Infinite Series, Finite Sum: Extending the bouncing-ball activity

If you teach intermediate algebra or precalculus, you might have done an activity where you drop balls from known heights and record the heights of the bounces. The "bounce height" will be a roughly constant fraction of the "drop height." If you plot bounce height against drop height, you get a straight line going through zero. And if you plot bounce height against bounce *number*, an exponential function will model the pattern. A quick search yields a wealth of lessons on this topic; you will find many activities from publishers and from individual teachers.

This is a great introduction to exponentials: it's active and hands-on; students see the bounce and experience it. But the activity has measurement problems. It's hard to measure the height of the drop and the height of the bounce. Despite our best efforts, our datasets are usually noisy and and often don't work too well.

Technology can help. You can, for example, film multiple bounces and measure the bounce heights from the video frames; or you can use sensors such as a sonic ranger. Although these can be fruitful and fun choices, in this paper we will avoid the bounce height altogether: instead, following Erickson and Cooley (2007), we'll record the *sound* and measure the *time* between the bounces.

We dropped a ping-pong ball, which makes a great "pock" sound, from an original height of about 7 cm, and used a Vernier microphone set to record the sound every millisecond. The system records the sound pressure at that instant, measured in arbitrary units. With a few tries, you can get data that looks more or less like the graph in Figure 1.

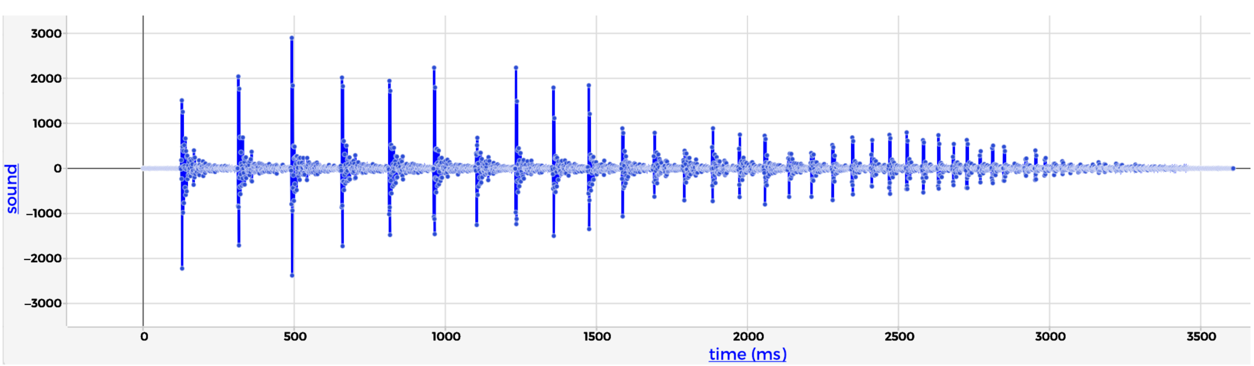

Figure 1. Sound from a bouncing ping-pong ball, about 3.5 seconds at 1000 points per second. The first bounce is at about 129 ms. We will call this "bounce zero." Bounce #1 is at about 317 ms in Figure 1. This graph is in CODAP.

You can listen to the sound at https://www.eeps.com/audio/pingpong.wav.

To give you an idea of what an individual bounce looks like zoomed in, Figure 2 shows the one at about 500 ms:

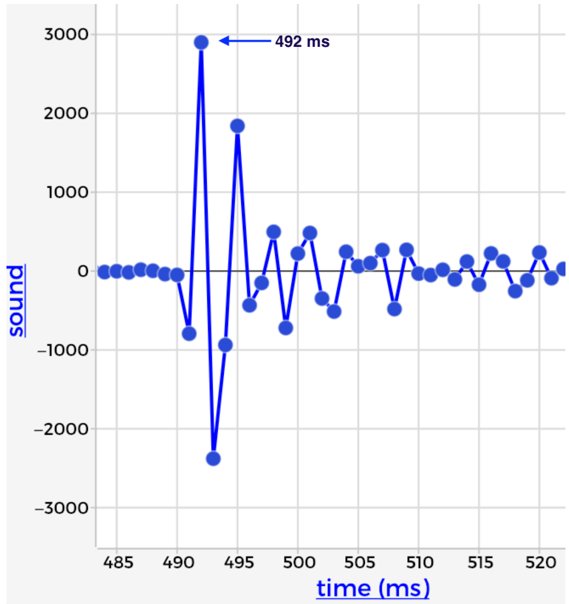

Figure 2: Data from the sound of a single bounce at just under 500 ms. This is the third bounce, which we label bounce #2.

