

Singular Moduli of Shimura Curves

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Final Phd Defense
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Modular Curve
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Shimura Curve
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My Method
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Summary
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Classical Set-Up

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- $\mathrm{GL}_2(\mathbb{R})$ acts on \mathfrak{h}^\pm , the union of the upper and lower half-planes:

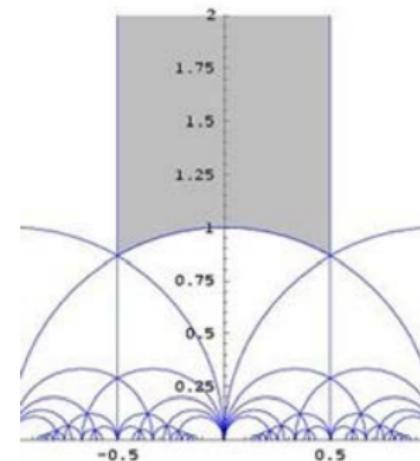
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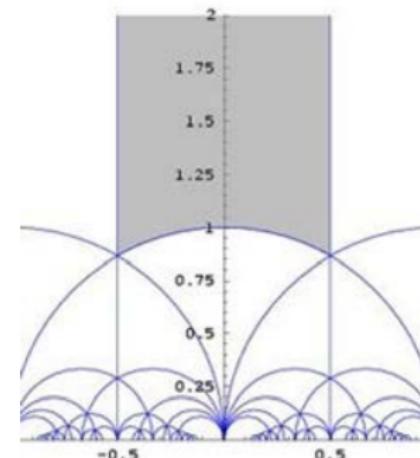


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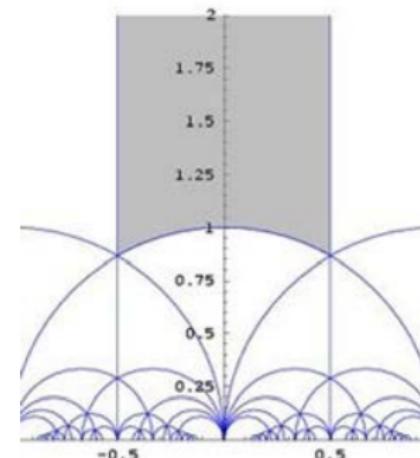
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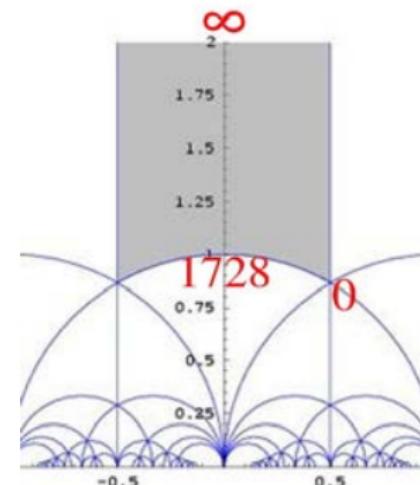
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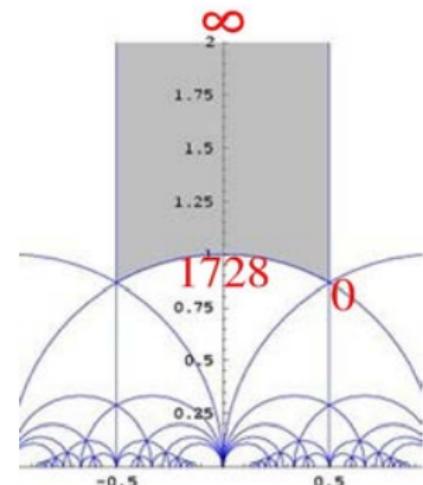
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where $\mathbf{q} = e^{2\pi i \tau}$. (Gauss, Dedekind, Klein, etc.)



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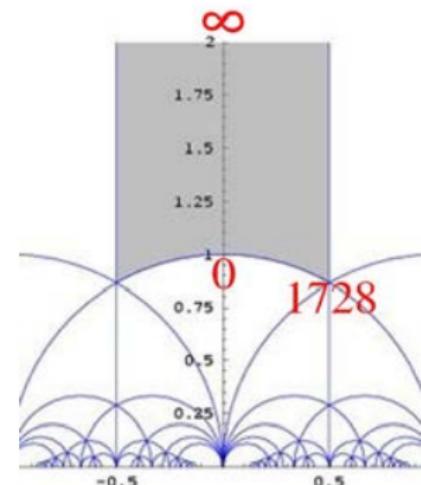
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- A CM-point τ is the solution to an integral quadratic equation with negative discriminant Δ .

