

Math 280 Solutions for November 7

Pythagoras Level

Problem 1: [Garden State Undergrad Math Conf 2006 Ind #9] Factor $10^5 = 2^5 \cdot 5^5$ so 10^5 has 36 divisors all of the form $2^i 5^j$, $0 \leq i, j \leq 5$. Their product is then

$$\begin{aligned} N &= \prod_{i=0}^5 \prod_{j=0}^5 2^i 5^j \\ &= \prod_{i=0}^5 \left(2^{6i} \prod_{j=0}^5 5^j \right) \\ &= \prod_{i=0}^5 2^{6i} 5^{15} \\ &= 5^{6 \cdot 15} \left(\prod_{i=0}^5 2^i \right)^6 \\ &= 5^{6 \cdot 15} 2^{6 \cdot 15} \\ &= 10^{90} \end{aligned}$$

So $\log_{10} N = 90$.

Problem 2: [Garden State Undergrad Math Conf 2007 Ind #3] Let $y = \log_2 x$. Then

$$\begin{aligned} 1 &= \log_x 4 - \log_2 x \\ &= \frac{\log_2 4}{\log_2 x} - \log_2 x \\ &= \frac{2}{y} - y \end{aligned}$$

Solving now for y yeilds $y = 1$ or $y = -2$. Since $0 \leq x = 2^y \leq 1$, we get $x = 1/4$.

Newton Level

Problem 3: [Garden State Undergrad Math Conf 2006 Ind #8] Rewrite as

$$\begin{aligned} \left(\frac{1}{3}\right)^{1/5} \left(\frac{1}{9}\right)^{1/25} \left(\frac{1}{27}\right)^{1/125} \cdots &= \left(\frac{1}{3}\right)^{1/5} \left(\frac{1}{3}\right)^{2/25} \left(\frac{1}{3}\right)^{3/125} \cdots \\ &= \prod_{i=1}^{\infty} \left(\frac{1}{3}\right)^{i/5^i} \\ &= \left(\frac{1}{3}\right)^{\sum_{i=1}^{\infty} i/5^i} \end{aligned}$$

So we need to evaluate this infinite sum. Consider the formula for a p -based geometric series

$$f(p) = \sum_{i=0}^{\infty} p^i = \frac{1}{1-p}.$$

Then

$$f'(p) = \sum_{i=0}^{\infty} ip^{i-1} = \frac{1}{(1-p)^2}.$$

So

$$\sum_{i=1}^{\infty} ip^i = \frac{p}{(1-p)^2}.$$

Applying this to the original product with $p = 1/5$ yeilds $\left(\frac{1}{3}\right)^{5/16}$.

Problem 4: [Garden State Undergrad Math Conf 2006 Ind #12] Rewrite the limit as

$$\lim_{x \rightarrow \infty} \int_0^x x e^{t^2 - x^2} dt = \lim_{x \rightarrow \infty} \frac{\int_0^x e^{t^2} dt}{e^{x^2}/x}$$

This has the indeterminant form of $\frac{\infty}{\infty}$ so apply L'Hospital's Rule and the Fundamental Theorem of Calculus to arrive at

$$\lim_{x \rightarrow \infty} \frac{e^{x^2}}{(2x^2 - 1)e^{x^2}/x^2} = \lim_{x \rightarrow \infty} \frac{x^2}{2x^2 - 1} = \frac{1}{2}$$

Wiles Level

Problem 5: [Garden State Undergrad Math Conf 2006 Ind #5] Let x and y be the respective two-digit and three-digit number. We are given the equation $1000x + y = 9xy$. Now $y(9x - 1) = 1000x$, so x divides $y(9x - 1)$. Since x and $9x - 1$ have no factors in common, x divides y . Writing $y = xk$, the equation becomes $1000 = k(9x - 1)$. Hence k and $9x - 1$ are factors of 1000. Since x is a two-digit number, $98 \leq 9x - 1 \leq 999$, and $9x - 1$ must then equal 100, 125, 200, 250, 500. Hence $9x - 1 = 125$, $x = 14$, $k = 8$, and $y = 112$. Then $x + y = 126$.

Problem 6: [Garden State Undergrad Math Conf 2006 Team #4] According to the first assumption, we can write

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = 2A \cdot \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then

$$\begin{aligned} \begin{pmatrix} a_{n+2008} \\ b_{n+2008} \end{pmatrix} &= 2A \cdot \begin{pmatrix} a_{n+2007} \\ b_{n+2007} \end{pmatrix} \\ &= 4A^2 \cdot \begin{pmatrix} a_{n+2006} \\ b_{n+2006} \end{pmatrix} \\ &\vdots \\ &= 2^{2008} A^{2008} \cdot \begin{pmatrix} a_n \\ b_n \end{pmatrix} \end{aligned}$$

Since $A^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ for all $m \in \mathbb{N}$, we have $\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 2^{2008} & 2008 \cdot 2^{2008} \\ 0 & 2^{2008} \end{pmatrix}$.