$:

$$b(t,s_0):=\hat{b}(t,g(s_0))=\frac{(a(t)-g(s_0))\,e^{\mathrm{i}s_0}}{(t-e^{\mathrm{i}s_0})(t^{-1}-e^{\mathrm{i}s_0})} \tag{4.6}$$

and as a function, the second argument of which is $\tau=e^{\mathrm{i}s_0}\in\mathbb{T}$:

$$\tilde{b}(t,\tau):=\frac{(a(t)-a(\tau))\,\tau}{(t-\tau)(t^{-1}-\tau)}. \tag{4.7}$$

Similarly, we will consider (2.12) in the form

$$b_n(t,s) := \frac{(a_n(t) - g_n(s))\, e^{\mathrm{i}s}}{(t - e^{\mathrm{i}s})(t^{-1} - e^{\mathrm{i}s})}, \qquad s \in \Pi_n(a) \tag{4.8}$$

and in the form

$$\tilde{b}_n(t,\tau) := \frac{(a_n(t) - a_n(\tau))\, \tau}{(t - \tau)(t^{-1} - \tau)}, \qquad \tau \in \mathcal{R}_n(a). \tag{4.9}$$

The following lemma gives conditions for the functions introduced above and for their partial derivatives to be in W^α.

Lemma 4.4. *Let $a \in \mathsf{CSL}^\alpha$, $\alpha \geq 2$. If $s_0 \in (0,\pi)$, and $s \in \Pi_n(a)$, then*

i) $\tilde{b}(\cdot,\tau) \in W^{\alpha-2}$, $\tilde{b}(t,\cdot) \in W^{\alpha-2}$ *and*

$$\|\tilde{b}(\cdot,\tau)\|_{\alpha-2} \leq \text{const}\, \|a\|_\alpha, \quad \|\tilde{b}(t,\cdot)\|_{\alpha-2} \leq \text{const}\, \|a\|_\alpha, \tag{4.10}$$

$$\|b_n(\cdot,s)\|_{\alpha-2} \leq \text{const}\, \|a\|_\alpha \tag{4.11}$$

and

$$\|b(\cdot,s_0) - b_n(\cdot,s_0)\|_{\alpha-2} \leq \text{const}\, n^{-(\alpha-2)}; \tag{4.12}$$

ii) for $\alpha \geq 2 + p + \ell$, we have that

$$\frac{\partial^{p+\ell}\tilde{b}(\cdot,\tau)}{\partial^p t \partial^\ell \tau} \in W^{\alpha-2-p-\ell}, \qquad \frac{\partial^{p+\ell}\tilde{b}(t,\cdot)}{\partial^p t \partial^\ell \tau} \in W^{\alpha-2-p-\ell},$$

and

$$\left\|\frac{\partial^{p+\ell}\tilde{b}(\cdot,\tau)}{\partial^p t \partial^\ell \tau}\right\|_{\alpha-2-p-\ell} \leq \text{const}\, \|a\|_\alpha; \qquad \left\|\frac{\partial^{p+\ell}\tilde{b}(t,\cdot)}{\partial^p t \partial^\ell \tau}\right\|_{\alpha-2-p-\ell} \leq \text{const}\, \|a\|_\alpha, \tag{4.13}$$

$$\left\|\frac{\partial^{p+\ell}b_n(\cdot,s)}{\partial^p t \partial^\ell s}\right\|_{\alpha-2-p-\ell} \leq \text{const}\, \|a\|_\alpha, \tag{4.14}$$

$$\left\|\frac{\partial^{p+\ell}b(\cdot,s_0)}{\partial^p t \partial^\ell s_0} - \frac{\partial^{p+\ell}b_n(\cdot,s_0)}{\partial^p t \partial^\ell s_0}\right\|_\alpha \leq \text{const}\, n^{-(\alpha-2-p-\ell)}; \tag{4.15}$$

iii) for $\alpha < 2 + p + \ell$, we have

$$\left\|\frac{\partial^{p+\ell}b_n(\cdot,s_0)}{\partial^p t \partial^\ell s_0}\right\|_0 \leq \text{const}\, n^{2+p+\ell-\alpha}\|a\|_\alpha. \tag{4.16}$$

Here all values of "const" do not depend on τ, t, s, s_0 and n.

Proof. We represent the function $b(t,s)$ in the form

$$b(t,s_0) = c_0(s_0)t\left(b_1(t,s_0) - b_2(t,s_0)\right),$$

where

$$b_1(t,s_0) := \frac{a(t)-g(s_0)}{t-e^{\mathrm{i}s_0}}, \quad b_2(t,s_0) := \frac{a(t)-g(s_0)}{t-e^{-\mathrm{i}s_0}}, \quad c_0(s_0) = \frac{1}{2\mathrm{i}\sin s_0}.$$

Since $g(s) = g(-s)$, we get

$$b(t,s_0) = c_0(s_0)t \sum_{j=-\infty, j\neq 0}^{\infty} a_j \left(\frac{t^j - e^{\mathrm{i}js_0}}{t-e^{\mathrm{i}s_0}} - \frac{t^j - e^{-\mathrm{i}js_0}}{t-e^{-\mathrm{i}s_0}}\right) := b^{(+)}(t,s_0) + b^{(-)}(t,s_0).$$

Here the $b^{(\pm)}(t,s_0)$ are responsible for summation over $j>0$ and $j<0$, respectively. We estimate the first term.

$$\begin{aligned} b^{(+)}(t,s_0) &= c_0(s_0)t\sum_{j=1}^{\infty} a_j \sum_{k=0}^{j-1} t^k \left(e^{\mathrm{i}(j-1-k)s_0} - e^{-\mathrm{i}(j-1-k)s_0}\right) \\ &= 2\mathrm{i}c_0(s_0)t\sum_{j=1}^{\infty} a_j \sum_{k=0}^{j-2} t^k \sin(j-1-k)s_0. \end{aligned}$$

Changing the order of summation, we have

$$b^{(+)}(t,s_0) = t\sum_{k=0}^{\infty}\left(\sum_{j=k+2}^{\infty} a_j \frac{\sin(j-1-k)s_0}{\sin s_0}\right)t^k. \tag{4.17}$$

Let us get the estimate of type (4.10). For this we use the inequality

$$\left|\frac{\sin(L\cdot s)}{\sin s}\right| \le \operatorname{const} L, \qquad L>0, \tag{4.18}$$

which is true for all real s (in particular for $s = s_0 \in [0,\pi]$), and complex s, such that

$$|L\cdot \operatorname{Im} s| \le M, \tag{4.19}$$

where $M>0$ is a fixed number. Then

$$\begin{aligned} \|b^{(+)}(\cdot,s_0)\|_{\alpha-2} &\le \operatorname{const}\sum_{k=0}^{\infty}\left(\sum_{j=k+2}^{\infty}|a_j|\cdot|j-1-k|\right)(2+k)^{\alpha-2} \\ &= \operatorname{const}\sum_{j=2}^{\infty}|a_j|\left(\sum_{k=0}^{j-2}|j-1-k|\cdot(2+k)^{\alpha-2}\right). \end{aligned}$$

It is not difficult to show that

$$\sum_{k=0}^{j-2} |j-1-k|(2+k)^{\alpha-2} \leq \text{const}\,(1+j)^{\alpha},$$

whence

$$\|b^{(+)}(\cdot, s_0)\|_{\alpha-2} \leq \text{const} \sum_{j=2}^{\infty} |a_j|(1+j)^{\alpha} = \text{const}\,\|a\|_{\alpha}. \tag{4.20}$$

Similarly, it can be shown that

$$\|b^{(-)}(\cdot, s_0)\|_{\alpha-2} \leq \text{const} \cdot \sum_{-\infty}^{j=-2} |a_j|(1+|j|)^{\alpha}.$$

Obviously, the last two inequalities imply the validity of the first inequality (4.10). The second relation (4.10) is true by symmetry $\tilde{b}(t,\tau) = \tilde{b}(\tau,t)$. The inequality (4.12) also follows from (4.10) if instead of $a(t)$ we take the difference $a(t) - a_n(t)$ and use the statement of Lemma 4.1, i). The inequality (4.11) is proved in the same way as in (4.20), one should only replace s_0 with s in (4.17), the infinite upper limits by j to $n-1$, and take into account that for $s \in \Pi_n(a)$ the condition (4.19) is satisfied because $|\operatorname{Im} s| \leq \dfrac{C}{n}$ and $(j-1-k) < n$.

