Addition with Carries

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Grade 2 Math: Addition with Carrying

Example

$$\begin{array}{r} 957 \\ +\ 218 \\ \hline \end{array}$$

Grade 2 Math: Addition with Carrying

Example

$$\begin{array}{rr} & {\scriptstyle 1} \\ & 957 \\ + & 218 \\ \hline & 5 \end{array}$$

Grade 2 Math: Addition with Carrying

Example

$$\begin{array}{r} {\scriptstyle 1} \\ 957 \\ +\ 218 \\ \hline 75 \end{array}$$

Grade 2 Math: Addition with Carrying

Example

$$\begin{array}{r} {\scriptstyle 1\ \ 1\ \ } \\ 957 \\ +\ \ 218 \\ \hline 175 \end{array}$$

Grade 2 Math: Addition with Carrying

Example

$$\begin{array}{r} {}^{1}\ {}^{1}\ \\ 957 \\ +\ 218 \\ \hline 1175 \end{array}$$

Grade 2 Math: Addition with Carrying

Example

$$\begin{array}{r} {}^{1}\ {}^{1} \\ 957 \\ +\ 218 \\ \hline 1175 \end{array}$$

Boring...

Grade 2 Fun: Addition with "Wrong" Carrying

Example

$$\begin{array}{rr} & 957 \\ + & 218 \\ \hline \end{array}$$

Grade 2 Fun: Addition with "Wrong" Carrying

Example

$$\begin{array}{r} {\scriptstyle 3} \\ 957 \\ +\quad 218 \\ \hline 5 \end{array}$$

Grade 2 Fun: Addition with "Wrong" Carrying

Example

$$\begin{array}{rr} & {}^{3} \\ & 957 \\ + & 218 \\ \hline & 95 \end{array}$$

Grade 2 Fun: Addition with "Wrong" Carrying

Example

$$\begin{array}{r} {}^{3}\;\;{}^{3}\;\;\; \\ 9\,5\,7 \\ +\quad 2\,1\,8 \\ \hline 1\,9\,5 \end{array}$$

Grade 2 Fun: Addition with "Wrong" Carrying

Example

$$\begin{array}{rr} & {\scriptstyle 3\;\;\;3} \\ & 957 \\ + & 218 \\ \hline & 3195 \end{array}$$

Grade 2 Fun: Addition with "Wrong" Carrying

Example

$$\begin{array}{r} {}^{3\ \ 3} \\ 957 \\ +_{3}\ \ 218 \\ \hline 3195 \end{array}$$

Grade 2 Fun: Addition with "Wrong" Carrying

Example

$$\begin{array}{r} {}^{3}\ \ {}^{3}\ \ \\ 957 \\ +_{3}\ \ 218 \\ \hline 3195 \end{array}$$

Cool, but I wonder...

What about subtraction?

Example

$$\begin{array}{r} 9\ 5\ 7 \\ -_3\ 2\ 1\ 8 \\ \hline \end{array}$$

What about subtraction?

Example

$$\begin{array}{r} \scriptstyle 2\ 17 \\ 9\ \cancel{5}\ \cancel{7} \\ -_3\ 2\ 1\ 8 \\ \hline \end{array}$$

What about subtraction?

Example

$$\begin{array}{r} {\scriptstyle 2\ 17} \\ 9\ \cancel{5}\ \cancel{7} \\ -_{3}\ 2\ 1\ 8 \\ \hline 7\ 1\ 9 \end{array}$$

What about subtraction?

Example

$$\begin{array}{r} \scriptstyle 2\ 17 \\ 9\ \cancel{5}\ \cancel{7} \\ -_3\ 2\,1\ 8 \\ \hline 7\,1\ 9 \end{array}$$

Example

$$\begin{array}{r} \scriptstyle 3\ \ \\ 7\,1\,9 \\ +_3\ 2\,1\,8 \\ \hline 9\,5\,7 \end{array}$$

Properties

- Commutative.

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- Inverses (through subtraction) and identity of 0

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It's an Abelian Group!

Properties

- Commutative.
- Inverses (through subtraction) and identity of 0
- Associative? ...Yes.

It's an Abelian Group!

Which one is it?

What size are my blocks?

- The digit positions represent generators of the group
- Ten of one generator is equivalent to three of the next generator.
- First (right-most) generator is still just $b_0 = 1$.

