APPLICATION FOR PROMOTION

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DEPARTMENT OF MATHEMATICS AND STATISTICS
WINONA STATE UNIVERSITY

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Note:

An electronic version of
this document is available at:

http://course1.winona.edu/eerrthum/promotion

INTRODUCTION

I respectfully submit the accompanying materials to support my application for promotion to the rank of Full Professor in the Department of Mathematics and Statistics at Winona State University. This is in accordance with the requirements of the IFO/MnSCU Master Agreement, Article 22 - Professional Development and Evaluation, and Article 25 - Tenure, Promotions, and Non-Renewal. This application contains both a curriculum vitae and a narrative that highlight my efforts in the five areas of faculty responsibility since last applying for promotion and tenure.

Teaching has been and always will be my main responsibility at Winona State and so it is in this area that the bulk of my time has been spent. Over the last 10 semester, I have taught a total of 137 semester hours (plus 13 more in summer sessions) in 13 different undergraduate mathematics courses (Math115, 120, 140, 160/212, 165/213, 242, 247, 280, 327, 347, 395/495, 462, 470/490). In order to provide a solid education for students, I have developed a rich teaching philosophy that continues to grow in response to my classroom experience and the ever-changing needs and backgrounds of the students. Many of my students consider my courses to be intellectually challenging, yet overall they have rated me well in their evaluations. I continue to reflect upon what works and what doesn't work to improve my teaching. One example of the success I've had comes in Math327 – Foundations of Mathematics where I "flipped" the classroom and started to emphasize quality over quantity, depth over breadth. This approach has shown to be a better preparation for subsequent upper-level math courses and the positive reception of my presentation on this approach at the Joint Mathematical Meetings in Seattle, WA indicates it fits within a larger movement in math education.

In the 5 years since my last promotion, I've continued to develop a research program of which I am proud. I have successfully published two research articles (one appearing in one of the top journals of my field, *Journal of Number Theory*) and one teaching-related article published in a widely-circulated journal for math educators. In addition, I continue to include undergraduates in a variety of short- and long-term research projects in topics that are both suitable for senior projects and interesting to the larger mathematical community.

At the same time I continue to learn more about my profession through attending conferences, reading articles, participating in online forums, and writing reviews for scholarly articles in my area. These activities have had a wide range of benefits, from leading me to examine my teaching from different perspectives to new and tantalizing directions to take in my research.

Underlying all of my work at Winona State, I've strived to provide students with a sound and comprehensive educational experience. I have been able to do this through offering undergraduate research opportunities, chaperoning trips to graduate schools, supervising math competitions, and helping students chart their path through their major. Of the seventeen students to whom I have been academic adviser and have graduated since Spring 2012, ten of them have successfully gone on to graduate studies.

I have also provided solid service to my department and the community as a whole. During my leadership role in the Math Subgroup, we created a stand-alone Math major to prepare students for Mathematics graduate school, reconfigured the mathematics curriculum to both align with other MnSCU institutions and to better handle the needs of the students in our department, started discussions around recruitment and retention that have resulted in departmental initiatives, and began a process for assessing of our program to identify future improvements. Since my last promotion, I also have acted as department webmaster during the transition of our website to the standard university format and have served on two search committees that were ultimately successful in hiring a tenure-track statistician and three fixed-term faculty. In addition, I led our department's successful hosting of the Spring 2015 Meeting of the Mathematical Association of America – North Central Section which brought to campus 72 attendees from 24 institutions in 5 states (MN, WI, IA, SD, ND). Meanwhile I have been the head coach of the Cotter High School Math Team and active in various roles in the larger Winona community and beyond. My off-campus service keeps me connected to the world outside of WSU and allows me to bring those experiences back to my work to enhance my performance.

I look forward to continuing many of these activities and finding more ways to grow professionally. My colleagues in the department and the university have provided an invaluable source of ideas and suggestions that are continuously beneficial to my growth. I am inspired by them and can't imagine finding a better group of colleagues. I hope that my successes at Winona State University will continue for many years to come and I'm grateful for all the opportunities that I've been given.

Respectfully submitted,

Eric Errthum, Ph.D.
Department of Mathematics and Statistics
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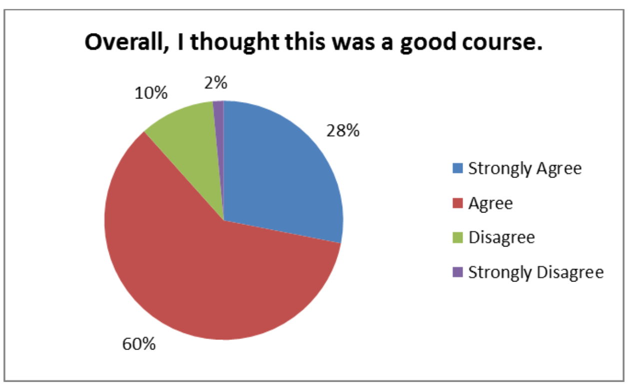
CRITERION I:
DEMONSTRATED ABILITY TO TEACH EFFECTIVELY

Over the last 10 semester, I have taught a total of 137 semester hours (plus 13 more in summer sessions) in 13 different undergraduate mathematics courses:

- Math115 – College Algebra
- Math120 – Precalculus
- Math140 – Applied Calculus
- Math160/212 – Calculus I
- Math165/213 – Calculus II
- Math242 – Linear Algebra
- Math247 – Discrete Mathematics
- Math280 – Special Topics: Problem Solving for Mathematical Competitions
- Math327 – Foundations of Mathematics
- Math347 – Number Theory
- Math395/495 – Professional Skills for Mathematics/Communication of Independent Project
- Math462 – Introduction to Topology
- Math470/490 – Various undergraduate research projects

Overall, 88% of the 523 students agreed on their evaluations that it was a good course.



Accomplishments (since Spring 2012)

1) Created materials for two new courses

   a) Created course materials (course schedule, lecture notes, classroom activities, assessments) for Math247 – Discrete Mathematics which I taught in Fall 2015 and Spring 2016. This was the first time such a course had been offered by our department in at least a decade.

b) Created (in conjunction with Chris Malone) course materials (course schedule, lecture notes, classroom activities, assessments) for Math395 – Professional Skills in Mathematics which I co-taught in Spring 2015 and Spring 2016. This was the first time such a course had been offered by our department.

2) Created course materials (course schedule, lecture notes, classroom activities, assessments) for Math242 – Linear Algebra which I taught for the first time in Fall 2014.

3) Improved upon previously taught courses

a) Created new materials for Math327 – Foundations of Mathematics designed to incorporate a more inquiry-based learning approached. These materials were put into use in Spring 2015 and in Fall 2016 and presented on at the Joint Mathematical Meetings in Seattle, WA in January 2016.

b) Created a comprehensive set of slides (i.e. gap notes) for Math140 – Applied Calculus that is utilized during lecture and available to students to download and take notes on. These have been so successful in this course that I've begun to create similar gap notes for other courses (e.g. Math212 – Calculus I).

c) Modified my existing approach for Math140 to include more technology and emphasize certain sections and chapters over others in an attempt to better serve the student population in this course.

d) Had success implementing online homework. In Math140 I've been using an online homework system hosted by the publisher of the text. In other service courses, I have used WebAssign or, more recently, the free open-source system WeBWorK. Since WeBWorK is not tied to a specific publisher or text, using the open-source option has significantly lowered costs for students both in the cost of the homework system and the cost of the text.

e) Had success in the classroom with students presenting homework problems at the board. In addition, I've moved to an inquiry-based-inspired version of Math327 that consists of students peer-reviewing each other's work.

f) Increased and diversified my in-class assessment library in all my classes.

4) Effectively communicated about course details

a) Kept detailed schedules for all my classes. During the course, these are available on my course1 webpage (and linked to from D2L) for students and colleagues to view and use.

b) Used a combination of D2L and my course1 website to post an online syllabus with complete schedule and to upload quiz solutions, sample exams, etc.

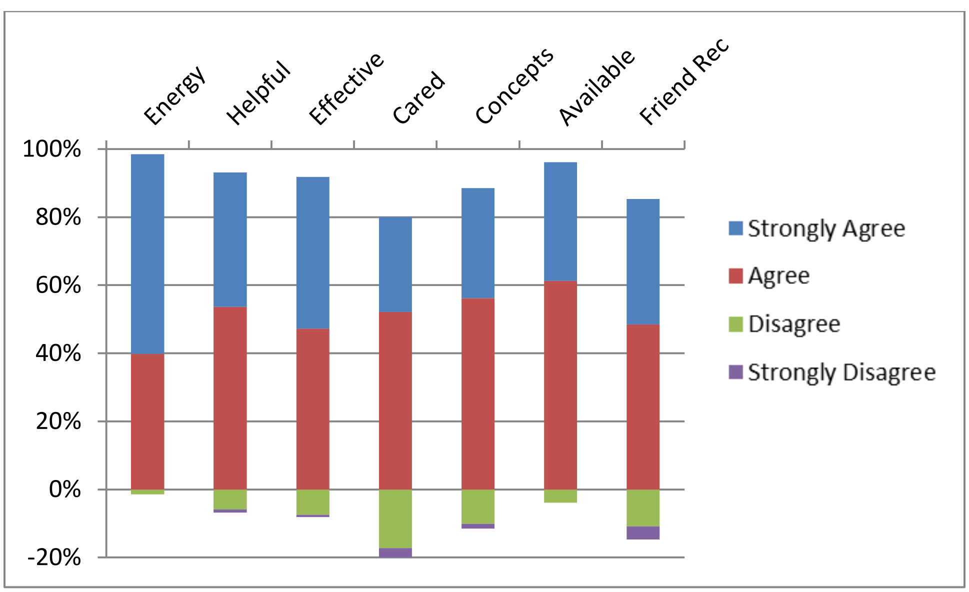
5) Since Spring 2014, I have been participating in our department's attempts at collecting data in our GEP courses to assist in future program/department reviews.

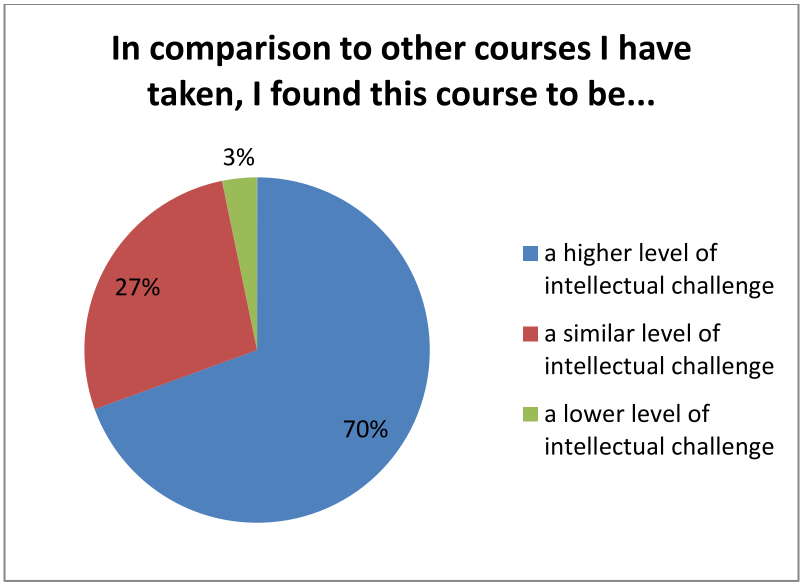
Summary of Evaluations

I collect end-of-semester course evaluations in all courses I instruct. While a large portion of the questions vary from course to course in order to get feedback on issues specific to the course, all course evaluations contain the following core questions to which students can respond "Strongly Agree", "Agree", "Disagree", or "Strongly Disagree":

- I thought the instructor had the appropriate level of enthusiasm and energy.
- I found the instructor's responses to my questions to be helpful.
- I found the instructor to be effective in teaching the subject matter.
- I thought the instructor cared about my progress in the course.
- The instructor clearly explained the concepts of the class.
- The instructor was available outside the class to help me.
- If a friend of mine asked me about this instructor, I would recommend taking a class from him.
- In comparison to other [MATH] courses I have taken, I found this course to be
    - a higher level of intellectual challenge.
    - a similar level of intellectual challenge.
    - a lower level of intellectual challenge .

The results of these questions for the past five years are summarized in the charts below.





I think these results point to that, while I challenge students, I do so in a friendly, effective, and energetic way that they appreciate.

More detailed course-by-course summary of evaluations can be found in Appendix B: Course Evaluations. Copies of the actual evaluations and/or raw data are available upon request.

Course Materials

Syllabi for all my courses are kept on my course1.winona.edu web space. This makes them accessible to students and other faculty members and keeps a public record of the course. These syllabi can be viewed by visiting: http://course1.winona.edu/eerrthum/PreviousSemesters.htm.

Other course materials, such as in-class activities, quizzes, exams, etc. are available upon request.

Criterion II:
Scholarly or Creative Achievement or Research

Since the time of my last promotion, I have continued a research program that is able to incorporate undergraduate students and have had some of the resulting work published in the *Journal of Number Theory*. I am proud of what I have accomplished especially considering the guidelines of the Mathematical Association of America:

> "Faculty for whom personnel decisions are based upon assessment of contributions in teaching, scholarship, and service should have teaching assignments that reflect these multiple expectations and allow for attention to non-classroom responsibilities. Teaching assignments above three courses [nine credit hours] per semester, when combined with other faculty responsibilities, do not allow the time needed to develop and maintain a program of sustained scholarship with the result that tenure and promotion might be effectively unattainable. For such faculty, teaching assignments above the level of three courses [nine credit hours] per semester must be avoided." (MAA Guidelines, Article C.5.1.2)

Since I have taught on average 13.7 credit hours per semester and still maintained enough of a research program able to incorporate undergraduates, I consider my accomplishments in Criterion II to be sufficient for promotion.

Accomplishments (since Spring 2012)

1) Two research articles appeared in peer-reviewed journals:

- E. Errthum. "A Division Algorithm Approach to p-Adic Sylvester Expansions." *J. Number Theory* 160 (2016), 1–10.

  Abstract: A method of constructing finite p-adic Sylvester expansions for all rationals is presented. This method parallels the classical Fibonacci–Sylvester (greedy) algorithm by iterating a p-adic division algorithm. The method extends to irrational p-adics that have an embedding in the reals.

  A copy of this article is included at the end of this section in Appendix A: Scholarly Achievements.

- E. Errthum. "Minimal Polynomials of Singular Moduli." *Math. Comp.* 83 (2014), no. 285, 411–420.

  Abstract: Given a properly normalized parametrization of a genus-0 modular curve, the complex multiplication points map to algebraic numbers called singular moduli. In the classical case, the maps can be given analytically.

However, in the Shimura curve cases, no such analytical expansion is possible. Fortunately, in both cases there are known algorithms for algebraically computing the rational norms of the singular moduli. We demonstrate a method of using these norm algorithms to algebraically determine the minimal polynomial of the singular moduli below a discriminant threshold. We then use these minimal polynomials to compute the algebraic *abc*-ratios for the singular moduli.

A copy of this article is included at the end of this section in Appendix A: Scholarly Achievements.

2) One teaching article appeared in a national journal:

- E. Errthum, D. Smith. "Finding Skewed Lattice Rectangles: The Geometry of $\boldsymbol{a^2} + \boldsymbol{b^2} = \boldsymbol{c^2} + \boldsymbol{d^2}$." *The Mathematics Teacher*, Vol. 106, No. 2 (September 2012), pp. 150-155.

Abstract: Many mathematics instructors try to insert guided exploration into their courses. However, the exploration task frequently comes across to the students as contrived, pertinent only to the recent section of the textbook. The students usually assume that the teacher already knows the answer. The true benefit of guided exploration comes when the students develop a real sense of ownership of the problem, work on it outside of class, and see the instructor as a companion on their journey of exploration. Such was the case in Smith's undergraduate Modern Geometry class. In this article we wish to illustrate how a very specific problem intended to teach a very specific concept grew into a larger, delving-deeper exploration with connections between algebra and geometry.

A copy of this article is included at the end of this section in Appendix A: Scholarly Achievements.

3) Two presentations at regional/national conferences:

- "Publishing or perishing in an intro-to-proof course." 2016 Joint Mathematical Meetings, Seattle, WA

Abstract: Many online forums rely on reputation to rank contributors and reward them for activity. Similarly, in the mathematical research community one builds up prestige through publishing and presenting material. In this talk, I will report on my experience with awarding reputation points and using a publish-or-perish scheme in an intro-to-proof class. I will explain how student work was evaluated like submissions for publication and how reputation was earned. Student feedback will also be shared.

- "A New Approach to Non-Integral Bases." 2016 Spring Meeting of the Mathematical Association of America – North Central Section, St. Paul, MN

  Abstract: A slight change in how we add two whole numbers leads to a new fractional base representation. This method is then extrapolated to bases that are algebraic numbers and the pros and cons of this type of representation (compared to the standard approach) will be discussed. Take home an idea that you can use to help your middle-school (grand)children compute in $\mathbb{Z}[\alpha]$.

4) Advised four students on their senior projects
   - Le Tang – "Theorems for Continued Fractions in $\mathbb{C}_p$." (in progress)
   - Jacob Beckel – "Alternative Carries in Base-p Addition." (technical report and presented at WSU Math/Stat Dept. seminar, April 2016)
   - Alex Klein – "Knot Analysis Algorithms." (technical report, April 2013)
   - Anthony Martino – "p-Adic Egyptian Fractions." (technical report, April 2013)
5) In the process of using WeBWorK in my classes, I have created numerous problems. Once properly prepared, these problems will be submitted to the WeBWorK Open Problem Library to be disseminated and available to anyone using WeBWorK.

Note: Any of the above materials are available upon request.

Appendix A:
Scholarly Achievements

Journal of Number Theory **article.**

Journal of Number Theory 160 (2016) 1–10

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A division algorithm approach to p-adic Sylvester expansions

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ABSTRACT

A method of constructing finite p-adic Sylvester expansions for all rationals is presented. This method parallels the classical Fibonacci–Sylvester (greedy) algorithm by iterating a p-adic division algorithm. The method extends to irrational p-adics that have an embedding in the reals.