Let us prove item ii). According to Lemma 4.1 ii), if $f(t) \in W^{\alpha}$, then

$$\frac{\partial^p f}{\partial^p t}(t) \in W^{\alpha-p} \quad \text{and}$$
$$\left\| \frac{\partial^p f}{\partial^p t} \right\|_{\alpha-p} \leq \text{const}\,\|f\|_p. \tag{4.21}$$

Therefore, in the case $\ell = 0$, the first of the relations (4.13) is proved. By the symmetry of $\tilde{b}(t,\tau)$ we can argue that the second of the inequalities is proved. Now let $p = 0$. In the case of the first of the relations (4.13), using (4.17) we have

$$\frac{\partial^{\ell}}{\partial^{\ell} s_0} b^{(+)}(t, s_0) = \sum_{k=0}^{\infty} \left(\sum_{j=k+2}^{\infty} a_j \frac{\partial^{\ell}}{\partial^{\ell} s_0} \left(\frac{\sin(j-1-k)s_0}{\sin s_0} \right) t^k \right). \tag{4.22}$$

Then, by analogy with (4.18), we can use the inequality

$$\frac{\partial^{\ell}}{\partial s^{\ell}} \left(\frac{\sin(L \cdot s)}{s} \right) \leq \text{const}\, L^{\ell+1} \tag{4.23}$$

which is true for all real $s = s_0$ and complex s because (4.19). Then we get

$$\left\|\frac{\partial^\ell}{\partial s^\ell} b^{(+)}(t,s)\right\|_{\alpha-2-\ell} \leq \text{const} \sum_{k=0}^{\infty} \sum_{j=k+2}^{\infty} |a_j| \cdot |j-1-k|^{\ell+1} (2+k)^{\alpha-2-\ell}$$
$$= \text{const} \sum_{j=2}^{\infty} |a_j| \left(\sum_{k=0}^{j-2} |j-1-k|^{\ell+1} (2+k)^{\alpha-2-\ell} \right)$$
$$= \text{const} \sum_{j=2}^{\infty} |a_j| (1+j)^{\alpha} \leq \text{const} \, \|a\|_{\alpha}.$$

The assertion for $b^{(-)}(t,s)$ and the other assertions ii) are proved by analogy with i) and by taking into account the property (4.21). We turn to the proof of iii). Differentiating (4.22) p times over t, we obtain

$$\frac{\partial^{p+\ell} b_n(t,s_0)}{\partial^p t \partial^\ell s_0} = \sum_{k=0}^{n-1} \left(\sum_{j=k+2}^{n-1} a_j \frac{\partial^\ell}{\partial^\ell s_0} \left(\frac{\sin(j-1-k)s_0}{\sin s_0} \right) \right) \left(\prod_{v=0}^{p-1} (k-v) \right) t^{k-p}.$$

Thus, using (4.23), (replacing ∞ with $n+1$), we get

$$\left\| \frac{\partial^{p+\ell} b_n(t,s_0)}{\partial^p t \partial^\ell s_0} \right\|_0 \leq \text{const} \sum_{k=p}^{n-1} \left(\sum_{j=k+1}^{n-1} |a_j| \cdot |j-1-k|^{\ell+1} \right) k^p$$
$$\leq \text{const} \sum_{j=2}^{n-1} |a_j| \sum_{k=p}^{j-2} (j-1-k)^{\ell+1} k^p$$
$$\leq \text{const} \sum_{j=2}^{n-1} |a_j| (j+1)^{p+\ell+2} \leq \text{const} \, n^{p+\ell+2-\alpha} \sum_{j=2}^{n-1} |a_j| (j+1)^{\alpha}$$
$$\leq \text{const} \, n^{p+\ell+2-\alpha} \|a\|_{\alpha}.$$

The case of the function $b^{(-)}(t)$ is treated similarly. □

Lemma 4.5. *Let $s = s_0 + \mathrm{i}\delta \in \Pi_n(a)$. If $a(t) \in W^\alpha$, $p+\ell \leq \alpha - 2$, $m = [\alpha - 2 - p - \ell]$ then*

$$\frac{\partial^{p+\ell}}{\partial^p t \partial^\ell s} b_n(t,s) = \frac{\partial^{p+\ell}}{\partial^p t \partial^\ell s} b(t,s_0) + \sum_{k=1}^{m} \frac{\partial^{p+\ell+k}}{\partial^p t \partial^{\ell+k} s} b(t,s_0)(\mathrm{i}\delta)^k + \sum_{k=0}^{m+1} \beta_{n,k}^{p,l}(t,s_0)(\mathrm{i}\delta)^k,$$

where $\beta_{n,k}^{p,l}(\cdot,s_0) \in W^0$ and besides

$$\|\beta_{n,k}^{p,l}(\cdot,s_0)\|_0 = o\left(n^{k-\alpha-2-p-\ell}\right), \quad k = 0,1,\ldots,m,$$
$$\|\beta_{n,m+1}^{p,l}(\cdot,s_0)\|_0 = O\left(n^{(m+1)-\alpha-2-p-\ell}\right)$$

The proof of this lemma is carried out on the basis of the previous lemma, similarly to the Lemma 4.2.

An important role in the theory of Toeplitz operators is played by the concept of the topological index of a function.

Definition 1. Let the function $a(t)$ be continuous on the unit circle $\mathbb{T}$, and $a(t) \neq 0$, $t \in \mathbb{T}$. Then the topological index of the function $a(t)$ with respect to the point $z = 0$ is the integer

$$\operatorname{wind} a(t) := \frac{1}{2\pi} \arg a(t)\Big|_{\mathbb{T}}$$

where $\arg a(t)|_{\mathbb{T}}$ is the increment of the continuous branch of the argument of the function $a(t)$ when the point t makes a full turn on the curve $\mathbb{T}$ in the positive direction.

We turn to the problem about the topological index of the functions $b(t, s)$ and $b_n(t, s)$.

Lemma 4.6. *Let $a(t) \in W^\alpha$, $\alpha \geq 2$. Then for each $s_0 \in (0, \pi)$ we have*

$$\operatorname{wind} b(t, s_0) = 0$$

and for each $s \in \Pi_n(a)$, we have

$$\operatorname{wind} b_n(t, s) = 0.$$

Proof. For each $s_0 \in (0, \pi)$ the function $b(t, s_0) \neq 0$. In addition, due to the symmetry $b(e^{\mathrm{i}\varphi}, s_0) = b(e^{\mathrm{i}(2\pi-\varphi)}, s_0)$, we can see that the image of $b(e^{\mathrm{i}\varphi}, s_0)$, $\varphi \in [0, 2\pi]$ represents a curve without interior, described twice: once and back, when φ describes the segments $[0, \pi]$ and $[\pi, 2\pi]$, respectively. Thus, the first relation in the formulation of Lemma 4.6 is proved. The second is proved similarly. □

Let us now consider the sequence of functions $\eta_n(s)$ of the type (2.13)-(2.14), (2.17) and compare it with the limit function (2.15)-(2.16) given by

$$\eta(s_0) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}} \frac{\log b(\tau, s_0)}{\tau - e^{\mathrm{i}s_0}} d\tau - \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}} \frac{\log b(\tau, -s_0)}{\tau - e^{-\mathrm{i}s_0}} d\tau, \quad s_0 \in (0, \pi). \tag{4.24}$$

It's obvious that $b(\tau, s_0) \neq 0$ for $\tau \in \mathbb{T}$ and $s_0 \in [0, \pi]$. Since $\mathbb{T} \times [0, \pi]$ is a compact set we have

$$\inf_{s_0 \in [0,\pi]} \inf_{t \in \mathbb{T}} |b(t, s_0)| = \Delta > 0. \tag{4.25}$$

Thus, according to Lemma 4.5, there is a large enough natural number N_0 such that

$$\inf_{n \ge N_0} \inf_{s \in \Pi_n(a)} \inf_{t \in \mathbb{T}} |b_n(t,s)| \ge \frac{\Delta}{2}. \tag{4.26}$$

To analyze the functions $\eta_n(s)$, $\eta(s)$, we need generalized Hölder classes. We say that $f(t) \in H^\mu(K)$, $0 < \mu \le 1$, where K is a compact domain of the complex plane $\mathbb{C}$, if the following condition is satisfied:

$$\|f\|_{H^\mu} := \sup_{t \in K} |f(t)| + \sup_{t_1,\, t_2 \in K} \frac{|f(t_2) - f(t_1)|}{|t_2 - t_1|^\mu} < \infty.$$

We define the class $\mathcal{C}^{m+\mu}(K)$. Let us say that $f(z) \in \mathcal{C}^{m+\mu}(K)$, $m = 0, 1, 2, \ldots$, $0 < \mu \le 1$ if $f^{(m)}(z) \in H^\mu(K)$. Moreover, the norm of the function $f(t)$ in this space is introduced by the formula

$$\|f\|_{\mathcal{C}^{m+\mu}(K)} = \sum_{k=0}^{m-1} \sup_{t \in K} |f^{(k)}(t)| + \|f^{(m)}(t)\|_{H^\mu}.$$

Note that $\mathcal{C}^{0+\mu}(K) = H^\mu(K)$. We also agree that $C^0 := W^0$. In the following, we use the following known result (see [20], Lemma 3.6).

Lemma 4.7. *If $f(t) \in W^\alpha$, $\alpha > 0$, then $f(t) \in \mathcal{C}^{m+\mu}(\mathbb{T})$, where $m = [\alpha]$, $\mu = \alpha - [\alpha]$.*

Introduce the following notation:

$$\Lambda(t, s_0) := \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}} \frac{\log b(\tau, s_0)}{\tau - t} d\tau, \qquad s_0 \in (0, \pi)$$

and

$$\Lambda_n(t, s) := \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}} \frac{\log b_n(\tau, s)}{\tau - t} d\tau, \qquad s \in \Pi_n(a).$$

Then we get that

$$\eta(s_0) = \Lambda(e^{\mathrm{i}s_0}, s_0) - \Lambda(e^{-\mathrm{i}s_0}, s_0), \qquad s_0 \in (0, \pi),$$

and

$$\eta_n(s) = \Lambda_n(e^{\mathrm{i}s}, s) - \Lambda_n(e^{-\mathrm{i}s}, s), \qquad s \in \Pi_n(a).$$

Lemma 4.8. *If $a(t) \in \mathsf{CSL}^\alpha$, $\alpha \ge 2$, then*

i) $\eta(s_0) \in \mathcal{C}^{m+\mu}([0, \pi])$, where $m = [\alpha - 2]$, $\mu = \alpha - 2 - m$;

ii) the point $s = s_0 + \mathrm{i}\delta \in \Pi_n(a)$ *allows the representation*

$$\eta_n(s) = \eta(s_0) + \sum_{k=1}^{m} \eta^{(k)}(s_0)(\mathrm{i}\delta)^k + \sum_{k=1}^{m+1} \gamma_{n,k}(s_0)(\mathrm{i}\delta)^k$$

where

$$|\gamma_{n,k}(s_0)| = o(n^{k-(\alpha-2)}), \qquad k = 0, 1, \ldots, m,$$
$$|\gamma_{n,m+1}(s_0)| = O(n^{m+1-(\alpha-2)}).$$

Proof. According to Lemma 4.6, for each $s_0 \in [0, \pi]$ we can choose a continuous branch of $\log b(\tau, s_0)$, moreover this function will be continuous in $s_0 \in [0, \pi]$. Further, according to a well-known theorem of the theory of Banach algebras $\log b(\cdot, s_0) \in W^{\alpha-2}$ since $b(\cdot, s_0) \in W^{\alpha-2}$ and the relation (4.25) is satisfied, and the norm $\|b(\cdot, s_0)\|_{\alpha-2}$ is continuous in s_0. Note that the function $\Lambda(t, s_0)$ also has these properties, since the Cauchy singular integral operator involved in the definition of the function $\Lambda(t, s)$ is bounded in the space $W^{\alpha-2}$.

