Linearization of a rational function

$f(X, Y) = \frac{1+Y}{1+X}$

$a = \frac{Y}{X} \rightarrow f(X) = \frac{1+aX}{1+X}$

$f(X) = \frac{g(X)}{h(X)}$ where $g(X) = 1 + aX$ and $h(X) = 1 + X$

$\frac{\partial f}{\partial g} = \frac{1}{h(X)}$

$\frac{\partial f}{\partial h} = -\frac{g(X)}{h(X)^2}$

$df(X) = \frac{\partial f}{\partial g} dg(X) + \frac{\partial f}{\partial h} dh(X)$

$\frac{df(X)}{dX} = \frac{\partial f}{\partial g}\frac{dg(X)}{dX} + \frac{\partial f}{\partial h}\frac{dh(X)}{dX}$

$f'(X) = \frac{\partial f}{\partial g} g'(X) + \frac{\partial f}{\partial h} h'(X)$

$f'(X) = \frac{g'(X)}{h(X)} - \frac{g(X)}{h(X)^2} h'(X)$

$g'(X) = a$

$h'(X) = 1$

$f'(X) = \frac{a}{1+X} - \frac{1+aX}{(1+X)^2} = \frac{a - \frac{1+aX}{1+X}}{1+X} = \frac{a + aX - 1 - aX}{(1+X)^2} = \frac{a-1}{(1+X)^2}$

Taylor series expansion (in the neighbourhood of $X_0 = 0$) :

$\tilde{f}(X) = f(0) + \frac{f'(0)}{1!}(X - 0) + \cdots$ where … ≈ 0

$= 1 + (a - 1)X$

$= 1 + \left(\frac{Y}{X} - 1\right) X$

$= 1 + \left(\frac{Y-X}{X}\right) X$

$= 1 + Y - X$