Math 280 Problems for September 12

Pythagoras Level

Problem 1: The sum of the reciprocals of two real numbers is -1, and the sum of their cubes is 4. What are the numbers?

Problem 2: Two students play a game based on the total roll of two standard dice. Student A says that a 12 will be rolled first. Student B says that two consecutive 7s will be rolled first. The students keep rolling until one of them wins. What is the probability that A will win?

Newton Level

Problem 3: Evaluate

$$\int_1^{2008} \frac{dx}{x + \lfloor \log_{10} x \rfloor}.$$

(For a real number u, $\lfloor u \rfloor$ denotes the greatest integer less than or equal to u.)

Problem 4: Express the product

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{2008^2}\right)$$

as simply as you can as a rational fraction in lowest terms. Justify your answer.

Wiles Level

Problem 5: Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points, A, B, and C, which are vertices of an equilateral triangle, and find its area.

Problem 6: Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)