Evolutionary computing

- Hessian -

Grupo 3CV11

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Chapter 1

Problem 01

Use the method discussed in class to characterize the critical points from the functions in the image.

$$f_1(x, y) = 4x + 2y - x^2 - 3y^2$$

1.1 Solution

Get the partial derivatives of $f_1(x, y)$.

$$\frac{\partial f_1(x, y)}{\partial x} = 4 - 2x$$

$$\frac{\partial f_1(x, y)}{\partial y} = 2 - 6y$$

As a vector we get:

$$\nabla f_1(x, y) = (4 - 2x)\hat{\imath} + (2 - 6y)\hat{\jmath}$$

Let's find the critical points first we solve $\frac{\partial f_1(x,y)}{\partial x} = 0$

$$4 - 2x = 0$$

$$2x = 4$$

$$x = \frac{4}{2}$$

$$\therefore x = 2$$

Let's solve $\frac{\partial f_1(x,y)}{\partial y} = 0$

$$2 - 6y = 0$$

$$6y = 2$$

$$y = \frac{2}{6}$$

$$\therefore y = \frac{1}{3}$$

Now we know that if we evaluate the function $f_1(x, y)$ in the values that we got before we now the coordinates of a critical point

$$f_1\left(2, \frac{1}{3}\right) \tag{1.1}$$

Now we need to check if our critical points are minimal or not so we must use a Hessian matrix to proof that.

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ \\ 0 & -6 \end{pmatrix}$$

Now we must evaluate the Hessian using the result of *eq.1*

$$H\left(2, \frac{1}{3}\right) = \begin{pmatrix} -2 & 0 \\ \\ 0 & -6 \end{pmatrix}$$

Let's get $\left|H\left(2, \frac{1}{3}\right)\right|$ using all the submatrix starting from the upper left corner, it means:

$$\begin{vmatrix} -2 \end{vmatrix} = -2$$

$$\begin{vmatrix} -2 & 0 \\ \\ 0 & -6 \end{vmatrix} = (-2)(-6) - (0)(0) = 12$$

And as we can see the first submatrix the result of the determinant is -2 and we know: *An mxm symetric matrix M is positive definite if* $|M_i| > 0 \ \forall_i \ M_i$ *is the ixi submatrix starting from the upper left corner of M* so we conclude that $\left(2, \frac{1}{3}\right)$ **is not a minimum**

Chapter 2

Problem 02

Use the method discussed in class to characterize the critical points from the functions in the image.

$$f_2(x, y, z) = e^{x-y} + e^{y-x} + e^{x^2} + z^2$$

2.1 Solution

Get the partial derivatives of $f_2(x, y, z)$. First of all we can rewrite the function:

$$e^x e^{-y} + e^y e^{-x} + e^{x^2} + z^2$$

$$\frac{\partial f_2(x, y, z)}{\partial x} = e^x e^{-y} - e^y e^{-x} + 2xe^{x^2} = e^{x-y} - e^{y-x} + 2xe^{x^2}$$

$$\frac{\partial f_2(x, y, z)}{\partial y} = -e^x e^{-y} + e^y e^{-x} = -e^{x-y} + e^{y-x}$$

$$\frac{\partial f_2(x, y, z)}{\partial z} = 2z$$

As a vector we get:

$$\nabla f_2(x, y, z) = (e^{x-y} - e^{y-x} + 2xe^{x^2})\hat{\imath} + (-e^{x-y} + e^{y-x})\hat{\jmath} + (2z)\hat{k}$$