This implies that the function $\Lambda(t, s_0)$ has partial derivatives with respect to t (with a fixed s_0), up to the order m and $\dfrac{\partial^m}{\partial^m t}\Lambda(\cdot, s_0) \in H^\mu(\mathbb{T})$. On the other hand, $\Lambda(t, s_0)$ has continuous partial derivatives with respect to s_0. Indeed:

$$\frac{\partial^{p+\ell}}{\partial^p t \partial^\ell s_0}\Lambda(t, s_0) = \frac{(p-1)!}{2\pi\mathrm{i}} \int\limits_{\mathbb{T}} \frac{\frac{\partial^\ell}{\partial^\ell s_0} \log b(\tau, s_0)}{(\tau - t)^p} d\tau.$$

Applying integration by parts p times to the last integral, we obtain

$$\frac{\partial^{p+\ell}}{\partial^p t \partial^\ell s_0}\Lambda(t, s_0) = \frac{1}{2\pi\mathrm{i}} \int\limits_{\mathbb{T}} \frac{\frac{\partial^{p+\ell}}{\partial^p \tau \partial^\ell s_0} \log b(\tau, s_0)}{\tau - t} d\tau.$$

Thus, we obtain that if $p + \ell \le m$ ($m = [\alpha - 2]$), then

$$\frac{\partial^{p+\ell}}{\partial^p t \partial^\ell s_0}\Lambda(\cdot, s_0) \in \mathcal{C}^{\alpha-2-p-\ell}(\mathbb{T}) \quad \text{and} \quad \frac{\partial^{p+\ell}}{\partial^p t \partial^\ell s_0}\Lambda(t, \cdot) \in \mathcal{C}^{\alpha-2-p-\ell}([0, \pi]).$$

From the last two relations it follows that for $k = 0, 1, \ldots, m$ we have

$$\frac{\partial^k \eta(s_0)}{\partial^k s_0} \in \mathcal{C}^{\alpha-2-k}([0, \pi]).$$

Indeed, taking for example, $k = 2$, we get

$$\frac{\partial^2\eta(s_0)}{\partial^2 s_0} = \left(\frac{\partial^2}{\partial^2 t}\Lambda(t,s_0)\bigg|_{t=e^{\mathrm{i}s_0}} - \frac{\partial^2}{\partial^2 t}\Lambda(t^{-1},s_0)\bigg|_{t=e^{-\mathrm{i}s_0}}\right)$$
$$+ 2\left(\frac{\partial^2}{\partial t\partial s_0}\Lambda(t,s_0)\bigg|_{t=e^{\mathrm{i}s_0}} - \frac{\partial^2}{\partial t\partial s_0}\Lambda(t^{-1},s_0)\bigg|_{t=e^{-\mathrm{i}s_0}}\right)$$
$$+ \left(\frac{\partial^2}{\partial^2 s_0}\Lambda(t,s_0)\bigg|_{t=e^{\mathrm{i}s_0}} - \frac{\partial^2}{\partial^2 s_0}\Lambda(t^{-1},s_0)\bigg|_{t=e^{-\mathrm{i}s_0}}\right).$$

Similarly

$$\frac{\partial^k\eta_n(s_0)}{\partial^k s_0} \in C^{\alpha-2-k}([0,\pi]),$$

moreover

$$\left\|\frac{\partial^k\eta(s_0)}{\partial^k s_0} - \frac{\partial^k\eta_n(s_0)}{\partial^k s_0}\right\|_{H^{\alpha-2-k}([0,\pi])} \le \operatorname{const} n^{-(\alpha-2-k)}, \tag{4.27}$$

where the "const" is independent of n.

Let now $s = s_0 + \mathrm{i}\delta$. Then

$$\eta_n(s) = \eta_n(s_0) + \sum_{k=1}^{m} \frac{\eta_n^{(k)}(s)}{k!}(\mathrm{i}\delta)^k + o_{m+1}(s_0,\delta), \tag{4.28}$$

where $m = [\alpha-2]$ and

$$|o_{m+1}(s_0,\delta)| \le \operatorname{const} \left|\eta_n^{(m+1)}(s_0)\right| \cdot |\mathrm{i}\delta|^{m+1}. \tag{4.29}$$

According to (4.27) we have

$$\eta_n^{(k)}(s_0) = \eta^{(k)}(s_0) + \gamma_{n,k}(s_0),$$

where $|\gamma_{n,k}(s_0)| = o(n^{k-\alpha+2})$. To estimate $o_{m+1}(\delta)$, we use an inequality of the form

$$\left\|\frac{d^{k+\ell}\left(\ln b(\cdot,s_0)\right)}{d^k t d^\ell s_0}\right\|_0 \le \operatorname{const} n^{(m+1)-\alpha+2-p-\ell},$$

where $m+1 = k+\ell > \alpha-2$ and "const" does not depend on n. Then it is easy to understand that

$$|\eta_n^{(m+1)}(s_0)| \le \operatorname{const} n^{(m+1)-(\alpha-2)}.$$

Thus, the Lemma 4.8 is proved. □

5. Equation for the eigenvalues

In this section we derive an equation for finding the eigenvalues of the considered Toeplitz matrices and subject this equation to asymptotic analysis when the parameter $n \to \infty$. So, we consider the standard equation for finding eigenvalues and eigenvectors:

$$T_n(a_n - \lambda)X_n = 0, \qquad \lambda \in \mathcal{R}_n(a) \tag{5.1}$$

in the space $L_2^{(n)}$. Let us present the expression $a_n(t) - \lambda$ as a product $p(t,\lambda)\hat{b}_n(\cdot,\lambda)$, where $\hat{b}_n(\cdot,\lambda)$ is a continuous non-degenerate index-zero function and $p(t,\lambda)$ is a Laurent polynomial with three terms, which inherits the zeros of the original function $a_n(t) - \lambda$. Further, through some transformations we reduce (5.1) to an equation with an invertible operator $T_{n+2}(\hat{b}_n(\cdot,\lambda))$ on the left-hand side. By applying the operator $T_{n+2}^{-1}(\hat{b}_n(\cdot,\lambda))$ to this equation and considering the zeros of $p(t,\lambda)$ we get a homogeneous system of linear equations, the main determinant of which gives us the above-mentioned equation for finding eigenvalues.

We first prove the following result.

Lemma 5.1. *Let $a \in \mathsf{CSL}^\alpha$, $\alpha \geq 2$. Then there is a natural N_0 independent of $\lambda \in \mathcal{R}_n(a)$ so that for all $\lambda \in \mathcal{R}_n(a)$ and for all $n \geq N_0$ the operator $T_{n+2}(\hat{b}_n(\cdot,\lambda))$ is invertible and besides*

$$\left\| T_{n+2}^{-1}(\hat{b}_n(\cdot,\lambda)) \right\|_{L_2} \leq M, \tag{5.2}$$

where M does not depend on n and $\lambda \in \mathcal{R}_n(a)$.

Proof. According to Lemma 4.6, wind $\hat{b}(\cdot,\lambda_0) = 0$ for all $\lambda_0 \in \mathcal{R}_n(a)$. Thus the finite section method (see, for example [2]) ensures the existence of a natural number $N_1(\lambda_0)$ such that for $n \geq N_1(\lambda_0)$ the operator $T_n(\hat{b}(\cdot,\lambda_0))$ is invertible. Since the set $\mathcal{R}(a)$ is compact, then, for all λ_0 from $\mathcal{R}(a)$, we can choose a single number $N_1 = \sup_{\lambda_0 \in \mathcal{R}(a)} N_1(\lambda) < \infty$ such that for $n > N_1$

$$\left\| T_{n+2}^{-1}(b(\cdot,\lambda_0)) \right\|_{L_2} \leq \frac{M}{2}, \tag{5.3}$$

where M does not depend on n and λ_0.

According to the Lemma 4.5 for all small enough $\varepsilon > 0$ there is a number N_2 such that for $n > N_2$

$$\|\hat{b}(\cdot,\lambda_0) - \hat{b}_n(\cdot,\lambda)\|_0 < \varepsilon, \tag{5.4}$$

where $\lambda = g(s)$, $\lambda_0 = g(s_0)$, $s = s_0 + \mathrm{i}\delta$ $(\in \Pi_n(a))$, and the numbers ε and N_2 do not depend on s_0 and δ. Obviously (5.3) and (5.4) imply (5.2), where $N_0 = \max(N_1, N_2)$. $\square$

We state the main result of this section. Let $\lambda = g_n(s)$, $s \in \Pi_n(a)$. Then the following theorem holds.