1. Introduction

In [2] and [3] A. Knopfmacher and J. Knopfmacher give algorithms for constructing Egyptian fraction expansions in a p-adic setting that are analogous to those given for the reals by Oppenheim [8]. However, for some positive rational inputs the Knopfmachers' algorithm fails to return a finite expansion. In Section 2 we review the basics of the Knopfmachers' Sylvester-type algorithm and give such an example. We then introduce a modification of their algorithm that will give finite Sylvester expansions for all rationals.

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The seemingly unnatural correction term in our algorithm is explained by the alternate approach detailed in the final two sections.

In Section 3 we briefly recall the Fibonacci–Sylvester Greedy Algorithm, especially its relationship to the classical division algorithm. This provides the main motivation for Section 4 wherein we define a new p-adic division algorithm and use the same relationship to construct finite rational p-adic Sylvester expansions. Lastly we show that the division algorithm approach and the given modification to Knopfmachers' algorithm yield the same output.

2. p-Adic expansions

2.1. Basics of p-adic numbers

We begin with some of the necessary basics of p-adic numbers. A more thorough exposition can be found in [4]. Let p be a prime and $\mathbb{Q}_p$ the completion of the rationals with respect to the p-adic absolute value $|\cdot|_p$ defined on $\mathbb{Q}$ by

$$|0|_p = 0 \text{ and } |r|_p = p^{-\nu} \text{ if } r = \frac{a}{b}p^{\nu}, \text{ where } p \nmid ab.$$

The exponent $\nu \in \mathbb{Z}$ is the p-adic valuation, or order, of r and will be denoted $\nu_p(r)$.

Proposition 2.1. *(Cf. [4].) Let nonzero $\zeta, \xi \in \mathbb{Q}_p$.*

1. *There exists a unique $\widehat{\zeta} \in \mathbb{Q}_p$ such that $\nu_p\left(\widehat{\zeta}\right) = 0$ and $\zeta = \widehat{\zeta}p^{\nu_p(\zeta)}$. We will call $\widehat{\zeta}$ the unit part of ζ.*
2. $\nu_p(\zeta\xi) = \nu_p(\zeta) + \nu_p(\xi)$.
3. $\nu_p(\zeta + \xi) \geq \min\{\nu_p(\zeta), \nu_p(\xi)\}$. *We have equality when $\nu_p(\zeta) \neq \nu_p(\xi)$.*

Every element of $\mathbb{Q}_p$ has the shape

$$\zeta = \sum_{n=\nu_p(\zeta)}^{\infty} c_n p^n \tag{1}$$

for $c_n \in \{0, 1, \ldots, p-1\}$. This representation is unique, however p-adic numbers can also be represented in the form

$$\zeta = a_0 + \sum_{n=1} \frac{1}{a_n} \tag{2}$$

where $a_n \in \mathbb{Z}[\frac{1}{p}]$ and the sum is either finite or converges p-adically. There are a variety of algorithms to decompose a p-adic into this form (cf. [2]). One reason to study such representations is that they can be considered the p-adic analogue to representing real numbers as Egyptian fractions, i.e. as the sum of unit fractions.

2.2. Sylvester-type series expansions of p-adic numbers

There are various methods of decomposing a rational number into an Egyptian fraction representation. One of the more naïve methods was given originally by Fibonacci and again in modern times by Sylvester [9]. This algorithm is commonly known as the Greedy Algorithm because at each inductive step of the decomposition one simply takes the largest unit fraction smaller than the value being decomposed. Later, Oppenheim [8] generalized this and other Egyptian fraction algorithms to real numbers given by their decimal representations.

The following inductive algorithm, which results in an expansion like (2), is presented by the Knopfmachers in [2] as a p-adic analogue to the Sylvester–Oppenheim algorithm on real numbers. Begin by defining the fractional part of a p-adic number ζ as in (1) by

$$\langle\zeta\rangle = \sum_{n=\nu_p(\zeta)}^{0} c_n p^n. \tag{3}$$

Algorithm 2.2. (See [2].) Let $\zeta \in \mathbb{Q}_p$. For the initial term, set $a_0 = \langle\zeta\rangle$. Then take $\zeta_1 = \zeta - a_0$ so that $\nu_p(\zeta_1) \geq 1$. Continuing as long as $\zeta_n \neq 0$, let

$$a_n = \left\langle \frac{1}{\zeta_n} \right\rangle \text{ and } \zeta_{n+1} = \zeta_n - \frac{1}{a_n}.$$

In [3] (see Proposition 5.3) it is stated that the above Sylvester-type algorithm and a similarly defined Engel-type algorithm terminate if and only if $\zeta \in \mathbb{Q}$. However in [1] Grabner and A. Knopfmacher give an example of a rational with nonterminating Engel-type algorithm. Likewise, there are rationals for which this Sylvester-type algorithm does not terminate.

Indeed, suppose $\zeta = \frac{a}{p+a} \in \mathbb{Q}$ with $p \nmid a$. Then $a_0 = 1$ and $\zeta_1 < 0$. Since all the a_n are defined to be positive, a finite sum in (2) leads to a contradiction. Although Laohakosol and Kanasri [6] give a complete characterization of the infinite Sylvester-type expansions from the Knopfmachers' algorithm that correspond to rational numbers, it is less obvious the necessary conditions under which a rational will result in a finite expansion.

2.3. A new Sylvester-type algorithm

We start by generalizing the definition in (3) in the following way. Let

$$\langle\zeta\rangle_k = \sum_{n=\nu_p(\zeta)}^{k-1} c_n p^n$$

so that $\langle\zeta\rangle_1 = \langle\zeta\rangle$. In other words, $\langle\zeta\rangle_k$ is the rational image of ζ under the mod p^k projection from $\mathbb{Q}_p$ to $\mathbb{Z}[\frac{1}{p}]$.

For real x, define the ceiling function, $\lceil x \rceil$, to be the least integer greater than x. Notice that since $\mathbb{Q}_p$ is not an ordered field, this function is not well-defined for all p-adics. However, if $\zeta \in \mathbb{Q}_p$ such that there exists an embedding of $\mathbb{Q}(\zeta)$ into $\mathbb{R}$, then the ceiling function pulls back to $\mathbb{Q}(\zeta) \subset \mathbb{Q}_p$.

We now state our modification to Algorithm 2.2.

Algorithm 2.3. Let nonzero $\zeta \in \mathbb{Q}_p$ such that there exists an embedding $\psi : \mathbb{Q}(\zeta) \to \mathbb{R}$ and $k \in \mathbb{Z}$ such that $k > -\nu_p(\zeta)$. Set $\zeta_0 = \zeta$. Inductively for $i \geq 0$ set $t_i = \left\langle \frac{1}{\zeta_i} \right\rangle_k$,

$$q_i = t_i + \left\lceil \frac{1 - t_i \psi(\zeta_i)}{p^k \psi(\zeta_i)} \right\rceil p^k,$$

and

$$\zeta_{i+1} = \zeta_i - \frac{1}{q_i}. \tag{4}$$

The algorithm terminates if any $\zeta_N = 0$.

Theorem 2.4. *Algorithm 2.3 produces a sequence of $q_i \in \mathbb{Z}[\frac{1}{p}]$ such that*

$$\zeta = \sum_{i=0} \frac{1}{q_i}$$

where the sum (if infinite) converges p-adically.

Proof. By (4), if the sum converges, it does so to ζ. It suffices to show that $|\zeta_i|_p \to 0$.

Suppose $\nu_p(\zeta_i) = s > -k$ so that $\zeta_i = \widehat{\zeta}_i p^s$. Then

$$t_i = \left\langle \frac{1}{\widehat{\zeta}_i} p^{-s} \right\rangle_k = \left\langle \widehat{\zeta}_1^{-1} \right\rangle_{k+s} p^{-s}$$

and

$$q_i = \left(\left\langle \widehat{\zeta}_1^{-1} \right\rangle_{k+s} + m p^{k+s} \right) p^{-s}$$

for some $m \in \mathbb{Z}$. Then

$$\zeta_i q_i - 1 = \widehat{\zeta}_i \left\langle \widehat{\zeta}_1^{-1} \right\rangle_{k+s} + \widehat{\zeta}_i m p^{k+s} - 1 \equiv 0 \mod p^{k+s}$$

so $\nu_p(\zeta_i q_i - 1) \geq k + s$. Then $\nu_p(\zeta_{i+1}) \geq k + s - \nu_p(q_i)$. Since $k > -s$, then $\nu_p(q_i) \geq -s$. Hence, $\nu_p(\zeta_{i+1}) \geq k + 2s > \nu_p(\zeta_i)$. Since the order of the ζ_i is strictly increasing, $|\zeta_i|_p \to 0$. $\square$

Example 2.5. Let $k = 1$ and consider $\xi \in \mathbb{Q}_7$ with $\xi^2 = \frac{1}{11}$ and $\xi \equiv 4 \bmod 7$. Then $\mathbb{Q}(\xi)$ embeds into $\mathbb{R}$ by either $\psi(\xi) = \frac{1}{\sqrt{11}}$ or $\psi(\xi) = \frac{-1}{\sqrt{11}}$. For the first choice, Algorithm 2.3 gives

$$\xi = \frac{1}{9} + \frac{7}{66} + \frac{7^3}{4709} + \frac{7^7}{72\,282\,453} + \cdots .$$

The second embedding yields

$$\xi = \frac{1}{2} + \frac{7}{12} + \frac{7^3}{617} + \frac{7^7}{1\,045\,103} + \cdots .$$

Algorithm 2.3 certainly isn't as elegant looking as Knopfmachers' Algorithm and has limitations for which p-adics it can be used on. However the importance of Algorithm 2.3 is in the following theorem.

Theorem 2.6. *Algorithm 2.3 terminates if and only if $\zeta \in \mathbb{Q}$.*

Instead of proving this theorem directly, in the following sections we will re-frame the approach to finding p-adic Sylvester expansions for rationals to mimic a more classical technique. In Section 4.3 we will show the link between the two and the proof of Theorem 2.6 will follow easily.

3. The Fibonacci–Sylvester Greedy Algorithm

The algorithm given by Fibonacci and Sylvester for Egyptian fractions of rationals can be interpreted as iterating a modified version of the classical division algorithm (cf. [7]). We review these classical methods now to provide reference and motivation for the techniques used later.

Theorem 3.1 *(Modified classical division algorithm). For all $a, b \in \mathbb{Z}$, $a > 0$, there exist unique q, $r \in \mathbb{Z}$ such that*

$$b = aq - r \tag{5}$$

with

$$0 \leq r < a. \tag{6}$$

Note that Theorem 3.1 is greedy in the sense that it finds the smallest q such that $aq > b$, i.e. so that $\frac{a}{b} > \frac{1}{q}$.

Algorithm 3.2 *(F-S Greedy Algorithm).* Let $-1 < \frac{a}{b} \in \mathbb{Q}$, with $a > 0$ and $\gcd(a, b) = 1$. Iterate Theorem 3.1 in the following way:

$$\begin{aligned} b &= aq_0 - r_0 \\ bq_0 &= r_0 q_1 - r_1 \\ &\vdots \\ bq_0 q_1 \cdots q_{i-1} &= r_{i-1} q_i - r_i \\ &\vdots \end{aligned} \tag{7}$$

The process terminates if any $r_N = 0$.

A straightforward computation (cf. [7]) then gives the following:

Theorem 3.3. *Algorithm 3.2 terminates in a finite number of steps and*

$$\frac{a}{b} = \sum_{i=0}^{N} \frac{1}{q_i}. \tag{8}$$

4. A p-adic division algorithm approach

4.1. The p^k-division algorithm

We begin the process of retracing the classical approach by generalizing Theorem 3.1. The p-adic division algorithm defined here is similar to the one given in [5] but differs considerably in the restrictions on the quotient and remainder.

Theorem 4.1 *(p^k-Division algorithm). Let p be prime and $k \in \mathbb{Z}$. For all $a, b \in \mathbb{Z}[\frac{1}{p}]$, $a > 0$, there exist unique $q, r \in \mathbb{Z}[\frac{1}{p}]$ such that*

$$b = aq - r \tag{9}$$

with

$$0 \leq r < ap^k \tag{10}$$

and

$$|r|_p \leq |ap^k|_p. \tag{11}$$

Proof. We first prove existence. Since $\widehat{a}$ and p are relatively prime, positive and negative powers of p are defined mod $\widehat{a}$. Let $\alpha = \nu_p(a)$ and $\beta = \nu_p(b)$ and take $0 \leq \overline{r} < \widehat{a}$ such that $\overline{r} \equiv -\widehat{b}p^{\beta-\alpha-k} \bmod \widehat{a}$.

(Case 1: $k > \beta - \alpha$.) For some $m \in \mathbb{Z}$ we have $\overline{r}p^{-\beta+\alpha+k} + \widehat{b} = \widehat{a}m$ which gives

$$\widehat{b}p^{\beta} = \widehat{a}p^{\alpha}mp^{\beta-\alpha} - \overline{r}p^{\alpha+k}.$$

Thus we can take $q = mp^{\beta-\alpha}$ and $r = \overline{r}p^{\alpha+k}$.

(Case 2: $k \le \beta - \alpha$.) For some $m \in \mathbb{Z}$ we have $\overline{r} + \widehat{b}p^{\beta-\alpha-k} = \widehat{a}m$ which gives

$$\widehat{b}p^{\beta} = \widehat{a}mp^{\alpha+k} - \overline{r}p^{\alpha+k}. \tag{12}$$

Thus we can take $q = mp^{k}$ and $r = \overline{r}p^{\alpha+k}$.

In both cases, since $\overline{r} < \widehat{a}$, both (10) and (11) are satisfied.

To show uniqueness, suppose that there exist q_1, r_1, q_2, r_2 satisfying (9), (10), and (11). Then

$$a(q_1 - q_2) = r_1 - r_2,$$

thus $r_1 \equiv r_2 \bmod \widehat{a}$. Also $r_1 \equiv r_2 \bmod p^{\alpha+k}$ since by assumption $\nu_p(r_i) \ge \nu_p(ap^k)$. Therefore, $r_1 \equiv r_2 \bmod ap^k$. Since both are between 0 and ap^k, $r_1 = r_2$. Thus $q_1 = q_2$ and we have uniqueness as desired. $\square$

Note that the value $\overline{r}$ may have nontrivial order. When $\nu_p(\overline{r}) \ge 1$ we say a jump occurs. Of course, if $p > \widehat{a}$ then $\overline{r} = \widehat{r}$ and there is no jump.

Also notice that one recovers Theorem 3.1 from Theorem 4.1 by setting $k = 0$ (or $p = 1$) and (redundantly) using the standard absolute value in (11). Additionally Theorem 4.1 generalizes the classical algorithm in the form of the following corollary.

Corollary 4.2. *Suppose $a, b, p, k \in \mathbb{Z}$ with p prime and $k \le \nu_p(b) - \nu_p(a)$. Let q_p denote the quotient from the p^k-division algorithm on a and b and let q_∞ denote the quotient from the modified classical division algorithm on ap^k and b. Then $q_p = q_\infty p^k$.*

Proof. Since $k \le \nu_p(b) - \nu_p(a)$, Case 2 in the proof of Theorem 4.1 applies and $q_p = mp^k$. So it suffices to show that $m = q_\infty$. By (12),

$$0 \le ap^k m - b < ap^k.$$

Thus (5) and (6) are satisfied. $\square$

In addition, note that the p^k-division algorithm can be extended to rational a and b in the following way. Suppose $b = \frac{s}{t}$ and $a = \frac{u}{v}$ for $r, s, t, u \in \mathbb{Z}$. To find the quotient and remainder, clear denominators and compute the desired division algorithm on $b' = sv$ and $a' = ut$ to find q' and r'. Then $q = q'$ and $r = \frac{r'}{vt}$ satisfy (9), (10) and (11). However, since we are mostly interested in the quotient $\frac{a}{b}$, we will assume $a, b \in \mathbb{Z}[\frac{1}{p}]$.

4.2. The p^k-Greedy Algorithm

Since we now have a generalization of Theorem 3.1, we can substitute it into the iterative process of Algorithm 3.2.

Algorithm 4.3 *(p^k-Greedy Algorithm).* Let $\frac{a}{b} \in \mathbb{Q}$, with $a > 0$, $\gcd(a,b) = 1$ and $k > -\nu_p\left(\frac{a}{b}\right)$. Iterate Theorem 4.1 as in (7). The process terminates if any $r_N = 0$.

The condition $k > -\nu_p\left(\frac{a}{b}\right)$ plays the synonymous role to $-1 < \frac{a}{b}$ in the F-S Greedy Algorithm: it prevents the division algorithm from returning a quotient equal to 0. This restriction on k is actually stronger than it needs to be for this alone. However, for reasons related to Corollary 4.2 and explained further below, $k \le -\nu_p\left(\frac{a}{b}\right)$ is generally undesirable. The Knopfmachers avoided this obstruction by defining their initial a_0 outside of the inductive pattern so that $\nu_p(\zeta_1) \ge 1$. A similar strategy could be used here as well, though we find it more desirable to instead simply choose a different k value.

Theorem 4.4. *Algorithm 4.3 terminates after a finite number of steps.*

Proof. As opposed to Algorithm 3.2, now we have $q_i, r_i \in \mathbb{Z}[\frac{1}{p}]$ instead of $\mathbb{Z}$. However

$$\widehat{r}_i \le \overline{r}_i < \widehat{r}_{i-1} \in \mathbb{Z},$$

so the sequence of $\widehat{r}_i$'s is a decreasing sequence of positive integers much like the classical remainders. □

Since the algorithm terminates, again we get that (8) holds.

Example 4.5. Consider $\frac{a}{b} = \frac{473}{25}$. Performing the 3-Greedy Algorithm yields,

$$\begin{aligned}
25 &= 473 \cdot 2 - 921 \\
25 \cdot 2 &= 921 \cdot \frac{5}{3} - 1485 \\
25 \cdot 2 \cdot \frac{5}{3} &= 1485 \cdot \frac{115}{81} - 2025 \\
25 \cdot 2 \cdot \frac{5}{3} \cdot \frac{115}{81} &= 2025 \cdot \frac{1150}{19\,683} - 0
\end{aligned}$$

and thus

$$\frac{473}{25} = \frac{1}{2} + \frac{3}{5} + \frac{3^4}{115} + \frac{3^9}{1150}.$$

However, performing the 3^4-Greedy Algorithm results in the sum

$$\frac{473}{25} = \frac{1}{23} + \frac{3^4}{5635} + \frac{3^{12}}{28\,175}.$$

Applying Corollary 4.2 to Algorithm 4.3 gives another relationship between the classical and p-adic algorithms.

