A formalization of Borel determinacy in Lean

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March 19, 2025

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- But the proof assistant is only as strong as the logic it uses internally!

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- (Lean is expressive enough to understand the second-order sentence $\forall P, P \vee \neg P$ and take is as an axiom, so you can use it without proving)
- Beware: these logics are not simple!

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- Alice and Bob alternately pick natural numbers $n_0, n_1, n_2, \ldots$. Alice is the first one to pick.
- That generates a sequence $\langle n_i \rangle_{i \in \mathbb{N}}$. Alice wins iff the generated sequence belongs to A, the set of winning sequences.

When Alice can win?

- If the winning set is finite, then the set of elements B_1 that Bob should choose in his first turn is also finite. So Bob can choose any number from $\mathbb{N} - B_1$ and the rest of the game doesn't matter - Alice loses.

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- Prove that for any suffix, Bob's sequence won't be winning

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- Regularity
- Choice (**)

Axiom of Extensionality

Two sets are equal if and only if they contain exactly the same members.
$\forall u(u \in X \iff u \in Y) \implies X = Y$

Axiom of the Unordered Pair

Given any two sets, we can form a new set containing precisely those two sets. $\forall a \forall b \exists c \forall x (x \in c \iff (x = a \lor x = b))$

Axiom of Specification (Separation / Comprehension)

We can form a subset of an existing set consisting of all elements that satisfy a given property. $\forall X \forall p \exists Y \forall u (u \in Y \iff (u \in X \wedge \phi(u, p)))$

Axiom of the Sum Set (Union)

Given a collection of sets (which is itself a set), we can form a new set containing all the elements that belong to at least one set in the collection.
$\forall X \exists Y \forall u (u \in Y \iff \exists z (z \in X \wedge u \in z))$

Axiom of the Power Set

For any set, we can form a new set that contains all possible subsets of the original set. $\forall X \exists Y \forall u (u \in Y \iff u \subseteq X)$

Axiom of Infinity

This axiom postulates the existence of at least one set with infinitely many elements. A common example is the set of natural numbers (or a set that can be put into one-to-one correspondence with it).
$\exists S[\emptyset \in S \land (\forall x \in S)[x \cup \{x\} \in S]]$

Axiom of Replacement

If F is a function, then for any X there exists a set $\{F(x) : x \in X\}$
$(\forall x \forall y \forall z[\phi(x, y, p) \wedge \phi(x, z, p) \implies y = z]) \implies (\forall X \exists Y \forall y[y \in Y \iff (\exists x \in X)\phi(x, y, p)])$

Axiom of Foundation (Regularity)

This axiom prevents the existence of infinite descending chains of set membership; every nonempty set has an $\in$-minimal element.
$\forall S[S \neq \emptyset \implies (\exists x \in S)[S \cap x = \emptyset]]$

Axiom of Choice

Given any collection of non-empty sets, it is possible to select one element from each set, even if the collection is infinite and there is no specific rule for making the selection. $\forall x \in a \exists y A(x, y) \implies \exists f \forall x \in a A(x, f(x))$

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- This set is not definable in Zermelo set theory, i.e. ZF without the axiom schema of replacement!
- Will we not run into troubles while proving statements about more complicted winning sets in our game?

Gale-Stewart

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- Discrete topology on $\mathbb{N}$: the topology where every subset of $\mathbb{N}$ is open.
- When we say ‘open game’, ‘closed game’, ‘Borel game’ we only consider e.g. openness of [T] in $\mathbb{N}^{\mathbb{N}}$ with the product topology

Box topology

- The topology on $\prod_{\alpha \in J} X_\alpha$ with basis $\prod_{\alpha \in J} U_\alpha$, where each U_α is open in X_α.

Product topology

- The topology on $\prod_{\alpha \in J} X_\alpha$ with basis $\prod_{\alpha \in J} U_\alpha$, where each U_α is open in X_α and all but finitely many $U_\alpha = X_\alpha$.

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- The key difference is that the box topology allows infinitely many factors in the basis element to be proper open subsets, while the product topology requires all but finitely many to be the entire space.

Intuition

- In an open game, if Player I is going to win, they will win after a finite number of moves. There will be a point in the game where, no matter what Player II does afterwards, the resulting infinite play will be in Player I's winning set. Player I's victory becomes guaranteed at some finite stage.

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- In a closed game, if Player II is going to win (meaning the play will *not* be in Player I's winning set P), they will win after a finite number of moves. There will be a point in the game where, no matter what Player I does afterwards, the resulting infinite play will *not* be in Player I's winning set. Player II's victory (Player I's loss) becomes guaranteed at some finite stage.

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- I.e. II has no winning strategy in the game $G(T|_p, P|_p)$, where $T|_p = \{s \mid ps \in T\}$, and $P|_p = \{y \mid py \in P\}$. So φ is not losing for I.

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- If for every possible move a_{2n} by Player I, there existed a response a_{2n+1} by Player II leading to a position from which Player I loses, then the initial position p would be losing for Player I, which contradicts our assumption.