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Examples

$$j\left(\frac{1+\sqrt{-3}}{2}\right) = 0, \quad j(\sqrt{-5}) = 2^3(25 + 13\sqrt{5})^3$$

$$j(\sqrt{-6}) = 12^3(1 + \sqrt{2})^2(5 + 2\sqrt{2})^3$$

$$j(\sqrt{-14}) = 2^3 \left(323 + 228\sqrt{2} + (231 + 161\sqrt{2})\sqrt{2\sqrt{2}-1} \right)^3$$

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- Recall $j\left(\frac{1+\sqrt{-3}}{2}\right) = 0$ and $j(\sqrt{-1}) = 12^3$.

Table 1. Factorizations of $N_{\mathbb{Q}(\sqrt{-d})}(j) = \pm a^3$, $N_{\mathbb{Q}(\sqrt{-d})}(j-1728) = \pm b^2 d$

$ d $	h	a	b
3	1	0	$2^3 3$
4	1	$2^2 3$	0
7	1	$3 \cdot 5$	3^3
8	1	$2^2 5$	$2^2 7$
11	1	2^5	$2^3 7$
19	1	$2^4 3$	$2^3 3^3$
43	1	$2^6 3 \cdot 5$	$2^3 3^4 7$
67	1	$2^5 3 \cdot 5 \cdot 11$	$2^3 3^2 7 \cdot 31$
163	1	$2^6 3 \cdot 5 \cdot 23 \cdot 29$	$2^3 3^2 7 \cdot 11 \cdot 19 \cdot 127$
23	3	$5^2 11 \cdot 17$	$7^3 11^2 19$
31	3	$3^3 11 \cdot 17 \cdot 23$	$3^{10} 11^2$
59	3	$2^{16} 11$	$2^9 11^2 23 \cdot 43$
83	3	$2^{16} 5^3$	$2^9 19 \cdot 47 \cdot 67 \cdot 79$
107	3	$2^{15} 5^3 17$	$2^9 7^3 43 \cdot 71 \cdot 103$
139	3	$2^{16} 3^2 23$	$2^9 3^{11} 103$
211	3	$2^{17} 3^2 17 \cdot 29$	$2^9 3^9 7^3 23 \cdot 67$
283	3	$2^{15} 3^2 5^3 53$	$2^9 3^{10} 19^2 31 \cdot 139$
307	3	$2^{17} 3^2 5^3 47$	$2^9 3^{11} 23 \cdot 163 \cdot 271$
331	3	$2^{15} 3^2 11 \cdot 23 \cdot 29 \cdot 59$	$2^9 3^{11} 7^3 11^2 59^2$
379	3	$2^{17} 3^2 11 \cdot 17 \cdot 53 \cdot 71$	$2^9 3^9 7^4 11^2 31 \cdot 47^2$
499	3	$2^{16} 3^2 17 \cdot 23 \cdot 41 \cdot 71 \cdot 83$	$2^9 3^{11} 7^3 71^2 463$
547	3	$2^{15} 3^2 5^3 17 \cdot 23 \cdot 101$	$2^9 3^{11} 7^3 31^2 59 \cdot 223$
643	3	$2^{15} 3^2 5^3 11 \cdot 17^2 113$	$2^9 3^{11} 11^2 43 \cdot 67 \cdot 71 \cdot 499 \cdot 607$
883	3	$2^{15} 3^2 5^3 11^2 41 \cdot 89 \cdot 113$	$2^9 3^{11} 7^3 11 \cdot 23 \cdot 43^2 307 \cdot 739$

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$$|j(\sqrt{-5})| = 2^{12} 5^3 11^3$$

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$$|j(a) - j(b)| = \prod_{n \in N(a,b)} n^{\varepsilon(n)}$$

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- Recall $j\left(\frac{1+\sqrt{-3}}{2}\right) = 0$ and $j(\sqrt{-1}) = 12^3$.
- The factorization is a lot of small primes to large powers.