The next step is to find and record the time of each bounce. In Figure 2, for example, we might choose 492 ms as the relevant time. It's clear from Figure 1 that we can find times for more than 20 individual bounces. Exactly how to do this will depend on your technology. Although some software tools make this step easier, students can always zoom in to the individual bounces (as in Figure 2) and type the times they see into their favorite data analysis software. Our first few bounce times appear in this CODAP table:

Bounces (30 cases)

index	time (ms)	sound	bounce (ms)	elapsed (ms)	interval (ms)	ratio
1	129	-2223.75	0	0	186	0.952
2	315	2042.05	1	186	177	0.938
3	492	2900.5	2	363	166	0.946
4	658	2017.11	3	529	157	0.943

*Figure 3: A CODAP table showing data for the first four bounces out of 30 in our dataset. In the table, we have also computed a variable called **elapsed**, which is the time since the first bounce, and **ratio**, which is the ratio of the next interval to the one in that row. Notice that the bounce number begins at zero. This makes the "A" coefficient in the exponential easier to understand.*

With a sequence of bounce times, students can compute the time between the bounces (the "interval" in the table), which they then model with the exponential. If they have the data in a spreadsheet, they can copy and paste it into modeling software, that is, software where you can plot data and functions together. Figure 4 shows that interval data plotted and modeled in Desmos.

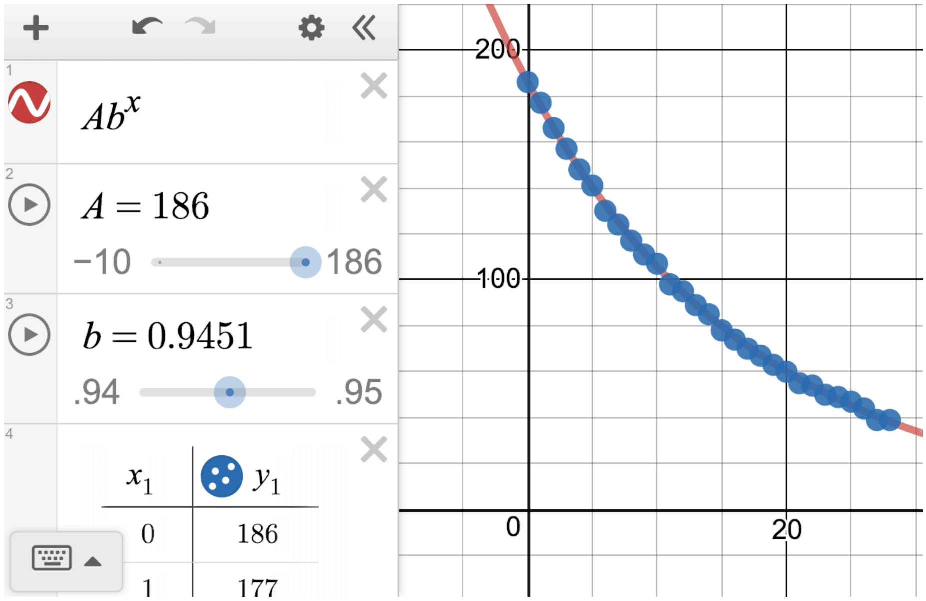

Figure 4. 29 bounce intervals with an exponential model in Desmos.

Notice:

- The x-variable is the bounce *number*, not the time!
- The bounce number starts with *zero*.
- Interval "zero" is the time between bounce #0 and bounce #1.

For the model in Figure 4, we have made sliders for the value for the first interval, A, and the base b—and then adjusted the b slider to better fit the data. Although it might be possible to find a "better" value for A using some least-squares procedure, we have kept its value equal to the first interval.