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- Likewise $3b_2 = 10b_1$. Hence $b_2 = \left(\frac{10}{3}\right)^2$.

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- The second generator must satisfy $3b_1 = 10 \cdot 1$. Hence $b_1 = \frac{10}{3}$.
- Likewise $3b_2 = 10b_1$. Hence $b_2 = \left(\frac{10}{3}\right)^2$.
- In general

$$b_n = \left(\frac{10}{3}\right)^n.$$

$957 +_3 218$

$$957 +_3 218 = \left(9\left(\frac{10}{3}\right)^2 + 5\left(\frac{10}{3}\right) + 7\right) + \left(2\left(\frac{10}{3}\right)^2 + 1\left(\frac{10}{3}\right) + 8\right)$$

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$$= 11\left(\frac{10}{3}\right)^2 + 6\left(\frac{10}{3}\right) + 15$$

$$
\begin{aligned}
957 +_3 218 &= \left(9\left(\frac{10}{3}\right)^2 + 5\left(\frac{10}{3}\right) + 7\right) + \left(2\left(\frac{10}{3}\right)^2 + 1\left(\frac{10}{3}\right) + 8\right) \\
&= 11\left(\frac{10}{3}\right)^2 + 6\left(\frac{10}{3}\right) + 15 \\
&= 11\left(\frac{10}{3}\right)^2 + 6\left(\frac{10}{3}\right) + 3\left(\frac{10}{3}\right) + 5
\end{aligned}
$$

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&= 11\left(\frac{10}{3}\right)^2 + 9\left(\frac{10}{3}\right) + 5 \\
&= \left(3\left(\frac{10}{3}\right) + 1\right)\left(\frac{10}{3}\right)^2 + 9\left(\frac{10}{3}\right) + 5
\end{aligned}
$$

$$
\begin{aligned}
957 +_3 218 &= \left(9\left(\frac{10}{3}\right)^2 + 5\left(\frac{10}{3}\right) + 7\right) + \left(2\left(\frac{10}{3}\right)^2 + 1\left(\frac{10}{3}\right) + 8\right)\\
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&= 11\left(\frac{10}{3}\right)^2 + 9\left(\frac{10}{3}\right) + 5\\
&= \left(3\left(\frac{10}{3}\right) + 1\right)\left(\frac{10}{3}\right)^2 + 9\left(\frac{10}{3}\right) + 5\\
&= 3\left(\frac{10}{3}\right)^3 + 1\left(\frac{10}{3}\right)^2 + 9\left(\frac{10}{3}\right) + 5
\end{aligned}
$$

Theorem

When you carry by c with $0 < c < 10$ with $\gcd(c, 10) = 1$, then you have $\mathbb{Z}\left[\frac{10}{c}\right]$.

Theorem

When you carry by c *with* $0 < c < 10$ *with* $\gcd(c, 10) = 1$*, then you have* $\mathbb{Z}\left[\frac{10}{c}\right]$.

Theorem

When you carry by c *in base-*b *with* $0 < c < b$ *with* $\gcd(c, b) = 1$*, then you have* $\mathbb{Z}\left[\frac{b}{c}\right]$.

Let's Get NUTS!

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Instead of carrying into one (the next) place, let's carry into a few.

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Example

Instead of carrying 1's, let's carry the sequence: 1, 7.

$$\begin{array}{rr} & 957 \\ +_{(1,\ 7)} & 218 \\ \hline \end{array}$$

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$$\begin{array}{r r} & {\scriptstyle 1\;7} \\ & 957 \\ +_{(1,\ 7)} & 218 \\ \hline & 5 \end{array}$$

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Example

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$$\begin{array}{rr}
 & 1\;7 \\
 & 1\;7 \\
 & 9\,5\,7 \\
+_{(1,\ 7)} & 2\,1\,8 \\
\hline
 & 3\,5
\end{array}$$

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Instead of carrying into one (the next) place, let's carry into a few.

Example

Instead of carrying 1's, let's carry the sequence: 1, 7.

$$
\begin{array}{rcccccc}
 & 1 & 7 & & & \\
 & & 1 & 7 & & \\
 & & & 1 & 7 & \\
 & & & 9 & 5 & 7 \\
+_{(1,\ 7)} & & & 2 & 1 & 8 \\
\hline
 & & & 9 & 3 & 5
\end{array}
$$

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Example

Instead of carrying 1's, let's carry the sequence: 1, 7.

$$\begin{array}{r} 1\ 7 \\ 1\ 7 \\ 1\ 7 \\ 957 \\ +_{(1,\ 7)} \quad 218 \\ \hline 18935 \end{array}$$

What size are my blocks?

