Mapping Cortical Surfaces Using Conformal Maps

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May 2013

Abstract

The Riemann Mapping Theorem is perhaps one of, if not the most, astounding conclusions in complex analysis. Its power as a tool in the field of topology cannot be understated. Unsurprisingly, one of the fields that obtains the most use from the Riemann Mapping Theorem is the field of image mapping. This paper specifically examines the use of the result of the Riemann Mapping Theorem, along with conformal mappings, to help produce canonical forms of cortical surfaces that can be used in the medical image registration of cortical surface maps. There are a variety of ways to accomplish this using conformal mapping. This paper examines two methods – the method of Least Squares Conformal Mapping(LSCM) and Harmonic Energy Minimization.

1   Introduction

The cerebrum is the largest part of the brain. The outermost layer of gray matter covering the cerebrum is called the cerebral cortex (Figure 1). The cerebral cortex is responsible for some of the higher level thought processes such as awareness, memory, decision-making, and language, and is also responsible for sensing and interpreting sensory input for hearing, touch, and sight. Since these functions are vital to most people, it is imperative that possible dangers to the cerebral cortex be detected and prevented or removed.
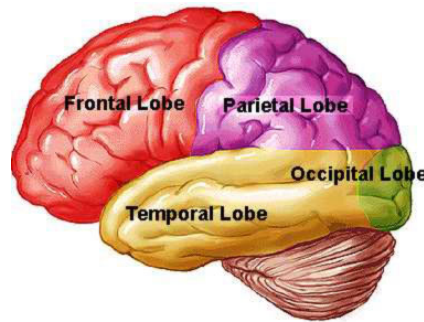

Figure 1: Example of the cerebral cortex with its four lobes labeled.

The first step in this process is detection. In order to detect threats, we need a way to image and analyze the cerebral cortex. Imaging of the cerebral cortex and the brain in general is handled by the field known as neuroimaging. Imaging of the brain is accomplished through a variety of methods such as Magnetic Resonance Imaging (MRI), Computed Tomography Scan (CT Scan), and Positron Emission Tomography (PET). These methods produce 3D models and images of the brain which can then be analyzed to detect dangers.

The entire 3D model may be analyzed, but many neurological disorders and diseases, such as Alzheimers Disease, autism, epilepsy, schizophrenia, and tumors cause changes to the surface of the cerebral cortex, which is called the cortical surface (Figure 2). In these cases, it is only necessary to analyze the cortical surface. However, the cortical surface is notoriously difficult to analyze due to its convoluted structure (Figure 3) of sulci (valleys) and gyri (ridges) [2]. A human can perform analysis of the surface, but this method is tedious and error prone since a human may miss fine details in the images produced. Analysis can be performed by computers as well. This method is faster and more correct, but the 3D models are still difficult for computer algorithms to efficiently perform analysis and comparison operations, due to things such as the resolution of the image, noise created by the imaging
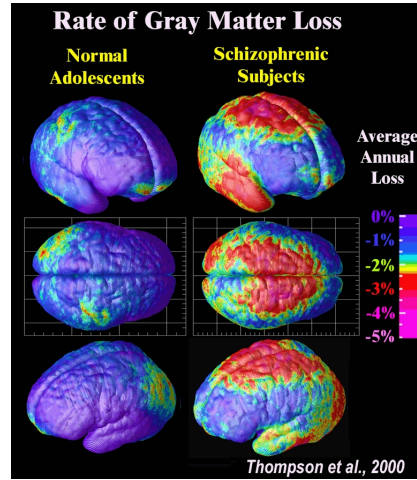
Figure 2: Loss of gray matter in schizophrenic brain and normal brain. Red and pink indicate areas of high loss.

method, or occlusion of the image, which produce errors in the models [2]. Thus, in order to make cortical surface analysis more convenient, we need methods to produce images of the cortical surface in a form that is convenient to analyze and also canonical so that is can be compared, all while resisting possible errors in the 3D models [2].
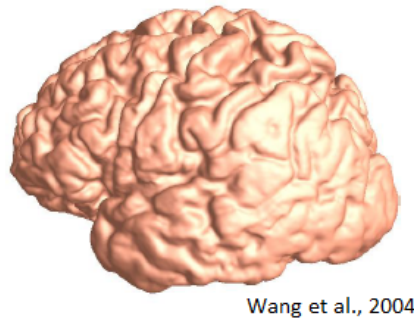

Figure 3: 3D model of the brain with convoluted surface.