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- There is an embedding $B \hookrightarrow M_2(\mathbb{Q}(\sqrt{b}))$ via

$$\alpha \mapsto \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}, \quad \beta \mapsto \begin{pmatrix} \sqrt{b} & 0 \\ 0 & -\sqrt{b} \end{pmatrix}$$

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Q: How do you compute $|t_D(\tau_\Delta)|$?

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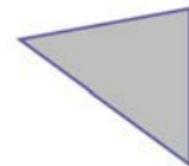
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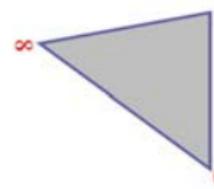
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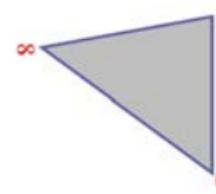
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- Elkies (1998) uses geometric involutions on the covering curves $\mathcal{X}_6^*(N)$.



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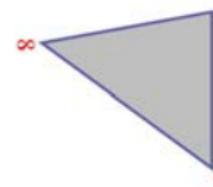
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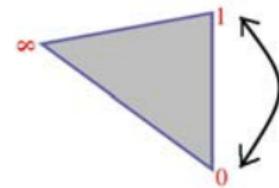
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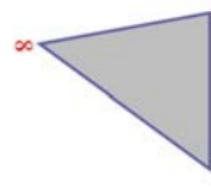
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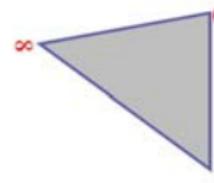
Elkies's Results

- Elkies is successful at algebraically determining the coordinates for **17 of the 27** rational CM points.
- Unable to prove the remaining 10 CM points, but makes **numerical approximations** and then recognizes them as rational numbers.
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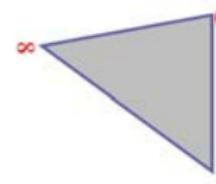
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Example

$$t_6(\tau_{-163}) \stackrel{?}{=} \frac{3^{11}7^419^423^4}{2^{10}5^611^617^6}.$$

Modular Curve
○○○○○

Shimura Curve
○○○○○

My Method
●○○○○

Summary
○○○○

Borcherds Forms

Borcherds Forms

Definition

Borcherds Form: Given a modular form $F : \mathfrak{h}^\pm \rightarrow \mathbb{C}[L^\vee/L]$,
Borcherds (1998) constructed

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Theorem

If F has Fourier expansion

$$F(\tau) = \sum_{\lambda \in L^\vee/L} \sum_{m \in \mathbb{Q}} c_\lambda(m) \mathbf{q}^m e_\lambda$$

then the divisor of $\Psi(F)$ is given in terms of the $c_\lambda(m)$ for $m < 0$ and rational quadratic divisors.

Borcherds Forms at CM Points

Theorem (J. Schofer, 2005)

$$\sum_{\substack{\text{Galois Orbit} \\ \text{of a CM Point}}} \log ||\Psi(F)|| = |Z_\Delta| \sum_{\lambda \in L^\vee / L} \sum_{m < 0} c_\lambda(m) \kappa_\lambda(m) \quad (1)$$

where $\kappa_\lambda(m)$ are *computable* coefficients of an Eisenstein series.

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Theorem

The map t_6 is a Borcherds form, too.

The map t_6 as a Borcherds Form

- Through a vectorization process, the scalar-valued $\Gamma_0(12)$ modular form

$$-6 \frac{\eta_2 \eta_3^2 \eta_4^4 \eta_6^4}{\eta_{12}^{10}} - 2 \frac{\eta_2^{12} \eta_3}{\eta_1^5 \eta_4^4 \eta_6 \eta_{12}^2} - 2 \frac{\eta_2^5}{\eta_1^2 \eta_4^2}$$

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- This implies there exists a nonzero constant k_6 such that

$$\Psi(F_6) = k_6 t_6.$$

Normalization



$$\sum_{\substack{\text{Galois Orbit} \\ \text{of } \tau_{-24}}} \log ||\Psi(F_6, \tau)|| = |Z_{-24}| \sum_{\lambda \in L^\vee / L} \sum_{m < 0} c_\lambda(m) \kappa_\lambda(m)$$

Normalization



$$\log ||\Psi(F_6, \tau_{-24})|| = |Z_{-24}| \sum_{\lambda \in L^\vee / L} \sum_{m < 0} c_\lambda(m) \kappa_\lambda(m)$$