Theorem 4. *Let $a \in \mathsf{CSL}^\alpha$, $\alpha \geq 2$, and $n > N_0$ where N_0 is a sufficiently large natural number. Then $\lambda \in \mathcal{R}_n(a)$ is an eigenvalue of $T_n(a)$ if and only if*

$$e^{-\mathrm{i}(n+1)s}\Theta_{n+2}\left(e^{\mathrm{i}s},\lambda\right)\hat{\Theta}_{n+2}\left(e^{-\mathrm{i}s},\lambda\right) - e^{\mathrm{i}(n+1)s}\Theta_{n+2}\left(e^{-\mathrm{i}s},\lambda\right)\hat{\Theta}_{n+2}\left(e^{\mathrm{i}s},\lambda\right) = 0, \tag{5.5}$$

where the functions Θ_{n+2} and $\hat{\Theta}_{n+2}$ are defined by the formulas

$$\Theta_{n+2}(t,\lambda) = T_{n+2}^{-1}\left(\hat{b}_n(\cdot,\lambda)\chi_0\right)(t); \qquad \hat{\Theta}_{n+2}(t,\lambda) = T_{n+2}^{-1}\left(\tilde{b}_n(\cdot,\lambda)\chi_0\right)(t^{-1}),$$

and $\tilde{b}_n(t,\lambda) = \hat{b}_n(1/t,\lambda)$.

Proof. Consider the equation

$$T_n(a-\lambda)X_n = 0, \quad X_n \in L_2^{(n)}.$$

Rewrite this equation in the form

$$T_n(a_n-\lambda)X_n = 0, \qquad \lambda \in \mathcal{R}_n(a). \tag{5.6}$$

By the (2.12) the above equation can be rewritten as follows:

$$P_n\hat{b}_n(\cdot,\lambda)p(\cdot,\lambda)X_n = 0, \tag{5.7}$$

where

$$p(t,\lambda) = \left(t - e^{\mathrm{i}s}\right)\left(t^{-1} - e^{\mathrm{i}s}\right).$$

(Recall that $g_n(s) = \lambda$.) Multiply equality (5.6) by the base vector $\chi_1 = t$. We obtain

$$(P_{n+1} - P_1)\hat{b}(\cdot,\lambda)p(\cdot,\lambda)\chi_1 X_n = 0. \tag{5.8}$$

Note that $P_{n+1} - P_1$ is a finite-dimensional orthogonal space projector from $L_2(\mathbb{T})$ to the linear hull of the vectors $\chi_1, \chi_2, \ldots, \chi_n$, where $\chi_j = t^j$. It is easy to see that $p(\cdot,\lambda)\chi_1 X_n \in L_2^{(n+2)}$. We set by definition

$$Y_{n+2} := P_{n+2}(a_n-\lambda)\chi_1 X_n = P_{n+2}\hat{b}_n(\cdot,\lambda)(\chi_1 p(\cdot,\lambda)X_n) = P_{n+2}(\hat{b}_n(\cdot,\lambda))(\chi_1 p(\cdot,\lambda)X_n).$$

Then equation (5.8) can be rewritten in the form

$$(P_{n+1} - P_1)Y_{n+2} = 0.$$

The last equality means that the values of Y have the representation

$$Y_{n+2} = y_0\chi_0 + y_{n+1}\chi_{n+1}.$$

According to the Lemma 5.1 the operator $T_{n+2}(\hat{b}_n(\cdot,\lambda))$ is invertible. So we get that

$$T^{-1}_{n+2}(\hat{b}_n(\cdot,\lambda))Y_{n+2} = \chi_1 p(\cdot,\lambda)X_n,$$

that is,

$$y_0[T^{-1}_{n+2}(\hat{b}_n(\cdot,\lambda))\chi_0](t) + y_{n+1}[T^{-1}_{n+2}(\hat{b}_n(\cdot,\lambda))\chi_{n+1}](t) = tp(t,\lambda)X_n(t). \tag{5.9}$$

We introduce the reflection operator acting on the space $L_2^{(n)}$ by the rule

$$(W_n f_n)(t) = \sum_{j=0}^{n-1} f_{n-1-j}t^j,$$

where

$$f_n(t) = \sum_{j=0}^{n-1} f_j t^j.$$

From the identity $W_{n+2}T_{n+2}(\hat{b}_n)W_{n+2} = T_{n+2}(\tilde{b}_n)$, which is easy to verify by simple calculation, we get

$$[T^{-1}_{n+2}(\hat{b}_n(\cdot,\lambda))\chi_{n+1}](t) = t^{n+1}T^{-1}_{n+2}(\tilde{b}_n(\cdot,\lambda))(t^{-1}).$$

Thus, equation (5.9) we can rewrite in the form

$$y_0\Theta_{n+2}(t,\lambda) + y_{n+1}t^{n+1}\hat{\Theta}_{n+2}(t,\lambda) = tp(t,\lambda)X_n(t). \tag{5.10}$$

Considering that the multiplier $p(t,\lambda)$ disappears when $t = e^{\mathrm{i}s}$ and when $t = e^{-\mathrm{i}s}$, we conclude that y_0 and y_{n+1} must satisfy the following homogeneous system of linear algebraic equations:

$$\begin{aligned}
&\Theta_{n+2}(e^{\mathrm{i}s},\lambda)y_0 + e^{\mathrm{i}(n+1)s}\hat{\Theta}_{n+2}(e^{\mathrm{i}s},\lambda)y_{n+1} = 0,\\
&\Theta_{n+2}(e^{-\mathrm{i}s},\lambda)y_0 + e^{-\mathrm{i}(n+1)s}\hat{\Theta}_{n+2}(e^{-\mathrm{i}s},\lambda)y_{n+1} = 0.
\end{aligned} \tag{5.11}$$

Note that if $y_0 = y_{n+1} = 0$ then by (5.10) we get $X_n \equiv 0$. Therefore, the original equation (5.6) has a non-trivial solution X_n if, and only if, the determinant of the system of equations (5.11) is zero. This is the form of the required equality (5.5). □

To investigate the asymptotic behavior of the functions Θ_n and $\hat{\Theta}_n$ when $n \to \infty$, we introduce the Toeplitz operator (infinite-dimensional) which corresponds to the matrix $(b_{i-j})_{i,j=0}^{\infty}$. Let $P:\, L_2(\mathbb{T}) \to H_2(\mathbb{T})$ be the projector defined by

$$(Pf)\,(t) = \sum_{j=0}^{\infty} f_j t^j, \quad \text{where} \quad \sum_{j=-\infty}^{\infty} f_j t^j \;\in\; L_2(\mathbb{T}),$$

and $H_2((T))$ be Hardy's famous space. Then

$$[T(b)f](t) = [Pbf](t) \tag{5.12}$$

is called the Toeplitz operator with symbol b (see [3]).

The subsequent reasoning is essentially based on the general theory of projection methods (see [2], [32], [33]). Recall the definition of Wiener-Hopf factorization. Let the function f belong to the Wiener class, i.e. $f \in W^{\alpha}$ and $f(t) \neq 0$, $t \in \mathbb{T}$. Then there is the representation of the function f as the following product:

$$f = f_+ t^{\kappa} f_-,$$

where $\kappa = \mathrm{wind}(f)$, $f_\pm \in W_\pm^\alpha$, and $\mathrm{wind}(f_\pm) = 0$. Here we assume

$$W_\pm^\alpha = \{f \in W^\alpha : f(t) = \sum_{j=0}^{\infty} f_{\pm j} t^{\pm j}\}.$$

By virtue of Lemma 4.6 and equation (4.25), the function $b(t,\lambda)$ is factorisable in the space $W^{\alpha-2}$ (see Lemma 4.4 i)), while the factorization factors $\hat{b}_\pm(t,\lambda)$ can be written in the form

$$\hat{b}_\pm(t,\lambda) = \exp\left(\frac{1}{2}\log(\hat{b}(t,\lambda)) \pm \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}} \frac{\log(\hat{b}(\tau,\lambda))}{\tau - t} d\tau\right), \tag{5.13}$$

where the integral is understood in the sense of the principal value. The functions $\hat{b}_\pm(t,\lambda)$ can be analytically continued inside and outside respectively of the unit circle $\mathbb{T}$ by the formula

$$\hat{b}_\pm(t,\lambda) = \exp\left(\pm \frac{1}{\pi \mathrm{i}} \int_{\mathbb{T}} \frac{\log(\hat{b}(\tau,\lambda))}{\tau - t} d\tau\right), \qquad |t^{\pm 1}| < 1.$$

Note that due to the symmetry $b(t^{-1},\lambda) = b(t,\lambda)$, the factorization factors satisfy the relation

$$\hat{b}_-(t,\lambda) = \hat{b}_+(t^{-1},\lambda)/\chi_b(\lambda), \tag{5.14}$$

where the number χ_b can be calculated by the formula

$$\chi_b(\lambda) = \exp\left\{\frac{1}{\pi}\int_0^{\pi} \log \hat{b}(e^{i\varphi}, \lambda) d\varphi\right\}. \tag{5.15}$$

Similarly, according to Lemma 4.6 and (4.26), the function $\hat{b}_n(t, \lambda)$ also has the Wiener-Hopf factorization

$$\hat{b}_n(t, \lambda) = \hat{b}_{n,+}(t, \lambda)\hat{b}_{n,-}(t, \lambda). \tag{5.16}$$

Note that the functions $b_{n,\pm}(t, \lambda)$ represent polynomials of degree $n-1$ of the variables t and t^{-1} respectively. In addition, due to (4.11) we have

$$\|\hat{b}_{n,\pm}(\cdot, \lambda)\|_{\alpha-2} \leq \text{const}\,\|a\|_{\alpha}. \tag{5.17}$$

Next we need to consider the functions $\hat{b}_n(t, \lambda)$ and $b_{n,\pm}(t, \lambda)$ in the annulus

$$K_n = \left\{z \in \mathbb{C} | 1 - \frac{C}{n} \leq |z| \leq 1 + \frac{C}{n}\right\}, \qquad C > 0,$$

(see the definition of $\Pi_n(a)$ in (2.8)). Consider the numbers $r_n^{\pm} = 1 \pm \frac{C}{n}$, $C > 0$.

