Tema 3
Ecuación 1D de Transporte

Ecuación 1D de transporte. Métodos explícitos e implícitos.

Referencias del Capítulo:

- Numerical Recipes. W.H. Press, B.P. Flannery, S.A. Teukolsky and W.T. Vetterling. Cambridge University Press (1988).
- Computational Techniques for Fluid Dynamics. C.A.J. Fletcher. Springer-Verlag (1991).

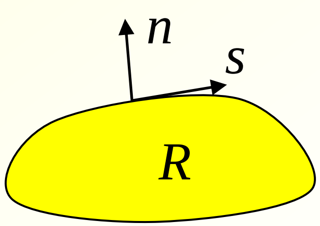
M3: Ecuación de Transporte.

Consideramos un sistema en el que el transporte de información puede ser difusivo y/o convectivo. La forma de ecuación más general tiene la forma:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} - \alpha\frac{\partial^2 T}{\partial x^2} = 0$$

donde T es la variable a estudiar (p.e.: temperatura) que se ve forzada con una velocidad de convección *u* y se difunde con una difusividad α.

Para tener un problema bien planteado necesitamos aportar:



- Condiciones iniciales (especificar $T(x)$ para un t_o y todo x).
- Condiciones de frontera para todo t.

∂R=frontera

1. Condiciones de Direchlet: $T=f$ en ∂R.
2. Condiciones de Neumann (de la derivada): $\frac{\partial T}{\partial n} = f$ o $\frac{\partial T}{\partial s} = g$ en ∂R
3. Condiciones de mezcla o de Robin:

$$\frac{\partial T}{\partial n} + kT = f \text{ con } k > 0 \text{ en } \partial R$$

M3: Ecuación de Transporte. Esquema FTCS

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} - \alpha\frac{\partial^2 T}{\partial x^2} = 0$$

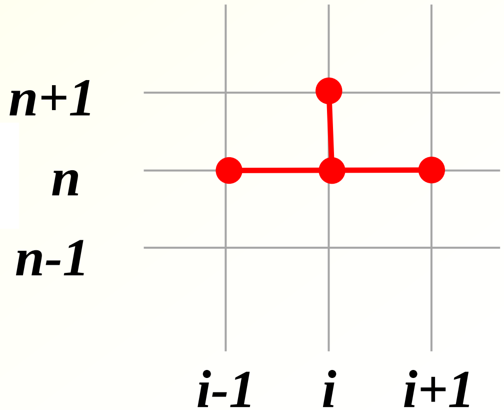
$$\frac{\partial T}{\partial t} = \frac{T_i^{n+1} - T_i^n}{\Delta t}$$

$$\frac{\partial T}{\partial x} = \frac{T_{i+1}^n - T_{i-1}^n}{2\Delta x} \qquad \Rightarrow \qquad \frac{T_i^{n+1} - T_i^n}{\Delta t} + u\frac{T_{i+1}^n - T_{i-1}^n}{2\Delta x} - \alpha\frac{T_{i-1}^n - 2T_i^n + T_{i+1}^n}{\Delta x^2} = 0$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i-1}^n - 2T_i^n + T_{i+1}^n}{\Delta x^2}$$

Consistencia:

$$E_i^n = Cu(\Delta x/2)\frac{\partial^2 T}{\partial x^2} - [C\alpha\Delta x - u(\Delta x^2/6)(1+2C^2)]\frac{\partial^3 T}{\partial x^3}$$



Estabilidad: Factor amplificación: $G = 1 - 2s(1-\cos\theta) - \mathrm{i}C\sin\theta$

Condición Estabilidad: $0 \leqq C^2 \leqq 2s \leqq 1$

$$|G| \leq 1 \quad \forall\theta$$

M3: Ecuación de Transporte.

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} - \alpha\frac{\partial^2 T}{\partial x^2} = 0$$

Table 9.3. Algebraic (discretised) schemes for the transport equation $\partial \bar{T}/\partial t + u\partial \bar{T}/\partial x - \alpha\partial^2 \bar{T}/\partial x^2 = 0$