Corollary 4.6. *Suppose $\frac{a}{b} > 0$ and the situation of $k \leq -\nu_p\left(\frac{a}{b}\right)$ holds. If no jumps occur than each term of the p^k-Greedy Algorithm for $\frac{a}{b}$ is equal to the corresponding term in the F-S Greedy Algorithm on $\frac{ap^k}{b}$ divided by p^k.*

Example 4.7. Consider the fraction $\frac{a}{b} = \frac{5}{121} = \frac{5}{11^2}$. If we take $p = 11$ and $k = 1$, then $k \leq -\nu_{11}\left(\frac{a}{b}\right) = 2$. Applying the 11-Greedy Algorithm to $\frac{5}{121}$ encounters no jumps and gives

$$\frac{5}{121} = \frac{1}{33} + \frac{1}{99} + \frac{1}{1089}.$$

On $\frac{5 \cdot 11}{121} = \frac{5}{11}$, the F-S Greedy Algorithm yields

$$\frac{5}{11} = \frac{1}{3} + \frac{1}{9} + \frac{1}{99}.$$

The restriction on jumps is sufficient but not necessary. For example, the 3-Greedy Algorithm on $\frac{22}{45}$ encounters a jump, yet the relation above to the classical algorithm on $\frac{22}{15}$ still holds.

Also notice that, again with $k \leq -\nu_p\left(\frac{a}{b}\right)$, if $\frac{ap^k}{b} > 1$ then the F-S Greedy Algorithm returns quotients $q_i = 1$ until the remaining value to be decomposed is less than 1. Hence, by Corollary 4.6 the p^k-Greedy Algorithm on $\frac{a}{b}$ produces a finite string of terms equal to $\frac{1}{p^k}$. For these reasons we only consider the cases where $k > -\nu_p\left(\frac{a}{b}\right)$.

This condition is minor, though, in the grand scheme. Each inductive step of Algorithm 4.3 is independent of each other with respect to the value of k. So if the desired k fails the order criteria, it is possible to temporarily use a sufficiently large k' in the initial step(s) and then switch back to the desired k value once the corresponding orders become large enough. In this way finite p-adic Sylvester expansions can be found for all rationals.

4.3. Connection between approaches

We are now in the position to return to the modification of the Knopfmachers' algorithm given in Section 2.

Theorem 4.8. *For $\zeta \in \mathbb{Q}$, Algorithm 2.3 and Algorithm 4.3 produce the same output.*

Proof. Suppose $\zeta_i \in \mathbb{Q}$ for some $i \geq 0$. Then $\zeta_i = \frac{a}{b}$ for $a, b \in \mathbb{Z}$. Algorithm 2.3 produces

$$q_i = \left\langle \frac{b}{a} \right\rangle_k + \left\lceil \frac{b/a - \langle b/a \rangle_k}{p^k} \right\rceil p^k.$$

Let $r = aq_i - b$ and $\sigma_k = \left\langle \frac{b}{a} \right\rangle_k - \frac{b}{a}$. Then

$$0 \leq \frac{r}{a} = \sigma_k + \left\lceil \frac{-\sigma_k}{p^k} \right\rceil p^k < p^k.$$

Since $\nu_p(\sigma_k) \geq k$, then $|r|_p \leq |ap^k|_p$. Hence both (10) and (11) are satisfied. □

The proof of Theorem 2.6 is simply a combination of Theorem 4.4 and Theorem 4.8. Further the condition of $k > -\nu_p(\zeta)$ in Algorithm 2.3 is explained in light of Corollary 4.6. Indeed, for negative or irrational ζ with $k \leq -\nu_p(\zeta)$, a statement analogous to Corollary 4.6 holds and thus Algorithm 2.3 produces a p-adically divergent result. Though, as mentioned above, this can be worked around by temporarily using alternate k-values.

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This work is dedicated to my children, Gedion and Eomji.

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MINIMAL POLYNOMIALS OF SINGULAR MODULI

ERIC ERRTHUM

This paper is dedicated to my wife Kate.

ABSTRACT. Given a properly normalized parametrization of a genus-0 modular curve, the complex multiplication points map to algebraic numbers called singular moduli. In both cases there are known algorithms for algebraically computing the rational norms of the singular moduli without relying on the the recognition of a decimal or p-adic expansion as a rational number. We demonstrate a method of extending these norm algorithms to determine the minimal polynomial of the singular moduli below a discriminant threshold. We then use these minimal polynomials to compute the algebraic abc-ratios for the singular moduli.

1. INTRODUCTION

The classical j-function is a modular function that has been studied since the late 1800s by the likes of Gauss, Hermite, Dedekind and Klein. Its values in the upper half-plane correspond to isomorphism classes of elliptic curves and at points corresponding to CM curves, the values of the j-function are called singular moduli. A Shimura curve is a generalization of the classical modular curve. Again there is a notion of singular moduli and in both cases singular moduli are algebraic numbers. Singular moduli have been computed using analytic methods [1] and in some cases [3] they have been computed algebraically. In this paper, we present a strictly algebraic method for determining the defining polynomials for a large class of singular moduli (both classical and Shimura) utilizing pre-existing algorithms that output their rational norms. It is worth noting that our method never relies on the recognition of a decimal or p-adic expansion as a rational number. This method works for singular moduli whose degree is strictly less than the number of rational singular moduli on the curve.

For instance, on the classical modular curve we compute that

$$\begin{aligned} &j\left(\frac{1}{2}(1+i\sqrt{39})\right) \\ &\quad = -\frac{27}{2}\left(6139474 + 1702799\sqrt{13} + 147\sqrt{39\left(89453213 + 24809858\sqrt{13}\right)}\right). \end{aligned}$$

Perhaps more importantly, we also compute minimal polynomials of singular moduli on the Shimura curve of discriminant 6. For example, the minimal polynomial of

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the singular modulus of discriminant 244 is

$$x^3 - \frac{2^2 3^7 31 \cdot 67 \cdot 37223 \cdot 235849}{17^6 29^6} x^2 + \frac{2^4 3^{14} 151 \cdot 1187 \cdot 163327}{17^6 29^6} x - \frac{2^6 3^{21} 19^4}{17^6 29^6}.$$

In Section 2 we give a brief review of modular curves both in the classical and Shimura cases. In Section 3 we review the Gross-Zagier formula for the algebraic norm of the difference of two singular moduli and demonstrate how it can be used to algebraically determine the minimal polynomial for a finite collection of singular moduli. In Section 4 we adapt this method to the Shimura curve defined by the quaternion algebra of discriminant 6. As suggested in [8], singular moduli have the potential to be good examples related to the *abc*-conjecture, so in Section 5 we review the basics for computing the algebraic *abc*-ratio and present the unencouraging data obtained from the singular moduli in both cases. Finally, in Section 6 we discuss the limitations of the method and ways to potentially overcome them.

Computations for this paper were performed in Mathematica (v7.0.1.0) and MAGMA (v2.13-14) provided by [12].

2. Modular curves and Singular Moduli

In this section we provide the basics of modular curves and singular moduli. For the purposes of this paper, we ignore level considerations. For a more complete description of these concepts, see [10], [16], [17], [18].

The classical modular curve $\mathcal{X}_1^*$ is the one-point compactification of the Riemann surface $\mathrm{GL}_2(\mathbb{Z})\backslash\mathfrak{h}^\pm$ where $\mathfrak{h}^\pm = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$. As $\mathcal{X}_1^*$ is a surface of genus 0, it is isomorphic to $\mathbb{P}^1$. Since any map $j : \mathcal{X}_1^* \to \mathbb{P}^1$ must be invariant under the $\mathrm{GL}_2(\mathbb{Z})$ action, it suffices to define the function by giving its values at three points. In the classical setting, the points $i = \sqrt{-1}$, $\omega = \frac{1+i\sqrt{3}}{2}$, and ∞ are chosen to be sent to 1728, 0, and ∞, respectively. This choice yields what is now commonly referred to as the j-function and it has a known Fourier series expansion (where $\mathbf{q} = e^{2\pi i \tau}$):

$$j(\tau) = \frac{1}{\mathbf{q}} + 744 + 196884\mathbf{q} + \cdots \in \frac{1}{\mathbf{q}}\mathbb{Z}[[\mathbf{q}]].$$

The points of $\mathcal{X}_1^*$ correspond to isomorphism classes of elliptic curves. Some of these classes support an extra endomorphism and are called CM curves. Hence, the corresponding points on $\mathcal{X}_1^*$ are called CM points. In the classical case, the CM points of $\mathcal{X}_1^*$ are the imaginary roots of quadratic equations. When τ is a CM point, $j(\tau)$ is called a singular moduli and is an algebraic integer.

A Shimura curve is a generalization of the modular curve. Let B be the indefinite quaternion algebra over $\mathbb{Q}$ with discriminant $D > 1$ and let $\Gamma^* = N_{B^\times}(\mathcal{O}) \subset B^\times$ be the normalizer of a maximal order $\mathcal{O} \subset B$. Since there is an algebra embedding $B \hookrightarrow \mathrm{M}_2(\mathbb{R})$, the discrete group Γ^* embeds into $\mathrm{GL}_2(\mathbb{R})$ and hence acts on $\mathfrak{h}^\pm$. The Shimura curve $\mathcal{X}_D^*$ is then given as

$$\mathcal{X}_D^* = \Gamma^*\backslash\mathfrak{h}^\pm.$$

When B is a division algebra, $\mathcal{X}_D^*$ is a compact Riemann surface without cusps [11].

Points on a Shimura curve can be identified with certain 2-dimensional abelian varieties and again there is the notion of CM points. As in the case of the modular curve, one may normalize a generator for the function field, $t : \mathcal{X}_D^* \to \mathbb{P}^1$, such that the images of CM points under the generator are all algebraic over $\mathbb{Q}$. However, since $\mathcal{X}_D^*$ has no cusps, such a map does not have a $\mathbf{q}$-expansion and calculations are more difficult than in the classical case.

3. Minimal polynomials of classical singular moduli

Let τ_1 and τ_2 be CM points on $\mathcal{X}_1^*$ with relatively prime negative discriminants $-d_1$ and $-d_2$. Let $h(-d_i)$ be the class number of the quadratic order of discriminant $-d_i$. If $-d_1, -d_2 < -4$, then $j(\tau_1) - j(\tau_2)$ is an algebraic integer over $\mathbb{Q}$ of degree $h(-d_1)h(-d_2)$. Let ω_i be the number of roots of unity in that order. Recall that in general the norm of an algebraic number is given by

$$|\alpha|_{\mathbb{Q}} = \prod_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})} \sigma(\alpha).$$

The main result of [10] is that the norm $|j(\tau_1) - j(\tau_2)|_{\mathbb{Q}}$ is given by

$$|j(\tau_1) - j(\tau_2)|_{\mathbb{Q}}^{\frac{2}{\omega_1 \omega 2}} = \pm \prod_{\substack{x, n, n' \in \mathbb{Z} \\ n, n' > 0 \\ x^2 + 4nn' = d_1 d_2}} n^{\epsilon(n')} \tag{1}$$

where $\epsilon(p)$ is a completely multiplicative function defined on primes p with $\left(\frac{d_1 d_2}{p}\right) \neq 1$ by

$$\epsilon(p) = \begin{cases} \left(\frac{-d_1}{p}\right) & \text{if } \gcd(p, -d_1) = 1, \\ \left(\frac{-d_2}{p}\right) & \text{if } \gcd(p, -d_2) = 1. \end{cases}$$

The computational benefit of (1) is that very little actually needs to be known about the algebraic integers $j(\tau_i)$ to perform the calculation.

We now proceed to describe our algorithm for computing the minimal polynomials of singular moduli. Suppose that $r \in \mathbb{Q}$ and τ is a CM point with discriminant $-d$. Then $\mathbb{Q}(r - j(\tau)) = \mathbb{Q}(j(\tau))$. Let $G = \mathrm{Gal}(\mathbb{Q}(j(\tau))/\mathbb{Q})$. Then

$$\begin{aligned} |r - j(\tau)|_{\mathbb{Q}} &= \prod_{\sigma \in G} \sigma(r - j(\tau)) \\ &= \prod_{\sigma \in G} r - \sigma(j(\tau)) \\ &= M_{j(\tau)}(r) \end{aligned}$$

where $M_\alpha(x) \in \mathbb{Q}[x]$ denotes the minimal polynomial of the algebraic number α.

Since the degree of $M_{j(\tau)}(r)$ is $h(-d)$, it suffices to know the value of the left-hand-side for $h(-d) + 1$ values of r to interpolate the polynomial. Hence, we only need to find $h(-d) + 1$ rational $j(\tau_i)$ with $\gcd(d_i, d) = 1$ and then use (1) to find (up to an issue of sign which we discuss in the example below) the value of $M_{j(\tau)}(j(\tau_i))$.

Unfortunately, the number of rational singular moduli is finite: they occur exactly at the CM points of discriminants -4, -8, -3, -7, -11, -19, -43, -67, and -163. This fact limits the previous method to only being possible for singular moduli of degree 8 or less. The largest achievable case is the singular moduli of discriminant -5923.

Also, it should be noted that this algorithm is in no way optimal or the most efficient from a computational viewpoint. Many of these facts can be discovered through much quicker analytic, floating-point calculations and the recognition of approximations. The importance of our method lies in its purely algebraic nature. This, in turn, allows it it be implemented in the Shimura curve case, as we will see in Section 4.

3.1. **Example:** $j(\frac{1}{2}(1+i\sqrt{39})$. Fix $\tau = \frac{1}{2}(1+i\sqrt{39})$. Then $d = 39$ and $h(-d) = 4$. Since $-d$ is relatively prime to the discriminants -4, -8, -7, -11, and -19, we can use (1) to compute the following:

$$\begin{aligned}
|j(\tau) - j(1+i)| &= 3^{12}7^8 19^4 23^2, \\
|j(\tau) - j(1+i\sqrt{2})| &= 7^8 13^2 23^2 29 \cdot 31^2 37^2 53, \\
|j(\tau) - j(\tfrac{1}{2}(1+i\sqrt{7}))| &= 3^{12}7^4 13^2 17^3 19^2 31^2, \\
|j(\tau) - j(\tfrac{1}{2}(1+i\sqrt{11}))| &= 7^8 13^2 17^3 19^2 29^2 101 \cdot 107, \\
|j(\tau) - j(\tfrac{1}{2}(1+i\sqrt{19}))| &= 3^{12}13^2 19^2 29 \cdot 31^2 37^2 \cdot 53 \cdot 113 \cdot 173 \cdot 179.
\end{aligned}$$

This gives the following 5 points on the curve $y = |M_{j(\tau)}(x)|$:

$$\begin{aligned}
(x_1, y_1) &= (12^3,\ 3^{12}7^8 19^4 23^2), \\
(x_2, y_2) &= (20^3,\ 7^8 13^2 23^2 29 \cdot 31^2 37^2 53), \\
(x_3, y_3) &= (-15^3,\ 3^{12}7^4 13^2 17^3 19^2 31^2), \\
(x_4, y_4) &= (-32^3,\ 7^8 13^2 17^3 19^2 29^2 101 \cdot 107), \\
(x_5, y_5) &= (-96^3,\ 3^{12}13^2 19^2 29 \cdot 31^2 37^2 \cdot 53 \cdot 113 \cdot 173 \cdot 179).
\end{aligned}$$

Since the absolute value obscures the polynomial's outputs, we fall back on an exhaustive search through the 16 different possibilities of choices for the signs of the y_i. Then using standard curve fitting we find that only one choice of signs, namely

$$(x_1, y_1),\ (x_2, y_2),\ (x_3, -y_3),\ (x_4, -y_4),\ (x_5, -y_5),$$

yields a monic polynomial. It is

$$\begin{aligned}
M_{j(\tau)}(x) &= x^4 + 331531596x^3 - 429878960946x^2 + 109873509788637459x \\
&\quad + 20919104368024767633 \\
&= x^4 + 2^2 3^3 11 \cdot 29 \cdot 9623x^3 - 2 \cdot 3^6 41 \cdot 1303 \cdot 5519x^2 \\
&\qquad + 3^{12}103 \cdot 2007246533x + 3^{15}17^3 23^3 29^3.
\end{aligned}$$

Since this is only a degree 4 polynomial over the reals, it is solvable by radicals and yields 4 roots. By comparing these to decimal approximations for $j(\tau)$ computed analytically we find that

$$\begin{aligned}
&j\left(\frac{1}{2}(1+i\sqrt{39})\right) \\
&= -\frac{27}{2}\left(6139474 + 1702799\sqrt{13} + 147\sqrt{39\left(89453213 + 24809858\sqrt{13}\right)}\right).
\end{aligned}$$

Note that as a corollary to this computation we can compute the following algebraic norm: $|j(\tau)| = 3^{15}17^3 23^3 29^3$. We could not have computed this from (1) alone since the discriminants -39 and -3 are not relatively prime.

4. Minimal polynomials of singular moduli from $\mathcal{X}_6^*$

For the Shimura curve $\mathcal{X}_6^*$, the idea is essentially the same. However, since there are no cusps on $\mathcal{X}_6^*$, there is no **q**-expansion for the function $t : \mathcal{X}_6^* \to \mathbb{P}^1$. Hence the analog to (1) in the Shimura curve case is considerably more complicated to compute. In [19], Schofer uses Whittaker coefficients to attain an explicit formula for the average of a Borcherds form over CM points associated to a quadratic form of signature $(n, 2)$. In the second half of [20] Schofer shows that this generalizes the Gross-Zagier formula in the classical modular curve case. In [6] an explicit modular form whose Taylor coefficients play the role of the q-expansion is given and Schofer's techniques are then applied to the Shimura curves $\mathcal{X}_6^*$ and $\mathcal{X}_{10}^*$ to algebraically compute the norms of CM points on these curves. We now further extend these methods in a manner similar to the previous section to compute the minimal polynomials of the singular moduli on $\mathcal{X}_6^*$.

