Math 280 Problems for September 28

Pythagoras Level

Problem 1: Suppose 100 people are each assigned a different seat on an airplane with 100 seats. The passengers are seated one at a time. The first person loses his boarding pass and sits in one of the 100 seats chosen at random. Each subsequent person sits in their assigned seat if it is unoccupied, and otherwise chooses a seat at random from among the remaining empty seats. Determine, with proof, the probability that the last person to board the plane is able to sit in her assigned seat.

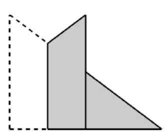
[Iowa-MAA 2011 #7] We will argue that the probability is 1/2 by strong induction, regardless of the number $n \geq 2$ of seats on the plane. The answer is clearly 1/2 for $n = 2$. In general, the first person takes his own seat with probability $1/n$, in which case everything works out. The first person takes the last persons seat also with probability $1/n$, which guarantees that she does not get her own seat. Otherwise the first person takes the seat of person k, for some $1 < k < n$. All people before person k sit in their assigned seat, then person k effectively takes the role of the first person, choosing a seat at random. By induction, the final person gets her seat with probability 1/2 once we reach this stage. Putting everything together gives an overall probability of 1/2 as well.

Problem 2: A jar contains 600 jelly beans, 100 red, 200 green, and 300 blue. These are drawn randomly from the jar, one at a time, without replacement. What is the probability that the first color to be exhausted is red?

[UMUMC 2007 #2] There will be a final string of beans of the same color (the color of the last bean drawn), and then a bean preceding it of a different color. The desired probability p is that these two colors are blue and green or green and blue, respectively. The probability that the last bean drawn is blue is $300/600 = 1/2$, and, no matter how long the final string of blue beans is, the probability that the bean of preceding color is green is $200/(100 + 200) = 2/3$. For green and blue these numbers become $200/600 = 1/3$ and $300/(100 + 300) = 3/4$. Thus, $p = (1/2)(2/3) + (1/3)(3/4) = (1/3) + (1/4) = 7/12$.

Newton Level

Problem 3: Consider a triangular sheet of paper with vertices at the points $(0, 0)$, $(4, 0)$ and $(0, 3)$. By making a single vertical crease in the paper and then folding the left portion of the triangle over the crease so that it overlaps the right portion we obtain a polygon with five sides, colored grey. Find the smallest possible area of this resulting polygon.



[Iowa-MAA 2011 #3] Minimizing the area corresponds to maximizing the region of overlap, which is a trapezoid. If the fold occurs at x then the trapezoid area is $3x - \frac{9}{8}x^2$, which has a maximum value of 2 at $x = 4/3$. Hence the minimum area is $6 - 2 = 4$.

Problem 4: A smooth curve crosses the y-axis at the point $(0, 2)$ and has the following curious property. Given any point P on the curve, the tangent line to the curve at P crosses the x-axis at a point Q exactly 2011 units to the right of P. (In other words, the x-coordinate of Q is 2011 more than the x-coordinate of P.) Determine the area of the region in the first quadrant bounded by the x-axis, the y-axis, and this curve, explaining how you found your answer.

[Iowa-MAA 2011 #4] The given property leads to the differential equation $\dfrac{dy}{dx} = -\dfrac{y}{2011}$ with initial condition $y(0) = 2$, hence solution $y = 2e^{x/2011}$. The desired area is given by

$$\int_0^\infty 2e^{x/2011}\, dx = 4022.$$

Wiles Level

Problem 5: Observe that $(1)(4)(7) = 28$ is one greater than a multiple of 9, while $(2)(5)(8) = 80$ is one less than a multiple of 9. Confirm that this phenomenon persists; in other words, prove that for all $n \geq 1$ we have

$$(1)(4)(7)(10)\cdots(3\cdot 3^{n-1}-2) \equiv 1 \mod 3^n,$$

$$(2)(5)(8)(11)\cdots(3\cdot 3^{n-1}-1) \equiv -1 \mod 3^n.$$

[Iowa-MAA 2011 #9] Suppose a and b are inverses mod 3^n. Then $ab \equiv 1 \mod 3^n$ implies $ab \equiv 1 \mod 3$ as well, meaning that either $a \equiv b \equiv 1 \mod 3$ or $a \equiv b \equiv 2 \mod 3$. Hence the factors in the product $(1)(4)(7)\cdots(3^n-2)$ cancel in pairs, leaving just 1. The same logic applies to the product $(2)(5)(8)\cdots(3^n-1)$, leaving just $(3^n-1) \equiv -1 \mod 3^n$. (Because 1 and 3^n-1 are their own inverse; all other inverses come in pairs.)

Problem 6: Prove that

$$e^{e^x} \geq e^{2x+1} - xe^{x+1}$$

for all real x.

[UMUMC 2007 #7] Let $f(x) = e^x - x - 1$, then $f'(x) = e^x - 1$. Note that $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$ so $f(x)$ is decreasing for $x < 0$ and increasing for $x > 0$. Since $f(x) = 0$ we get $f(x) \geq 0$ for all $x \in \mathbb{R}$. This implies

$$f(f(x)) = e^{e^x-x-1} - (e^x - x - 1) - 1 = e^{e^x-x-1} - e^x + x \geq 0$$

Multiplying with e^{x+1} gives us

$$e^{e^x} - e^{2x+1} + xe^x + 1 \geq 0.$$