Lemma 5.2. *Let the function $f_n(t)$ be in $L_2^{(n)}$. Then*

$$\|f_n(r_n^{\pm}t)\|_{\alpha} \leq \text{const}\,\|f_n(t)\|_{\alpha}.$$

Proof. Let

$$f_n(t) = \sum_{j=-(n-1)}^{n-1} f_j t^j.$$

Then

$$\|f_n(r_n^{\pm}t)\|_{\alpha} = \sum_{j=-(n-1)}^{n+1} |f_j| \cdot |r_n^{\pm}|^j (1 + |j|)^{\alpha}.$$

Since $|j| \leq n-1$,

$$|r_n^{\pm}|^j = \left(1 + \frac{C}{n}\right)^j \leq \exp\left(\frac{|j|C}{n}\right) \leq \exp(C).$$

Thus

$$\|f_n(r_n^{\pm}t)\|_\alpha \le \exp(C) \sum_{j=-(n-1)}^{n+1} |f_j| \cdot (1+|j|)^\alpha = \mathrm{const}\, \|f_n(t)\|_\alpha. \quad \square$$

The following statement follows from the above lemma.

Lemma 5.3. *Let $a \in \mathrm{CSL}^\alpha$, $\alpha \ge 2$. Then the functions $\hat{b}(r_n^{\pm}t, \lambda)$ belong to $W^{\alpha-2}$, with*

$$\sup_{\lambda \in \mathcal{R}_n(a)} \|\hat{b}_n(r_n^{\pm}t, \lambda)\|_{\alpha-2} \le \mathrm{const}\, \|a\|_\alpha. \tag{5.18}$$

Besides, $\hat{b}_{n,\pm}(r_n^{\pm}t, \lambda) \in W^{\alpha-2}$ and

$$\sup_{\lambda \in \mathcal{R}_n(a)} \|\hat{b}_{n,\pm}(r_n^{\pm}t, \lambda)\|_{\alpha-2} \le \mathrm{const}\, \|a\|_\alpha. \tag{5.19}$$

Proof. According to Lemma 5.2 we get (5.18). Further, by Lemma 4.5 we obtain that for sufficiently large n, $\hat{b}_n(r_n^{\pm}t, \lambda) \ne 0$ and $\mathrm{wind}\, \hat{b}_n(r_n^{\pm}t, \lambda) = 0$ for all $t \in \mathbb{T}$. Thus, the functions $\hat{b}_n(r_n^{\pm}t, \lambda)$ admit a Wiener-Hopf factorization in the space $W^{\alpha-2}$ and the following inequalities hold:

$$\|\hat{b}_{n,\pm}(r_n^{\pm}t, \lambda)\|_{\alpha-2} \le \mathrm{const}\, \|\hat{b}_n(r_n^{\pm}t, \lambda)\|_{\alpha-2}.$$

Applying Lemma 5.2 and inequality (5.18) we get (5.19). $\square$

Now we are ready to get an asymptotic representation of the functions $\Theta_{n+2}(t, \lambda)$ from Theorem 4. To do this, we note that the inverse of the Toeplitz operator (5.12) calculated by the formula

$$T^{-1}(b) = b_+^{-1}(t) P b_-^{-1}(t),$$

where $b(t) = b_+(t) b_-(t)$ is a Wiener-Hopf factorization in the space W^α (see (5.13)). Thus

$$T^{-1}(\hat{b}_n)\chi_0 = \hat{b}_{n,+}^{-1}(t) P \left(\hat{b}_{n,-}^{-1}(\cdot) \cdot 1 \right)(t) = \hat{b}_{n,+}^{-1}(t). \tag{5.20}$$

Lemma 5.4. *Let $a \in \mathsf{CSL}^\alpha$, $\alpha \ge 2$. Then the following asymptotic representation holds:*

$$\Theta_{n+2}(t, \lambda) = \hat{b}_{n,+}^{-1}(t, \lambda) + \tilde{R}_1^{(n)}(t, \lambda), \quad \hat{\Theta}_{n+2}(t, \lambda) = \hat{b}_{n,-}^{-1}(t^{-1}, \lambda)/\chi_b(\lambda) + \tilde{R}_2^{(n)}(t, \lambda),$$

where, for $n \to \infty$

$$\sup \left\{ \left| \tilde{R}_j^{(n)}(z, \lambda) \right| : (z, \lambda) \in K_n \times \mathcal{R}_n(a) \right\} = o\left(\frac{1}{n^{\alpha-2}} \right), \qquad j = 1, 2,$$

and $\chi_b(\lambda)$ is given by (5.15).

Proof. From the definition of $\Theta_n(t,\lambda)$ (see (5.20)) it follows that

$$\Theta_n(t,\lambda) = \hat{b}_{n,+}^{-1}(t,\lambda) + R_1^{(n)}(t,\lambda),$$

where

$$\begin{aligned}\tilde{R}_1^{(n)}(t,\lambda) &= T_{n+2}^{-1}\left(\hat{b}_n(\cdot,\lambda)\right)\chi_0 - T^{-1}\left(\hat{b}_n(\cdot,\lambda)\chi_0\right) \\ &= T_{n+2}^{-1}\left(\hat{b}_n(\cdot,\lambda)\right)\left[T\left(\hat{b}_n(\cdot,\lambda)\right) - T_{n+2}\left(\hat{b}_n(\cdot,\lambda)\right)\right]T^{-1}\left(\hat{b}_n(\cdot,\lambda)\right)\chi_0 \\ &\quad - Q_{n+2}T^{-1}\left(\hat{b}_n(\cdot,\lambda)\right)\chi_0.\end{aligned}$$

By using the obvious equalities

$$PbP = P_n b P_n + P_n b Q_n + Q_n b P_n + Q_n b Q_n$$

and (5.20) we get

$$\begin{aligned}\tilde{R}_1^{(n)}(t,\lambda) = &\left[T_{n+2}^{-1}\left(\hat{b}_n(\cdot,\lambda)\right)P_{n+2}\hat{b}_n(\cdot,\lambda)Q_{n+2}\hat{b}_n^{-1}(\cdot,\lambda)\right](t) \\ &- \left[Q_{n+2}\left(\hat{b}_{n,+}^{-1}(\cdot,\lambda)\right)\right](t), \qquad t\in\mathbb{T}.\end{aligned} \tag{5.21}$$

Thus

$$\begin{aligned}\|\tilde{R}_1^{(n)}(\cdot,\lambda)\|_{\alpha-2} \le &\left(\|T_{n+2}^{-1}\left(\hat{b}_n(\cdot,\lambda)\right)\|_{\alpha-2}\cdot\|P_{n+2}\left(\hat{b}_n(\cdot,\lambda)\right)\|_{\alpha-2}+1\right) \\ &\cdot\left\|Q_{n+2}\left(\hat{b}_{n,+}^{-1}(\cdot,\lambda)\right)\right\|_{\alpha-2}.\end{aligned}$$

From (5.2)

$$\|T_{n+2}^{-1}\left(\hat{b}_n(\cdot,\lambda)\right)\|_{\alpha-2} \le M.$$

Further

$$\|P_{n+2}\left(\hat{b}_n(\cdot,\lambda)\right)\|_{\alpha-2} \le \text{const}\,\|\hat{b}_n(\cdot,\lambda)\|_{\alpha-2} \le \text{const}\,\|\hat{b}(\cdot,\lambda)\|_{\alpha-2}.$$

Finally, Lemma 4.1 i) gives

$$\|Q_{n+2}\left(\hat{b}_{n,+}^{-1}(\cdot,\lambda)\right)\|_{\alpha-2} = o\left(\frac{1}{n^{\alpha-2}}\right). \tag{5.22}$$

Thus

$$\|\tilde{R}_1^{(n)}\|_{\alpha-2} = o\left(\frac{1}{n^{\alpha-2}}\right). \tag{5.23}$$

In the above calculations $t \in \mathbb{T}$. Consider the case $z \in K_n$, i.e.,

$$z = rt, \qquad r \in [r_n^-, r_n^+]. \tag{5.24}$$

Denote the first and second term in (5.21), respectively, $R_{1,1}^{(n)}(t,\lambda)$ and $R_{1,2}^{(n)}(t,\lambda)$. Note that $R_{1,1}^{(n)}(t,\lambda) \in L_2^{(n+2)}$ and with (5.22)–(5.23), we get

$$\|\tilde{R}_{1,1}^{(n)}(t,\lambda)\|_{\alpha-2} = o\left(\frac{1}{n^{\alpha-2}}\right), \qquad t \in \mathbb{T}.$$

Thus, Lemma 5.2 implies that

$$\sup_{r\in[r_n^-, r_n^+]} \|\tilde{R}_{1,1}^{(n)}(rt,\lambda)\|_{\alpha-2} = o\left(\frac{1}{n^{\alpha-2}}\right).$$

Consider now $\tilde{R}_{1,2}^{(n)}(t,\lambda) := \left[Q_{n+2}\left(\hat{b}_{n,+}^{-1}(\cdot,\lambda)\right)\right](t)$. The function $b_{n,+}(t,\lambda)$ is in $L_2^{(n+2)}$ and by Lemma 5.3 we have

$$\sup_{r\in[r_n^-, r_n^+]} \|b_{n,+}(rt,\lambda)\|_{\alpha-2} \le \text{const}\, \|a\|_\alpha.$$