An area calculation [5] shows that $\mathcal{X}_6^*$ has genus 0 and so there exists a parameterization $t : \mathcal{X}_6^* \xrightarrow{\sim} \mathbb{P}^1$ over $\mathbb{Q}$. Such a map giving the isomorphism is only well defined up to a PGL_2 action on $\mathbb{P}^1$. However, the map is uniquely determined once the value at three points of $\mathcal{X}_6^*$ are chosen. Thus it suffices to assign its value at three CM points. Let s_d denote the CM point of discriminant d. In [5], Elkies shows that the triangle group Γ^* is generated by s_3, s_4 and s_{24} and makes the arbitrary, albeit natural, choice of defining the function's zeros and poles at those points. In [6], Errthum follows suit, defining a map such that

$$\begin{aligned} t(s_3) &= \infty, \\ t(s_4) &= 0, \\ t(s_{24}) &= 1. \end{aligned} \tag{2}$$

Then, using Schofer's techniques, he determines $|t(s_d)|$, the rational norm of the singular moduli of $\mathcal{X}_6^*$.

We can now construct the minimal polynomial for $t(s_d)$ much as we did for $j(\tau)$ in the previous section. In general, the algebraic degree of a singular modulus on a Shimura curve, $h(d)$, can be computed using genus theory [5]. For $\mathcal{X}_6^*$, there are exactly 27 rational singular moduli. Let ζ_d denote a rational singular moduli (e.g., ζ_d only makes sense for the 27 specific values of d for which the singular moduli is, in fact, rational). Then we can define a new parametrization t_d by making different choices than those in (2). Specifically, if we choose instead, for $d \neq 3, 4$,

$$\begin{aligned} t_d(s_3) &= \infty, \\ t_d(s_d) &= 0, \\ t_d(s_4) &= \zeta_d, \end{aligned} \tag{3}$$

it yields the relationship

$$t_d(x) = \zeta_d - t(x). \tag{4}$$

Note that although (3) alone does not uniquely define t_4, in light of (4) we can define $t_4 = -t$.

Again, for a given $s_{d'}$ we can compute $|t_d(s_{d'})|$ in two different ways—via the calculations in [6] and by definition:

$$\begin{aligned}|t_d(s_{d'})| &= \left|\prod_\sigma \sigma(t_d(s_{d'}))\right| \\ &= \left|\prod_\sigma \sigma(\zeta_d - t(s_{d'}))\right| \\ &= \left|\prod_\sigma (\zeta_d - \sigma(t(s_{d'})))\right| \qquad \text{(since } \zeta_d \text{ is rational)} \\ &= |M_{t(s_{d'})}(\zeta_d)|.\end{aligned}$$

Repeating this for $h(d')+1$ choices of r_d gives us sufficiently many points to exhaustively search the $2^{h(d')}$ possibilities for the monic polynomial $M_{t(s_{d'})}(x)$. Notice that since the calculations in [6] are not constrained by a property analogous to the relative primeness of discriminants, this method of calculation works for any singular modulus with $h(d') \leq 26$. This yields a much larger collection of points than in the classical case.

4.1. **Example:** $t_6(s_{244})$**.** We now consider the case of the CM point $s_{244} \in \mathcal{X}_6^*$. Genus theory shows that the image of s_{244} is an algebraic number of degree 3. Since the norm calculator in the Shimura curve case does not require us to work with relatively prime discriminants, we can use the algorithms in [6] to compute the following:

$$\begin{aligned}|t_4(s_{244})| &= \frac{2^6 3^{21} 19^4}{17^6 29^6}, \\ |t_{24}(s_{244})| &= \frac{19^4 37^2 47^2 61}{17^2 29^6}, \\ |t_{40}(s_{244})| &= \frac{3^{22} 83 \cdot 101 \cdot 107 \cdot 163}{5^9 17^2 29^4}, \\ |t_{52}(s_{244})| &= \frac{2^{18} 3^{23} 37^2 103 \cdot 131 \cdot 179 \cdot 199 \cdot 263}{5^{18} 17^6 29^6}.\end{aligned}$$

This gives the following 4 points on the curve $y = |M_{t(s_{d'})}(x)|$:

$$\begin{aligned}(x_1, y_1) &= \left(0, \frac{2^6 3^{21} 19^4}{17^6 29^6}\right), \\ (x_2, y_2) &= \left(1, \frac{19^4 37^2 47^2 61}{17^2 29^6}\right), \\ (x_3, y_3) &= \left(-\frac{3^7}{5^3}, \frac{3^{22} 83 \cdot 101 \cdot 107 \cdot 163}{5^9 17^2 29^4}\right), \\ (x_4, y_4) &= \left(\frac{2^2 3^7}{5^6}, \frac{2^{18} 3^{23} 37^2 103 \cdot 131 \cdot 179 \cdot 199 \cdot 263}{5^{18} 17^6 29^6}\right).\end{aligned}$$

Using standard curve fitting, there are 8 different possibilities depending on the choices of sign for y_i. Only one choice of signs, namely

$$(x_1, -y_1),\ (x_2, y_2),\ (x_3, -y_3),\ (x_4, y_4),$$

yields a monic polynomial, which is

$$\begin{aligned} M_{t(s_{d'})}(x) &= x^3 - \frac{159511016412629892}{14357588953446649}x^2 + \frac{2240284633411688496}{14357588953446649}x \\ &\quad - \frac{87245036145162432}{14357588953446649} \\ &= x^3 - \frac{2^2 3^7 31 \cdot 67 \cdot 37223 \cdot 235849}{17^6 29^6}x^2 + \frac{2^4 3^{14} 151 \cdot 1187 \cdot 163327}{17^6 29^6}x - \frac{2^6 3^{21} 19^4}{17^6 29^6}. \end{aligned}$$

This is only a degree 3 polynomial over the rationals and is thus solvable by radicals. As in the classical case, a floating point approximation of the CM point is required to determine which of the 3 roots it is. Although it is not easy to achieve an approximation by working complex analytically, Greenberg [9] has demonstrated an approach that uses the Ceredník-Drinfeld p-adic uniformization to compute a p-adic approximation.

5. Algebraic abc-ratios of singular moduli

Oesterle and Masser's well-known abc-conjecture states that for relatively prime positive integers that satisfy $a + b = c$ there is a bound on how large c can be in terms of the product of all the primes involved. More explicitly the conjecture asserts that given any $\epsilon > 0$ there is a constant C_ϵ such that

$$c \leq C_\epsilon(\mathrm{rad}(abc))^{1+\epsilon}$$

where $\mathrm{rad}(n)$ is the product of all prime divisors of n.

Taking the constant as 1, one can measure the quality of an abc-example by the necessary size of ϵ. For this reason we consider the abc-ratio

$$\alpha(a,b,c) = \frac{\ln(c)}{\ln(\mathrm{rad}(abc))}.$$

To this date, the largest known abc-ratio is

$$\alpha(2, 3^{10}109, 23^5) \approx 1.62991.$$

For comparison, the median value of $\alpha(a,b,c)$ for $1 \leq a,b \leq 100$ is approximately 0.429 with the maximum being $\alpha(1,80,81) \approx 1.29203$. A standard threshold for the quality is 1.4 [4], [7], [14], so an abc-example with $\alpha(a,b,c) > 1.4$ is called good.

Vojta [21] generalized the abc-conjecture to number fields in the following way (as described in [14]). Let K be an algebraic number field and let V_K denote the set of primes on K. Then any $v \in V_K$ gives an equivalence class of nontrivial norms on K (finite or infinite). Let $||x||_v = |P|^{-v_P(x)}$ if v is a prime defined by a prime ideal P of the ring of integers O_K in K and v_P is the corresponding valuation, where $|\cdot|$ is the absolute norm. Let $||x||_v = |g(x)|^e$ for all nonconjugate embeddings $g : K \to \mathbb{C}$ with $e = 1$ if g is real and $e = 2$ if g is complex. Define the height of any triple a, b, $c \in K^\times$ to be

$$H_K(a,b,c) = \prod_{v \in V_K} \max(||a||_v, ||b||_v, ||c||_v),$$

and the radical (or conductor) of (a,b,c) by

$$\mathrm{rad}_K(a,b,c) = \prod_{P \in I_K(a,b,c)} |P|,$$

where $I_K(a,b,c)$ is the set of all prime ideals P of O_K for which $||a||_v$, $||b||_v$, $||c||_v$ are not equal. Let $D_{K/\mathbb{Q}}$ denote the discriminant of K.

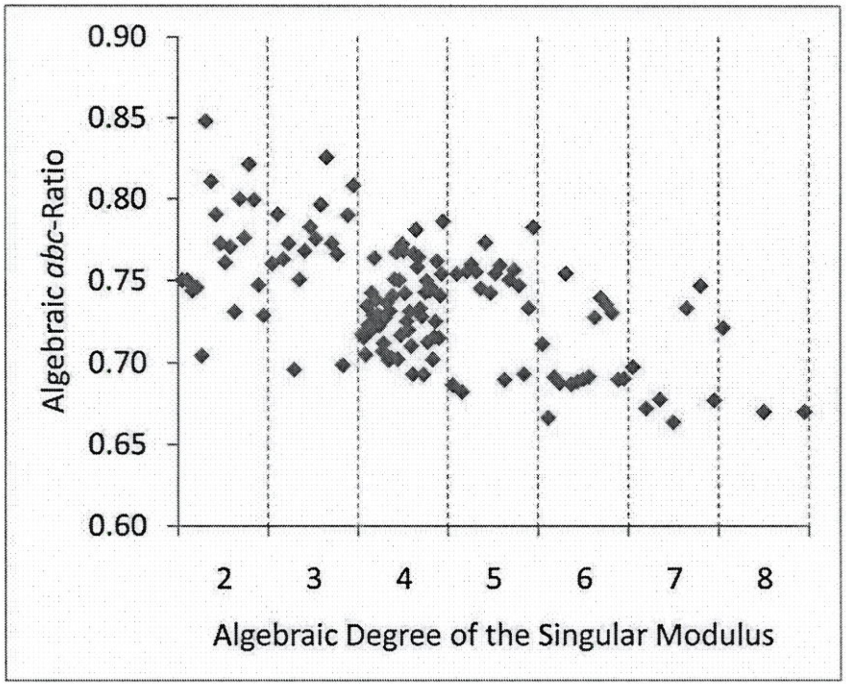

FIGURE 1. Algebraic *abc*-ratio for classical singular moduli

The (uniform) algebraic *abc*-conjecture then states that for any $\epsilon > 0$, there exists a positive constant C_ϵ such that for all a, b, $c \in K^\times$ satisfying $a+b+c=0$, we have

$$H_K(a,b,c) < C_\epsilon^{[K:\mathbb{Q}]}(|D_{K/\mathbb{Q}}|\mathrm{rad}_K(a,b,c))^{1+\epsilon}.$$

Again, constraining the constant leads to a measure of the quality of an algebraic *abc*-example given by the algebraic *abc*-ratio

$$\gamma(a,b,c) = \frac{\ln(H_K(a,b,c))}{\ln(D_{K/\mathbb{Q}}) + \ln(\mathrm{rad}_K(a,b,c))}.$$

The largest known algebraic *abc*-ratio is

$$\gamma(w,(w+1)^{10}(w-1),2^9(w+1)^5) \approx 2.029229$$

where $w^2 - w - 3 = 0$. Since algebraic *abc*-ratios are in general slightly larger than *abc*-ratios, any example with $\gamma(a,b,c) > 1.5$ is considered good [2], [14].

In [8] the authors suggest that there might exist a large number of good *abc*-examples in the collection of singular moduli. For instance, (1) indicates that the norms of singular moduli are small primes to large powers. In [5] the *abc*-ratios of the rational singular moduli on $\mathcal{X}_6^*$ are computed, noting that none of them are close to being good. In fact, most of the rational singular moduli on $\mathcal{X}_6^*$ have an *abc*-ratio less than 1. Since the algebraic *abc*-ratio is Galois-invariant, it is sufficient to know only the minimal polynomial for the algebraic number involved. Thus, with the minimal polynomial available for singular moduli of the classical modular curve and the Shimura curve $\mathcal{X}_6^*$, we can continue this search by computing their algebraic *abc*-ratios. Figure 1 presents a plot of $\gamma(a,b,c)$ versus $[K:\mathbb{Q}]$ for the classical singular moduli. Figure 2 does the same for the Shimura curve $\mathcal{X}_6^*$. (Note: Data points with common degree were lexicographically ordered according to the coefficients of the minimal polynomial. Not all data points were available for the classical singular moduli due to software reaching computational limits. For $\mathcal{X}_6^*$, all data points of singular moduli up to the discriminant of 4744 are plotted.)

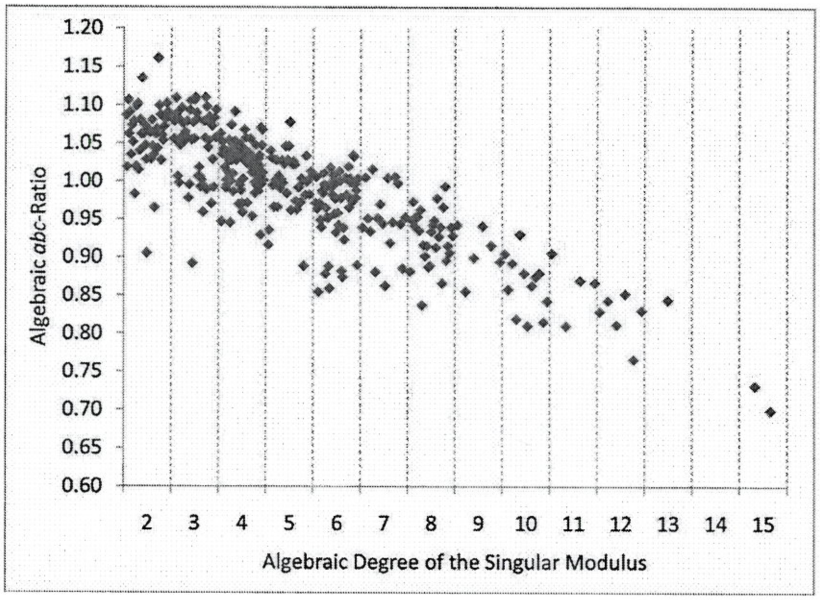

FIGURE 2. Algebraic abc-ratio for $\mathcal{X}_6^*$ singular moduli

Similar to [5], in neither case are any good abc-examples found. Furthermore, the abc-ratios appear to be following a decreasing trend as the degree increases, putting doubt on the likelihood of finding good high-degree examples in singular moduli. In fact, it may be possible to use the general formulas in [19] to construct lower and upper bounds on the algebraic abc-ratios of singular moduli.

6. Extensions and Limitations

As noted in Section 3, there is a definite cut-off in the degree of the minimal polynomials that are able to be calculated. This cut-off is dictated by the number of rational CM points on the classical modular curve and is a by-product of the Gross-Zagier formula's limitation to only compute rational norms. If one had a method of computing the norm down to any extension $F/\mathbb{Q}$ of $j(\tau_1) - j(\tau_2)$, then the methods in this paper could be salvaged in the following way. Fix a CM point τ of class number h and find the smallest extension field F such that $j(\tau_i) \in F$ for at least $h+1$ CM points. Then the minimal polynomial for τ could be interpolated over F. For example, in the classical case there are six additional singular moduli in $\mathbb{Q}(\sqrt{5})$. Knowing a Gross-Zagier formula for this field would allow the methods of this paper to extend to finding the minimal polynomials over $\mathbb{Q}(\sqrt{5})$ of singular moduli of degree 14 or less. Likewise, extending the generalization of Gross-Zagier given by Schofer to extension fields would allow the methods shown here to apply to higher degree singular moduli of Shimura curves.

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The Mathematics Teacher article

Dick J. Smith and Eric F. Errthum

Finding Skewed Lattice Rectangles

Many mathematics instructors attempt to insert guided exploration into their courses. However, exploration tasks frequently come across to students as contrived, pertinent only to the most recently covered section of the textbook. In addition, students usually assume that the teacher already knows the answers to these explorations.

The true benefit of guided exploration comes when students develop a real sense of ownership of a problem, work on it outside class, and see the teacher as a companion on their journey of exploration. Such was the case with the undergraduate modern geometry class taught by coauthor Smith. We will show how a specific problem intended to teach a specific concept grew into a larger exploration, with connections between algebra and geometry.

Delving Deeper offers a forum for classroom teachers to share the mathematics from their own work with the journal's readership; it appears in every issue of *Mathematics Teacher*. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more background information on the department and guidelines for submitting a manuscript, visit http://www.nctm.org/publications/content.aspx?id=10440#delving.

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THE PROBLEM

Ask carpenters how they know whether the corners of the rectangular frame they constructed are "square," and they will respond that they simply check that the diagonals have the same measure. This ancient concept, which predates the ideas of analytic geometry and slope, can be stated as a theorem of geometry: If the diagonals and both pairs of opposing sides of a quadrilateral are congruent, the quadrilateral is a rectangle.

This geometry theorem was the topic at hand the day Smith asked his students to verify that the four coordinates M (5, –7), N (1, –13), O (28, –31), and P (32, –25) were the vertices of a rectangle (see **fig. 1**). Students were familiar with the distance formula and computed the lengths of the four sides and the two diagonals. Smith had used this problem for many years, but this time something different happened. After class that day, a student stopped by to discuss the problem. Tony's curiosity was piqued when he checked the diagonal measures and found that

$$\begin{aligned} MO &= \sqrt{23^2+24^2} \\ &= NP = \sqrt{31^2+12^2} = \sqrt{1105}. \end{aligned}$$

He did not expect to see two different pairs of squared integers yield the same number. By reflecting on the solution of a routine problem, Tony became motivated to explore the problem further.

Pólya (2004) is explicit about the opportunities for learning mathematics evoked by this simple act of "thinking back." He writes: "Even fairly good students, when they have obtained the solution of the problem and written down neatly the argument, shut their books and look for something else. Doing so, they miss an important and instructive phase of the work" (Pólya 2004, p. 14). In the journey we are about to describe, we came to understand more fully what Pólya meant when he said, "A good teacher should understand and impress on his students the view that no problem whatever is completely exhausted" (Pólya 2004, p. 15).

THE ACTIVITY

Mathematics teachers become enthusiastic when students become enthusiastic. Tony tried to find other sets of four numbers that would solve $a^2 + b^2 = c^2 + d^2$. The first numbers he found were $a = 1$, $b = 7$, $c = 5$, and $d = 5$. He then attempted to find vertices at lattice points that produced a skewed rectangle with diagonals that measured $\sqrt{50}$. Although Tony had trouble with this task, he was tenacious. His persistence encouraged Smith to start playing around with the problem and share it with coauthor Errthum.

