A formalization of Borel determinacy in Lean

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- But the proof assistant is only as strong as the logic it uses internally!

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- Beware: these logics are not simple!

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- Alice and Bob alternately pick natural numbers $n_0, n_1, n_2, \ldots$. Alice is the first one to pick.
- That generates a sequence $\langle n_i \rangle_{i \in \mathbb{N}}$. Alice wins iff the generated sequence belongs to A, the set of winning sequences.

When Alice can win?

- If the winning set is finite, then the set of elements B_1 that Bob should choose in his first turn is also finite. So Bob can choose any number from $\mathbb{N} - B_1$ and the rest of the game doesn't matter - Alice loses.

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- Prove that $\mathbb{N} - B_1$ is not empty
- Prove that for any suffix, Bob's sequence won't be winning

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- Relate these axioms to the proof above to convince reader that the existence of the proof above is actually not that obvious as it seems

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- This set is not definable in ZF without the axiom schema of replacement!
- Will we not run into troubles while proving statements about more complicted winning sets in our game?

Gale-Stewart

- A Gale-Stewart game is a pair $G = (T, P)$, where T is a nonempty pruned tree and $P \subseteq [T]$ is the winning set. Define topology, open, closed and Borel sets of sequences; open, closed and Borel games.

Is the game determined when the winning set is open or closed?

- todo: a proof, needs transfinite induction?

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- This is much more difficult!
- Harvey Friedman showed that determinacy for Gale-Stewart games where the winning set is only Borel, is not provable in ZF without the axiom schema of replacement!
- But will we be able to prove it in Lean 4?

ZFC version used in Lean 4

Tarski-Grothendieck set theory

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- ZFC + Tarski's axiom, which implies existence of inaccessible cardinals
- enough to define category theory, in contrast to ZFC
- Mizar: a Polish theorem prover. In 2009 its mathlib was the biggest body of formalized maths in the world!
- the underlying theory of Mizar is precisely first-order logic with Tarski-Grothendieck set theory

Alexander Grothendieck (1928-2014)

Throwback: How can you expect tax-payers to believe in this? (Inter-universal Teichmüller theory)

$$
\begin{array}{ccccccccccccccccccccc}
\{\pm 1\} & \curvearrowright & (-l^* & < & \ldots & < & -2 & < & -1 & < & 0 & < & 1 & < & 2 & < & \ldots & < & l^*) \\
 & & (/^{\pm} & & & & /^{\pm} & & /^{\pm} & & /^{\pm} & & /^{\pm} & & /^{\pm} & & & & /^{\pm})
\end{array}
$$

$$\mathfrak{D}_T$$

$$\Downarrow \ \phi_{\pm}^{\Theta^{\mathrm{ell}}}$$

$$
\begin{array}{ccccc}
 & \pm & \longrightarrow & \pm & \\
 \nearrow & & & & \searrow \\
\pm & & \mathbb{F}_l^{\rtimes\pm} \curvearrowright & & \pm \\
\uparrow & & \mathcal{D}^{\circledcirc\pm} = & & \downarrow \\
\pm & & \mathcal{B}(\underline{X}_K)^0 & & \pm \\
 \nwarrow & & & & \swarrow \\
 & \pm & \ldots & \pm &
\end{array}
$$

Grothendieck, 1970

Grothendieck, Lasserre, France, 2013