Math 280 Solutions for October 17

Pythagoras Level

Problem 1: [Nick's Math Puzzles #30] There are four possible outcomes, all equally likely.

- Solid added; original solid drawn.
- Solid added; new solid drawn.
- Stripe added; original solid drawn.
- Stripe added; new stripe drawn.

Since we know a solid was drawn from the bag we can exclude the final outcome. In two out of the three remaining outcomes the other ball is a solid. Therefore the probability that the ball remaining in the bag is also a solid is 2/3.

Problem 2: [Iowa MAA 2005 #1] It is $100°$. From triangle BCD we see that angle BDC is $40°$, and thus angle DBA is likewise $40°$. But triangle ABD is isosceles, with angle DAB equal to angle DBA, which is $40°$. Thus the angle ADB is $100°$.

Newton Level

Problem 3: [Putnam 2006 A1] We change to cylindrical coordinates, i.e., we put $r = \sqrt{x^2 + y^2}$. Then the given inequality is equivalent to

$$r^2 + z^2 + 8 \leq 6r,$$

or

$$(r - 3)^2 + z^2 \leq 1.$$

This defines a solid of revolution (a solid torus); the area being rotated is the disc $(x - 3)^2 + z^2 \leq 1$ in the xz-plane. By Pappus's theorem, the volume of this equals the area of this disc, which is π, times the distance through which the center of mass is being rotated, which is $(2\pi)3$. That is, the total volume is $6\pi^2$.

Problem 4: [Putnam 2005 A5] We make the substitution $x = \tan\theta$, rewriting the desired integral as

$$\int_0^{\pi/4} \log(\tan(\theta) + 1)\, d\theta.$$

Write

$$\begin{aligned} &\log(\tan(\theta) + 1) \\ &= \log(\sin(\theta) + \cos(\theta)) - \log(\cos(\theta)) \end{aligned}$$

and then note that $\sin(\theta) + \cos(\theta) = \sqrt{2}\cos(\pi/4 - \theta)$. We may thus rewrite the integrand as

$$\frac{1}{2}\log(2) + \log(\cos(\pi/4 - \theta)) - \log(\cos(\theta)).$$

But over the interval $[0, \pi/4]$, the integrals of $\log(\cos(\theta))$ and $\log(\cos(\pi/4 - \theta))$ are equal, so their contributions cancel out. The desired integral is then just the integral of $\frac{1}{2}\log(2)$ over the interval $[0, \pi/4]$, which is $\pi \log(2)/8$.

Wiles Level

Problem 5: [Iowa MAA 2005 #5] The answer is $M = 48$. Since $f(1) = -48$, it is clear that M must be a divisor of 48. Using the factorization $u^n - 1 = (u-1)(u^{n-1} + u^{n-2} + \cdots + u + 1)$ we may write

$$f(n) = 25^n - 1 - 72n = 24(25^{n-1} + 25^{n-2} + \cdots + 25 + 1) - 24(3n).$$

Whether n is even or odd, this is an even multiple of 24, so is divisible by 48. Thus, $M = 48$ is the largest common divisor of all the values of $f(n)$.

ALTERNATE SOLUTION

As above we see from $f(1)$ that M must be a divisor of 48. Now examine $f(n) = 25^n - 72n - 1$ modulo 3 and modulo 16 separately: Modulo 3, $f(n) \equiv 1^n - 0 - 1 = 0$, and modulo 16, $25 \equiv 9$ and $25^2 \equiv 9^2 \equiv 1$, while $72 \equiv 8$. Thus, for even $n = 2m$, $f(2m) \equiv 1^m - 8(2m) - 1 = 1 - 0 - 1 = 0$, and for odd $n = 2m+1$, $f(2m+1) \equiv 25^{2m}25 \equiv 1^{2m}(9) - 8(2m+1) - 1 \equiv 9 - 8 - 1 = 0$. Hence $f(n)$ is a multiple of $3 \cdot 16 = 48$ for all n, and therefore $M = 48$.

Problem 6: [Nick's Math Puzzles #72] Grouping terms according to the number of digits in their denominator, we have

$$S_n = (1/1 + ... + 1/8) + (1/10 + ... + 1/18 + 1/20 + ... + 1/88) + (1/100 + ... + 1/888) + ... + (1/10^{n-1} + ... + 1/8...8)$$

where the last term is the reciprocal of the n-digit number of all 8s. Now form another series, which is term-for-term greater than or equal to S_n, by setting each term with k digits in the denominator to $1/10^{k-1}$:

$$T_n = (1/1 + ... + 1/1) + (1/10 + ... + 1/10) + (1/100 + ... + 1/100) + ... + (1/10^{n-1} + ... + 1/10^{n-1})$$

Note that the number of k-digit numbers without a 9 is $8 \cdot 9^{k-1}$. (There are 8 choices for the leading digit, and 9 choices for each of the other digits.) Hence

$$T_n = 8 \cdot 1 + (8 \cdot 9)/10 + (8 \cdot 9^2)/10^2 + ... + (8 \cdot 9^{n-1})/10^{n-1}$$

This is a geometric series with first term 8, common ratio 9/10, and n terms. Therefore $T_n = 80(1 - (9/10)^n)$. As n tends to infinity, T_n tends to 80. That is, the infinite series, T, converges to 80. By the Comparison Test, and since at least one term in S_n is strictly less than the corresponding term in T_n, the infinite series,

$$S = (1/1 + ... + 1/8) + (1/10 + ... + 1/18 + 1/20 + ... + 1/88) + (1/100 + ... + 1/888) + ... < 80.$$

That is, S converges, to a value less than 80.