Together we decided to build a series of conversations with Smith's students to guide them down a path of discovery. We wanted students to think about the geometry related to the integer solutions of $a^2 + b^2 = c^2 + d^2$. We wanted them to think deeply, for a longer period of time than we could devote to this problem in class. So we developed a set of prompts to extend the investigation into rectangles and encourage students to discover strategies for finding integer solutions to the equation $a^2 + b^2 = c^2 + d^2$.

Analytic geometry allows students to perform calculations and algebraic manipulations related to geometric deductions and conjectures. However, the quantities are often fractional or irrational. Some students have difficulty finding rectangles whose vertices are at lattice points—that is, points with integer coordinates—unless the sides of the rectangles are parallel to axes. In our guided exploration, we stipulated that the vertices of the rectangles be lattice points.

Our question and the answers it evoked engaged students in the kind of mathematical thinking called for in the Standards for Mathematical Practice listed in the Common Core State Standards Initiative (CCSSI 2010). In particular, this exploration engaged students in using specific examples to formulate generalizations and make use of algebraic structure to model geometric situations. As students brought their own discoveries to class, they learned to explain their work and criticize others' work productively.

By design, the investigation supplemented topics covered in our regular geometry classes. Our hints and student conjectures would span months. We asked students to immerse themselves in a problem that required a long time and a diligent attitude. In this way, students would have the chance to take ownership of the problem and realize the nature of true mathematical investigation.

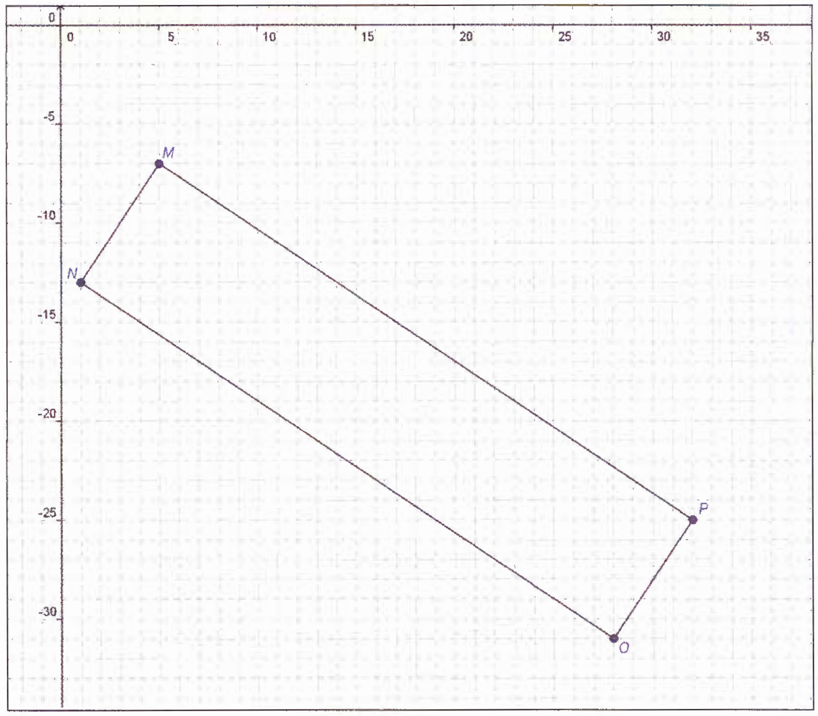


Fig. 1 Students were given the following task: Verify that this quadrilateral is a rectangle. The answer produces an interesting equality.

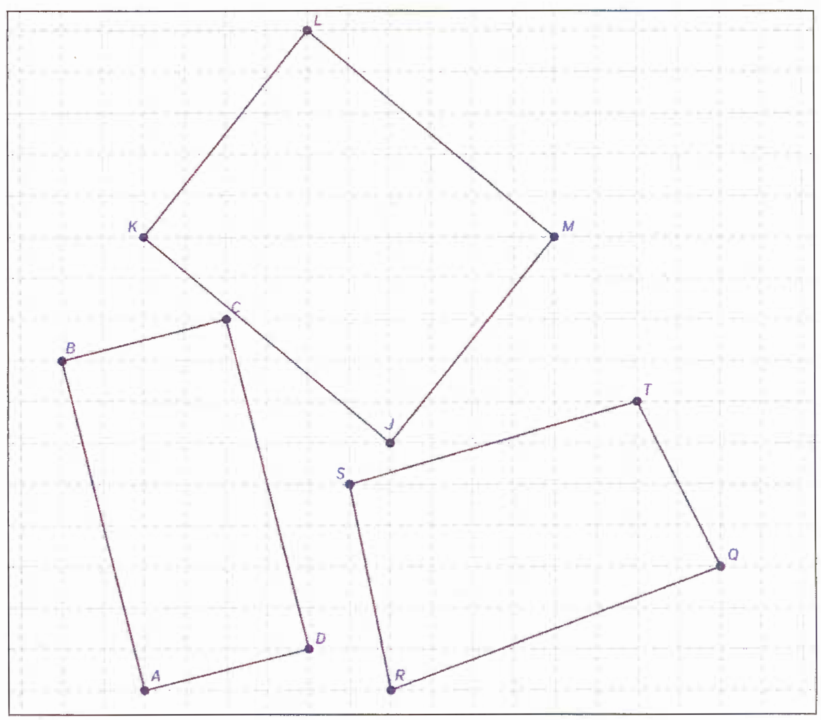


Fig. 2 Which of these quadrilaterals are rectangles?

To initiate the journey in the classroom, Smith presented three quadrilaterals (see **fig. 2**) and asked students to determine which are rectangles. He reminded students that quadrilaterals with equal diagonals and equal opposite sides are rectangles.

Naturally, students decided to focus mainly on the diagonal measures and see what they could discover. In their initial discussion of the problem, students decided against considering slope, trigonometry, or other tools of analytical geometry, but they agreed to use the distance formula and the basic algebra of the integer coordinate plane.

During the next class, we discussed students' work. Most had done a good job of using the distance formula to show that quadrilateral *QRST* is not a rectangle. Even though the diagonals are both equal to $\sqrt{85}$, the sides are not equal. Randi said, "I just looked and knew that it was not a rectangle."

That quadrilateral *JKLM* is not a rectangle is not as visually obvious. Although the two pairs of sides are equal, respectively, to $\sqrt{41}$ and $\sqrt{61}$, the diagonals are $LJ = \sqrt{104}$ and $MK = 10$. Randi mentioned that she had to do the mathematics to examine this one.

Students were pleased to find that quadrilateral *ABCD* is a rectangle, even though its sides are not parallel to the x- and y-axes. This quadrilateral passes the test because the two pairs of sides are equal, respectively, to $\sqrt{68}$ and $\sqrt{17}$ and the diagonals are both equal to $\sqrt{85}$.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	2	5	10	17	26	37	50	65	82	101	122	145	170	197
2		8	13	20	29	40	53	68	85	104	125	148	173	200
3			18	25	34	45	58	73	90	109	130	153	178	205
4				32	41	52	65	80	97	116	137	160	185	212
5					50	61	74	89	106	125	146	169	194	221
6						72	85	100	117	136	157	180	205	232
7							98	113	130	149	170	193	218	245
8								128	145	164	185	208	233	260
9									162	181	202	225	250	277
10										200	221	244	269	296
11											242	265	290	317
12												288	313	340
13													338	365
14														392

Fig. 3 Students use Excel to find instances of $a^2 + b^2 = c^2 + d^2$.

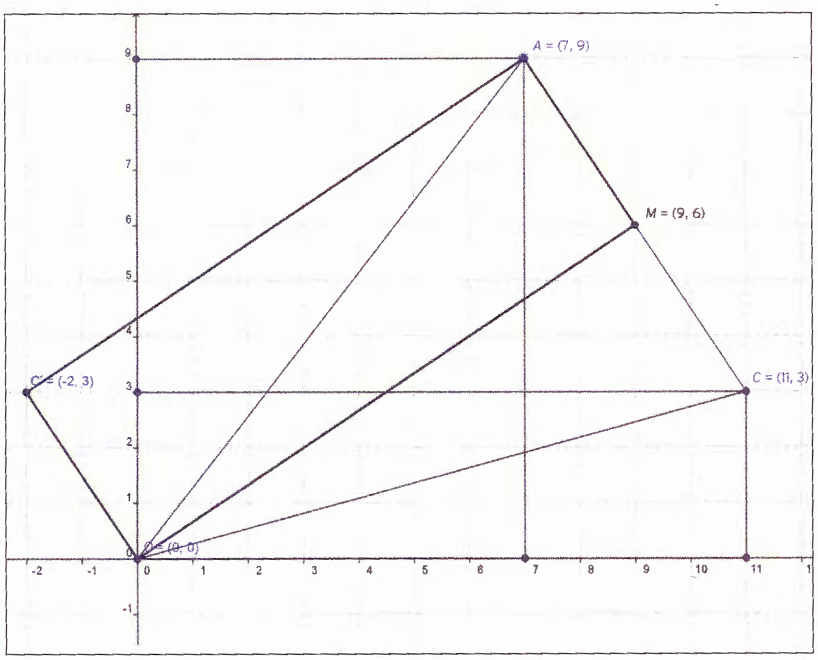


Fig. 4 Stacking congruent right triangles produces a skewed rectangle.

OTHER STUDENTS JOIN TONY'S PURSUIT

A week had now passed, and Tony had still not found lattice vertices of a skewed rectangle with a diagonal of $\sqrt{50}$. Also, he was frustrated that he could not find many integer solutions to

$$\sqrt{a^2 + b^2} = \sqrt{c^2 + d^2}.$$

We encouraged other students to help in the search.

Our students are required to be proficient with spreadsheets. Without prompting, Tracy developed a spreadsheet (see **fig. 3**) to investigate the values of $a^2 + b^2$. Using the algebraic commands of the spreadsheet, she set the top row and the right column to be whole numbers and the cells to be the sum of the square of these whole numbers. She then visually inspected the spreadsheet to find solutions to the equation $a^2 + b^2 = c^2 + d^2$.

With the discovery of a method for generating numerous integer solutions to $a^2 + b^2 = c^2 + d^2$, Tony and Tracy joined forces and attempted to form skewed rectangles whose diagonal measures corresponded to those integer solutions. They decided to start with the cells for $7^2 + 9^2$ and $11^2 + 3^2$ (those containing 130 in **fig. 3**). They drew the two rectangles having sides on the x- and y-axes and corners at the points *A* (7, 9) and *C* (11, 3), respectively.

Tracy drew $\overline{AC}$ and found the midpoint *M* (9, 6) (see **fig. 4**). Tracy and Tony noticed that triangles *MOA* and *MOC* were congruent by SSS. They could now stack these two congruent right triangles to form a skewed rectangle with vertices at lattice points and diagonals of length $\sqrt{130}$. The fourth point they constructed was C' (–2, 3). When they showed their results to Smith, he challenged them to generalize this approach by using the variables a, b, c, and d. Three days later, Tracy demonstrated the following for A (a, b) and C (c, d) where $a^2 + b^2 = c^2 + d^2$:

$$OA = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2}$$

$$C'M = \sqrt{\left(\frac{a+c}{2} - \frac{a-c}{2}\right)^2 + \left(\frac{b+d}{2} - \frac{b-d}{2}\right)^2}$$
$$= \sqrt{c^2 + d^2}$$

Tony also noticed that he could create many rectangles at different lattice points in the plane by translating the one that he and Tracy had discovered and also by stacking the smaller rectangles to create a larger one (see **fig. 5**). They enjoyed demonstrating these rectangles to the class, and everyone seemed to want to investigate the problem in more depth.

Several days later, Kelli came to class with another discovery. She had used Tracy's coordinates,

$$\left(\frac{a+c}{2},\frac{b+d}{2}\right) \text{ and } \left(\frac{a-c}{2},\frac{b-d}{2}\right),$$

and rearranged their y-coordinates to get

$$\left(\frac{a+c}{2},\frac{b-d}{2}\right) \text{ and } \left(\frac{a-c}{2},\frac{b+d}{2}\right),$$

noting that this rearrangement does not change the computation for the length $C'M$. The rearrangement created points (–2, 6) and (9, 3). Kelli still used (0, 0) and $(a, b) = (7, 9)$ as the two other vertices. Taking these ideas another step forward, she then wrote

$$\left(\frac{a+c}{2},\frac{d-b}{2}\right) \text{ and } \left(\frac{c-a}{2},\frac{b+d}{2}\right)$$

and plotted (9, –3) and (2, 6) along with (0, 0) and $(c, d) = (11, 3)$. Her rectangles were not developed by stacking previous rectangles on top of each other, but both had diagonals that measured

$$\sqrt{7^2+9^2}=\sqrt{11^2+3^2}=\sqrt{130}.$$

Kelli then drew a circle with center (0, 0) containing points (7, 9) and (11, 3) and challenged other students to look for rectangles with vertices at lattice points intersected by the circle (see **fig. 6**).

At this juncture, students were becoming excited. Karissa took one of Kelli's rectangles and started stacking copies of it to the right and up. **Figure 5** leads to the following integer solutions:

$$22^2+6^2=EI=AG=14^2+18^2$$
$$20^2+0^2=EH=DG=16^2+12^2$$
$$13^2+9^2=EB=AF=5^2+15^2$$

Students started using Karissa's technique to produce other examples of the algebraic phenomenon $a^2+b^2=c^2+d^2$. But they noticed that they still needed an original skewed rectangle to start the process. This was an opportune moment to move students away from the spreadsheet-motivated solution to a more efficient and general method for finding solutions to $a^2+b^2=c^2+d^2$.

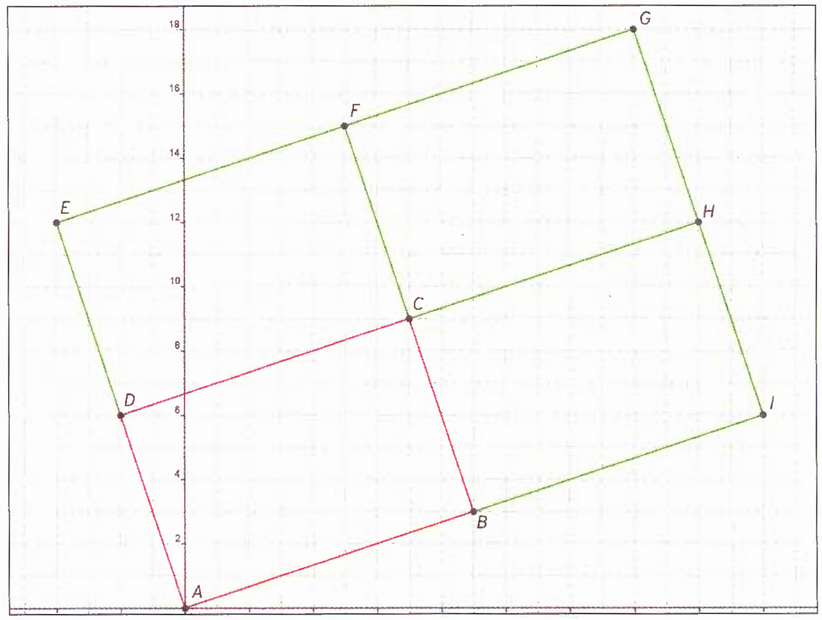


Fig. 5 Translations of the original (red) rectangle will produce others.

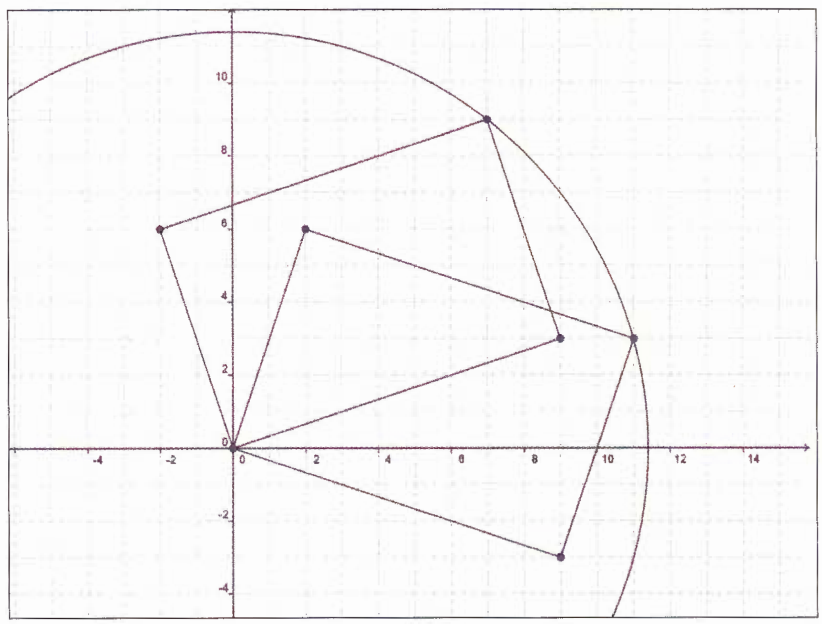


Fig. 6 When the circle intersects a lattice point, a new skewed rectangle is formed.

HELPFUL ALGEBRA

Students were now ready to be taught the algebra that fit the problem we were investigating. We introduced them to the following method of creating a, b, c, and d that satisfy $a^2+b^2=c^2+d^2$. This technique comes from rearranging the equation into $a^2-c^2=d^2-b^2$ and then factoring to obtain $(a+c)(a-c)=(d+b)(d-b)$. Hence, a clue for finding a, b, c, and d is to study numbers that have two different factorizations.

Choose a positive whole number N that can be written as a product of two distinct whole numbers in two different ways: $N = ef = gh$. (For simplicity, to generate positive solutions, choose $e > f$ and $g > h$.) Now, for each of these pairs of factors, set $a = e + f$ and $c = g + h$ and then set $d = e - f$ and $b = g - h$, making $a^2 + b^2 = c^2 + d^2$.