Scheme	Algebraic form	Truncation error[a] (E) (leading terms)	Amplification factor G $(\theta = m\pi\Delta x)$	Stability Restrictions	Remarks
FTCS 	$\frac{\Delta T_j^{n+1}}{\Delta t} + uL_x T_j^n - \alpha L_{xx} T_j^n = 0$	$Cu(\Delta x/2)\frac{\partial^2 T}{\partial x^2}$ $-[C\alpha\Delta x - u(\Delta x^2/6)(1+2C^2)]\frac{\partial^3 T}{\partial x^3}$	$1-2s(1-\cos\theta) - iC\sin\theta$	$0 \leqq C^2 \leqq 2s \leqq 1$	$R_{cell} \ll 2/C$ for accuracy
Upwind 	$\frac{\Delta T_j^{n+1}}{\Delta t} + u\frac{(T_j^n - T_{j-1}^n)}{\Delta x} - \alpha L_{xx} T_j^n = 0$	$-u(\Delta x/2)(1-C)\frac{\partial^2 T}{\partial x^2} - [C\alpha\Delta x$ $-u(\Delta x^2/6)(1-3C+2C^2)]\frac{\partial^3 T}{\partial x^3}$	$1-(2s+C)(1-\cos\theta) - iC\sin\theta$	$C + 2s \leqq 1$	$R_{cell} \ll 2/(1-C)$ for accuracy
DuFort-Frankel 	$(T_j^{n+1} - T_j^{n-1})/2\Delta t + uL_x T_j^n$ $-\frac{\alpha}{\Delta x^2}\{T_{j-1}^n - (T_j^{n-1} + T_j^{n+1})$ $+T_{j+1}^n\} = 0$	$\alpha C^2\frac{\partial^2 T}{\partial x^2} + (1-C^2)[u\Delta x^2/6$ $-2\alpha^2 C^2/u]\frac{\partial^3 T}{\partial x^3}$	$\frac{B \pm [B^2 - 8s(1+2s)]^{\frac{1}{2}}}{(2+4s)}$ where $B = 1 + 4s\cos\theta - i2C\sin\theta$	$C \leqq 1$	$C^2 \ll 1$ for accuracy

M3: Ecuación de Transporte.
Esquema completamente implícito

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} - \alpha\frac{\partial^2 T}{\partial x^2} = 0$$

$$\frac{\partial T}{\partial t} = \frac{T_i^{n+1} - T_i^n}{\Delta t}$$

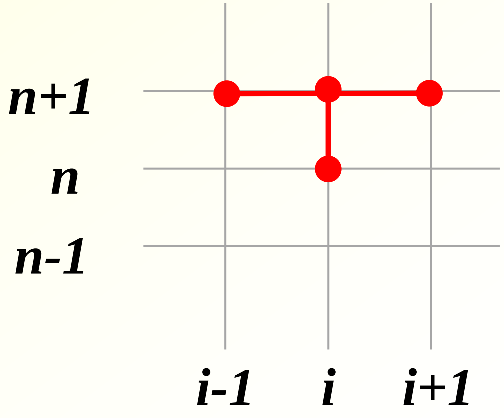
$$\frac{\partial T}{\partial x} = \frac{T_{i+1}^{n+1} - T_{i-1}^{n+1}}{2\Delta x} \quad \Rightarrow \quad \frac{T_i^{n+1} - T_i^n}{\Delta t} + u\frac{T_{i+1}^{n+1} - T_{i-1}^{n+1}}{2\Delta x} - \alpha\frac{T_{i-1}^{n+1} - 2T_i^{n+1} + T_{i+1}^{n+1}}{\Delta x^2} = 0$$

$i = 1 \;\; ... \;\; N-1$

+ 2 cond. frontera

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i-1}^{n+1} - 2T_i^{n+1} + T_{i+1}^{n+1}}{\Delta x^2}$$

Consistencia: $E_i^n \propto \dfrac{\partial^3 T}{\partial T^3}$



Estabilidad: incondicionalmente estable

M3: Ecuación de Transporte.

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} - \alpha\frac{\partial^2 T}{\partial x^2} = 0$$

Lax-Wendroff 	$\frac{\Delta T_j^{n+1}}{\Delta t} + uL_x T_j^n - \alpha^* L_{xx} T_j^n = 0$ where $\alpha^* = \alpha + 0.5uC\Delta x$	$-[C\alpha\Delta x - u(\Delta x^2/6)(1-C^2)]\frac{\partial^3 T}{\partial x^3}$ $+[C\alpha^2/u(\Delta x/2) - \alpha\Delta x^2/12$ $-uC(\Delta x^3/8)(C^2-1)]\frac{\partial^4 T}{\partial x^4}$	$1-2s^*(1-\cos\theta) - iC\sin\theta$ where $s^* = \alpha^*\Delta t/\Delta x^2$	$0 \leqq C^2 \leqq 2s^* \leqq 1$	$R_{cell} \leqq 2$ to avoid spatial oscillations
Crank-Nicolson 	$\frac{\Delta T_j^{n+1}}{\Delta t} + \{uL_x - \alpha L_{xx}\}\left\{\frac{T_j^n + T_j^{n+1}}{2}\right\} = 0$	$u(\Delta x^2/6)(1+0.5C^2)\frac{\partial^3 T}{\partial x^3}$ $-\alpha(\Delta x^2/12)(1+3C^2)\frac{\partial^4 T}{\partial x^4}$	$\frac{1-s(1-\cos\theta) - i\,0.5C\sin\theta}{1+s(1-\cos\theta)+i\,0.5C\sin\theta}$	None	$R_{cell} \leq 2$ to avoid spatial oscillations
Three-level fully implicit 	$\frac{3}{2}\frac{\Delta T_j^{n+1}}{\Delta t} - \frac{1}{2}\frac{\Delta T_j^n}{\Delta t}$ $+\{uL_x - \alpha L_{xx}\} T_j^{n+1} = 0$	$u(\Delta x^2/6)(1+2C^2)\frac{\partial^3 T}{\partial x^3}$ $-\alpha(\Delta x^2/12)(1+12C^2)\frac{\partial^4 T}{\partial x^4}$	$\frac{1 \pm \frac{1}{3}i[3+16s(1-\cos\theta)+i8C\sin\theta]^{\frac{1}{2}}}{2(1+\frac{2}{3}[2s(1-\cos\theta)+iC\sin\theta])}$	None	$R_{cell} \leq 2$ to avoid spatial oscillations
Linear F.E.M./ Crank-Nicolson 	$M_x\frac{\Delta T_j^{n+1}}{\Delta t} + uL_x\left\{\frac{T_j^n + T_j^{n+1}}{2}\right\}$ $-\alpha L_{xx}\left\{\frac{T_j^n + T_j^{n+1}}{2}\right\} = 0$	$uC^2(\Delta x^2/12)\frac{\partial^3 T}{\partial x^3}$ $+\alpha(\Delta x^2/12)(1-3C^2)\frac{\partial^4 T}{\partial x^4}$	$\frac{2+3\cos\theta - 3s(1-\cos\theta) - i1.5C\sin\theta}{2+3\cos\theta+3s(1-\cos\theta)+i1.5C\sin\theta}$	None	$R_{cell} \leq 2$ to avoid spatial oscillations