Don't let students use an automatic fit for the exponential. The purpose, after all, is for them to learn how exponential functions work. One idea is to have them compute the ratio of each interval time to its predecessor. This yields a good estimate for the base of the exponential; our ratios clustered mostly between 0.94 and 0.95, a satisfyingly high bounce efficiency. We did that in the table above; a dot plot showing those ratios appears in Figure 5.

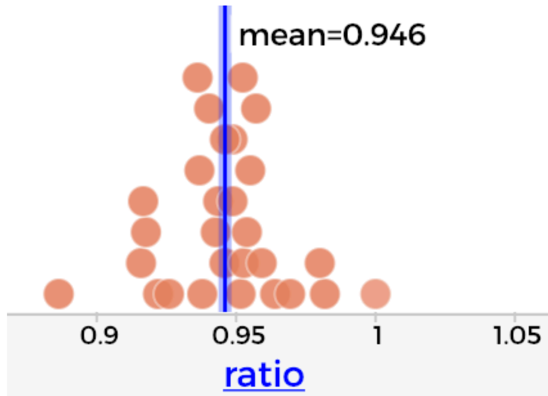

Figure 5: the 28 ratios of adjacent intervals, also showing the average value. The more extreme intervals are from later bounces.

Hello, Zeno

As an experienced teacher, you will fill in the gaps in the description so far, adjusting the students' path to an exponential model to match their experience and your constraints.

But now let's venture where you might not have gone before and ask: how long does the ball bounce?

If you are like me, you probably thought, if everything were perfectly mathematical, it would bounce an infinite number of times, each bounce getting shorter—but never quite getting to zero. In the real world, however (you and I both reasoned), there are probably intermolecular, small-scale effects that take over when the bounces get small. These frictional forces change the physics of the situation, dominate the motion, and stop the ball—after a finite number of bounces.

But the *number* of bounces was not the question. We wanted to know *how long* the ball would bounce. To shed some light on this, let's make the extreme assumption that it *is* perfectly mathematical, with no small-scale forces. In that case, the ball would bounce an infinite number of times.

But does that mean it would bounce forever? No. To see why, let's stop using an exponential function as our model and instead use a geometric sequence. The basics are the same, of course, but we traditionally use different letters. The initial value A becomes a_0, and the base b becomes r. With that notation, our sequence is:

$$a_0,\ a_0 r,\ a_0 r^2,\ a_0 r^3,\ a_0 r^4,\ \ldots$$

The total time, then, is the sum of all those terms, an infinite geometric *series* with $-1 < r < 1$. We can actually find that sum (and here we just give the formula, not its derivation):

$$T_{inf} = \sum_{i=0}^{\infty} a_0 r^i = \frac{a_0}{1-r}$$

Plugging in 186 for a_0 and 0.9451 for r (which are the values in Figure 4), we get a sum—the total time for an infinite number of bounces—of 3388 ms, or almost 3.4 seconds. Infinite bounces, finite time. Even if there were no difficult physics, *the ball would still stop bouncing*. Not only that, but when we look back at

Figure 1, or imagine listening to a bouncing ping-pong ball, we find that the sound stops at just about the time the infinite sum predicts.

Think about this for a moment. Our exponential-function activity, recast this way—looking at time instead of height—is also a way to demonstrate a geometric series with real data, and show what its convergence means.

An extension: Partial Sums

And we can go further. Let's interpret the elapsed cumulative bounce times—rather than the intervals—as partial sums. To find the total time T_n to bounce n, we must sum the first $(n - 1)$ terms of that same geometric series. Its sum is

$$T_n = \sum_{i=0}^{n} a_0 r^i = \frac{a_0(1 - r^n)}{1 - r}.$$

We can use that formula as a model for the bounce times, which we do in Figure 6, which also shows T_{inf} at about 3.4 seconds. In the figure, the data are the "elapsed" times—the times since the first bounce—that appeared in Figure 3. That way the first bounce (bounce zero) is at the origin (time zero), simplifying our model.

