# Singular Moduli of Shimura Curves

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# Modular Curve

## Classical Set-up

- $\mathrm{GL}_2(\mathbb{R})$ acts on $\mathfrak{h}^{\pm}$, the union of the upper and lower half-planes:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$
- $\mathcal{X}_1^* = \mathrm{GL}_2(\mathbb{Z})\backslash\mathfrak{h}^{\pm}$ has genus 0.
- There is an isomorphism
$$j : \mathcal{X}_1^* \xrightarrow{\sim} \mathbb{P}^1$$
- $j(\tau) = 1/\mathbf{q} + 744 + 196884\mathbf{q} + \ldots \in \frac{1}{\mathbf{q}}\mathbb{Z}[[\mathbf{q}]]$ where $\mathbf{q} = e^{2\pi i\tau}$.

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## Singular Moduli

- $\mathcal{X}_1^* \stackrel{\sim}{\leftrightarrow} \mathcal{E}$, where $\mathcal{E}$ is the space of isomorphism classes of elliptic curves.
- If $\tau$ is associate with an elliptic curve with Complex Multiplication, $\tau$ is called a CM-point.
- If $\tau_{\mathrm{CM}}$ is a CM point, $j(\tau_{\mathrm{CM}})$ is called a singular modulus.
- Theorem: Singular moduli are algebraic integers.
- Examples:

$$j(i) = 12^3, \quad j\left(\frac{1+i\sqrt{3}}{2}\right) = 0$$

$$j(i\sqrt{6}) = 12^3(1+\sqrt{2})^2(5+2\sqrt{2})^3$$

$$j(\sqrt{-14}) = 2^3\left(323+228\sqrt{2}+(231+161\sqrt{2})\sqrt{2\sqrt{2}-1}\right)^3$$

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## Gross-Zagier Theorem

- Since a singular modulus is an algebraic integer, it has norm in $\mathbb{Z}$.
- Theorem (Gross-Zagier):

$$|j(a) - j(b)| = \prod_{n \in N(a,b)} n^{\epsilon_n}$$

  where $n$, $\epsilon_n \in \mathbb{Z}$.
- Recall $j\left(\frac{1+i\sqrt{3}}{2}\right) = 0$.
- The factorization is a lot of small primes to large powers.

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# Quaternion Algebras

- A quaternion algebra is $B = \mathbb{Q}(\alpha, \beta)$ where $\alpha^2 = a$, $\beta^2 = b$ and $\alpha\beta = -\beta\alpha$.
- There is an embedding $B \hookrightarrow M_2(\mathbb{Q}(\sqrt{a}))$ via

$$\alpha \mapsto \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}, \qquad \beta \mapsto \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$$

- Example: $M_2(\mathbb{Q})$ where $a = 1$, $b = -1$.
- Example: Hamiltonians where $a = b = -1$.
- A maximal order $\mathcal{O}$ is a maximal $\mathbb{Z}$-module such that $\mathcal{O} \otimes \mathbb{Q} = B$.
- Example: $M_2(\mathbb{Z}) \subset M_2(\mathbb{Q})$.
- Example: $\mathbb{Z}\left[1, \alpha, \beta, \frac{1+\alpha+\beta+\alpha\beta}{2}\right]$ in the Hamiltonians.

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- $j : \mathcal{X}_1^* \xrightarrow{\sim} \mathbb{P}^1$
- $j = 1/\mathbf{q} + 744 + \ldots$
- CM Points
- $j(\tau_{\mathrm{CM}})$ algebraic integer
- Gross-Zagier
  Factorization of the Norm

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- $B^\times \hookrightarrow \mathrm{GL}_2(\mathbb{Q}(\sqrt{d}))$ action
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- No cusps, so no $\mathbf{q}$ expansion
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Q: How do you compute $|t_B(\tau_{\mathrm{CM}})|$ ?

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# Elkies Attempt

- Considers the rational CM points on $\mathcal{X}_6^*$.
- Uses geometric involutions
- Is successful at computing 17 of the 27 rational CM points.
  Example:
  $$t_6(\mathcal{P}_{-312}) = \frac{7^4 23^4}{5^6 11^6}$$
- Unable to compute remaining 10 CM points, but makes numerical approximations.
  Example:
  $$t_6(\mathcal{P}_{-163}) \stackrel{?}{=} \frac{3^{11} 7^4 19^4 23^4}{2^{10} 5^6 11^6 17^6}.$$
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# Our Method

## Borcherds Forms

- Borcherds Form: Given a modular form $F : \mathbb{C} \to \mathbb{C}[\Lambda]$, Borcherds constructs

$$\Psi(F) : \mathcal{X}_B^* \to \mathbb{P}^1$$

- If $F$ has Fourier expansion

$$F(\tau) = \sum_{\lambda \in \Lambda} \sum_{m \in \mathbb{Q}} c_\lambda(m) \mathbf{q}^m e_\lambda$$

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$$\sum_{\substack{\text{Galois Orbit} \\ \text{of a CM Point}}} \log ||\Psi(F)|| = \sum_{\lambda \in \Lambda} \sum_{m<0} c_\lambda(m) \kappa_\lambda(m) \tag{1}$$

  where $\kappa_\lambda(m)$ are computable coefficients of an Eisenstein series.

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- Theorem (Jarad Schofer): $j = \Psi(F)$ for some $F$ and Gross-Zagier is a specific case of (1).

# Our Method

Computing $|t_6(\tau_{\mathrm{CM}})|$

- **Theorem:** $\Psi(F_6) = 6^6 t_6$.
- Find an $F_6$ such that

$$\mathrm{div}(\Psi(F_6)) = \mathrm{div}(t_6)$$

- Then $\Psi(F_6) = k_6 t_6$.
- Use (1) to compute

$$\sum_{\substack{\text{Galois Orbit} \\ \text{of a CM Point}}} \log ||\Psi(F_6)||$$

  at a point where $t_6(\tau_{\mathrm{CM}})$ is known to find the normalization constant $k_6$.
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# Results

- We proved the conjectural values in Elkies' table of rational CM points of $\mathcal{X}_6^*$, including

$$t_6(\mathcal{P}_{-163}) = \frac{3^{11}7^4 19^4 23^4}{2^{10}5^6 11^6 17^6}.$$

- We also proved the conjectural values in Elkies' table of rational CM points of $\mathcal{X}_{10}^*$.
- We can compute examples far beyond the scope of his work, such as norms of irrational CM points of arbitrary discriminant.

  Example:

$$|t_6(\mathcal{P}_{-996})| = \frac{2^{16}7^{12}71^4 83^2}{17^6 29^6 41^6}.$$

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# Thanks

Questions?