In addition, Lemma 4.5 implies that

$$\sup_{z\in K_n} |b_{n,+}(z,\lambda)| \ge \delta,$$

where δ does not depend on n and λ. According to a standard theorem of the theory of Banach algebras, for all $r \in [1 - C/n, 1 + C/n]$, $b_{n,+}^{-1}(rt,\lambda) \in W^{\alpha-2}$, and in addition

$$\begin{aligned}\|b_{n,+}^{-1}(rt,\lambda)\|_{\alpha-2} &\le \text{const}\, \delta^{-1} \|b_{n,+}(rt,\lambda)\|_{\alpha-2} \\ &\le \text{const}\, \delta^{-1} \|a(t,\lambda)\|_\alpha, \qquad (|t| = 1).\end{aligned}$$

Thus, according to Lemma 4.1 i) it is possible to show that

$$\sup_{r\in[r_n^-, r_n^+]} \|\tilde{R}_{1,2}^{(n)}(rt,\lambda)\|_{\alpha-2} = o\left(\frac{1}{n^{\alpha-2}}\right)$$

and therefore

$$\sup_{r\in[r_n^-, r_n^+]} \|\tilde{R}_{1}^{(n)}(rt,\lambda)\|_{\alpha-2} = o\left(\frac{1}{n^{\alpha-2}}\right).$$

Since

$$\|f\|_0 \le \|f\|_{\alpha-2},$$

this lemma is proved for the case $j = 1$. The case $j = 2$ can be treated similarly. □

Denote

$$\hat{b}_n(t, g_n(s)) := b_n(t, s), \qquad \hat{b}_{n,\pm}(t, g_n(s)) := b_{b,\pm}(t, s),$$
$$\tilde{R}_j(t, g_n(s)) := R_j(t, s), \qquad j = 1, 2.$$

Note that as in (5.14) we have

$$b_{n,-}(t^{-1}, s) = b_{n,+}(t, s)/\chi_b. \tag{5.25}$$

We introduce a continuous function $\eta_n(s)$ satisfying the relation

$$\frac{b_{n,+}(e^{\mathrm{i}s}, s)}{b_{n,+}(e^{-\mathrm{i}s}, s)} = e^{-\mathrm{i}\eta_n(s)}, \quad s \in \Pi_n(a). \tag{5.26}$$

We take the continuous branch of the function $\eta_n(s)$ specified by $\eta_n(0) = 0$. It is not difficult to see that

$$\eta_n(\pi) = \eta_n(0) = 0. \tag{5.27}$$

Lemma 5.5. *Let $\alpha \geq 2$ and $a \in \mathsf{CSL}^\alpha$. Then there is such natural N_0, larger enough, that for all $n \geq N_0$ the number $\lambda = g_n(s)$ is an eigenvalue of $T_n(a)$ if and only if $s \in \Pi_n(a)$ satisfy the equation*

$$(n+1)s + \eta_n(s) + R_6^{(n)}(s) = \pi j, \qquad j = 1, 2, \ldots, n \tag{5.28}$$

where

$$R_6^{(n)}(s) = o(1/n^{\alpha-2}) \tag{5.29}$$

uniformly in j.

Proof. Considering the results of Lemma 5.4, we rewrite equality (5.5) in the form

$$e^{-\mathrm{i}(n+1)s}\left(b_{n,+}(e^{\mathrm{i}s}, s) + R_1^{(n)}(e^{\mathrm{i}s}, s)\right)\left(b_{n,-}(e^{-\mathrm{i}s}, s) + R_2^{(n)}(e^{-\mathrm{i}s}, s)\right)$$
$$= e^{\mathrm{i}(n+1)s}\left(b_{n,+}(e^{-\mathrm{i}s}, s) + R_1^{(n)}(e^{-\mathrm{i}s}, s)\right)\left(b_{n,-}(e^{\mathrm{i}s}, s) + R_2^{(n)}(e^{\mathrm{i}s}, s)\right),$$
$$e^{2\mathrm{i}(n+1)s} = \frac{b_{n,+}(e^{\mathrm{i}s}, s)b_{n,-}(e^{-\mathrm{i}s}, s)\left(1 + R_3^{(n)}(s)\right)}{b_{n,+}(e^{-\mathrm{i}s}, s)b_{n,-}(e^{\mathrm{i}s}, s)\left(1 + R_4^{(n)}(s)\right)}$$

where

$$R_3^{(n)}(s) = b_{n,+}^{-1}(e^{\mathrm{i}s}, s)R_1^{(n)}(e^{\mathrm{i}s}, s) + b_{n,-}^{-1}(e^{-\mathrm{i}s}, s)R_2^{(n)}(e^{-\mathrm{i}s}, s)$$
$$+ b_{n,+}^{-1}(e^{\mathrm{i}s}, s)b_{n,-}^{-1}(e^{-\mathrm{i}s}, s)R_1^{(n)}(e^{\mathrm{i}s}, s)R_2^{(n)}(e^{-\mathrm{i}s}, s),$$
$$R_4^{(n)}(s) = R_3^{(n)}(-s).$$

Considering (5.25), we get

$$e^{2\mathrm{i}(n+1)s} = e^{-2\mathrm{i}\eta_n(s)} \frac{1 + R_3^{(n)}(s)}{1 + R_4^{(n)}(s)}.$$

Let

$$R_5^{(n)}(s) := \log\left(\frac{1 + R_3^{(n)}(s)}{1 + R_4^{(n)}(s)}\right).$$

Then

$$e^{2\mathrm{i}(n+1)s} = e^{-2\mathrm{i}\eta_n(s) + R_5^{(n)}(s)}.$$

The last equation is equivalent to the following set of equations:

$$2\mathrm{i}(n+1)s = -2\mathrm{i}\eta_n(s) + R_5^{(n)}(s) + 2\mathrm{i}\pi j, \qquad j \in \mathbb{Z}.$$

Assuming

$$R_6^{(n)}(s) := -\frac{R_5^{(n)}(s)}{2\mathrm{i}}$$

we get

$$(n+1)s + \eta_n(s) + R_6^{(n)}(s) = \pi j, \qquad j \in \mathbb{Z}.$$

Considering the relations connecting $R_6^{(n)}(s)$ with $R_1^{(n)}(e^{\pm \mathrm{i}s}, s)$ and $R_2^{(n)}(e^{\pm \mathrm{i}s})$, we get the asymptotic expansion (5.29). □

6. Solvability analysis of equation (5.28)

Rewrite the equation (5.28) in the form

$$F_n(s) + \frac{R_6^{(n)}(s)}{n+1} = d_{j,n}, \tag{6.1}$$

where

$$F_n(s) := s + \frac{\eta_n(s)}{n+1}, \tag{6.2}$$

and

$$d_{j,n} := \frac{\pi j}{n+1}.$$

Along with (6.1), consider the approximating equation

$$F_n(s) = d_{j,n}, \quad j = 1, 2, \ldots, n. \tag{6.3}$$

We introduce the notion of the modulus of continuity in the complex domain. Let a function $f(z)$ be continuous in some bounded domain G of the complex plane. Then the modulus of continuity $f(z)$ is the function

$$w_f(\delta) := \sup_{z_{1,2} \in G,\ |z_1 - z_2| \le \delta} |f(z_1) - f(z_2)|, \qquad 0 < \delta \le \delta_0.$$

Let us introduce the domains

$$\Pi_{j,n}(a) := \left\{ s \in \Pi_n(a) \,:\, |s - e_{j,n}| \le \frac{c_n}{n+1} \right\}, \tag{6.4}$$

where

$$e_{j,n} := d_{j,n} - \frac{\eta_n(d_{j,n})}{n+1} \text{ and } c_n := 2 \left\| R_6^{(n)}(s) \right\|_\infty + w_{\eta_n}\left(\frac{2\|\eta_n\|_\infty}{n+1} \right), \tag{6.5}$$

and the norm $\|\cdot\|_\infty$ is defined in the standard way on the set G, where $G = \Pi_n(a)$. Recall that

$$\Pi_n(a) = \left\{ s = s_0 + \mathrm{i}\delta \,|\, s_0 \in [cn^{-1}, \pi - cn^{-1}],\ \delta \in [-Cn^{-1}, Cn^{-1}] \right\},$$

where c, C are some fixed positive numbers such that c is small enough and C is large enough.

In the following statement we will apply the principle of contractive mappings to the analysis of the solvability of (6.1), (6.3).

Let's introduce the mappings

$$\Phi_{j,n}(s) := d_{j,n} - \frac{\eta_n(s)}{n+1}.$$

Lemma 6.1. *Let the function a be in CSL^α, $\alpha \ge 2$. Then, if $s \in \Pi_{j,n}(a)$,*

i) $\Phi_{j,n}(s) \in \Pi_{j,n}(a)$,

ii) $\left(\Phi_{j,n}(s) + \dfrac{R_6^{(n)}(s)}{n+1}\right) \in \Pi_{j,n}(a).$

Proof. We prove ii). Let $s \in \Pi_{j,n}(a)$. Then for sufficiently large n, we get

$$\begin{aligned}\left|\Phi_{j,n}(s) + \frac{R_6^{(n)}(s)}{n+1} - e_{j,n}\right| &\le \frac{|\eta_n(s) - \eta_n(d_{j,n})|}{n+1} + \frac{|R_6^{(n)}(s)|}{n+1} \\ &\le \frac{w_{\eta_n}(|s-d_{j,n}|)}{n+1} + \frac{\|R_6^{(n)}\|_\infty}{n+1} \le \frac{w_{\eta_n}\left(|s-e_{j,n}| + \frac{|\eta_n(d_{j,n})|}{n+1}\right)}{n+1} + \frac{\|R_6^{(n)}\|_\infty}{n+1} \\ &\le \frac{w_{\eta_n}\left(\frac{c_n}{n+1} + \frac{\|\eta_n\|_\infty}{n+1}\right)}{n+1} + \frac{\|R_6^{(n)}\|_\infty}{n+1} \le \frac{w_{\eta_n}\left(2\frac{\|\eta_n\|_\infty}{n+1}\right)}{n+1} + \frac{\|R_6^{(n)}\|_\infty}{n+1} \le \frac{c_n}{n+1}.\end{aligned}$$

Thus, item ii) of the lemma is proved.