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$$t_6 = 6^{-6} \Psi(F_6)$$

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Theorem

$$t_{10} = 2^{-2} \Psi(F_{10})$$

Computing $|t_6(\tau_\Delta)|$

- $$\sum_{\substack{\text{Galois Orbit} \\ \text{of a CM Point}}} \log ||\Psi(F_6)|| = |Z_\Delta| \sum_{\lambda \in L^\vee / L} \sum_{m < 0} c_\lambda(m) \kappa_\lambda(m)$$

Computing $|t_6(\tau_\Delta)|$

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- Use this to compute $|t_6(\tau_\Delta)|$ for any CM point.
- Calculation of the $\kappa_\lambda(m)$ is intensive and was programmed in Mathematica.

Results

- The maps t_6 and t_{10} are Borcherds Forms.

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Coordinates of Rational CM Points on \mathcal{X}_6^*

Δ	Numerator	Denominator
-40	3^7	5^3
-52	$2^2 3^7$	5^6
-19	3^7	2^{10}
-84	$-2^2 7^2$	3^3
-88	$3^7 7^4$	$5^6 11^3$
-100	$2^4 3^7 7^4 5$	11^6
-120	7^4	$3^3 5^3$
-132	$2^4 11^2$	5^6
-148	$2^2 3^7 7^4 11^4$	$5^6 17^6$
-168	$-7^2 11^4$	5^6
-43	$3^7 7^4$	$2^{10} 5^6$
-51	-7^4	2^{10}
-228	$2^6 7^4 19^2$	$3^6 5^6$
-232	$3^7 7^4 11^4 19^4$	$5^6 23^6 29^3$
-67	$3^7 7^4 11^4$	$2^{16} 5^6$
-75	11^4	$2^{10} 3^3 5$
-312	$7^4 23^4$	$5^6 11^6$
-372	$-2^2 7^4 19^4 31^2$	$3^3 5^6 11^6$
-408	$-7^4 11^4 31^4$	$3^6 5^6 17^3$
-123	$-7^4 19^4$	$2^{10} 5^6$
-147	$-11^4 23^4$	$2^{10} 3^3 5^6 7$
-163	$3^{11} 7^4 19^4 23^4$	$2^{10} 5^6 11^6 17^6$
-708	$2^8 7^4 11^4 47^4 59^2$	$5^6 17^6 29^6$
-267	$-7^4 31^4 43^4$	$2^{16} 5^6 11^6$

Coordinates of Rational CM Points on \mathcal{X}_{10}^*

Δ	Numerator	Denominator
-40	3^3	1
-52	$-2 \cdot 3^3$	5^2
-72	5^3	$3 \cdot 7^2$
-120	-3^3	7^2
-88	$3^3 5^3$	$2 \cdot 7^2$
-27	$-2^6 3$	5^2
-35	2^6	7
-148	$2 \cdot 3^3 11^3$	$5^2 7^2 13^2$
-43	$2^6 3^3$	$5^2 7^2$
-180	$-2 \cdot 11^3$	13^2
-232	$3^3 11^3 17^3$	$2^2 5^2 7^2 23^2$
-67	$-2^6 3^3 5^3$	$7^2 13^2$
-280	$3^3 11^3$	$2 \cdot 7 \cdot 23^2$
-340	$2 \cdot 3^3 23^3$	$7^2 29^2$
-115	$2^9 3^3$	$13^2 23$
-520	$3^3 29^3$	$2^3 7^2 13 \cdot 47^2$
-163	$-2^9 3^3 5^3 11^3$	$7^2 13^2 29^2 31^2$
-760	$3^3 17^3 47^3$	$7^2 31^2 71^2$
-235	$2^6 3^3 17^3$	$7^2 37^2 47$

Results

- The maps t_6 and t_{10} are Borcherds Forms.
- Proved all the conjectural values in Elkies's table of rational CM points of \mathcal{X}_6^* .
- Also proved all the conjectural values in Elkies's table of rational CM points of \mathcal{X}_{10}^* .
- Can compute examples far beyond the scope of Elkies's work, such as norms of irrational CM points of arbitrary discriminant on \mathcal{X}_6^* and \mathcal{X}_{10}^* .

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Example

$$|t_6(\tau_{-996})| = \frac{2^{16}7^{12}71^483^2}{17^629^641^6}.$$

Modular Curve
○○○○○

Shimura Curve
○○○○○

My Method
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Summary
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Thanks

Questions?