A formalization of Borel determinacy in Lean

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- But the proof assistant is only as strong as the logic it uses internally!

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- Beware: these logics are not simple!

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- Alice and Bob alternately pick natural numbers $n_0, n_1, n_2, \ldots$. Alice is the first one to pick.
- That generates a sequence $\langle n_i \rangle_{i \in \mathbb{N}}$. Alice wins iff the generated sequence belongs to A, the set of winning sequences.

When Alice can win?

- If the winning set is finite, then the set of elements B_1 that Bob should choose in his first turn is also finite. So Bob can choose any number from $\mathbb{N} - B_1$ and the rest of the game doesn't matter - Alice loses.

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- Prove that for any suffix, Bob's sequence won't be winning

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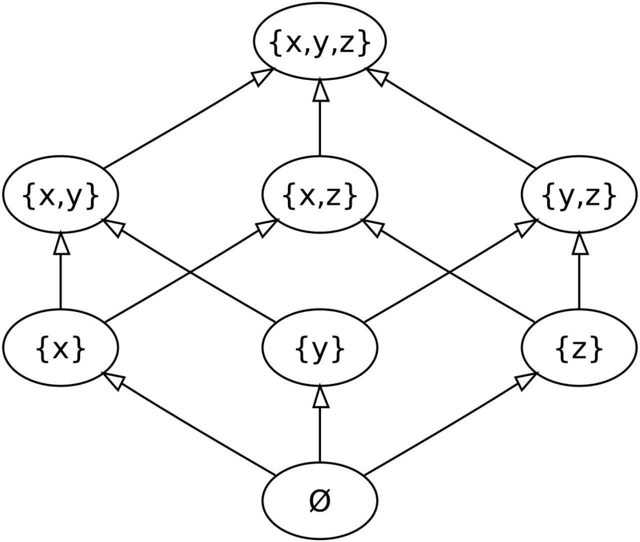
- Logical symbols $\neg, \wedge, \vee, \forall, \exists, =, \in, ()$

Set theory as a graph



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- Choice (**)

Axiom of the Sum Set (Union)

Given a collection of sets (which is itself a set), we can form a new set containing all the elements that belong to at least one set in the collection.
$\forall X \exists Y \forall u (u \in Y \iff \exists z (z \in X \wedge u \in z))$

Axiom of the Power Set

For any set, we can form a new set that contains all possible subsets of the original set. $\forall X \exists Y \forall u (u \in Y \iff u \subseteq X)$

Axiom of Infinity

This axiom postulates the existence of at least one set with infinitely many elements. A common example is the set of natural numbers (or a set that can be put into one-to-one correspondence with it).
$\exists S[\emptyset \in S \land (\forall x \in S)[x \cup \{x\} \in S]]$

Axiom of the Unordered Pair

Given any two sets, we can form a new set containing precisely those two sets. $\forall a \forall b \exists c \forall x (x \in c \iff (x = a \lor x = b))$

Axiom of Extensionality

Two sets are equal if and only if they contain exactly the same members.
$\forall u(u \in X \iff u \in Y) \implies X = Y$

Axiom of Foundation (Regularity)

This axiom prevents the existence of infinite descending chains of set membership; every nonempty set has an $\in$-minimal element.
$\forall S[S \neq \emptyset \implies (\exists x \in S)[S \cap x = \emptyset]]$

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- We can form a subset of an existing set consisting of all elements that satisfy a given property. Note that we can only take a subset and not form arbitrary sets - the latter leads to Russel's paradox.

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- $\forall X \forall p \exists Y \forall u (u \in Y \iff (u \in X \land \phi(u, p)))$

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- $\forall p(\forall x \forall y \forall z[\phi(x, y, p) \wedge \phi(x, z, p) \implies y = z]) \implies (\forall X \exists Y \forall y[y \in Y \iff (\exists x \in X)\phi(x, y, p)])$

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- $\forall x \in a \exists y A(x, y) \implies \exists f \forall x \in a A(x, f(x))$

$\{\mathbb{N}, \mathcal{P}(\mathbb{N}), \mathcal{P}(\mathcal{P}(\mathbb{N})), \dots\}$

- This set is not definable in Zermelo set theory, i.e. ZF without the axiom schema of replacement!

$\{N, P(N), P(P(N)), \ldots\}$

- This set is not definable in Zermelo set theory, i.e. ZF without the axiom schema of replacement!
- Will we not run into troubles while proving statements about more complicated winning sets in our game?

Gale-Stewart game

- A Gale-Stewart game is a game like above. For simplicity, say that Alice and Bob choose elements from $\{0, 1\}$, so that they construct a sequence from $\{0, 1\}^\omega$. Denote the winning set as $P \subseteq \{0, 1\}^\omega$.