The item i) is proved similarly if we put $R_6^{(n)}(s) \equiv 0$. □

Theorem 5. *Let the function a belong to CSL^α. Then:*

i) For $\alpha \ge 2$ the equation (6.1) has a unique solution $s_{j,n} \in \Pi_{j,n}(a)$, $j = 1, 2, \dots, n$, and all $s_{j,n}$ are different.

ii) Let $s^*_{j,n}$ be a solution of the equation (6.3) belonging to $\Pi_{j,n}(a)$. Then for $\alpha \ge 3$

$$\|s_{j,n} - s^*_{j,n}\| = O\left(\frac{1}{n^{\alpha-1}}\right),$$

where the estimate is uniform in n and j.

Proof. Let us prove statement i). For this purpose, consider the sequence

$$s_{j,n}^{(0)} = e_{j,n}, \qquad s_{j,n}^{(k+1)} = \Phi_{j,n}(s_{j,n}^{(k)}) + \frac{R_6^{(n)}(s_{j,n})}{n+1}, \quad k = 1, 2, \dots.$$

According to Lemma 6.1 ii), the sequence $\{s_{j,n}^{(k)}\}_{k=1}^\infty$ is contained in the domain $\Pi_{j,n}(a)$. Choose from it some convergent subsequence and denote its limit by $\tilde{s}_{j,n}$. Obviously, $\tilde{s}_{j,n}$ satisfies (6.1). Note that for any $j_1 \ne j_2$

$$\Pi_{j_1,n}(a) \cap \Pi_{j_2,n}(a) = \varnothing,$$

because $|e_{j_1,n} - e_{j_2,n}| \ge \frac{\Delta}{n+1}$, $(\Delta > 0)$, while $\operatorname{diam} \Pi_{j,n} = o(1/n)$. Thus, according to the Lemma 5.5, the numbers $g(\tilde{s}_{j,n})$, $j = 1, 2, \dots, n$ are eigenvalues of the matrix $T_n(a)$. Since this matrix has at most n, we conclude that $\tilde{s}_{j,n}$ is a unique solution of equation (6.1) in the domain $\Pi_{j,n}(a)$. Denoting $\tilde{s}_{j,n} := s_{j,n}$, we completed the proof of i).

Let us prove ii). Substituting into the equations (6.1) and (6.3) respectively after the two numbers $s_{j,n}$ and $s^*_{j,n}$, and subtracting the second expression from the first one, we get

$$(s_{j,n} - s^*_{j,n}) + \frac{\eta_n(s_{j,n}) - \eta_n(s^*_{j,n})}{n+1} = -\frac{R_6^{(n)}(s_{j,n})}{n+1}. \tag{6.6}$$

Since $\alpha \geq 3$, according to Lemma 4.9, $\eta_n(s)$ has a derivative that is bounded uniformly in n. In this way we obtain

$$|\eta_n(s_{j,n}) - \eta_n(s^*_{j,n})| \leq \eta_1 |s_{j,n} - s^*_{j,n}|,$$

where

$$\eta_1 = \sup_{n \in \mathbb{N}} \sup_{s \in \Pi_n(a)} |\eta_n'(s)| < \infty.$$

From (6.6) we get

$$|s_{j,n} - s^*_{j,n}| \leq \frac{\eta_1 |s_{j,n} - s^*_{j,n}|}{n+1} + \frac{\left|R_6^{(n)}(s_{j,n})\right|}{n+1}.$$

From (5.29) it follows that

$$|s_{j,n} - s^*_{j,n}| \left(1 - \frac{\eta_1}{n+1}\right) \leq \frac{\left|R_6^{(n)}(s_{j,n})\right|}{n+1}$$

and finally it results that

$$|s_{j,n} - s^*_{j,n}| = o\left(\frac{1}{n^{\alpha-1}}\right). \quad \square$$

The statement proved above shows that the roots $s_{j,n}$ of the equation (6.1) can be approximated by the roots $s^*_{j,n}$ of the equation (6.3) for large values of n. Besides, the values $s^*_{j,n}$ can be approximated using the method of successive approximations by the values $s^{*(k)}_{j,n}$ defined in the following way:

$$s^{*(0)}_{j,n} = e_{j,n}, \quad s^{*(k+1)}_{j,n} = \Phi_n(e^{*(k)}_{j,n}), \quad k = 0, 1, \ldots. \tag{6.7}$$

Lemma 6.2. *Let the function a be in CSL^α, $\alpha \geq 3$. Then the equation (6.3) has a unique solution $s^*_{j,n} \in \Pi_{j,n}(a)$, $j = 1, 2, \ldots, n$ and for sufficiently large n, the following estimate is valid:*

$$|s^*_{j,n} - s^{*(k)}_{j,n}| \leq 2\frac{\eta}{\eta_1^2}\left(\frac{\eta_1}{n+1}\right)^{k+2}, \tag{6.8}$$

where

$$\eta = \sup_{n\in\mathbb{N}} \sup_{s\in\Pi_n(a)} |\eta_n(s)|,$$

$$\eta_1 = \sup_{n\in\mathbb{N}} \sup_{s\in\Pi_n(a)} |\eta_n'(s)|.$$

Proof. We show that sequence $\left\{s_{j,n}^{*(k)}\right\}_{k=1}^{\infty}$ is convergent. Indeed, according to Lemma 4.8, the functions $\eta_n(s)$ and $\eta_n'(s)$ are bounded uniformly respect to n. That is,

$$\sup_{n\in\mathbb{N}} \sup_{s\in\Pi_n(a)} |\eta_n'(s)| = \eta_1 < \infty.$$

Then we have

$$\begin{aligned}|s_{j,n}^{*(1)} - s_{j,n}^{*(0)}| &= \frac{|\eta_n(e_{j,n}) - \eta_n(d_{j,n})|}{n+1} \le \eta_1 \frac{|e_{j,n} - d_{j,n}|}{n+1} \\ &= \eta_1 \frac{|\eta_n(d_{j,n})|}{(n+1)^2} \le \frac{\eta_1 \eta}{(n+1)^2}.\end{aligned}$$

Further

$$\begin{aligned}|s_{j,n}^{*(2)} - s_{j,n}^{*(1)}| = |\Phi_{j,n}(s_{j,n}^{*(1)}) - \Phi_{j,n}(s_{j,n}^{*(0)})| &= \frac{|\eta_n(s_{j,n}^{*(1)}) - \eta_n(s_{j,n}^{*(0)})|}{n+1} \\ \le \eta_1 \frac{|s_{j,n}^{*(1)} - s_{j,n}^{*(0)}|}{n+1} &\le \frac{\eta_1^2 \eta}{(n+1)^3}.\end{aligned}$$

Similarly

$$|s_{j,n}^{*(k+1)} - s_{j,n}^{*(k)}| \le \frac{\eta \eta_1^k}{(n+1)^{k+2}}. \tag{6.9}$$

Since $\eta_1/(n+1) < 1$ for sufficiently large n, the sequence $\left\{s_{j,n}^{*(k)}\right\}_{k=1}^{\infty}$ converges to $s_{j,n}^*$.

From the estimate (6.9) it follows that

$$|s_{j,n}^{*(k+m)} - s_{j,n}^{*(k)}| \le \frac{\eta}{\eta_1^2} \left(\frac{\eta_1}{n+1}\right)^{k+2} \frac{1 - \left(\frac{\eta_1}{n+1}\right)^{m+1}}{1 - \frac{\eta_1}{n+1}}.$$

Assuming in this inequality that $n + 1 > \eta_1$ and passing to the limit when $m \to \infty$, we get that

$$|s_{j,n}^{*} - s_{j,n}^{*(k)}| \le \frac{\eta}{\eta_1^2} \left(\frac{\eta_1}{n+1}\right)^{k+2} \frac{1}{1 - \frac{\eta_1}{n+1}}.$$

The assertion of the lemma obviously follows from this inequality. □

Now we are ready to prove the main results of the work.

7. Proof of the main results

Theorem 1 follows from Lemma 5.5 and Theorem 5.

7.1. Proof of Theorem 2

Let $2 \le \alpha < 3$. We estimate the error term

$$\Delta_2^{(n)}(j) := s_{j,n} - e_{j,n}.$$

We express $s_{j,n}$ from equation (6.1) and obtain

$$|s_{j,n} - e_{j,n}| = \left| \frac{\eta_n(d_{j,n}) - \eta_n(s_{j,n})}{n+1} + \frac{R_n^{(6)}(s)}{n+1} \right| \le \frac{w_{\eta_n}(|s_{j,n} - d_{j,n}|)}{n+1} + \frac{|R_n^{(6)}(s)|}{n+1}$$
$$\le \frac{w_{\eta_n}\left(|s_{j,n} - e_{j,n}| + \frac{\eta_n(d_{j,n})}{n+1}\right)}{n+1} + \frac{|R_n^{(6)}(s)|}{n+1} \le \frac{w_{\eta_n}\left(\frac{c_n + \|\eta\|_\infty}{n+1}\right)}{n+1} + \frac{\|R_n^{(6)}\|_\infty}{n+1},$$

where the value c_n is given in (6.5). Thus, for sufficiently large n we get

$$|s_{j,n} - e_{j,n}| \le \frac{w_{\eta_n}\left(\frac{2\eta}{n+1}\right) + \|R_n^{(6)}\|_\infty}{n+1}, \tag{7.1}$$

where η_n is given by formula (5.26).

