BOLETÍN T8 - EJERCICIO 11

CÁLCULO DE LA ESTABILIDAD Y CONSISTENCIA DEL MÉTODO <u>UPWIND</u> (CONVECCIÓN)

$$(1)\quad \frac{T_i^{n+1}-T_i^n}{\Delta t} + u\,\frac{T_i^n - T_{i-1}^n}{\Delta x} = 0$$

$$(2)\quad T_i^{n+1} = T(x, t+\Delta t) = T_i^n + \Delta t\frac{\partial T}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 T}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 T}{\partial t^3}$$

$$(3)\quad T_{i-1}^n = T(x-\Delta x, t) = T_i^n - \Delta x\frac{\partial T}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 T}{\partial x^2} - \frac{\Delta x^3}{6}\frac{\partial^3 T}{\partial x^3}$$

USAMOS $C = \frac{u\Delta t}{\Delta x}$ y SUSTITUIMOS EN LA EC. ORIGINAL (1)

$$T_i^{n+1} = T_i^n - C\left(T_i^n - T_{i-1}^n\right)$$

SUSTITUIMOS (2) Y (3) EN (1)

$$\cancel{T_i^n} + \Delta t\left(\frac{\partial T}{\partial t}\right) + \frac{\Delta t^2}{2}\frac{\partial^2 T}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 T}{\partial t^3} = \cancel{(1-C)}\,T_i^n + C\cdot\left(\cancel{T_i^n} - \Delta x\frac{\partial T}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 T}{\partial x^2} - \frac{\Delta x^3}{6}\frac{\partial^3 T}{\partial x^3}\right)$$

SABEMOS QUE $\frac{\partial^2 T}{\partial t^2} = \frac{\partial}{\partial t}\left(-u\frac{\partial T}{\partial x}\right) = u^2\frac{\partial}{\partial x}\frac{\partial T}{\partial x} = u^2\frac{\partial^2 T}{\partial x^2}$

TAMBIÉN DESPRECIAMOS LOS TÉRMINOS CON Δt^2 O MÁS

$$\Delta t\frac{\partial T}{\partial t} + C\Delta x\frac{\partial T}{\partial x} - C\frac{\Delta x^2}{2}\frac{\partial T^2}{\partial x^2} + C\frac{\Delta x^3}{6}\frac{\partial T^3}{\partial x^3} + O(\Delta x^4, \Delta t^2) = 0$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} - u\Delta x\,\frac{1}{2}\frac{\partial T^2}{\partial x^2} + \Delta t\,\frac{1}{2}\frac{\partial^2 T}{\partial t^2} + O(\Delta x^2, \Delta t^2) = 0$$

$$\Delta t\frac{\partial T}{\partial t} + C\Delta x\frac{\partial T}{\partial x} - C\Delta x^2\frac{1}{2}\frac{\partial^2 T}{\partial x^2} + \Delta t^2\frac{1}{2}\frac{\partial^2 T}{\partial t^2} + O(\Delta x^3, \Delta t^3) = 0$$

$$E_i^n = \frac{-C\Delta x^2}{2\Delta t}\frac{\partial^2 T}{\partial x^2} + \frac{\Delta t}{2}\frac{\partial^2 T}{\partial t^2} + O(\Delta x^3, \Delta t^2)$$

TAMBIÉN TENEMOS QUE $\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x}$, $\frac{\partial^2 T}{\partial t^2} = u^2\frac{\partial^2 T}{\partial x^2}$

$$E_i^n = \frac{-u\Delta x}{2}\frac{\partial^2 T}{\partial x^2} + \frac{u^2\Delta t}{2}\frac{\partial^2 T}{\partial x^2} + \dots = -\frac{\Delta x^2}{2\Delta t^2}C(1-C)\frac{\partial^2 T}{\partial x^2} + \dots =$$

$$= -\frac{u\Delta x}{2}(1-C)\frac{\partial^2 T}{\partial x^2} + O(\Delta x^2, \Delta t^2)$$

SI $C = 1 \rightarrow E_i^{n+1} << E_i^n$

ESTABILIDAD

$T_j^n \rightarrow \xi_j^n = G^n e^{i\theta j}$ Modo arbitrario

$$G^{n+1}e^{i\theta j} = (1-C)G^n e^{i\theta j} + C G^n e^{i\theta(j-1)}$$

$$G = (1-C) + Ce^{-i\theta} = (1-C) + C(\cos\theta - i\,\text{sen}\,\theta) = 1 - C(1-\cos\theta) - iC\,\text{sen}\,\theta$$

$|G| \le 1 \rightarrow |G|^2 \le 1$

$|G|^2 = (1-C+C\cos\theta)^2 + C^2 \operatorname{sen}^2\theta = (1-C)^2 + C^2 - 2C(1-C)\cos\theta$

$C \geq 0 \rightarrow |G| \le 1 \; \forall\theta \Leftrightarrow 0 \le (1-C)^2 + C^2 - 2C(1-C)\cos\theta \le 1$

$\rightarrow (1-C)^2 + C^2 + 2C(1-C) \le 1 \rightarrow \cancel{C^2} + \cancel{C^2} - \cancel{2C^2} + 1 - \cancel{2C} + \cancel{2C} \le 1$ ✓ $\forall C$

$\rightarrow (1-C)^2 + C^2 - 2C(1-C) \geq 0 \rightarrow C^2 + 1 - 2C + C^2 - 2C + 2C^2 = 4C^2 - 4C + 1 \geq 0$

Se verifica si $C \le 1$