Math 280 Solutions for December 5

Pythagoras Level

Problem 1: [Putnam 2004 A1] Yes. Suppose otherwise. Then there would be an N such that $S(N) < 80\%$ and $S(N+1) > 80\%$; that is, O'Keal's free throw percentage is under 80% at some point, and after one subsequent free throw (necessarily made), her percentage is over 80%. If she makes m of her first N free throws, then $m/N < 4/5$ and $(m+1)/(N+1) > 4/5$. This means that $5m < 4n < 5m+1$, which is impossible since then $4n$ is an integer between the consecutive integers $5m$ and $5m+1$.

Problem 2: [Putnam 2002 B1] The probability is 1/99. In fact, we show by induction on n that after n shots, the probability of having made any number of shots from 1 to $n-1$ is equal to $1/(n-1)$. This is evident for $n = 2$. Given the result for n, we see that the probability of making i shots after $n+1$ attempts is

$$\begin{aligned} \frac{i-1}{n}\frac{1}{n-1} + \left(1 - \frac{i}{n}\right)\frac{1}{n-1} &= \frac{(i-1)+(n-i)}{n(n-1)} \\ &= \frac{1}{n}, \end{aligned}$$

as claimed.

Newton Level

Problem 3: [Putnam 1999 A1] Note that if $r(x)$ and $s(x)$ are any two functions, then

$$\max(r,s) = (r + s + |r - s|)/2.$$

Therefore, if $F(x)$ is the given function, we have

$$\begin{aligned} F(x) &= \max\{-3x-3, 0\} - \max\{5x, 0\} + 3x + 2 \\ &= (-3x - 3 + |3x+3|)/2 \\ &\quad - (5x + |5x|)/2 + 3x + 2 \\ &= |(3x+3)/2| - |5x/2| - x + \frac{1}{2}, \end{aligned}$$

so we may set $f(x) = (3x+3)/2$, $g(x) = 5x/2$, and $h(x) = -x + \frac{1}{2}$.

Alternate Solution (Errthum): Assume, for no good reason other than it works, that f, g, and h are linear functions and that $f(-1) = g(0) = 0$ and both have positive slope. Thus $|f(x)| = -f(x)$ for $x < -1$ and $|g(x)| = -g(x)$ for $x < 0$. Then set up the equations:

$$\begin{aligned} -f(x) + g(x) + h(x) &= -1 \\ f(x) + g(x) + h(x) &= 3x + 2 \\ f(x) - g(x) + h(x) &= -2x + 2 \end{aligned}$$

Solving this system of linear equations yeilds $f(x) = 3(x+1)/2$, $g(x) = 5x/2$ and $h(x) = 1/2 - x$.

Problem 4: [Putnam 1999 A4] Denote the series by S, and let $a_n = 3^n/n$. Note that

$$\begin{aligned} S &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{a_m(a_m+a_n)} \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{a_n(a_m+a_n)}, \end{aligned}$$

where the second equality follows by interchanging m and n. Thus

$$\begin{aligned} 2S &= \sum_m \sum_n \left(\frac{1}{a_m(a_m+a_n)} + \frac{1}{a_n(a_m+a_n)} \right) \\ &= \sum_m \sum_n \frac{1}{a_m a_n} \\ &= \left(\sum_{n=1}^{\infty} \frac{n}{3^n} \right)^2. \end{aligned}$$

But

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{3}{4}$$

since, e.g., it's $f'(1)$, where

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{3^n} = \frac{3}{3-x},$$

and we conclude that $S = 9/32$.

Wiles Level

Problem 5: [Putnam 2002 A3] Note that each of the sets $\{1\}, \{2\}, \ldots, \{n\}$ has the desired property. Moreover, for each set S with integer average m that does not contain m, $S \cup \{m\}$ also has average m, while for each set T of more than one element with integer average m that contains m, $T \setminus \{m\}$ also has average m. Thus the subsets other than $\{1\}, \{2\}, \ldots, \{n\}$ can be grouped in pairs, so $T_n - n$ is even.

Problem 6: [Putnam 2002 A2] Draw a great circle through two of the points. There are two closed hemispheres with this great circle as boundary, and each of the other three points lies in one of them. By the pigeonhole principle, two of those three points lie in the same hemisphere, and that hemisphere thus contains four of the five given points.