The field that is responsible for creating a standardized form for the images is called medical image registration, which is the field of image registration applied to medicine. In image registration, different sets of data are transformed into one common coordinate system. When medical image registration is applied to neuroimaging, many methods arise to create convenient canonical forms of the cortical surface. These include methods such as circle packing and straight line embeddings. While these methods create

a convenient canonical representation of the cortical surface, they are not resistant to noise, resolution, or occlusion. One method that can resist those errors while providing a useful representation is called a conformal mapping.
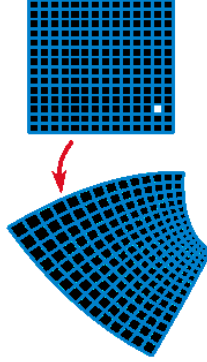

Figure 4: Example of a conformal mapping on a grid. Notice the angles between intersecting lines are preserved.

Conformal mappings are useful since their transformations preserve the angles from the source image to the result image (Figure 4). In addition, we are guaranteed that a conformal mapping to a more convenient representation exists for the cortical surface of any given 3D model of the brain. This is because the cortical surface is topologically equivalent to a genus-zero surface, such as a sphere, which is also topologically equivalent to a disk [3]. Because of this, we can use the Riemann Mapping theorem, a profound result from complex analysis which says that a conformal map exists for any subset of the complex plane that maps the subset to the unit disk. However, even though a mapping is guaranteed to exist, we cannot always find them. As such, methods developed to provide conformal mappings only find very close approximations. These approximate maps are called quasi-conformal maps.

There are a variety of methods that utilize conformal maps to achieve the desired effect of a conformal or quasi-conformal parameterization of a cortical surface. Techniques that do so include harmonic energy minimization, least squares conformal mapping, Laplacian operator linearization, angle based methods, and circle packing [4]. There are also a variety of methods that can be used to accomplish these techniques. This paper looks at an example of Least Squares Conformal Mapping as proposed by Samaras et al., and an example of Harmonic Energy Minimization, as proposed by Wang et al.

The rest of the paper examines two techniques of finding this mapping. The first, the Least Squares Conformal Mapping with Spring Energy, is in section 2. The second, Harmonic Energy Minimization, is in section 3. Section 4 concludes the paper and briefly summarizes the information.

2   Least Squares Conformal Mapping

If U is an open subset of $\mathbb{C}$, then a conformal mapping is a function $f : U \to \mathbb{C}$ that is holomorphic everywhere in U and whose derivative $f' \neq 0$. Because it is holomorphic, it satisfies the Cauchy-Riemann equations [3].

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1}$$

It also satisfies the Laplace equations, meaning it is harmonic.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{2}$$

Now, given a discrete three-dimensional triangular mesh of a cortical surface S and a mapping $f : S \to \mathbb{C}$, f is conformal on S if and only if the Cauchy-Riemann equations hold true for f on S [1]. However, we cannot guarantee this condition will be satisfied on S, so the conformal mapping is instead approximated in the least squares sense [3].

$$C(S) = \sum_{d \in S} \|\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\|^2 A(d) \tag{3}$$

where d is a triangle on the mesh S with the area A(d). A triangle in S is made up of any three points that are adjacent to each other, that is, for triangle $\{1, 2, 3\}$, 1 is adjacent to 2 and 3, and 2 and 3 are adjacent to each other. Furthermore, let $\alpha_j = u_j + iv_j$ and $\beta_j = x_j + iy_j$, so $f(\beta_j) = \alpha_j$ for $j = 1, 2, \ldots, n$. Then we rearrange the vector α such that $\alpha = (\alpha_f, \alpha_p)$ where α_f consists of $n - p$ free coordinates and α_p consists of p constrained point coordinates [3]. Therefore, equation 3 can be rewritten as

$$C(S) = \|M_f \alpha_f + M_p \alpha_p\|^2 \tag{4}$$

where $M = (M_f, M_p)$, a sparse $m \times n$ complex matrix.