In general, for any e, f, g, and h that satisfy $ef = gh$, the following algebraic derivation provides a direct proof of the method:

$$\begin{aligned} ef &= gh \\ 4ef &= 4gh \\ 2ef - 2gh &= 2gh - 2ef \\ e^2 + 2ef + f^2 + g^2 - 2gh + h^2 &= g^2 + 2gh + h^2 + e^2 - 2ef + f^2 \\ (e+f)^2 + (g-h)^2 &= (g+h)^2 + (e-f)^2 \\ a^2 + b^2 &= c^2 + d^2 \end{aligned}$$

As an example, choose $N = 8 = 8 \cdot 1 = 4 \cdot 2$. This leads to $a = 8 + 1 = 9$; $b = 4 - 2 = 2$; $c = 4 + 2 = 6$; and $d = 8 - 1 = 7$.

CONTINUING THE EXPLORATION

Apparently, the revelation of an algebraic method for generating integer solutions to the equation $a^2 + b^2 = c^2 + d^2$ was not enough. Students began meeting outside class to discuss the problem. Six class members worked with the method presented above using $N = 144$. They found 28 different solutions for $a^2 + b^2 = c^2 + d^2$. They also discovered that $a^2 + b^2 = c^2 + d^2 = 25{,}925$ was largest when $N = 1 \cdot 144 = 2 \cdot 72$ and that $a^2 + b^2 = c^2 + d^2 = 625$ was the smallest when $N = 9 \cdot 6 = 12 \cdot 12$. Randi noticed that one-fourth of the sums were themselves squares: 625, 676, 900, 1600, 2601, 5476, and 21025. She further noticed that they had all occurred when one of the factored forms was $12 \cdot 12$. Without realizing it, Randi had stumbled on a method for generating Pythagorean triples: Let $N = g^2$, a perfect square with factorization $ef = gg$, and apply the method given above.

At this point, students were pleased with the state of the problem. Eventually, however, the most industrious students started looking for shortcuts. They wanted to know if there were a way simply to substitute numbers. So we showed them a more general method.

Consider N in terms of its prime factorization. Expressing N as the product of two numbers is essentially taking some prime factors to make a "left" factor and using the rest to make a "right" factor. To factorize N as a different product, some prime factors just have to swap sides. Algebraically, this formulation looks like

$$N = (wx)(yz) = (wy)(xz)$$

where each quantity in parenthesis represents a "left" or "right" factor and x and y are the parts they swap to make a new factoring of N. (Of course, this means that x cannot equal y and w cannot equal z but some of the variables may equal 1.)

Using this thought process together with our algorithm above produces the following:

$$\begin{aligned} a &= wx + yz \\ b &= wy - xz \\ c &= wx - yz \\ d &= wy + xz \end{aligned}$$

A little algebra then reveals that

$$a^2 + b^2 = (wx)^2 + (wy)^2 + (xz)^2 + (yz)^2 = c^2 + d^2.$$

We then asked students to choose four numbers at random. They chose $w = 3$, $x = 5$, $y = 4$, and $z = 8$ and then computed $a = 47$, $b = -28$, $c = -17$, and $d = 52$ to produce the identity $47^2 + (-28)^2 = (-17)^2 + 52^2$. We returned to the geometrical aspect of the problem and used these values to create the rectangle with vertices at $(0, 0)$, $(15, 12) = ((a + c)/2, (b + d)/2)$, $(-17, 52)$, and $(-32, 40) = ((c - a)/2, (d - b)/2)$.

We all had now arrived at the point at which we could produce as many examples of $a^2 + b^2 = c^2 + d^2$ as we wanted just by substituting new numbers. Similarly, we could create as many skewed lattice rectangles as we wanted.

ADDING ANOTHER TERM

Of course, once one hits the jackpot, greed sets in. Now the authors quickly began to ask themselves how to find solutions to $a^2 + b^2 + c^2 = d^2 + e^2 + f^2$. Neither of us could readily arrive at a simple technique, so we offered this problem as a challenge to the students. We confessed that we did not know the answer either, and for a while everyone—students and instructors—were stumped.

Smith later realized that the same factorization technique used earlier could be applied in this situation. If $a^2 + b^2 + c^2 = d^2 + e^2 + f^2$, then $a^2 - d^2 + b^2 - e^2 = f^2 - c^2$. Hence,

$$(a + d)(a - d) + (b + e)(b - e) = (f + c)(f - c).$$

Smith stopped here and asked students what they would do now. They randomly chose integers a, d, b, and e to compute the sum on the left side. They wrote this sum as a product of two factors and solved for the necessary f and c to complete the identity. (Note: In general, depending on our choices for a, d, b, and e, we may have to multiply everything by 2 to ensure that f and c turn out to be integers.)

As an example, let $a = 7$, $d = 2$, $b = 6$, and $e = 3$. Then $(9)(5) + (9)(3) = 72 = (f + c)(f - c)$. If we factor 72 as $18 \cdot 4$, solve $f + c = 18$ and $f - c = 4$ to arrive at $f = 11$ and $c = 7$. Therefore, $7^2 + 6^2 + 7^2 = 3^2 + 2^2 + 11^2$. Alternatively, we could have factored 72 as $36 \cdot 2$ and arrived instead at $7^2 + 6^2 + 17^2 = 3^2 + 2^2 + 19^2$.

The students got excited about the additional challenge of adding a term. Without teacher input, they made three-dimensional boxes and measured the oblique diagonals. They considered this extension as belonging to us—teachers and students. They claimed ownership because they were involved in the whole project.

A POSTSCRIPT

This experience was more profoundly beneficial for students than any of the other regular exercises or activities. Because students considered this to be *their* problem, they listened to our input and were motivated to learn. Tracy, now a middle school teacher, says that her students can follow what she is doing when using a simplified version of the same problem. During her student teaching, Karissa showed her students what she called "her problem." Kelli and Randi plan to incorporate this problem in their preservice teaching.

Our students learned that deep mathematical understanding develops over time. They now see mathematics as something that can be long and messy. However, with time, insight, and teamwork, interesting connections can sometimes be found.

As teachers, we found that not knowing the answers all the time actually helped students. We listened to their questions, communicated with them, and, when the moment was right, shared some of our thoughts with them.

REFERENCES

Common Core State Standards Initiative (CCSSI). 2010. *Common Core State Standards for Mathematics.* Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf.

Pólya, George. *How to Solve It*. 2004. Expanded Princeton Science Library Edition. Princeton, NJ: Princeton University Press.

DICK J. SMITH, dsmith@dbq.edu, is an assistant professor of mathematics at the University of Dubuque in Iowa, where he teaches college algebra, modern geometry, and discrete mathematics. ERIC F. ERRTHUM, eerrthum@winona.edu, is an assistant professor of mathematics at Winona State University in Minnesota. His interests are number theory and working with students on undergraduate research.

Criterion III:
Evidence of continuing preparation and study

Since I truly love both mathematics and teaching, it is natural for me to embody the WSU ideal of a lifelong learner.

Accomplishments (since Spring 2012)

1) In addition to attending departmental seminars and colloquia, I have attended the following regional and/or national conferences:

    - MAA-NCS Fall Meeting, Minneapolis, MN, October 2016
    - MAA-NCS Spring Meeting, St. Paul, MN, April 2016
    - MAA-AMS Joint Mathematics Meetings, Seattle, WA, January 2016
    - MAA-NCS Spring Meeting, Winona, MN, April 2015
    - MAA-NCS Spring Meeting, St. Cloud, MN, April 2014

2) I have had 7 reviews published to Mathematical Reviews, a division of the American Mathematical Society's online catalog MathSciNet. I have benefited in reviewing these articles as they are in my field of research but I otherwise might not have been aware of them since they are typically for lesser-tier/smaller-market journals.

3) I have been an AP Calculus Reader (i.e. grader) during June of 2013, 2014, 2015, and 2016 in Kansas City, MO. It has been extremely beneficial to meet and discuss with other Calculus instructors (both college-level and high school) from across the country. Some of these discussions have led to new activities and approaches in the classroom.

4) I routinely read and participate in the following online forums:

    - MathEducators.stackexchange.com – intended for discussions about teaching mathematics and related pedagogy, current reputation: 2,578 (top 10% of 5,127 users)
    - MathOverflow.net – research-level discussions in mathematics, current reputation: 1,375 (top 23% of 61,827 users)

5) Two solutions (awaiting verification) to problems posed in publications of the MAA. While these are not research-level questions, these problems are still a challenge and a great way to practice problem-solving and mathematical writing skills.

Note: Any of the above materials are available upon request.

CRITERION IV:
CONTRIBUTION TO STUDENT GROWTH AND DEVELOPMENT

Students choose Winona State University so they can get to know the faculty and experience college in a more individually meaningful way. The Department of Mathematics and Statistics prides itself on the opportunities and activities that we offer our students outside the classroom and I enjoy the chance to engage students beyond my classes.

Accomplishments (since Spring 2012)

1) Advised four students on their senior projects

   - Le Tang – "Theorems for Continued Fractions in $\mathbb{C}_p$." (in progress)
   - Jacob Beckel – "Alternative Carries in Base-p Addition." (technical report and presented at WSU Math/Stat Dept. seminar, April 2016)
   - Alex Klein – "Knot Analysis Algorithms." (technical report, April 2013)
   - Anthony Martino – "p-Adic Egyptian Fractions." (technical report, April 2013)

2) Served as primary academic advisor to 30 students majoring in our department.

   - 10 are current students
   - 17 have graduated
   - 10 attended graduate school after graduating from WSU

3) Have written letters of recommendations for 11 students applying to internships and/or graduate schools

4) Chaperoned (with Chris Malone) students on trips to visit regional graduate schools:

   - University of Iowa. November 2015. (12 students)
   - Iowa State University. November 2014. (8 students)
   - University of Minnesota, Twin Cities. November 2013. (8 students)
   - University of Nebraska, Lincoln. November 2012. (10 students)

5) Coached and supervised students for regional and national mathematics competitions:

   - William Lowell Putnam Mathematical Competition (National Competition)
     - 2013 (3 students)
     - 2012 (12 students)
   - MAA-NCS Team Competition (Regional Competition)
     - 2013 (3 students, top team receiving 58th place)
     - 2012 (18 students, top team receiving 17th place)

Criterion V:
Service to the University and Community

Since my last promotion I have taken on numerous leadership roles within our department and have provided service both in the greater Winona community and to my professional community. I enjoy being in the subset of people who get things done and I like to work together with others to produce positive results.

Accomplishments (since Spring 2012)

1) Chaired the departmental MATH subgroup (Fall 2012 – present)

   The main function of this committee to oversee all issues concerning MATH courses, MATH and MTHM majors, and MATH minors. (Note: Meeting minutes and/or other materials/documents are available upon request.)

   a) Main Accomplishments for AY2012-13:

      - Updated the MATH curriculum with a course numbering scheme to be more consistent with MnSCU,
      - Created clearer advising materials for our majors and minors and to be used at freshman registration
      - Updated course outlines.

   b) Main Accomplishments for AY2013-14:

      - Developed the framework for MATH395 (Professional Skills in Mathematics) and MATH495 (Communication of Independent Project), which started running in Spring 2015.
      - Revisited the requirements for our Math Education majors and, in response to deficiencies there and in other service areas, created two new courses, MATH117 and MATH247.

   c) Main Accomplishments for AY2014-15:

      - Renewed Writing Intensive designations
      - Completed paperwork for new courses
      - Revamped our honors program,
      - Created materials for the department webpage,
      - Began fixing some MnSCU transfer issues.

d) Main Accomplishments for AY2015-16:

- Worked with the physics department to create a new upper-level course to serve their majors and our MATH majors,
- Reviewed MnSCU transfer equivalent MATH courses to verify they were correctly matched
- Worked on strategies to lower the cost to students through open-source texts and free online homework systems
- Restructured the systematic offering of our upper-level electives and required courses to better serve students who declare the major during their junior year
- Compiled information about our program's alumni to be used in recruiting materials
- Began gathering program data for assessment purposes.

e) Main Accomplishments for AY2016-17 (in progress):

- Renewed Critical Analysis Intensive designations
- Created MATH program-level learning outcomes
- Analyzed program data from some courses and have begun to look for ways to improve these courses and our assessment strategies

2) Chaired the Local Planning Committee for hosting the MAA-NCS regional conference which took place at WSU in Spring 2015. The conference had 72 attendees from 24 institutions in 5 states (MN, WI, IA, SD, ND).

3) Served on two departmental search committees

   a) Statistics Search Committee (Spring 2013 – January 2014). Results: Hired Dr. Silas Bergen.

   b) Fixed-Term Search Committee (Fall 2015 – Spring 2016). Results: Hired Sue Florin, Chris Phan, and Samuel Schmidt

4) Served on other departmental committees

   a) Seminar, Library and Colloquium (Fall 2008 – present; Co-Chair, Fall 2016 – present)
   The main function of this committee is to handle the logistics for outside speakers, students, and faculty presenting to the department and our students.

b) Communications Committee (Fall 2013 – present)

My main duties on this committee have come in the way of revamping the departmental webpage via communicating and working with the web services office on campus. Our department's webpage has finally been modernized but will continue to see improvements.

c) Student Opportunities and Social Activities Committee (Fall 2012 – Spring 2016).

This committee is in charge of overseeing department scholarships, student travel to conferences, graduate schools, and industry tours, and organizing social events for alumni and current students. This committee serves an especially important role for our department's relation with past, current, and future students.

5) Reviewed and commented on departmental PDPs and PDRs for faculty members: Silas Bergen, Nic Gilbertson, Todd Iverson, Chris Phan, Sam Schmidt, Samuel Tsegai, and Lee Windsperger.

6) Served on the university-level Grade Appeals Committee

7) Service to my professional community

a) Member-At-Large Officer of the North Central Section of the MAA (Spring 2016 – present)

b) North Central Section MAA Governor's Selection committee, member (Spring 2016)

c) 7 reviews for the Mathematical Reviews, a division of the American Mathematical Society's online catalog MathSciNet.

d) AP Calculus Reader (i.e. grader) during June of 2013, 2014, 2015, and 2016 in Kansas City, MO.

e) Provided consultation for the book "Learning Qlikview Data Visualization" by Karl Pover. (Summer 2013)

8) Service to the broader Winona Community

a) Coach for the Cotter High School Math Team (Fall 2009 – present)

b) Chief Coordinator of Display & Safety for the Annual Southeastern Minnesota and Western Wisconsin Regional Science Fair (Spring 2012 – present)

c) Winona's Human Rights Commission (Fall 2010 – Fall 2013; Vice-President, October 2011 – September 2013).

CURRICULUM VITAE

Education:

Ph.D. in Mathematics: University of Maryland, 2007.
Dissertation title: Singular Moduli of Shimura Curves

B.S. in Mathematics with Honors and Distinction and in Physics with Distinction:
University of Iowa, 2002.

Professional Experience:

Associate Professor: August 2012 – Present.
Department of Mathematics and Statistics, Winona State University, Winona, MN.

Assistant Professor: August 2007 – August 2012.
Department of Mathematics and Statistics, Winona State University, Winona, MN.

Graduate Teaching/Research Assistant: August 2002 – May 2007.
Department of Mathematics, University of Maryland, College Park, MD.

Mathematical Research Intern: Summers 2002, 2003.
National Security Agency, Fort George G. Meade, MD.

Teaching Assignments at Winona State University:

Fall 2016 (16 s.h., 119 students):

- MATH 140-01: Applied Calculus (3 s.h., 35 students)
- MATH 212-01: Calculus I (4 s.h., 34 students)
- MATH 212-02: Calculus I (4 s.h., 33 students)
- MATH 327-01: Foundations of Mathematics (4 s.h., 16 students)
- MATH 490-01: Independent Problems in Mathematics: Theorems on Continued Fractions (1 s.h., 1 students)

Spring 2016 (13 s.h., 53 students):

- MATH 213-01: Calculus II (4 s.h., 19 students)
- MATH 247-01: Discrete Mathematics (3 s.h., 19 students)
- MATH 347-01: Number Theory (3 s.h., 12 students)
- MATH 395-01: Professional Skill Development for Mathematics (1 s.h., 1 students)
- MATH 490-01: Independent Problems in Mathematics-Alternate forms of base-p addition (1 s.h., 1 students)
- MATH 495-01: Communication of Independent Project Outcomes (1 s.h., 1 students)

Fall 2015 (14 s.h., 122 students):

- MATH 140-01: Applied Calculus (3 s.h., 34 students)
- MATH 140-04: Applied Calculus (3 s.h., 36 students)
- MATH 213-01: Calculus II (4 s.h., 22 students)
- MATH 247-01: Discrete Mathematics (3 s.h., 29 students)
- MATH 490-01: Independent Problems in Mathematics-Cohomology of base-p (1 s.h., 1 students)

Summer 2015 (3 s.h., 13 students):

- MATH 140-01: Applied Calculus (3 s.h., 13 students)

Spring 2015 (13 s.h., 33 students):

- MATH 213-03: Calculus II (4 s.h., 9 students)
- MATH 327-01: Foundations of Mathematics (4 s.h., 11 students)
- MATH 347-01: Number Theory (3 s.h., 10 students)
- MATH 395-01: Professional Skill Development for Mathematics (1 s.h., 2 students)
- MATH 495-01: Communication of Independent Project Outcomes (1 s.h., 1 students)

Fall 2014 (14 s.h., 120 students):

- MATH 140-01: Applied Calculus (3 s.h., 40 students)
- MATH 140-04: Applied Calculus (3 s.h., 38 students)
- MATH 213-01: Calculus II (4 s.h., 23 students)
- MATH 242-01: Linear Algebra (4 s.h., 19 students)

Summer 2014 (3 s.h., 19 students):

- MATH 115-01: College Algebra (3 s.h., 19 students)

Spring 2014 (11 s.h., 55 students):

- MATH 212-04: Calculus I (4 s.h., 30 students)
- MATH 213-02: Calculus II (4 s.h., 18 students)
- MATH 462-01: Introduction to Topology (3 s.h., 7 students)

Fall 2013 (11 s.h., 99 students):

- MATH 140-01: Applied Calculus (3 s.h., 37 students)
- MATH 212-03: Calculus I (4 s.h., 40 students)
- MATH 213-01: Calculus II (4 s.h., 22 students)

Summer 2013 (3 s.h., 20 students):

- MATH 115-01: College Algebra (3 s.h., 20 students)

Spring 2013 (16 s.h., 100 students):

- MATH 120-04: Precalculus (4 s.h., 31 students)
- MATH 160-04: Calculus I (4 s.h., 36 students)
- MATH 310-01: Number Theory (3 s.h., 24 students)
- MATH 445-01: Abstract Algebra II (3 s.h., 7 students, first 9 weeks)
- MATH 490-02: Independent Problems in Mathematics: Algorithms on Knots (1 s.h., 1 student)
- MATH 490-03: Independent Problems in Mathematics: p-Adic Egyptian Fractions III (1 s.h., 1 student)