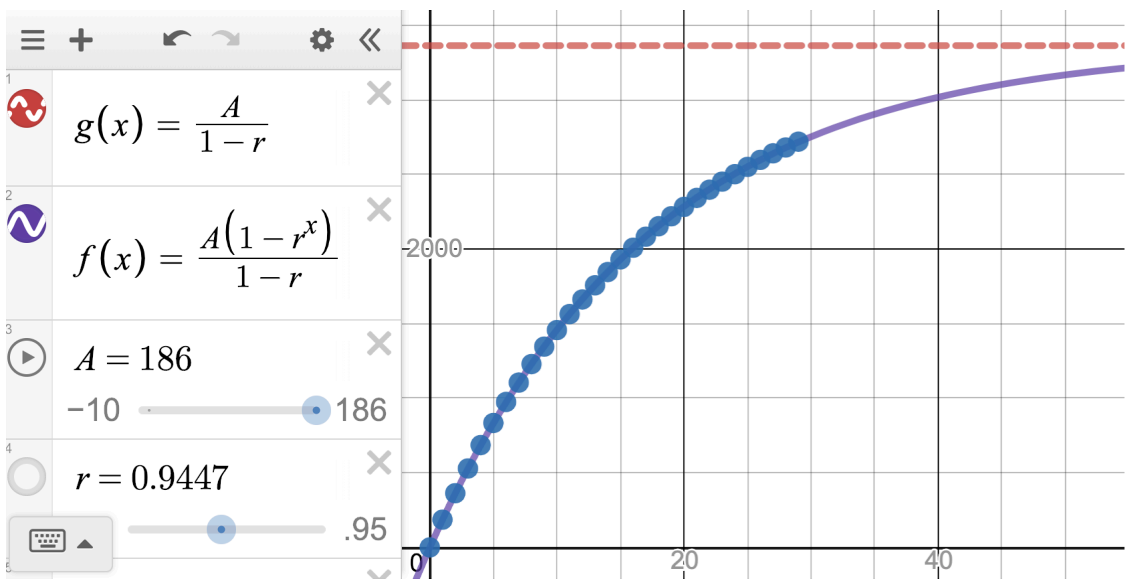

Figure 6: Using Desmos, cumulative times with a model for partial sums of a geometric series. Here we use $f(x)$ as the model for the data, and $g(x)$—the dotted red line—as the asymptote, which is equal to the infinite sum.

Conclusions, Caveats, and Resources

As I mentioned above, the description here is not intended as a complete lesson plan for bouncing balls—there are plenty— but rather a suggestion that if you use time instead of distance as the thing that decreases exponentially, you can use the activity to illuminate geometric series as well as exponential functions.

As to tools, I used Vernier equipment and software to collect the data, and CODAP and Desmos (both free) to do the analysis. If you cannot collect your own data, I have provided the data used in this paper in a repository at https://github.com/eepsmedia/ping-pong-bounce, along with additional commentary in the wiki.

But if you can collect your own data, a tip: drop the ball from a height of less than about 10 cm. Any higher, and the pattern of times will be so affected by air resistance that the exponential will not fit as well.

It's worth reflecting that this is an intriguing use of technology. You do not have to settle for data that only "sort of" show an exponential pattern. Using the computer to aid in measurement creates new opportunities—precise times for 30 bounces in sequence!—but also new challenges, for example, deciding which point best represents the time of the bounce.

This is also a triumph for mathematical modeling in the classroom. We have real data from a physical phenomenon. The data more or less follow a pattern, but not exactly. This is, after all, a hallmark of modeling: a model is an approximation, an idealization of the real world.

Even though the model is imperfect, we can still use the tools we develop in the abstract, perfect universe of mathematics to draw conclusions and expand our understanding of real-world events. We may not understand the detailed intermolecular forces between ping-pong balls and kitchen counters, but we totally understand why the ball does not bounce forever. Remember George Box's (1987) deathless comment: *all models are wrong, but some are useful.*

References

Box, G. E. P. and Draper, N.R. 1987. *Empirical Model-Building and Response Surfaces.* New York: Wiley.

Erickson, T. and Cooley, B. 2007. "Bouncing Balls and Energy Loss" in *Den of Inquiry: Data-Rich Labs for Introductory Physics, Volume 2*. volume 2. Page 147.