Let's find the critical points first we solve $\frac{\partial f_2(x,y,z)}{\partial z} = 0$

$$2z = 0$$

$$\therefore z = 0$$

Then let's solve $\frac{\partial f_2(x,y,z)}{\partial y} = 0$

$$-e^{x-y} + e^{y-x} = 0$$

$$-e^{x}e^{-y} + e^{y}e^{-x} = 0$$

$$e^{y}e^{-x} = e^{x}e^{-y}$$

$$\frac{e^{x}}{e^{-x}} = \frac{e^{y}}{e^{-y}}$$

$$e^{2x} = e^{2y}$$

$$ln\left(e^{2x}\right) = ln\left(e^{2y}\right)$$

$$2x = 2y$$

$$\therefore x = y$$

Finally we solve $\frac{\partial f_2(x,y,z)}{\partial x} = 0$ we already know that $x = y$ and $z = 0$ so we replace those values and we get:

$$e^{y-y} - e^{y-y} + 2ye^{y^2} = 0$$

$$e^{0} - e^{0} + 2ye^{y^2} = 0$$

$$2ye^{y^2} = 0$$

if $y = 0$ we can see that the condition is true so:

$$\therefore y = 0$$

Now we know that if we evaluate the function $f_2(x,y,z)$ in the values that we got before we now the coordinates of a critical point

$$f_2(0,0,0) \tag{2.1}$$

Then we need to check if our critical points are minimal or not, so we must use a Hessian matrix to proof that.

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$$

So let's calculate each element of the Hessian matrix:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial(e^{x-y} - e^{y-x} + 2xe^{x^2})}{\partial x} = \frac{\partial(e^x e^{-y} - e^y e^{-x} + 2xe^{x^2})}{\partial x} = e^{x-y}+e^{y-x}+2e^{x^2}+4x^2 e^{x^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial(e^{x-y} - e^{y-x} + 2xe^{x^2})}{\partial y} = \frac{\partial(e^x e^{-y} - e^y e^{-x} + 2xe^{x^2})}{\partial y} = -e^x e^{-y} - e^y e^{-x} = -(e^{x-y}+e^{y-x})$$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2(e^x e^{-y} - e^y e^{-x} + 2xe^{x^2})}{\partial z} = 0$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2(-e^x e^{-y} + e^y e^{-x})}{\partial x} = -(e^{x-y} + e^{y-x})$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial(-e^x e^{-y} + e^y e^{-x})}{\partial y} = e^{x-y} + e^{y-x}$$

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial(-e^x e^{-y} + e^y e^{-x})}{\partial z} = 0$$

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial(2z)}{\partial x} = 0$$

$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial(2z)}{\partial y} = 0$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial(2z)}{\partial z} = 2$$

Then build the Hessian matrix

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} = \begin{pmatrix} e^{x-y} + e^{y-x} + 2e^{x^2} + 4x^2 e^{x^2} & -(e^{x-y} + e^{y-x}) & 0 \\ \\ -(e^{x-y} + e^{y-x}) & e^{x-y} + e^{y-x} & 0 \\ \\ 0 & 0 & 2 \end{pmatrix}$$

Now we evaluate the Hessian matrix using the critical point that we find before.

$$H(0,0,0) = \begin{pmatrix} e^{0-0} + e^{0-0} + 2e^{0^2} + 40^2 e^{0^2} & -(e^{0-0} + e^{0-0}) & 0 \\ \\ -(e^{0-0} + e^{0-0}) & e^{0-0} + e^{0-0} & 0 \\ \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 0 \\ \\ -2 & 2 & 0 \\ \\ 0 & 0 & 2 \end{pmatrix}$$

Now let's get all the determinats from all the possible matrix of mxm from the upper left corner

$$\begin{vmatrix} 4 \end{vmatrix} = 4$$

$$\begin{vmatrix} 4 & -2 \\ \\ -2 & 2 \end{vmatrix} = 8 - 4 = 4$$

$$\begin{pmatrix} 4 & -2 & 0 \\ \\ -2 & 2 & 0 \\ \\ 0 & 0 & 2 \end{pmatrix} = (4)(2)(2) + (-2)(0)(0) + (0)(-2)(0) - [(0)(2)(0) + (0)(0)(4) + (2)(-2)(-2)]$$

$$(4)(2)(2) - (2)(-2)(-2) = 16 - 8 = 8$$

As we can see all the values that we got calculating the determinants are greater than zero so we can conclude that the crical point $f_2(0,0,0)$ **is a minimum**.