Player I's Strategy Construction

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- II then plays some a_1. I responds with a_2 such that for all responses a_3, (a_0, a_1, a_2, a_3) is not losing for I, **etc.**

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- There is a k such that $N_{(a_0,\ldots,a_{2k-1})} \cap [T] \subseteq W$

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- Since P is closed in $[T]$, $P^c is open.
- Suppose that $(a_0, \ldots) \notin P$.
- There exists k such that the neighborhood $N(a_0, \ldots, a_{2k-1}) \cap [T] \subseteq P^c$.

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- Therefore, the assumption that $(a_0, ...) \notin P$ must be false
- So Player I wins.

Remarks on Axiom of Choice (from Kechris)

- Theorem 20.1 (Determinacy of Open/Closed Games) generally requires the **Axiom of Choice** due to the single-valuedness condition in the definition of a strategy. A strategy specifies a unique move for each possible history.

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- Harvey Friedman showed that determinacy for Gale-Stewart games where the winning set is only Borel, is not provable in ZF without the axiom schema of replacement!
- But will we be able to prove it in Lean 4?

Lean 4: hierarchy of universes

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- There are the two universes Type and Prop at the top
- There are the types and the theorem statements one level below them
- Then there are the terms and the theorem proofs at the bottom.
- The type of all real numbers $\mathbb{R}$ is a type, so $\mathbb{R}$ lives at the middle level, and real numbers like 7 are terms; we write $7 : \mathbb{R}$ to indicate that 7 is a real number.

ZFC version used in Lean 4: infinite hierarchy of universes

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- the type of `Type` is `Type 1`, etc.

ZFC version used in Lean 4: pre-sets

Set some specific universe `u`. Define a notion of **pre-set** inductively:

```
inductive PSet : Type (u + 1)
  | mk (a : Type u) (A : a → PSet) : PSet
```

ZFC version used in Lean 4: sets as pre-set quotients

Define extensional equivalence on pre-sets. Define ZFC sets by quotienting pre-sets by extensional equivalence.

ZFC version used in Lean 4: classes

Define classes as sets of ZFC sets. Then the rest is usual set theory.

```
def Class :=
  Set ZFSet
```

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- In the HoTT book, there is a whole section on ZFC. Requires in-depth HoTT knowledge, so also category theory and algebraic topology.

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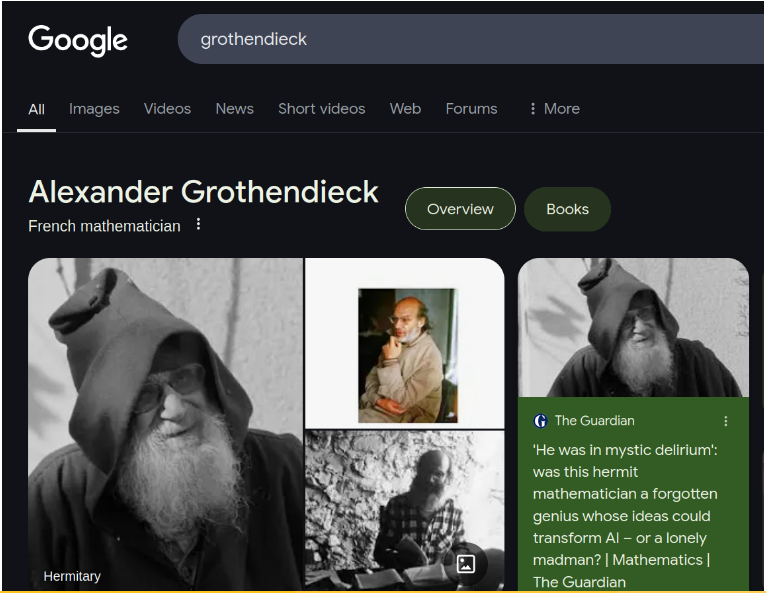
- the underlying theory of Mizar is precisely first-order logic with Tarski-Grothendieck set theory
- if you were to formalize that ALL games are determined, Mizar would certainly not be your best assistant: determinacy of all games implies Axiom of Determinacy, which is known to contradict the Axiom of Choice.

Alexander Grothendieck (1928-2014)



Throwback: How can you expect tax-payers to believe in this? (Inter-universal Teichmüller theory)

$$\{\pm 1\} \quad \curvearrowright \quad (\,-l^* \;<\; \dots \;<\; -2 \;<\; -1 \;<\; 0 \;<\; 1 \;<\; 2 < \;\dots\; < l^*\,)$$

$$(\,/^{\pm} \qquad\qquad\qquad /^{\pm} \quad /^{\pm} \quad /^{\pm} \quad /^{\pm} \quad /^{\pm} \qquad\qquad /^{\pm}\,)$$

$$\mathfrak{D}_T$$

$$\Downarrow \;\; \phi_{\pm}^{\Theta^{\mathrm{ell}}}$$

$$\begin{array}{ccccc}
 & \pm & \longrightarrow & \pm & \\
\nearrow & & & & \searrow \\
\pm & & \mathbb{F}_l^{\rtimes\pm} \curvearrowright & & \pm \\
\uparrow & & \mathcal{D}^{\circledcirc\pm} = & & \downarrow \\
\pm & & \mathcal{B}(\underline{X}_K)^0 & & \pm \\
\nwarrow & & & & \swarrow \\
 & \pm & \dots & \pm &
\end{array}$$

Grothendieck, 1970

Grothendieck, Lasserre, France, 2013