MATH 242: Exam D & E (32 points + 10 bonus to be scaled to 50 points + 31 bonus.)

Name:

1. (4 points) Give the characteristic polynomial of

$$A = \begin{pmatrix} 4 & -3 \\ 2 & 5 \end{pmatrix}$$

2. (5 points) Find the determinant of the matrix

$$D = \begin{pmatrix} 0 & 5 & 6 & 0 \\ 1 & 0 & 0 & 4 \\ 0 & 15 & 26 & 9 \\ -2 & 0 & 0 & 2 \end{pmatrix}$$

3. (3 points) The matrix

$$B = \begin{pmatrix} -12 & -15 \\ 6 & 7 \end{pmatrix}$$

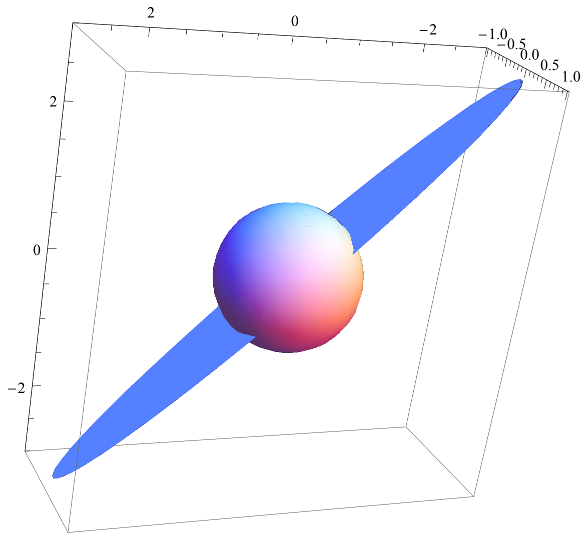
has eigenvector $v = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$. Determine the associated eigenvalue.

4. (5 points) The matrix

$$C = \begin{pmatrix} 9 & 8 & 0 \\ -10 & -9 & 0 \\ 20 & 20 & 1 \end{pmatrix}$$

has eigenvalue 1. Give a basis of the eigenspace $\mathcal{E}_C(1)$.

5. (5 points) A 3×3 real matrix G is applied to all the points on the unit sphere in $\mathbb{R}^3$. The image of the sphere is a flat elliptical disc:



(a) Describe everything you know about the eigenvalues of G.

(b) Describe the limit of the sphere under repeated transformations by G.

6. (5 points) Prove that if $\lambda \neq \gamma$ are two distinct eigenvalues of a matrix F, then the intersection of the eigenspaces is trivial, i.e. $\mathcal{E}_F(\lambda) \cap \mathcal{E}_F(\gamma) = \{\vec{0}\}$

7. (5 points) Prove that if matrix H is similar to matrix K then $\det(H) = \det(K)$.

8. (5 points) The matrix $L = \begin{pmatrix} \frac{-1+i}{2} & \frac{-1-i}{2} \\ \frac{-1-i}{2} & \frac{-1+i}{2} \end{pmatrix}$ has eigenvalues $\lambda_1 = i$ and $\lambda_2 = -1$ for eigenvectors $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Compute L^6.

Bonus. (5 points) Suppose M is a diagonalizable matrix such that there exists a positive integer n with $M^n = M$. Show that all the eigenvalues of M are either 0 or of the form $\lambda_i = e^{2\pi i k_i/(n-1)}$ for integers k_i.

MATH 242: Exam LT (50 points + 12 bonus.)

Name:

1. (6 points) Suppose U is a vector space with operations of vector addition and scalar multiplication denoted by $\oplus$ and $\otimes$, respectively. Suppose V is a vector space V with operations of vector addition and scalar multiplication denoted by $\vee$ and $\star$, respectively. Define what it means to say T is a linear transformation from U to V.

2. (3 points) Define: A linear transformation $T : U \to V$ is 1-to-1 (also known as injective):

3. (3 points) Define: A linear transformation $T : U \to V$ is onto (also known as surjective):

4. (15 points) Suppose $T : \mathbb{C}^4 \to \mathbb{C}^3$ is given by

$$T\left(\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}\right) = \begin{pmatrix} a - b + 3c - 2d \\ 2b + 4c - 6d \\ a + b + 8c - 8d \end{pmatrix}$$

(a) Which do you know immediately: T is not 1-to-1 T is not onto.
Why do you know this?

(b) Give the matrix representation of T using the standard basis for both $\mathbb{C}^4$ and $\mathbb{C}^3$.

(c) In part (a) you gave a reason why T is <u>not</u> one of those things. Explain how we know/prove that T <u>is</u> the other one.

(d) Give a basis for $\ker(T)$.

5. (4 points) Give an example, in pictures/cartoons of blobs and arrows, of two functions $f : U \to V$ and $g : V \to W$ such that f is 1-to-1 and g is onto, but $g \circ f$ is neither.

6. (4 points) Give an example, in pictures/cartoons of blobs and arrows, of two functions $f : U \to V$ and $g : V \to W$ such that f is onto and g is 1-to-1, but $g \circ f$ is neither.