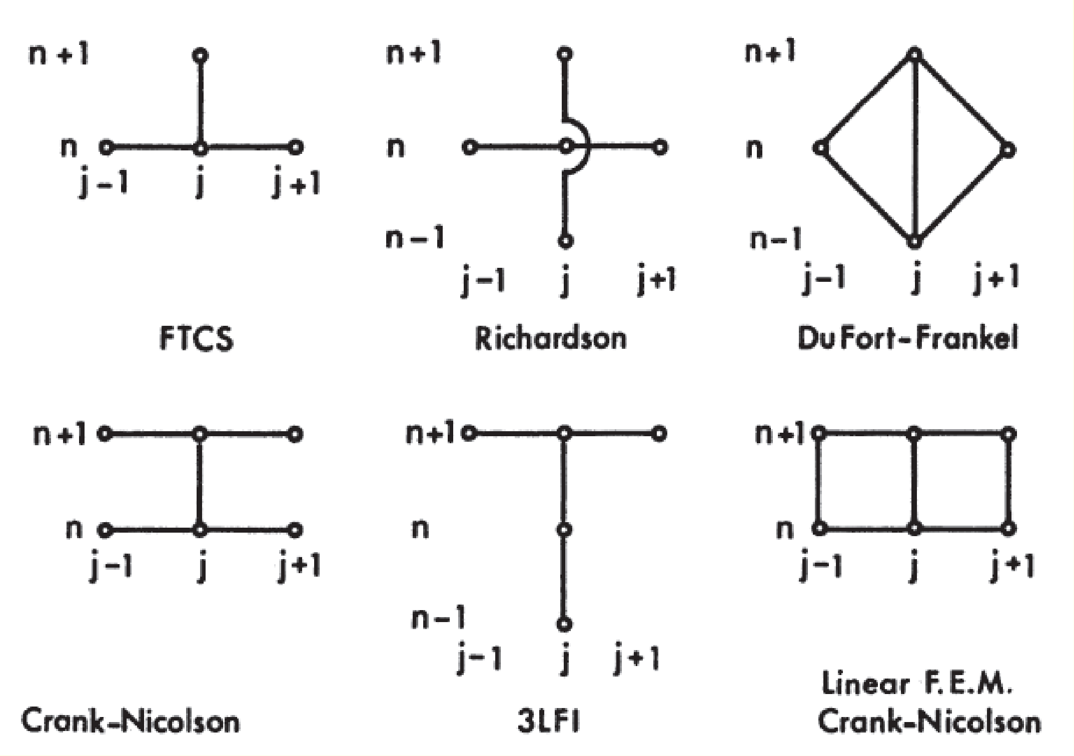
[a] The algebraic scheme is equivalent to $\partial T/\partial t + u\partial T/\partial x - \alpha\partial^2 T/\partial x^2 + E(T) = 0$

$L_x = \frac{1}{2\Delta x}\{-1, 0, 1\}$, $L_{xx} = \frac{1}{\Delta x^2}\{1, -2, 1\}$, $M_x = \{\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\}$, $C = u\Delta t/\Delta x$, $s = \alpha\Delta t/\Delta x^2$, $R_{cell} = C/s = u\Delta x/\alpha$

M3: Ecuación de Transporte.

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} - \alpha\frac{\partial^2 T}{\partial x^2} = 0$$

Esquemas de nodos activos para los principales esquemas de integración considerados:



Ecuación Unidimensional de Transporte

Implementar los siguientes algoritmos en un programa que resuelva la ecuación unidimensional de transporte:

Métodos Explícitos:

6*.- Esquema Forward in Time Centered in Space (FTCS).

7*.- Esquema upstream.

8*.- Esquema DuFort-Frankel.

Métodos Implícitos:

9.- Esquema totalmente implícito a dos niveles.

10.- Esquema Crank-Nicolson.

11*.- Calcular la estabilidad y la consistencia de uno de los métodos anteriores.