- In this case, $b_n + 7b_{n-1} = 10b_{n-2}$, $b_0 = 1$.
- This is a homogeneous linear recursion with characteristic equation $r^2 + 7r - 10 = 0$.

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- Hence

$$b_n = A\left(\frac{-7+\sqrt{89}}{2}\right)^n + (1-A)\left(\frac{-7-\sqrt{89}}{2}\right)^n.$$

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- Hence

$$b_n = A\left(\frac{-7+\sqrt{89}}{2}\right)^n + (1-A)\left(\frac{-7-\sqrt{89}}{2}\right)^n.$$

- If we force b_1 to be the positive root, then

$$b_n = \left(\frac{-7+\sqrt{89}}{2}\right)^n$$

and again we have a base-like representation.

Theorem

When you carry by sequence $(c_n, c_{n-1}, \ldots, c_1)$, *then you have* $\mathbb{Z}[\alpha]$ *where* α *satisfies*

$$c_n\alpha^n + c_{n-1}\alpha^{n-1} + \cdots + c_1\alpha - 10 = 0.$$

Theorem

When you carry by sequence $(c_n, c_{n-1}, \ldots, c_1)$ *and use base* b *digits, then you have* $\mathbb{Z}[\alpha]$ *where* α *satisfies*

$$c_n\alpha^n + c_{n-1}\alpha^{n-1} + \cdots + c_1\alpha - b = 0.$$

Bad α's

Example

In base $\alpha = \frac{1}{10}\left(\sqrt{249} - 7\right)$ you carry the sequence: 5, 7.

$$\frac{\quad +(5,\ 7) \qquad\qquad}{}$$

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$$\begin{array}{r} 5\,7 \\ +_{(5,\ 7)} \qquad 5 \\ \hline \end{array}$$

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In base $\alpha = \frac{1}{10}\left(\sqrt{249} - 7\right)$ you carry the sequence: 5, 7.

$$\begin{array}{rr} & {\scriptstyle 5\ 7} \\ & 5\,7 \\ +_{(5,\ 7)} & 5 \\ \hline & 2 \end{array}$$

Bad α's

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In base $\alpha = \frac{1}{10}\left(\sqrt{249} - 7\right)$ you carry the sequence: 5, 7.

$$
\begin{array}{lrrrrrr}
 & 5 & 7 & & & & \\
 & & 5 & 7 & & & \\
 & & & 5 & 7 & & \\
+_{(5,\ 7)} & & & & 5 & & \\
\hline
 & & & 2 & 2 & &
\end{array}
$$

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 & 5 & 7 & & & & \\
 & & 5 & 7 & & & \\
 & & & 5 & 7 & & \\
 & & & & 5 & 7 & \\
+_{(5,\ 7)} & & & & & 5 & \\
\hline
 & & & 2 & 2 & 2 &
\end{array}
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\begin{array}{rcccccc}
 & & 5 & 7 & & & \\
 & & & 5 & 7 & & \\
 & & & & 5 & 7 & \\
 & & & & & 5 & 7 \\
+_{(5,\ 7)} & & & & & & 5 \\
\hline
\dots & 2 & 2 & 2 & 2 & 2 &
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$$

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 & & 5 & 7 & & & & \\
 & & & 5 & 7 & & & \\
 & & & & 5 & 7 & & \\
 & & & & & 5 & 7 \\
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\hline
\ldots & 2 & 2 & 2 & 2 & 2
\end{array}$$

But $\alpha \approx 0.877973 \cdots < 1$

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In base $\alpha = \frac{1}{10}\left(\sqrt{249} - 7\right)$ you carry the sequence: 5, 7.

$$
\begin{array}{rrrrrrr}
 & & 5 & 7 & & & \\
 & & & 5 & 7 & & \\
 & & & & 5 & 7 & \\
 & & & & & 5 & 7 \\
+_{(5,\,7)} & & & & & & 5 \\
\hline
 & \dots & 2 & 2 & 2 & 2 & 2
\end{array}
$$

But $\alpha \approx 0.877973 \cdots < 1$

Theorem

For $c_i \geq 0$, if $c_n + c_{n-1} + \cdots + c_1 > b$, then $0 < \alpha < 1$.