Let now $2 < \alpha < 3$. Then, according to Lemma 4.8, $\eta_n \in H^{\alpha-2}(\Pi_n(a))$ with norm bounded uniformly in n. Thus

$$|s_{j,n} - e_{j,n}| = O\left(\left(\frac{2\eta}{n+1}\right)^{\alpha-2} \cdot \frac{1}{n+1}\right) + \frac{o(\frac{1}{n^{\alpha-2}})}{n+1} = O\left(\frac{1}{n^{\alpha-1}}\right), \tag{7.2}$$

where the estimate is uniform in n. Let $\alpha = 2$. Then $\eta(s) \in C$ and $\eta_n(s)$ has a modulus of continuity with evaluation uniform in n. Thus, (7.1) implies

$$\Delta_2^{(n)}(j) = o\left(\frac{1}{n}\right).$$

And we have that

$$s_{j,n} = d_{j,n} - \frac{\eta_n(d_{j,n})}{n+1} + \Delta_2^{(n)}(j). \tag{7.3}$$

Thus, from the formulas (7.2), (7.3) we get the equality

$$|\Delta_2^{(n)}(j)| = \begin{cases} o(1/n), & \alpha = 2, \\ O\left(n^{-(\alpha-1)}\right), & 2 < \alpha < 3. \end{cases}$$

Take into account that

$$\eta_n(d_{j,n}) = o\left(\frac{1}{n^{\alpha-2}}\right),$$

(7.2) gives Theorem 2 for case $0 \le \alpha < 3$.

Now suppose $3 \le \alpha < 4$. Then consider the difference

$$\tilde{\Delta}_2^{(n)}(j) := s_{j,n} - s_{j,n}^{*(1)}.$$

From Theorem 5, ii)

$$|s_{j,n} - s_{j,n}^*| = o\left(\frac{1}{n^{\alpha-1}}\right).$$

On the other hand, the estimate (6.8) gives

$$|s_{j,n}^* - s_{j,n}^{*(1)}| = O\left(\frac{1}{n^3}\right).$$

In this way we have

$$|\tilde{\Delta}_2^{(n)}(j)| = o\left(\frac{1}{n^{\alpha-1}}\right), \tag{7.4}$$

and we can write

$$\begin{aligned} s_{j,n} = s_{j,n}^{*(1)} + \tilde{\Delta}_2^{(n)}(j) &= d_{j,n} - \frac{\eta_n(s_{j,n}^{*(0)})}{n+1} + \tilde{\Delta}_2^{(n)}(j) \\ &= d_{j,n} - \frac{\eta_n\left(d_{j,n} - \frac{\eta_n(d_{j,n})}{n+1}\right)}{n+1} + \tilde{\Delta}_2^{(n)}(j). \end{aligned}$$

Since the function $\eta_n(d_{j,n})$ has a derivative, according to Lemma 4.8, with $H^{\alpha-3}(\Pi_n(a))$-norm bounded uniformly on n, then we have that

$$s_{j_n} = d_{j,n} - \frac{\eta_n(d_{j,n})}{n+1} + \frac{\eta_n'(d_{j,n})\eta_n(d_{j,n})}{(n+1)^2} + O\left(\frac{1}{n^3}\right) + \tilde{\Delta}_n^{(2)}(j).$$

Using the Lemma 4.8 again and the relation (7.4), we obtain the statement of Theorem 2 for the case $3 \le \alpha < 4$.

The case of $\ell \le \alpha < \ell + 1$, $\ell \ge 4$ is treated in a similar way, using the iteration $\varphi_{j,n}^{*(\ell-1)}$ as an approximating expression.

7.2. Proof of Theorem 3

From the proved Theorem 2 and the definition of the function g we obtain the assertions of Theorem 3. Indeed, since $\lambda_j^{(n)} = g_n(\varphi_j^{(n)})$, consider the Taylor series decomposition at the point $d_{j,n}$ for the function g_n.

We prove formula (2.22) for the first two terms of the expansion in the case $[\alpha] = 2$. As an increment of the argument Δx consider the expression $-\dfrac{\eta(d_{j,n})}{n+1} + \Delta_2^{(n)}(j)$. According Taylor's formula we have

$$g_n(x_0 + \Delta x) = g_n(x_0) + g_n'(x_0)\Delta x + O(\Delta x^2).$$

Hence, taking into account the definitions of the functions g_n, we obtain the decomposition (2.22):

$$\begin{aligned}\lambda_j^{(n)} &= g_n(d_{j,n}) - g_n'(d_{j,n})\left(\frac{\eta(d_{j,n})}{n+1} + \Delta_2^{(n)}(j)\right) + O\left(\frac{\eta(d_{j,n})}{n+1} + \Delta_2^{(n)}(j)\right)^2 \\ &= g_n(d_{j,n}) - \frac{g_n'(d_{j,n})\eta(d_{j,n})}{n+1} - g_n'(d_{j,n})\Delta_2^{(n)}(j) + O\left(\frac{\eta(d_{j,n})}{n+1} + \Delta_2^{(n)}(j)\right)^2.\end{aligned}$$

Note now that when $\alpha = 2$,

$$g_n'(d_{j,n}) = O(d_{j,n}(\pi - d_{j,n})), \qquad \Delta_2^{(n)}(j) = o(1/n), \qquad \eta(d_{j,n}) = O(d_{j,n}(\pi - d_{j,n})).$$

The error term is

$$o\left(\frac{d_{j,n}(\pi - d_{j,n})}{n}\right) + O\left(\frac{\eta(d_{j,n})}{n+1} + \Delta_2^{(n)}(j)\right)^2 = o\left(\frac{d_{j,n}(\pi - d_{j,n})}{n}\right).$$

It now remains to note that for points φ lying on the real line, by virtue of Lemma 4.2, we have the equality

$$g_n^{(k)}(\varphi) = g^{(k)}(\varphi) + o(n^{-(\alpha-k)}), \qquad k = 0, 1, \ldots, [\alpha].$$

Thus, we obtain that

$$\lambda_j^{(n)} = g(d_{j,n}) - \frac{g'(d_{j,n})\eta(d_{j,n})}{n+1} + o\left(\frac{d_{j,n}(\pi - d_{j,n})}{n}\right) + o\left(\frac{1}{n^2}\right) + o\left(\frac{1}{n^2}\right).$$

We now consider the case of $2 < \alpha < 3$. Repeating the above reasoning, we obtain the required estimate of the remainder:

$$O\left(\frac{d_{j,n}(\pi - d_{j,n})}{n^{\alpha-1}}\right).$$

Consider the case of $[\alpha]=3$:

$$\begin{aligned}
\lambda_j^{(n)} &= g_n(d_{j,n}) + g_n'(d_{j,n})\left(-\frac{\eta(d_{j,n})}{n+1} + \frac{\eta(d_{j,n})\eta'(d_{j,n})}{(n+1)^2} + O\left(\frac{1}{n^{\alpha-1}}\right)\right) \\
&\quad + \frac{g_n''(d_{j,n})}{2}\left(-\frac{\eta(d_{j,n})}{n+1} + \frac{\eta(d_{j,n})\eta'(d_{j,n})}{(n+1)^2} + O\left(\frac{1}{n^{\alpha-1}}\right)\right)^2 \\
&\quad + O\left(\frac{d_{j,n}(\pi-d_{j,n})}{n}\right)^3 \\
&= g_n(d_{j,n}) - \frac{g_n'(d_{j,n})\eta(d_{j,n})}{n+1} + \frac{\frac{1}{2}g_n''(d_{j,n})\eta^2(d_{j,n}) + g_n'(d_{j,n})\eta(d_{j,n})\eta'(d_{j,n})}{n+1} \\
&\quad + O\left(\frac{d_{j,n}(\pi-d_{j,n})}{n^{\alpha-1}}\right).
\end{aligned}$$

Also, as above, using Lemma 4.2, we obtain that

$$\begin{aligned}
\lambda_j^{(n)} &= g(d_{j,n}) + g'(d_{j,n})\left(-\frac{\eta(d_{j,n})}{n+1} + \frac{\eta(d_{j,n})\eta'(d_{j,n})}{(n+1)^2} + O\left(\frac{1}{n^{\alpha-1}}\right)\right) \\
&\quad + \frac{g''(d_{j,n})}{2}\left(-\frac{\eta(d_{j,n})}{n+1} + \frac{\eta(d_{j,n})\eta'(d_{j,n})}{(n+1)^2} + O\left(\frac{1}{n^{\alpha-1}}\right)\right)^2 \\
&\quad + O\left(\frac{d_{j,n}(\pi-d_{j,n})}{n}\right)^3 + o\left(\frac{1}{n^{\alpha}}\right) \\
&= g(d_{j,n}) - \frac{g'(d_{j,n})\eta(d_{j,n})}{n+1} + \frac{\frac{1}{2}g_n''(d_{j,n})\eta^2(d_{j,n}) + g_n'(d_{j,n})\eta(d_{j,n})\eta'(d_{j,n})}{n+1} \\
&\quad + O\left(\frac{d_{j,n}(\pi-d_{j,n})}{n^{\alpha-1}}\right).
\end{aligned}$$

Thus, we obtain the formula (2.22) and the estimate

$$\Delta_3^{(n)}(j) = O\left(\frac{d_{j,n}(\pi-d_{j,n})}{n^{\alpha-1}}\right).$$

The general case is proved similarly.

Declaration of Competing Interest

No competing interest.

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