BOLETÍN 7, ~~E~~EJERCICIO 7

CALCULAR LA CONSISTENCIA Y LA ESTABILIDAD (EJ. 2, 3 NIVELES TEMPORALES)

$$\frac{0{,}5T_j^{n-1} - 2T_j^n + 1{,}5T_j^{n+1}}{\Delta t} - \alpha\left(\frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{\Delta x^2}\right) = 0 \qquad s = \frac{\alpha \Delta t}{\Delta x^2}$$

$$\rightarrow T_j^{n+1} = \frac{2}{3}s\left[T_{j+1}^n - 2T_j^n + T_{j-1}^n\right] - \frac{1}{3}T_j^{n-1} + \frac{4}{3}T_j^n$$

DESARROLLO EN TORNO A $\Delta x = 0$

$$T_{j\pm1}^n = T(x\pm\Delta x, t) = T_j^n \pm \Delta x\left(\frac{\partial T}{\partial x}\right)_j^n + \frac{\Delta x^2}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)_j^n \pm \frac{\Delta x^3}{6}\left(\frac{\partial^3 T}{\partial x^3}\right)_j^n + \frac{\Delta x^4}{24}\left(\frac{\partial^4 T}{\partial x^4}\right)_j^n \pm O(\Delta x^5)$$

DESARROLLO EN TORNO A $\Delta t = 0$

$$T_j^{n\pm1} = T(x, t\pm\Delta t) = T_j^n \pm \Delta t\left(\frac{\partial T}{\partial t}\right)_j^n + \frac{\Delta t^2}{2}\left(\frac{\partial^2 T}{\partial t^2}\right)_j^n \pm \frac{\Delta t^3}{6}\left(\frac{\partial^3 T}{\partial t^3}\right)_j^n + \frac{\Delta t^4}{24}\left(\frac{\partial^4 T}{\partial t^4}\right)_j^n \pm O(\Delta t^5)$$

TÉRMINOS TEMPORALES

$$T_j^{n+1} + \frac{1}{3}T_j^{n-1} - \frac{4}{3}T_j^n = \cancel{T_j^n\left(1+\frac{1}{3}-\frac{4}{3}\right)} + \Delta t\left(\frac{\partial T}{\partial t}\right)_j^n\left(1-\frac{1}{3}\right) + \frac{\Delta t^2}{2}\left(\frac{\partial T}{\partial t}\right)_j^n\left(1+\frac{1}{3}\right) +$$
$$+ \frac{\Delta t^3}{6}\left(\frac{\partial^3 T}{\partial t^3}\right)_j^n\left(1-\frac{1}{3}\right) + \frac{\Delta t^4}{24}\left(\frac{\partial^4 T}{\partial t^4}\right)_j^n\left(1+\frac{1}{3}\right) + O(\Delta t^5) =$$
$$= \frac{2}{3}\Delta t\left(\frac{\partial T}{\partial t}\right) + \frac{2}{3}\Delta t^2\left(\frac{\partial^2 T}{\partial t^2}\right) + \frac{1}{9}\Delta t^3\left(\frac{\partial^3 T}{\partial t^3}\right) + \frac{1}{18}\Delta t^4\left(\frac{\partial^4 T}{\partial t^4}\right) + O(\Delta t^5)$$

TÉRMINOS ESPACIALES

$$\frac{2}{3}s\left[T_{j+1}^n - 2T_j^n + T_{j-1}^n\right] = \frac{2}{3}s\left[\cancel{T_j^n(1-2+1)} + \cancel{\Delta x\left(\frac{\partial T}{\partial x}\right)(1-1)} + \frac{\Delta x^2}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)(1+1) +\right.$$
$$\left. + \cancel{\frac{\Delta x^3}{6}\left(\frac{\partial^3 T}{\partial x^3}\right)(1-1)} + \frac{\Delta x^4}{24}\left(\frac{\partial^4 T}{\partial x^4}\right)(1+1) + \cancel{O(\Delta x^5)} + O(\Delta x^6)\right] =$$
$$= \frac{2}{3}s\Delta x^2\left(\frac{\partial^2 T}{\partial x^2}\right) + \frac{1}{18}s\Delta x^4\left(\frac{\partial^4 T}{\partial x^4}\right) + O(\Delta x^6)$$