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- $\{0, 1\}^\omega$: an infinite product of topological spaces. What is the topology? On the next slide!

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- $\{0, 1\}^\omega$: an infinite product of topological spaces. What is the topology? On the next slide!
- When we say 'open game', 'closed game', 'Borel game' we only consider e.g. openness of P in $\{0, 1\}^\omega$ with the product topology

Product topology

- The topology on $\prod_{\alpha \in J} X_\alpha$ with basis $\prod_{\alpha \in J} U_\alpha$, where each U_α is open in X_α and all but finitely many $U_\alpha = X_\alpha$.

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- Any open set in this topology can be expressed as a union of some family of elements from the basis
- Intuitively, an open set in the product topology is such that it only specifies a finite number of conditions on coordinates

Intuition

- In a closed game, if Player II is going to win (meaning the play will *not* be in Player I's winning set P), they will win after a finite number of moves. There will be a point in the game where, no matter what Player I does afterwards, the resulting infinite play will *not* be in Player I's winning set. Player II's victory (Player I's loss) becomes guaranteed at some finite stage.

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- If Player II doesn't win after any finite number of steps, then necessarily, Player I wins.

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- I.e. II has no winning strategy in the game on $P|_p$, where $P|_p = p^{-1}P = \{y \mid py \in P\}$. So φ is not losing for I.

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- If for every possible move a_{2n} by Player I, there existed a response a_{2n+1} by Player II leading to a position from which Player I loses, then the initial position p would be losing for Player I, which contradicts our assumption.

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- II then plays some a_1. I responds with a_2 such that for all responses a_3, (a_0, a_1, a_2, a_3) is not losing for I, **etc.**

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- There is a k such that $(a_0, ..., a_{2k-1}) * \{0,1\}^\omega \subseteq W$

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- Since P is closed, P^c is open in $\{0,1\}^\omega$.
- Suppose that $(a_0, \dots) \notin P$.
- Then there exists k such that the neighborhood $(a_0, \dots, a_{2k-1})\{0,1\}^\omega \subseteq P^c$.

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- Therefore, the assumption that $(a_0, \ldots) \notin P$ must be false
- So Player I wins.

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- Take any countably many sets A_i. Their union and intersection is also Borel.
- Iterate these operations transfinitely. The result is the **Borel σ-algebra**.

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- Harvey Friedman showed that determinacy for Gale-Stewart games where the winning set is only Borel, is not provable in ZF without the axiom schema of replacement!
- But will we be able to prove it in Lean 4?

Lean 4: hierarchy of universes

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- There are the two universes Type and Prop at the top
- There are the types and the theorem statements one level below them
- Then there are the terms and the theorem proofs at the bottom.
- The type of all real numbers $\mathbb{R}$ is a type, so $\mathbb{R}$ lives at the middle level, and real numbers like 7 are terms; we write $7 : \mathbb{R}$ to indicate that 7 is a real number.

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- it leads to paradoxes to set the type of `Type` to be also `Type`
- the type of `Type` is `Type 1`, etc.

Lean 4: inductive types

Type of binary trees over a type *a* in universe *u*:

```
inductive Tree. {u} (a : Type u) : Type u
  | nil : Tree a
  | node : a → Tree a → Tree a → Tree a
```

ZFC version used in Lean 4: pre-sets

A pre-set is a set without an equality relation. Type of a **pre-set** over a Lean type a lives in a universe of a higher level. Function A is the embedding of objects of type a into pre-sets, i.e. if we want to have a pre-set of `Nat` objects, we need the type `Nat` in Lean and we need a way to represent objects $n : Nat$ as pre-sets:

```
inductive PSet : Type (u + 1)
  | mk (a : Type u) (A : a → PSet) : PSet

def empty_pset : PSet :=
  PSet.mk Empty (fun x : Empty => nomatch x)
```

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- Lean's type theory is said to be equiconsistent to ZFC + "there are countably many inaccessible cardinals" (Mario Carneiro)

Modeling ZFC in Coq

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- I honestly don't know the details. You define that a ZFC-set is a Type, then define some properties. Coq has no quotient types by default, unlike Lean. It is very very subtle

Theory of Mizar: Tarski-Grothendieck set theory

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- enough to define category theory, in contrast to ZFC
- if you were to formalize that ALL games are determined, Mizar would certainly not be your best assistant: using the Axiom of Choice, you can construct a not determined winning set! (see end remark)

A formalization of Borel determinacy in Lean

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- https://www.arxiv.org/pdf/2502.03432

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- https://www.mimuw.edu.pl/~niwinski/Prace/ed.pdf

Alexander Grothendieck (1928-2014)

Grothendieck, 1970

Grothendieck, Lasserre, France, 2013