New Approach to Nonintegral Bases

Example

Put 38 into base $\alpha = \dfrac{\sqrt{37} - 2}{3}$.

New Approach to Nonintegral Bases

Example

Put 38 into base $\alpha = \dfrac{\sqrt{37}-2}{3}$.

The α satisfies $3\alpha^2 + 4\alpha - 11 = 0$. So use base-11 digits, e.g. $0,\ 1,\ \ldots, 8,\ 9,\ a$ and carry the sequence $(3,\ 4)$

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				38

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		9	12	
				38
				5

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Put 38 into base $\alpha = \dfrac{\sqrt{37}-2}{3}$.

The α satisfies $3\alpha^2 + 4\alpha - 11 = 0$. So use base-11 digits, e.g. 0, 1, …, 8, 9, *a* and carry the sequence (3, 4)

$$\begin{array}{c|c|c|c|c}
 & 3 & 4 & & \\
 & & 9 & 12 & \\
 & & & & 38 \\
\hline
 & & & 1 & 5
\end{array}$$

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$$
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3 & 4 & & & \\
 & 3 & 4 & & \\
 & & 9 & 12 & \\
 & & & & 38 \\
\hline
 & & 2 & 1 & 5
\end{array}
$$

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$$
\begin{array}{c|c|c|c|c}
3 & 4 & & & \\
 & 3 & 4 & & \\
 & & 9 & 12 & \\
 & & & & 38 \\
\hline
3 & 7 & 2 & 1 & 5
\end{array}
$$

New Approach to Nonintegral Bases

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Put 38 into base $\alpha = \dfrac{\sqrt{37}-2}{3}$.

The α satisfies $3\alpha^2 + 4\alpha - 11 = 0$. So use base-11 digits, e.g. $0,\ 1,\ \ldots, 8,\ 9,\ a$ and carry the sequence $(3,\ 4)$

3	4			
	3	4		
		9	12	
				38
3	7	2	1	5

Homework: Check $3\alpha^4 + 7\alpha^3 + 2\alpha^2 + \alpha + 5 = 38$.

Standard Approach to Nonintegral Bases

Example

Put 38 into base $\alpha = \dfrac{\sqrt{37} - 2}{3} \approx 1.36092...$

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Example

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$$\alpha^{11} < 38 < \alpha^{12}$$
$$\alpha^{6} < 38 - \alpha^{11} < \alpha^{7}$$

Standard Approach to Nonintegral Bases

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Put 38 into base $\alpha = \dfrac{\sqrt{37}-2}{3} \approx 1.36092...$

$$\alpha^{11} < 38 < \alpha^{12}$$
$$\alpha^{6} < 38 - \alpha^{11} < \alpha^{7}$$
$$\alpha^{2} < 38 - \alpha^{11} - \alpha^{6} < \alpha^{3}$$

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$$\alpha^{11} < 38 < \alpha^{12}$$
$$\alpha^{6} < 38 - \alpha^{11} < \alpha^{7}$$
$$\alpha^{2} < 38 - \alpha^{11} - \alpha^{6} < \alpha^{3}$$
$$\alpha^{-7} < 38 - \alpha^{11} - \alpha^{6} - \alpha^{2} < \alpha^{-6}$$
$$\vdots$$

Standard Approach to Nonintegral Bases

Example

Put 38 into base $\alpha = \dfrac{\sqrt{37}-2}{3} \approx 1.36092...$

$$\begin{gathered}
\alpha^{11} < 38 < \alpha^{12} \\
\alpha^{6} < 38 - \alpha^{11} < \alpha^{7} \\
\alpha^{2} < 38 - \alpha^{11} - \alpha^{6} < \alpha^{3} \\
\alpha^{-7} < 38 - \alpha^{11} - \alpha^{6} - \alpha^{2} < \alpha^{-6} \\
\vdots
\end{gathered}$$

$$38 = 100001000100.0000001000001\ldots$$

Standard Approach to Nonintegral Bases

Example

Put 38 into base $\alpha = \dfrac{\sqrt{37}-2}{3} \approx 1.36092...$

$$\alpha^{11} < 38 < \alpha^{12}$$
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$$\alpha^{2} < 38 - \alpha^{11} - \alpha^{6} < \alpha^{3}$$
$$\alpha^{-7} < 38 - \alpha^{11} - \alpha^{6} - \alpha^{2} < \alpha^{-6}$$
$$\vdots$$

$$38 = 100001000100.0000001000001\ldots$$

And the addition rules are even worse.

