Eric Errthum
Winona State University
175 West Mark Street
Winona, MN 55987
eerrthum@winona.edu

College Mathematics Journal, Vol 47, No 1, January 2016

1069. Let $\{u_n\}_{n\geq 0}$ be a sequence defined recursively by $u_0 \geq 0$, $u_1 \geq 0$, and $u_{n+1} = \sqrt{u_n \cdot u_{n-1}}$, for $n \geq 1$. Determine $\lim\limits_{n\to\infty} u_n$ in terms of u_0, u_1.

Solution, by Eric Errthum, Winona State University, Winona, MN

If $u_0 \cdot u_1 = 0$, then trivially the limit is zero. Otherwise, all the terms of the sequence are positive and so we can define the auxiliary sequence $v_n = \ln(u_n)$. Then for $n \geq 1$,

$$v_{n+1} = \frac{1}{2}v_n + \frac{1}{2}v_{n-1}.$$

This is a standard homogeneous linear recursion with characteristic polynomial

$$r^2 - \frac{1}{2}r - \frac{1}{2} = (r-1)\left(r + \frac{1}{2}\right).$$

Hence $v_n = A + B(-\frac{1}{2})^n$ for constants A and B and $\lim\limits_{n\to\infty} v_n = A$. Using the initial conditions, gives

$$A = \frac{1}{3}(v_0 + 2v_1) = \ln \sqrt[3]{u_0 \cdot u_1^2}.$$

Thus

$$\lim_{n\to\infty} u_n = \sqrt[3]{u_0 \cdot u_1^2}.$$