This is the least squares minimization problem, which can be solved using the Conjugate Gradient Method, providing a quasi-conformal mapping [3].
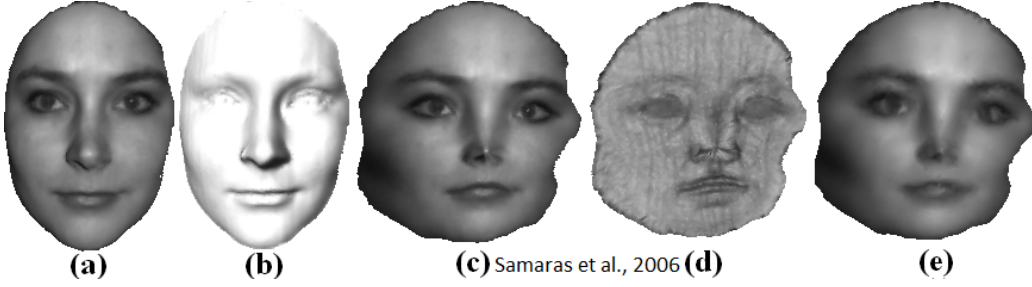
Figure 5: Example of LSCM. (a) is the original surface with texture. (b) is the original surface without texture. (c) is the LSCM with texture. (d) is the shape image of the mapping. (e) is the LCSM of the same surface, but with the resolution subsampled by a factor of 4. Notice (e) is very similar to (c), demonstrating the insensitivity of a conformal mapping to resolution changes [3]

3 Harmonic Energy Minimization

Suppose K is a simplicial complex and mapping $f : |K| \to \mathbb{R}^3$ embeds K in $\mathbb{R}^3$, then $M = (K, u)$ is called a mesh, which is made up of triangles. Given two genus zero meshes M_1, M_2, there are many conformal mappings between them.

A mapping between two genus zero surfaces $f : S_1 \to S_2$ is conformal if and only if f is harmonic [4]. In order to create a conformal parameterization of $S_1 \to S_2$, a homeomorphism h between them must be found. Once found, h can be deformed in such a way as to minimize the harmonic energy of h [4].

All piecewise linear functions defined on K form a linear space denoted by $C^{PL}(K)$ [4]. Suppose a set of string constants $k(u, v)$ are assigned to each edge $\{u, v\}$ in K. Then, the inner product on C^{PL} is defined as

$$< f, g >= \frac{1}{2} \sum_{\{u,v\} \in K} k(u, v)(f(u) - f(v)(g(u) - g(v))) \tag{5}$$

where $f, g \in C^{PL}$ [4].

The string energy of a piecewise linear function f is defined as the norm of the inner product of f on $C^{'PL}$ [4].

$$E(f) =< f, f >= \frac{1}{2} \sum_{\{u,v\} \in K} k(u, v)\|(f(u) - f(v)\|^2 \tag{6}$$

If the string constants used are $k(u, v) = 1$, then the resulting string energy is known as the Tutte energy of f [4]. Now, suppose edge $\{u, v\}$ has

6

two adjacent triangular faces T_α and T_β. This means T_α and T_β share vertices u and v. Defined T_α by the three-tuple of vertices $\{v_1, v_2, v_3\}$ and define the parameters

$$a^\alpha_{v_1,v_2} = \frac{1}{2}\frac{(v_1 - v_3)\cdot(v_2 - v_3)}{(v_1 - v_3)\times(v_2 - v_3)} \tag{7}$$

$$a^\alpha_{v_2,v_3} = \frac{1}{2}\frac{(v_2 - v_1)\cdot(v_3 - v_1)}{(v_2 - v_1)\times(v_3 - v_1)} \tag{8}$$

$$a^\alpha_{v_3,v_1} = \frac{1}{2}\frac{(v_3 - v_2)\cdot(v_1 - v_2)}{(v_3 - v_2)\times(v_1 - v_2)} \tag{9}$$

for T_α. Parameters for T_β are defined similarly. If $k(u,v) = a^\alpha_{u,v} + a^\beta_{u,v}$, then the string energy obtained from equation 6 is called the harmonic energy [4]

