Math 280 Solutions for October 3

Pythagoras Level

Problem 1: [Nick's Math Puzzles #23] Let $\angle BAE = x$, and $\angle BCD = y$. Then

$$AB = 9\cos x = AC\cos 2x \text{ and } BC = 8\sqrt{2}\cos y = AC\cos 2y.$$

Eliminating AC yeilds

$$\frac{9\cos x}{\cos 2x} = \frac{8\sqrt{2}\cos y}{\cos 2y}.$$

Then $y = 45 - x$. Also, $2y = 90 - 2x$, and so $\cos 2y = \sin 2x$. Therefore:

$$\frac{9\cos x}{\cos 2x} = \frac{8\sqrt{2}\cos(45 - x)}{\sin 2x}.$$

Using the trigonometric identity $\cos(a - b) = \cos a\cos b + \sin a\sin b$ then $\cos(45 - x) = (\cos x + \sin x)/\sqrt{2}$. Rearranging gives

$$\tan 2x = \frac{8(\cos x + \sin x)}{9\cos x}.$$

Using the trigonometric identity $\tan 2a = 2\tan a/(1 - \tan^2 a)$, and letting $t = \tan x$ gives

$$\frac{2t}{1 - t^2} = \frac{8(1 + t)}{9}.$$

Therefore

$$\frac{9t}{4} = (1 + t)(1 - t^2) = 1 + t - t^2 - t^3.$$

Hence $t^3 + t^2 + (5/4)t - 1 = 0$. By inspection, one root is $t = 1/2$. Therefore $(t - 1/2)(t^2 + 3t/2 + 2) = 0$. The quadratic factor has no real roots and so $t = 1/2$ is the only real root. Then $AC = 9\cos x/\cos 2x$. Using the trigonometric identities $\cos x = 1/\sqrt{1 + t^2}$ and $\cos 2x = (1 - t^2)/(1 + t^2)$ yeilds $AC = 9\sqrt{(1 + t^2)/(1 - t^2)}$.

Therefore $AC = 9(\sqrt{5}/2)/(3/4) = 6\sqrt{5}$ inches.

Problem 2: [Iowa MAA 2005 #9] From the equation

$$(a + bi)^3 = (a^3 - 3ab^2) + (b^3 - 3a^2b)i = 39 + i\sqrt{487}$$

we see that

$$a^2 + b^2 = |a + bi|^2 = |39 + i\sqrt{487}|^{2/3} = (\sqrt{2008})^{2/3} = \sqrt[3]{2008}$$

Newton Level

Problem 3: [Putnam 2001 B-2] By adding and subtracting the two given equations, we obtain the equivalent pair of equations

$$\begin{aligned} 2/x &= x^4 + 10x^2y^2 + 5y^4 \\ 1/y &= 5x^4 + 10x^2y^2 + y^4. \end{aligned}$$

Multiplying the former by x and the latter by y, then adding and subtracting the two resulting equations, we obtain another pair of equations equivalent to the given ones,

$$3 = (x + y)^5, \qquad 1 = (x - y)^5.$$

It follows that $x = (3^{1/5} + 1)/2$ and $y = (3^{1/5} - 1)/2$ is the unique solution satisfying the given equations.

Problem 4: [Putnam 2007 B-1] Write $f(n)=\sum_{i=0}^{d} a_i n^i$ with $a_i>0$. Then

$$\begin{aligned} f(f(n)+1) &= \sum_{i=0}^{d} a_i (f(n)+1)^i \\ &\equiv f(1) \pmod{f(n)}. \end{aligned}$$

If $n=1$, then this implies that $f(f(n)+1)$ is divisible by $f(n)$. Otherwise, $0<f(1)<f(n)$ since f is nonconstant and has positive coefficients, so $f(f(n)+1)$ cannot be divisible by $f(n)$.

Wiles Level

Problem 5: [MAA-NCS 1997 #10] Modulo r, there are at most r^3 different triples $(a_{k+1},a_{k+2},a_{k+3})$, so the sequence must eventually be periodic mod r. But since the recursion is reversible ($a_n = a_{n+3} - a_{n+1}a_{n+2}$), it must be truly periodic mod r and hence eventually return to its initial state. At that point, when $a_{s+1}\equiv a_{s+2}\equiv a_{s+3}\equiv 1 \mod r$, we have $a_s = a_{s+3} - a_{s+1}a_{s+2}\equiv 0 \mod r$.

Problem 6: [Iowa MAA 2006 #10] The integrating factor for the differential equation is any function of the form $e^{\int 2tdt}$, say e^{t^2}. Multiplying the differential equation by e^{t^2} yeilds

$$\frac{d}{dt}\left(e^{t^2}y\right) = t^2 e^{t^2}.$$

We choose

$$\int_0^t \tau^2 e^{\tau^2}\,d\tau$$

for the antiderivative of $t^2e^{t^2}$, so that

$$e^{t^2}y = \int_0^t \tau^2 e^{\tau^2}\,d\tau + C.$$

Substituting $t=0$ gives $C=y(0)$. Therefore,

$$\frac{y}{t} = \frac{\int_0^t \tau^2 e^{\tau^2}\,d\tau + y(0)}{te^{t^2}}.$$

Using L'Hospital's Rule and the Fundamental Theorem of Calculus gives

$$\lim_{t\to\infty}\frac{y}{t} = \lim_{t\to\infty}\frac{t^2e^{t^2}}{(2t^2+1)e^{t^2}} = \lim_{t\to\infty}\frac{t^2}{2t^2+1} = \frac{1}{2}.$$