Fall 2012 (15 s.h., 107 students):

- MATH 120-02: Precalculus (4 s.h., 37 students)
- MATH 160-03: Calculus I (4 s.h., 36 students)
- MATH 165-01: Calculus II (4 s.h., 26 students)
- MATH 280-01: Problem Solving for Math Competitions (1 s.h., 6 students)
- MATH 470-09: Independent Study: Algorithms on Knots (1 s.h., 1 student)
- MATH 470-10: Independent Study: p-Adic Egyptian Fractions II (1 s.h., 1 student)

Summer 2012 (4 s.h., 14 students):

- MATH 120-01: Precalculus (4 s.h., 14 students)

Spring 2012 (14 s.h., 132 students):

- MATH 140-01: Applied Calculus (3 s.h., 37 students)
- MATH 140-02: Applied Calculus (3 s.h., 38 students)
- MATH 165-01: Calculus II (4 s.h., 34 students)
- MATH 310-01: Number Theory (3 s.h., 22 students)
- MATH 470-04: Independent Study: Rational Tangles (1 s.h., 1 student)

Fall 2011 (15 s.h., 111 students):

- MATH 120-07: Precalculus (4 s.h., 38 students)
- MATH 165-01: Calculus II (4 s.h., 35 students)
- MATH 280-01: Problem Solving for Math Competitions (1 s.h., 3 students)
- MATH 440-02: Abstract Algebra (4 s.h., 33 students)
- MATH 470-05: Independent Study: Padé Approximants II (1 s.h., 1 student)
- MATH 470-06: Independent Study: p-Adic Egyptian Fractions (1 s.h., 1 student)

Spring 2011 (12 s.h., 92 students):

- MATH 160-02: Calculus I (4 s.h., 38 students)
- MATH 160-04: Calculus I (4 s.h., 39 students)
- MATH 450-01: Introduction to Topology (3 s.h., 14 students)
- MATH 470-06: Independent Study: Padé Approximants (1 s.h., 1 student)

Fall 2010 (13 s.h., 99 students):

- MATH 165-02: Calculus II (4 s.h., 35 students)
- MATH 210-01: Foundations of Mathematics (4 s.h., 21 students)
- MATH 210-02: Foundations of Mathematics (4 s.h., 24 students)
- MATH 280-01: Problem Solving for Math Competitions (1 s.h., 6 students)

Summer 2010 (3 s.h., 12 students):

- MATH 140-01: Applied Calculus (3 s.h., 12 students)

Spring 2010 (11 s.h., 61 students):

- MATH 120-05: Pre-calculus (4 s.h., 20 students)
- MATH 160-04: Calculus I (4 s.h., 34 students)
- MATH 410-01: History of Mathematics (3 s.h., 7 students)

Fall 2009 (12 s.h., 124 students):

- MATH 140-02: Applied Calculus (3 s.h., 32 students)
- MATH 165-02: Calculus II (4 s.h., 38 students)
- MATH 210-01: Foundations of Mathematics (4 s.h., 34 students)
- MATH 280-01: Problem Solving for Math Competitions (1 s.h., 8 students)

Spring 2009 (12 s.h., 84 students):

- MATH 120-01: Pre-calculus (4 s.h., 37 students)
- MATH 120-05: Pre-calculus (4 s.h., 37 students)
- MATH 310-01: Number Theory (3 s.h., 9 students)
- MATH 470-01: Independent Study: p-Adic Continued Fractions II (1 s.h., 1 students)

Fall 2008 (14 s.h., 144 students):

- MATH 119-01: Pre-Calculus Enrichment (1 s.h., 9 students)
- MATH 140-02: Applied Calculus (3 s.h., 39 students)
- MATH 140-03: Applied Calculus (3 s.h., 38 students)
- MATH 165-02: Calculus II (4 s.h., 34 students)
- MATH 166-02: Calculus Companion II (1 s.h., 15 students)
- MATH 280-01: Problem Solving for Math Competitions (1 s.h., 7 students)
- MATH 470-01: Independent Study: p-Adic Continued Fractions (1 s.h., 2 students)

Spring 2008 (11 s.h., 89 students):

- MATH140-04: Applied Calculus (3 s.h., 21 students)
- MATH160-04: Calculus I (4 s.h., 36 students)
- MATH161-04: Calculus Companion I (1 s.h., 16 students)
- MATH310-01: Number Theory (3 s.h., 16 students)

Fall 2007 (11 s.h., 100 students):

- MATH120-02: Pre-calculus (4 s.h., 37 students)
- MATH140-03: Applied Calculus (3 s.h., 33 students)
- MATH440-01: Abstract Algebra (4 s.h., 30 students)

Print Publications

- E. Errthum. "A Division Algorithm Approach to p-Adic Sylvester Expansions." J. Number Theory 160 (2016), 1–10.

- E. Errthum. "Minimal Polynomials of Singular Moduli." Math. Comp. 83 (2014), no. 285, 411–420.

- E. Errthum, D. Smith. "Finding Skewed Lattice Rectangles: The Geometry of $\boldsymbol{a^2 + b^2 = c^2 + d^2}$." The Mathematics Teacher, Vol. 106, No. 2 (September 2012), pp. 150-155

- E. Errthum. "Singular Moduli of Shimura Curves." Canadian Journal of Mathematics 63(2011), no. 4, 826-861. doi:10.4153/CJM-2011-023-7

Online Publications

- E. Errthum. "p-Adic Continued Fractions." Wolfram Demonstrations Project, 2009. http://demonstrations.wolfram.com/PAdicContinuedFractions/

- E. Errthum. "Addition of Points on an Elliptic Curve over the Reals" Wolfram Demonstrations Project, 2007. http://demonstrations.wolfram.com/AdditionOfPointsOnAnEllipticCurveOverTheReals/

Presentations:

- E. Errthum. "Addition with Carries." MAA North Central Section Meeting, St. Paul, MN, April 2016.

- E. Errthum. "Publish or Perish in an Intro to Proofs Course." Joint Mathematical Meetings, Seattle, WA, January 2016.

- E. Errthum. "Minimal Polynomials of Singular Moduli." Joint Mathematical Meetings, New Orleans, LA, January 2011.

- E. Errthum. "A p-Adic Euclidean Algorithm." MAA Iowa Section Meeting, Cedar Falls, IA, October 2009.

- E. Errthum. "Learning Your ABC." Bethany Lutheran College Colloquium Series, Bethany, MN, April 2009.

- E. Errthum. “Learning Your ABC.” Winona State Department of Mathematics and Statistics Seminar, January 2009.

- B. Deppa, E. Errthum, D. Wrolstad, J. Wrolstad and A.Ylvisaker. “A Panel Discussion about Graduate Schools in Mathematics and Statistics.” Winona State Department of Mathematics and Statistics Seminar, November 2008.

- E. Errthum. “Finding Minimal Polynomials with a Norm Calculator.” MAA North Central Section Meeting, Moorhead, MN, October 2008.

- E. Errthum. “Singular Moduli of Shimura Curves.” University of Wisconsin – Madison Department of Mathematics Number Theory Seminar, Madison, WI, March 2008.

- E. Errthum. “Singular Moduli of Shimura Curves.” The Ohio State University Department of Mathematics Number Theory Seminar, Columbus, OH, February 2008.

- E. Errthum. “Singular Moduli of Shimura Curves.” Joint Mathematical Meetings, San Diego, CA, January 2008.

Grants:

- “MAA North Central Section Meeting” Winona State University Foundation Grant. November 2014. (funded at $1200)
- “WSU Students Attend and Present at PME Undergraduate Math Conference.” Winona State University Foundation Grant. April 2011. (funded at $375)
- “WSU Students Attend and Present at the Pi Mu Epsilon Conference at St. Johns University.” Winona State University Foundation Grant, submitted with Joyati Debnath. April 2010. (funded at $420)
- “Undergraduate Research Project in Mathematics.” Professional Improvement Fund. March 2010. (funded at $1086)
- “Increasing STEM enrollment and math success through effective implementation of WeBWorK at many Institutions.” Center for Teaching and Learning Instructional Development Grants, submitted with Aaron Wangberg, Tisha Hooks, and Chris Malone. April 2009 (not funded)
- “Undergraduate Research Project in Mathematics.” Professional Improvement Fund. March 2009. (funded at $1275)
- ”Undergraduate Research Project in Mathematics.” Professional Improvement Fund. October 2008. (not funded)
- “Project NExT Fellowship Support.” Professional Improvement Fund, March 2008. (funded at $1280)
- “Mathematical Scholarship: Undergraduate and Beyond.” Professional Improvement Fund. October 2007. (funded at $1200)

Professional Conferences Attended:

- MAA North Central Section Meeting, Minneapolis, MN, October 2016
- MAA North Central Section Meeting, St. Paul, MN, April 2016 (contributed presentation)
- Joint Mathematical Meetings, Seattle, WA, January 2016 (contributed presentation)
- MAA North Central Section Meeting, Winona, MN, April 2015 (chair of organizational committee)
- MAA North Central Section Meeting, St. Cloud, MN, April 2014
- Minnesota Pi Mu Epsilon Conference, Collegeville, MN, April 2011 (student presenting)
- Joint Mathematical Meetings, New Orleans, LA, January 2011 (contributed presentation)
- Minnesota Pi Mu Epsilon Conference, Collegeville, MN, April 2010 (student presenting)
- MAA Iowa Section Meeting, Cedar Falls, IA, October 2009 (contributed presentation)
- Minnesota State High School Mathematics League Summer Conference, St. Paul, MN, July 2009
- MAA Wisconsin Section Meeting, La Crosse, WI, April 2009 (student presenting)
- MAA North Central Section Meeting, Moorhead, MN, October 2008 (contributed presentation)
- MAA North Central Section NExT Meeting, Moorhead, MN, October 2008
- MathFest, Madison, WI, August 2008
- MAA North Central Section Meeting, Collegeville, MN, April 2008
- MAA North Central Section NExT Meeting, Collegeville, MN, April 2008
- Joint Mathematical Meetings, San Diego, CA, January 2008 (contributed presentation)
- MAA North Central Section Meeting, Bemidji, MN, October 2007
- MAA North Central Section NExT Meeting, Bemidji, MN, October 2007

Workshops/Seminars Attended:

- "A Beginner's Guide to the Scholarship of Teaching and Learning in Mathematics" MAA mini-course, San Diego, CA, January 2008
- "Directing Undergraduate Research" MAA mini-course, San Diego, CA, January 2008
- "New Faculty Community of Practice" seminar series, Winona, MN, Fall 2007

Recognitions:

- MAA North Central Section NExT Fellow, 2007 – 2008.

Memberships

- Mathematics Association of America (MAA)
- MathEducators.StackExchange.com, MathOverflow.net

Student Research Presentations and Publications:

- Jacob Beckel. "Alternative Carries in Base-p Addition." WSU Department of Mathematics and Statistics Seminar, April 2016
- Anthony Martino. "p-Adic Egyptian Fractions."
  - Winona State University Celebration of Research and Creative Scholarship, Winona, MN, April 2011.
  - Minnesota Pi Mu Epsilon Conference, Collegeville, MN, April 2011.
  - Winona State Department of Mathematics and Statistics Seminar, Winona, MN, April 2011.
- Michael Pilla. "Generalized Factorials and Taylor Expansions." Minnesota Pi Mu Epsilon Conference, Collegeville, MN, April 2010.
- Cortney Lager. "A p-Adic Euclidean Algorithm."
  - A p-Adic Euclidean Algorithm , *Rose-Hulman Institute for Technology Undergraduate Math Journal,* **(10)** no. 2, 2009
  - MathFest, Portland, OR, August 2009.
  - MAA Wisconsin Section Meeting, La Crosse, WI, April 2009.
- Erica Fremstad and Cortney Lager. "A p-Adic Euclidean Algorithm." Winona State Department of Mathematics and Statistics Seminar, Winona, MN, December 2008.

Other Student Research Mentored:

- Le Tang. Theorems for Continued Fractions in $\mathbb{C}_p$. Fall 2016 – Spring 2017
- Alex Klein. Knot Analysis Algorithms. Fall 2012 - Spring 2013
- Dan Buerman. Padé Approximants. Spring 2011
- Michael Pilla. Curvature of Space-Time. Spring 2009
- Dason Kurkiewicz. Fibonacci Cycles Modulo n. Spring 2008

Student Trip Advisor

- Graduate School Visit to the University of Iowa, Iowa City, IA. With Chris Malone. November 2015. (12 students)
- Graduate School Visit to Iowa State University, Ames, IA. With Chris Malone. November 2014. (8 students)
- Graduate School Visit to the University of Minnesota, Twin Cities, Minneapolis, MN. With Chris Malone. November 2013. (8 students)
- Graduate School Visit to the University of Nebraska, Lincoln, NE. With Chris Malone. November 2012. (10 students)
- Graduate School Visit to the University of Wisconsin, Madison, WI. With Chris Malone. October 2011. (8 students)
- Pi Mu Epsilon Undergraduate Mathematics Conference, Collegeville, MN. With Joyati Debnath and Sam Schmidt. April 2011. (16 students)
- Graduate School Visit to the University of Iowa, Iowa City, IA. With Chris Malone and Sam Schmidt. November 2010. (9 students)

- Pi Mu Epsilon Undergraduate Mathematics Conference, Collegeville, MN. With Joyati Debnath. April 2010. (16 students)
- Graduate School Visit to Iowa State University, Ames, IA. With Chris Malone. November 2009. (9 students)
- MAA Wisconsin Section Meeting, La Crosse, WI. With Joyati Debnath and Aaron Wangberg. April 2009. (12 students)
- Graduate School Visit to the University of Wisconsin – Milwaukee and the Medical College of Wisconsin, Milwaukee, WI. With Chris Malone. November 2008. (8 students)
- Graduate School Visit to the University of Nebraska, Lincoln, NE. With Tisha Hooks and Chris Malone. November 2007. (9 students)

Student Competitions Advised and Proctored

- William Lowell Putnam Mathematical Competition (National Competition)
    - 2013 (3 students, top student receiving 3088.5th place)
    - 2012 (12 students, top student receiving 1360.5th place)
    - 2011 (10 students, top student receiving 3407th place)
    - 2010 (10 students, two students receiving 883rd place)
    - 2009 (12 students, top student receiving 478.5th place)
    - 2008 (9 students, top student receiving 619th place)
- MAA-NCS Team Competition (Regional Competition)
    - 2013 (3 students, top team receiving 58th place)
    - 2012 (18 students, top team receiving 17th place)
    - 2011 (13 students, top team receiving 23rd place)
    - 2010 (18 students, top team receiving 18th place)
    - 2009 (17 students, top team receiving 30th place)
    - 2008 (15 students, top team receiving 10th place)

Other Service to Students

- September 2007 – present; Student advising
- January 2008 – present; Letters of Recommendation
- January 2008 – May 2012; Co-advisor for Math/Stat Club
- Fall2011; organized a COSE cohort program with Nathan Moore
- Spring 2009, Spring 2010; organizer of the Winona State Problem Solvers

Service to the Department

- MATH Subgroup (Fall 2007 – present; Chair, Fall 2012 – present)
- Communications Committee/Departmental Webmaster (Fall 2013 – present)
- Seminar, Library and Colloquium Committee (Fall 2008 – present; Co-Chair, Fall 2016 – present)

- Student Opportunities and Social Activities Committee (Fall 2008 – Spring 2015; Chair Fall 2008 – Spring 2012)
- WileyPlus course administrator for MATH140, MATH150, and MATH155 (Fall 2010 – Spring 2011)
- Various Course Groups (Fall 2007 – Spring 2010)
- Math Achievement Center Task Force (Fall 2008 – Spring 2009)
- Program Improvement Committee (Fall 2007 – Spring 2008)
- Scholarship Committee member (Fall 2007 – Spring 2008)
- Student Lounge Task Force member(Fall 2007 – Spring 2008)

Service to the University

- Grade Appeal Committee, member AY2008/09 – present
- Cultural Diversity Committee, member AY2008/09 – AY2010/11, chair AY2010/11
- Library Committee, member AY2008/09

Service to the Scientific Community

- Mathematical Association of America – North Central Section, At-Large Board Member (Spring 2016 – present)
- Cotter High School Minnesota Math League coach (Fall 2009 – present)
    - Winona State students involved:
        - Anthony Martino, Fall 2011 – Spring 2012
- Annual Southeastern Minnesota and Western Wisconsin Regional Science Fair
    - Display & Safety Coordinator, 2012 – present
    - Created and provided a cryptography activity for 6^{th} graders (2011)
    - Judge, 2008 – 2011

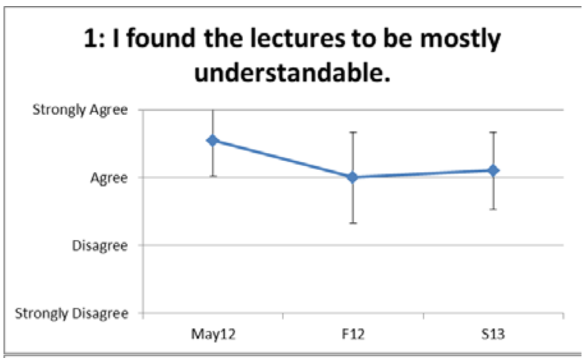
Service to the Community

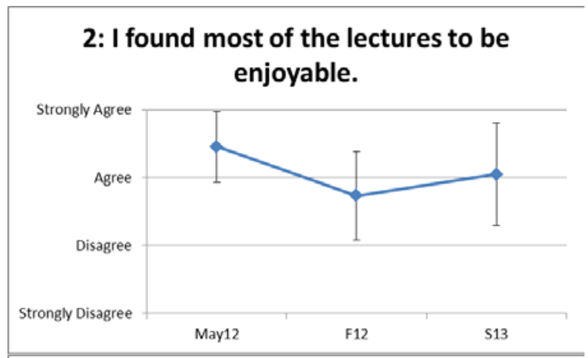
- City of Winona Human Rights Commission, 2010 – 2013; Vice-President, October 2011 – September 2013.
- Des Moines Catholic Worker internship program, summer 2008.
- Offered "Faith and Reason" high school faith formation mini-course through Saint Mary's Catholic Church, winter 2008.
- Saint Mary's Catholic Church member, 2007 – present.