So the new way is better

- Pros

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    - Every integer has a finite expression

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Example

$$
\begin{array}{rrr}
 & 2 & 0 \\
-_3 & & 1 \\
\hline
\end{array}
$$

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Example

$$\begin{array}{rrr} & {-1} & {10} \\ & \cancel{2} & \cancel{0} \\ -_3 & & 1 \\ \hline \end{array}$$

So the new way is better

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Example

$$
\begin{array}{rrr}
 & -1 & 10 \\
 & \cancel{2} & \cancel{0} \\
-_3 & & 1 \\
\hline
 & & 9
\end{array}
$$

So the new way is better

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Example

$$
\begin{array}{rrr}
 & 9 & \\
-3 & -\cancel{1} & 10 \\
 & \cancel{2} & \cancel{0} \\
-3 & & 1 \\
\hline
 & & 9
\end{array}
$$

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Example

$$
\begin{array}{rrrr}
 & & 9 & \\
 & -3 & -\cancel{1} & 10 \\
 & & \cancel{2} & \cancel{0} \\
-3 & & & 1 \\
\hline
 & & 9 & 9
\end{array}
$$

So the new way is better

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Example

$$\begin{array}{ccccc} & & 7 & 9 & \\ & -3 & -\cancel{3} & -\cancel{1} & 10 \\ & & & \cancel{2} & \cancel{0} \\ {}_{-3} & & & & 1 \\ \hline & & 7 & 9 & 9 \end{array}$$

So the new way is better

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Example

$$
\begin{array}{cccccc}
 & \cdots & 7 & 7 & 9 & \\
 & \cdots & -\cancel{3} & -\cancel{3} & -\cancel{1} & 10 \\
 & & & & \cancel{2} & \cancel{0} \\
 & {}^{-}3 & & & & 1 \\
\hline
 & \ldots 7 & 7 & 7 & 9 & 9
\end{array}
$$

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Example

$$\begin{array}{rrrrr}
\cdots & 7 & 7 & 9 & \\
\cdots & -\cancel{3} & -\cancel{3} & -\cancel{1} & 10 \\
 & & & \cancel{2} & \cancel{0} \\
{}_{-3} & & & & 1 \\
\hline
\ldots 7 & 7 & 7 & 9 & 9
\end{array}$$

Note: $2 \cdot \frac{10}{3} - 1 = \frac{17}{3} = 9 + 9 \cdot \frac{10}{3} + 7\left(\frac{10}{3}\right)^2 \left(\frac{1}{1-\frac{10}{3}}\right)$

So the new way is better

- Pros
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- Cons
  - Subtraction might require infinite digits to the <u>left</u>.

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  - Not easily extended to represent all of $\mathbb{Q}$

So the new way is better ... right?

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  - Every integer has a finite expression
  - Easy to add
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- Cons
  - Subtraction might require infinite digits to the left.
  - Not easily extended to represent all of $\mathbb{Q}$

So the new way is better ... right?

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  - Every integer has a finite expression
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- Cons
  - Subtraction might require infinite digits to the <u>left</u>.
  - Not easily extended to represent all of $\mathbb{Q}$

Take something elementary and explore mathematical silliness.

So the new way is better ... right?

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  - Every integer has a finite expression
  - Easy to add
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- Cons
  - Subtraction might require infinite digits to the <u>left</u>.
  - Not easily extended to represent all of $\mathbb{Q}$

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Applications: Fun with kids.

So the new way is better ... right?

- Pros
  - Every integer has a finite expression
  - Easy to add
  - IT'S FUN!
- Cons
  - Subtraction might require infinite digits to the left.
  - Not easily extended to represent all of $\mathbb{Q}$

Take something elementary and explore mathematical silliness.

Applications: Fun with kids.

Credit to Daniel C. Isaksen "A Cohomological Viewpoint on Elementary School Arithmetic"

Thanks for staying late on Saturday!