Math 280 Problems for October 2

Pythagoras Level

1. Find 2×2 matrices A and B such that

$$AB - BA = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},$$

or show that such matrices do not exist.

2. Given n integers, prove that either one of them is a multiple of n, or a number of them add up to a multiple of n.

Newton Level

3. Find a non-zero real number a and a positive integer n such that

$$\lim_{x \to 0} \frac{1 - \cos(1 - \cos x)}{x^n} = a.$$

4. Let $F_0(x) = \ln x$. For $n \geq 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)\, dt$. Evaluate

$$\lim_{n \to \infty} \frac{n! F_n(1)}{\ln n}.$$

Wiles Level

5. Suppose that f is a real-valued function defined on the set of all real numbers and satisfying

$$f(x + y) = f(x)f(1 - y) + f(y)f(1 - x)$$

for all real x, y. Also assume that $f(0) = 1/2$. Prove that f must be a constant function.

6. If A is a subset of a group G and the order of A is more than half the order of G (i.e., $|A| > |G|/2$), prove that any element $g \in G$ can be expressed as a product of two elements of A.