SUSTITUÍMOS $\left(\partial T_t^n = \frac{\partial^n T}{\partial t^n} \quad \partial T_x^n = \frac{\partial^n T}{\partial x^n}\right)$

$$\frac{2}{3}\Delta t\,\partial T_t + \frac{2}{3}\Delta t^2\,\partial T_t^2 + \frac{1}{9}\Delta t^3\,\partial T_t^3 + O(\Delta t^4) = \frac{2}{3}s\Delta x^2\,\partial T_x^2 + \frac{1}{18}s\Delta x^4\,\partial T_x^4 + O(\Delta x^6)$$

$$\rightarrow \Delta t\,\partial T_t - s\Delta x^2\,\partial T_x^2 = -\Delta t^2\,\partial T_t^2 - \frac{1}{6}\Delta t^3\,\partial T_t^3 + \frac{s}{12}\Delta x^4\,\partial T_x^4 + O(\Delta x^6, \Delta t^4)$$

$$\rightarrow \alpha = \frac{s\Delta x^2}{\Delta t} \rightarrow \cancel{\Delta t}\,\partial T_t - \alpha\cancel{\Delta t}\,\partial T_x^2 = -\Delta t^{\cancel{2}}\,\partial T_t^2 - \frac{1}{6}\Delta t^{\cancel{3}\,2}\,\partial T_t^3 + \frac{1}{12}s\Delta x^4\,\partial T_x^4 + O(\Delta x^6, \Delta t^{\cancel{4}\,3})$$

$$E_j^n = -\Delta t\,\partial T_t^2 - \frac{1}{6}\Delta t^2\,\partial T_t^3 + \frac{s\Delta x^4}{12\,\Delta t}\,\partial T_x^4 + O(\Delta x^6, \Delta t^3)$$

$$\frac{\partial^2 T}{\partial t^2} = \frac{\partial}{\partial t}\left(\alpha\frac{\partial^2 T}{\partial x^2}\right) = \alpha\frac{\partial^2}{\partial x^2}\left(\alpha\frac{\partial^2 T}{\partial x^2}\right) = \alpha^2\frac{\partial^4 T}{\partial x^4}$$

$$E_j^n = \left(-\Delta t\,\alpha^2 + \frac{s\Delta x^4}{12\,\Delta t}\right)\left(\frac{\partial^4 T}{\partial x^4}\right) + O(\Delta t^2, \Delta x^6)$$

IGUALAMOS A 0

$$-\Delta t\,\alpha^2 + \frac{s\Delta x^4}{12\,\Delta t} = 0 = -\cancel{\Delta t^2}\,\frac{s^2\cancel{\Delta x^4}}{\cancel{\Delta t^2}} + \frac{\cancel{s}\,\cancel{\Delta x^4}}{12} \longrightarrow s = \frac{1}{12}$$ Error consistencia mínimo $(E = O(\Delta t^2, \Delta x^6))$

ESTABILIDAD

$$\zeta_j^n = G^n e^{i\theta j}$$

$$G^{n+1} e^{i\theta j} = \frac{2}{3} s \left(G^n e^{i\theta (j+1)} - 2 G^n e^{i\theta j} + G^n e^{i\theta (j-1)} \right) - \frac{1}{3} G^{n-1} e^{i\theta j} + \frac{4}{3} G^n e^{i\theta j}$$

$$G = \frac{2}{3} s \left(e^{i\theta} - 2 + e^{-i\theta} \right) - \frac{1}{3} G^{-1} + \frac{4}{3}$$

$$G = \frac{4}{3} s \cos\theta - \frac{4}{3} s - \frac{1}{3} G^{-1} + \frac{4}{3} \rightarrow 3G = 4 s \cos\theta - G^{-1} + 4(1-s)$$

$$3G^2 - 4G\left(s\cos\theta + 1 - s\right) + 1 = 0 \rightarrow 3G^2 - 4G\left(s(\cos\theta - 1) + 1\right) + 1 = 0$$

$$3G^2 - 4G\left(1 - 2 \operatorname{sen}^2\frac{\theta}{2}\right) + 1 = 0 \rightarrow G = \frac{4\left(1 - 2s \operatorname{sen}^2\frac{\theta}{2}\right) \pm \sqrt{16\left(1 - 2s \operatorname{sen}^2\frac{\theta}{2}\right)^2 - 12}}{6}$$

$$G = \frac{4 - 8s \operatorname{sen}^2\frac{\theta}{2} \pm 2\sqrt{1 + 16s \operatorname{sen}^2\frac{\theta}{2}\left(s \operatorname{sen}^2\frac{\theta}{2} - 1\right)}}{6} \rightarrow |G| \leq 1 \ \forall \theta$$

CONDICIÓN ESTABILIDAD

$$-1 \leq \frac{2}{3} - \frac{4}{3} \operatorname{sen}^2\frac{\theta}{2} \pm \frac{1}{3}\sqrt{16 s \operatorname{sen}^2\frac{\theta}{2}\left(s \operatorname{sen}^2\frac{\theta}{2} - 1\right) + 1} \leq 1$$