The piecewise Laplacian is the linear operator $\Delta_{PL} : C^{PL} \to C^{PL}$ [4]. On $C^{PL}(K)$, it is defined as

$$\Delta_{PL}(f) = \sum_{\{u,v\}\in K} k(u,v)(f(u) - f(v)). \tag{10}$$

If f minimizes the string energy, then f satisfies the condition that $\Delta_{PL}(f) = 0$

Suppose S_1 and S_2 are two genus zero meshes and a mapping $\vec{f} : S_1 \to S_2$, then $\vec{f}$ can be treated as a map $\vec{f} : S_1 \to \mathbb{R}^3$

Now, suppose there exists a vector valued function $\vec{f} = (f_1, f_2, f_3)$ that maps $S_1 \to \mathbb{R}^3$. The energy of such a function is the norm of $\vec{f}$, defined as

$$E(\vec{f}) = \|\vec{f}\|^2 = \sum_{i=1}^3 \|f_i\|^2. \tag{11}$$

Then, the piecewise Laplacian for $\vec{f}$ is defined in a similar fashion [4].

$$\Delta_{PL}(\vec{f}) = (\Delta_{PL}(f_1), \Delta_{PL}(f_2), \Delta_{PL}(f_3)) \tag{12}$$

Finally, $\vec{f}$ is harmonic if and only if it has a normal component and the tangenital component is 0 [4].

$$\Delta_{PL}(\vec{f}) = (\Delta_{PL}(\vec{f}))^\perp \tag{13}$$

To compute a mapping $\vec{f} : M_1 \to M_2$ that minimizes the string energy $E(\vec{f})$, the steepest descent algorithm can be used [4].

$$\frac{d\vec{f}(t)}{dt} = -\Delta \vec{f}(t) \tag{14}$$

However, the solution the algorithm provides is not unique, but instead forms a *Möbius* group, that is, all solutions are *Möbius* transformations.

To identify a unique solution, additional constraints must be added. This is accomplished by applying a zero mass-center constraint and a landmark constraint [4]. A mapping $\vec{f} : M_1 \rightarrow M_2$ satisfies the zero mass-center constraint if and only if

$$\int_{M_2} \vec{f} d\sigma_{M_1} = 0 \tag{15}$$

where σ_{M_1} is the area element on M_1 [4].

Finally, sulcal curves can be manually identified as significant geometric landmarks on the cortical surface mesh, providing additional constraints [4].

As input to the spherical mapping algorithm, a gauss mapping N is created between M_1 and M_2, defined as

$$N(v) = \vec{n}(v), v \in M_1 \tag{16}$$

where $\vec{n}(v)$ is the normal at v [4].

The input of the algorithm is a mesh M_1, the step length δt, and the energy difference threshold δE.

First a spherical Tutte mapping is created.

1. Compute N

2. Let $\vec{t} = N$ and compute the Tutte energy of $\vec{t}$, which is $E_0(\vec{t})$

3. For each vertex $v \in M_1$, compute the absolute derivative $D\vec{t}(v)$

4. Update $\vec{t}(v)$ by $\delta t(\vec{v}) = -D\vec{t}(v)\delta t$

5. Compute new Tutte energy $E(\vec{t})$

6. If $E(\vec{t}) - E_0(\vec{t}) < \delta E$ return $\vec{t}$

7. If $E(\vec{t}) - E_0(\vec{t}) \not< \delta E$, then set $E_0(\vec{t})$ to $E(\vec{t})$ and repeat steps 2 through 5.

Now $\vec{t}$ minimizes the Tutte energy. We can use it as input to the second part of the algorithm which computes a spherical conformal mapping $\vec{h}$