7. (4 points) The linear transformation $T : P_1 \to \mathbb{C}^2$ given by

$$T(ax + b) = \begin{pmatrix} 3a + 7b \\ 2a + 5b \end{pmatrix}$$

is a bijection. Give $T^{-1}\left(\begin{pmatrix} r \\ s \end{pmatrix}\right)$.

8. (5 points) Suppose $S: U \to V$ is a linear transformation and that

$$S(3u_1 - 4u_3) = v_1,\ S(u_1 + u_3) = v_2,\ S(u_2) = v_3 \text{ and } S(u_2 + u_3) = v_4.$$

Show that the set $\{v_1, v_2, v_3, v_4\}$ is linearly dependent by exhibiting a linear combination that gives the zero vector. Make sure to justify your answer completely.

9. (6 points) The linear transformation $R: P_2 \to \mathbb{C}^4$ has the properties:

$$R(5) = \begin{pmatrix} 10 \\ 5 \\ -5 \\ 15 \end{pmatrix},\ R(x+1) = \begin{pmatrix} 3 \\ 1 \\ 4 \\ 1 \end{pmatrix} \text{ and } R(x^2 + x + 1) = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 2 \end{pmatrix}.$$

Give the general formula for $R(ax^2 + bx + c)$.

Bonus. Let U be a vector space. Recall that

$$\mathcal{L}(U) = \{\text{All linear transformations } T : U \to U\}$$

is also a vector space (with operations of $(T_1 + T_2)(u) = T_1(u) + T_2(u)$ and $(\alpha T)(u) = \alpha T(U)$.).

Now suppose $S : U \xrightarrow{\cong} V$ is a bijective linear transformation, i.e. $U \cong V$. Show that $\mathcal{L}(U) \cong \mathcal{L}(V)$.

Steps: For $T \in \mathcal{L}(U)$ define $\phi_S(T) \in \mathcal{L}(V)$ by $(\phi_S(T))(v) = S(T(S^{-1}(v)))$

(a) (4 points) Show that ϕ_S satisfies the properties of a linear transformation.

(b) (4 points) Show that ϕ_S is 1-to-1. Do this by supposing for all $v \in V$ that $(\phi_S(T_1))(v) = (\phi_S(T_2))(v)$ and then showing $T_1(u) = T_2(u)$ for all $u \in U$.

(c) (4 points) Show that ϕ_S is onto. Do this by supposing $R \in \mathcal{L}(V)$ and finding $T \in \mathcal{L}(U)$ such that $\phi_S(T) = R$.

MATH 242: Exam SLE (62 points + 10 bonus points to be scaled to 50 points + 8 bonus points)

Name:

1. (15 points) Give as good of a definition as you can for each bolded term:

   (a) The linear system, $\mathbf{LS(A, b)}$, associated to the matrix A and vector of constants b.

   (b) The **Solution Set** of a system of equations.

   (c) The **null space** of a matrix A.

   (d) A **nonsingular** matrix A.

   (e) **Equivalent** linear systems of equations.

2. (28 points) Label the following statements as 'True' or 'False' (1 point each). Explain why (3 points each). (Note: An example where it is true is <u>not</u> enough to justify a 'True' answer.)

   (a) If two linear systems have the same solution set then the two associated augmented matrices are row-equivalent.

   (b) If $r > n$ then the system is inconsistent.

   (c) It's possible for a matrix to have $D = \{1, 3, 4, 7\}$ and $F = \{2, 5\}$.

   (d) The identity matrix is consistent.

(e) If the augmented matrix $[A|b]$ row-reduces to the identity, then there is a unique solution to the associated system of equations.

(f) It's possible for the zero vector to be a solution to an inhomogeneous system of equations.

(g) If A reduces to the identity matrix, then the null space of A contains only the zero vector.

3. (3 points) Suppose you have a linear system $LS(A, b)$ of 5 equations with 4 complex variables. You put the augmented matrix, $[A|b]$, into SAGE and have return the row-reduced echelon form. The output from SAGE is

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 3+I & 0 & 2 \\ 0 & 0 & 0 & 1 & 4*I \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the solution set using proper written/human notation.

4. (4 points) Suppose you know a system of equations has 5 equations in 9 variables. Say **as much as possible** about what you know about the system (especially regarding the solution set and any definitions/terms that apply).

5. (4 points) Put the following matrix in row-reduced echelon form. Make sure to annotate your intermediate steps and box your final answer.

$$\begin{pmatrix} 7 & 3 & 1 \\ 2 & 8 & 2 \\ 0 & 5 & 0 \end{pmatrix}$$

6. (4 points) Prove/explain why: if A is row-equivalent to B and B is row equivalent to C, then we know A is row-equivalent to C.

7. (4 points) Suppose that A is a nonsingular matrix and A is row-equivalent to the matrix B. Prove/explain why: B must be nonsingular.

Bonus 1. (5 points)

(a) Solve $2x + 3 = 4 \mod 7$.

(b) Explain why $2x + 3 = 4 \mod 6$ doesn't have a solution.

Bonus 2. (5 points) Suppose that the coefficient matrix of a consistent system of linear equations has two columns that are identical. Prove that the system has infinitely many solutions.