1. Compute the Tutte embedding of $\vec{t}$

2. Let $\vec{h} = \vec{t}$ and compute the Tutte energy $E_0(\vec{h})$

3. For each vertex $v \in M_1$, compute the absolute derivative $D\vec{h}(v)$

4. Update $\vec{h}(v)$ by $\delta\vec{h}(v) = -D\vec{h}(v)\delta t$

5. Compute $M\ddot{o}bius$ transformation $\vec{\rho}_0 : M_1 \to M_2$ such that

$$\Gamma(\vec{\rho}) = \int_{M_2} \vec{\rho} \circ \vec{h} d\eta_{M_1}, \vec{\rho} \in M\ddot{o}bius(CP^l) \tag{17}$$

$$\vec{\rho}_0 = \min_{\vec{\rho}} ||\Gamma(\vec{\rho})||^2 \tag{18}$$

6. Compute the harmonic energy $E(\vec{h})$

7. If $E(\vec{h}) - E_0(\vec{h}) < \delta E$ return $\vec{h}$

8. If $E(\vec{h}) - E_0(\vec{h}) \not< \delta E$, then set $E_0(\vec{h})$ to $E(\vec{h})$ and repeat steps 2 through 6.

The $\vec{h}$ that is returned minimizes the harmonic energy and satisfies the zero-mass center constraint, resulting in a quasi-conformal map from M_1 to M_2
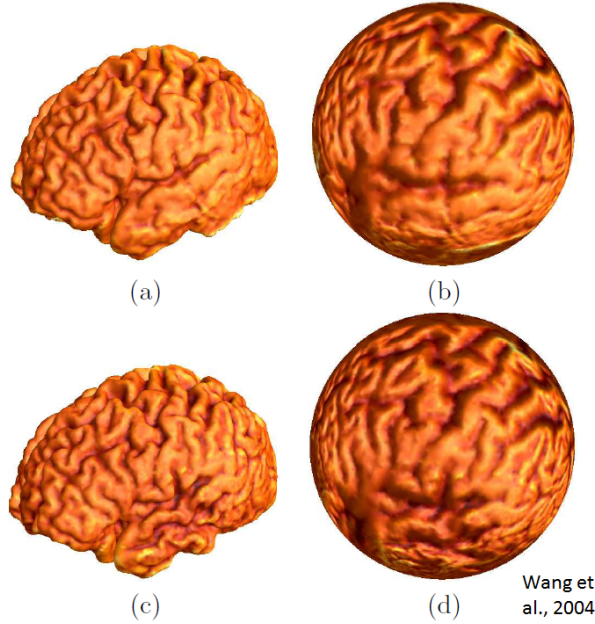
Figure 6: Reconstructed brain meshes and their spherical harmonic mappings. (a) and (b) are the reconstructed surfaces for the same brain scanned at different times. Due to scanner noise and inaccuracy in the reconstruction algorithm, there are visible geometric differences. (c) and (d) are the spherical conformal mappings of (a) and (b) respectively. Notice that despite the geometric differences due to noise, the information from both surfaces was preserved [5].

4   Conclusion

In this paper, we examined two methods to create quasi-conformal mappings of cortical surfaces modeled as three-dimensional triangular meshes. These mappings allow the cortical surface to be mapped to a canonical form such as a disk or sphere, providing a solution to the problem of medical image registration for cortical surfaces. While it is impossible to avoid metric distortion or curvature distortion when using the conformal maps, the angles between curves on the surface are preserved, which allows the major geometric landmarks of a brain surface to be preserved on the mapping, while making the mapping resistant to noise and occlusion in the source cortical model, as well as making the mapping insensitive to the resolution of the source cortical model [3, 4]. The importance and difficulty of analyzing the cortical surface was also examined.

In addition to methods using conformal maps as described above, there

also ad-hoc methods, and metric methods of mapping the cortical surface. Ad-hoc methods recognize that the cortical surface meshes used above are actually a type of planar graph and utilize the theory of straight-line embeddings of planar graphs to map them [3]. Metric methods attempt to iteratively unfold the mesh into a flat map while applying statistical methods to minimize geometric distortion caused by the unfolding [3].

In general, there is no best method to use when registering images. This is because, as in most fields, different methods produce different results which are useful for certain things, so the usefulness of any particular algorithm will change depending on the needs of the analyst. Interestingly, on April 2nd, 2013, the President of the United States, Barack Obama, announced a federal project aimed at combatting neurological diseases and disorders that will contribute $100 million dollars to brain mapping research. Since neuroimaging is a subset of the field of brain mapping and the effort is aimed at neurological disorders and diseases, it is extremely likely that the topic of creating convenient cortical surface images will get much attention. In this case, perhaps a general method that has desirable properties in most situations can be found, providing a standardized algorithm that sees widespread usage.

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