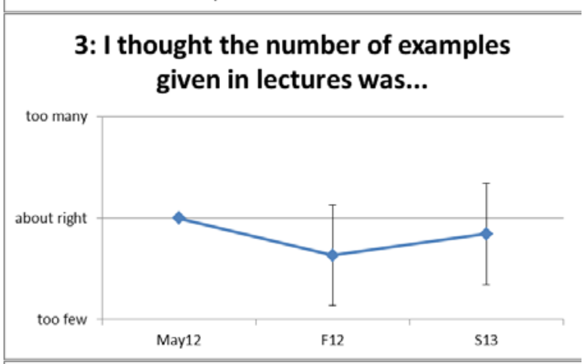
Appendix B:
Course Evaluations

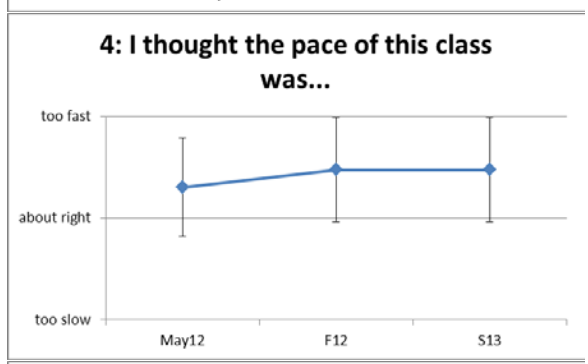
Math 120 – Precalculus

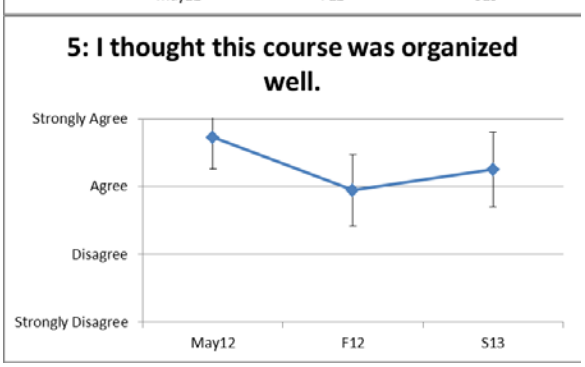
The charts below give the average (with standard deviation) for responses on course evaluations for Math120 – Precalculus. After the charts are representative replies to the free response questions.

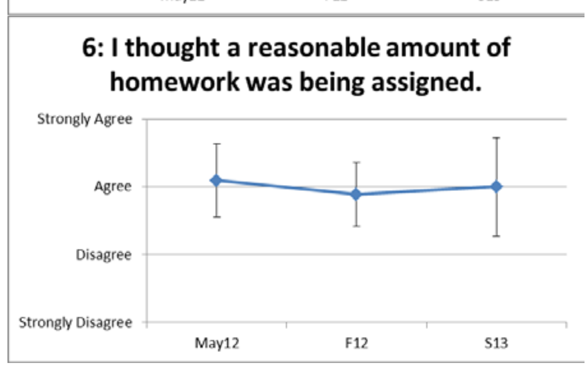


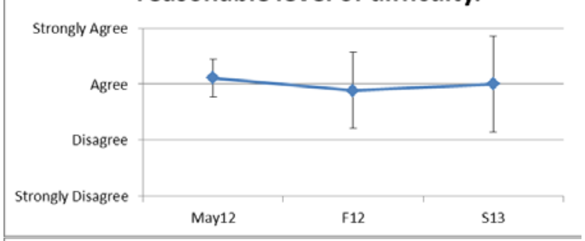




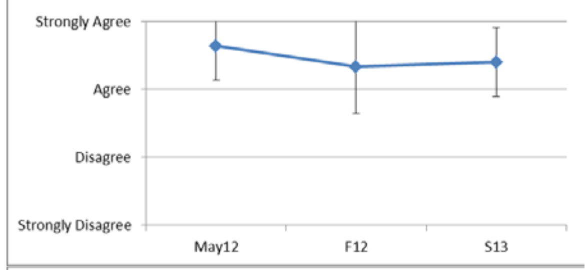




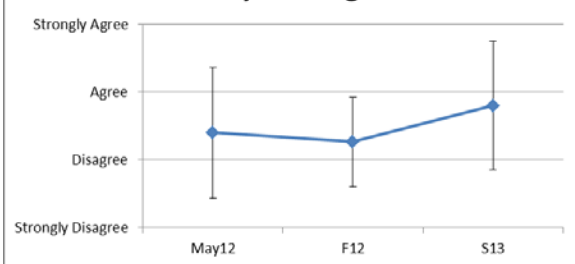




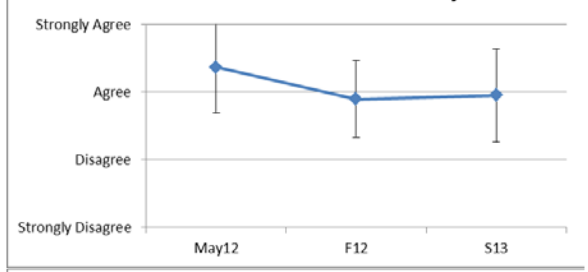
7: I thought the individual homework problems assigned were at a reasonable level of difficulty.



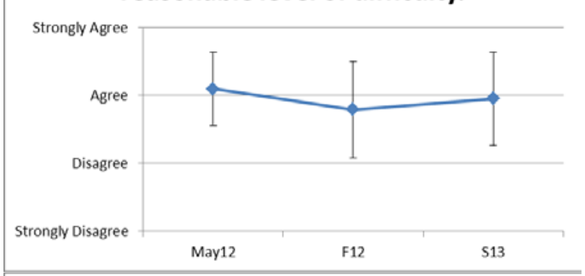
8: I thought the homework days were beneficial to my learning.



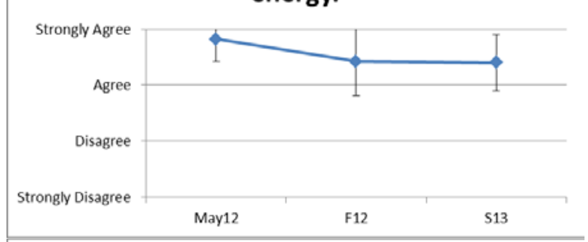
9: I thought ALEKs was beneficial to my learning.



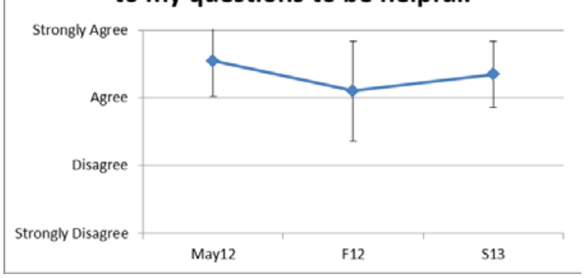
10: I thought the quizzes were at a reasonable level of difficulty.



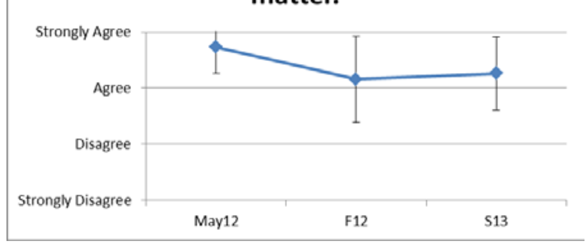
11: I thought the exams were at a reasonable level of difficulty.



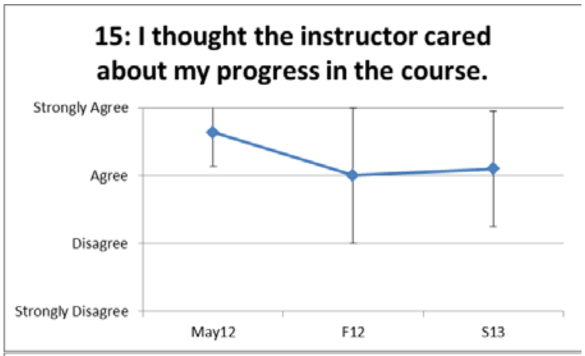
12: I thought the instructor had the appropriate level of enthusiasm and energy.

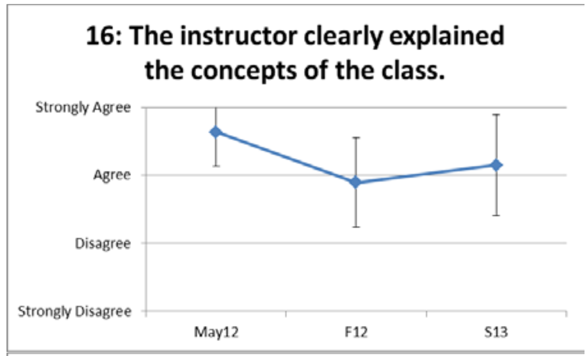


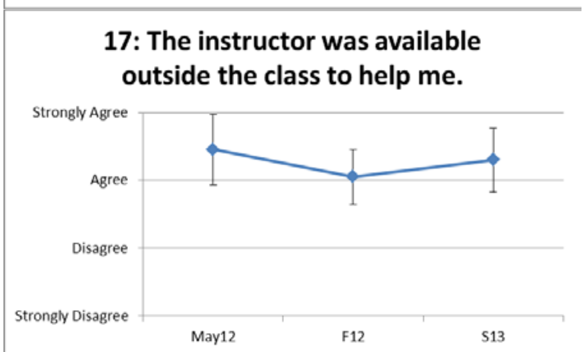
13: I found the instructor's responses to my questions to be helpful.

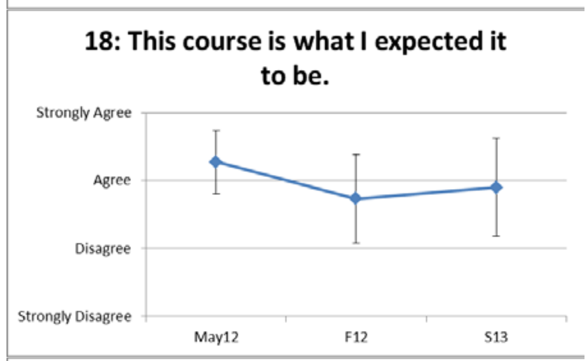


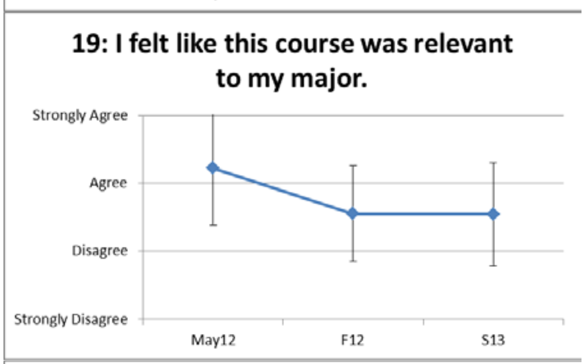
14: I found the instructor to be effective in teaching the subject matter.

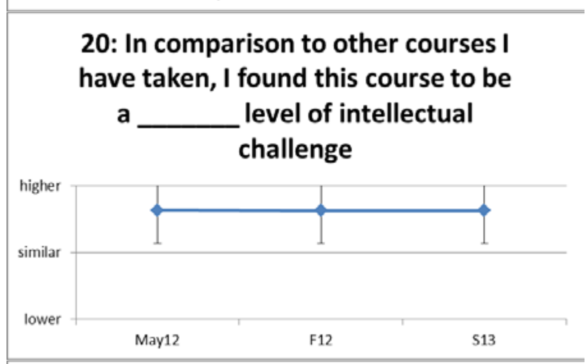

15: I thought the instructor cared about my progress in the course.
Strongly Agree
Agree
Disagree
Strongly Disagree
May12
F12
S13

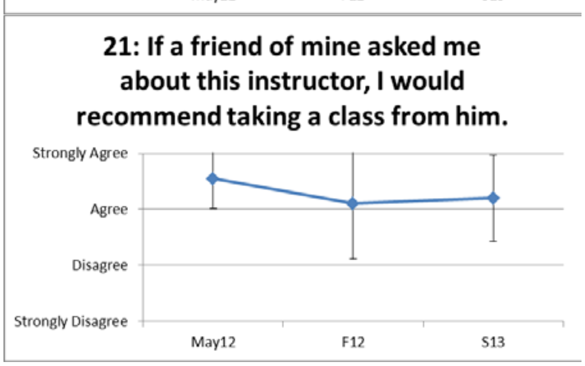

16: The instructor clearly explained the concepts of the class.
Strongly Agree
Agree
Disagree
Strongly Disagree
May12
F12
S13

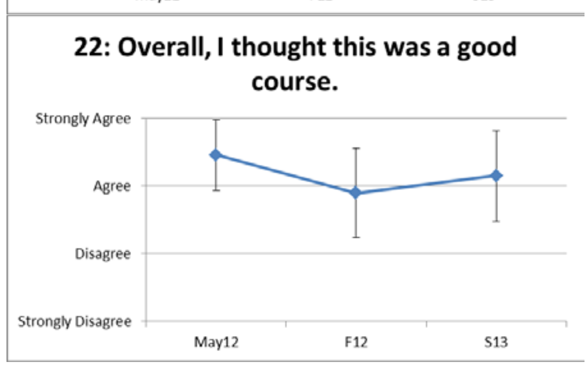

17: The instructor was available outside the class to help me.
Strongly Agree
Agree
Disagree
Strongly Disagree
May12
F12
S13

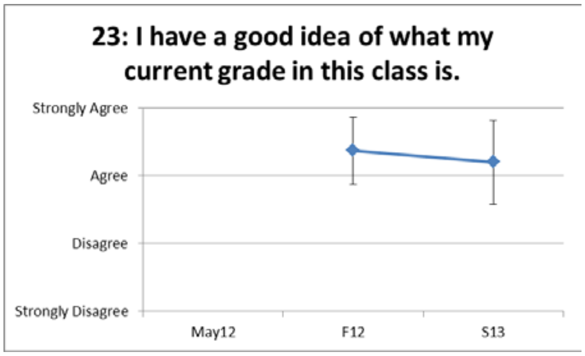

18: This course is what I expected it to be.
Strongly Agree
Agree
Disagree
Strongly Disagree
May12
F12
S13

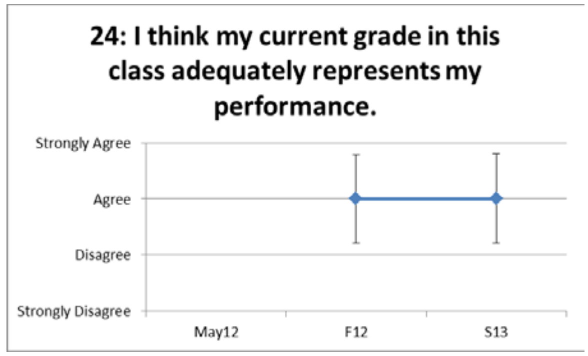

19: I felt like this course was relevant to my major.
Strongly Agree
Agree
Disagree
Strongly Disagree
May12
F12
S13


20: In comparison to other courses I have taken, I found this course to be a _______ level of intellectual challenge
higher
similar
lower
May12
F12
S13


21: If a friend of mine asked me about this instructor, I would recommend taking a class from him.
Strongly Agree
Agree
Disagree
Strongly Disagree
May12
F12
S13


22: Overall, I thought this was a good course.
Strongly Agree
Agree
Disagree
Strongly Disagree
May12
F12
S13





Representative Comments (May 2012)

23: What do you think most contributed to your success in this class?

- Going over <u>all</u> the homework really helped me. The Aleks was beneficial.
- Work hard.
- Going over the quizzes in class.
- Studying the practice exams, homework points and attendance points
- Asking questions instead of thinking "oh I'll figure it out later."

24: Besides the pace of the course, what do you think most contributed to any troubles you had in this class?

- The homework literally took 5+ hours to complete, and it was so hard to study the night before a test b/c of the hw.
- Previous knowledge, if you weren't introduced to it before the course it was harder to grasp.
- Examples in class were not like the homework out of the book.
- Terrible math background. Also haven't taken a math class for years.

25: Do you have any suggestions for the instructor that may be helpful for the next time this course is taught?

- I loved how this course was taught; Dr. Errthum is very upbeat and makes the class enjoyable so keep that up!
- Do not take it on summer.
- Larger variety of example problems.
- Try to "dumb" it down a little more. Some concepts and examples are passed over or given to fast.
- Probably use ALEKS a bit more for precalc stuff and not just to test algebra knowledge.

26: Do you have any advice you would give to students in the first week/day of this class in future semesters?

- Keep up with the homework
- Go to class and pay attention. If you pay attention and stay involved you will understand everything, and pass.
- Read the chapter(s) before you come to class

27: Any additional comments?

- I hate math but I enjoyed this class. Sometimes it is easier to go through a week in one day b/c the information fits together better.
- Great job. It was actually enjoyable! Win!
- For it being so long each day, it was still alright to sit through and he made it interesting.
- Great class. I enjoyed it very much. You are a great teacher – I highly recommend you!!!
- Very nice instructor, will be taking classes from him in the future.

Representative Comments (Fall 2012)

25: What do you think most contributed to your success in this class?

- Doing the homework
- Discussions and in class examples
- Homework sessions

26: What do you think most contributed to any troubles you had in this class?

- The exams were hard
- My lack of effort towards learning new material
- Not paying attention in class
- Lectures were sometimes confusing or hard to follow
- Maybe a little fast-paced near the end of the semester

27: Do you have any suggestions that may be helpful for the next time this course is taught?

- Don't use ALEKS. It is a terrible program that gave me a grade that was far below what it should have been.
- More examples in class
- Give more info on which formulas need to be memorized
- Go a little bit slower and explain steps of a problem more

28: Do you have any advice to give to students in the first week of this class in future semesters?

- Study hard
- Actually do the homework and look at quiz solutions online.
- Do the ALEKS because it can really make a difference on your grade
- Don't fall behind in homework. If you show up & take notes, you'll do fine! ☺
- Come to class every day. Lectures are way more helpful than the book.

29: Any additional comments?

- I enjoyed you as a professor!
- Good class! Very much enjoyed.
- Okay class.

Representative Comments (Spring 2013)

25: What do you think most contributed to your success in this class?

- Going to every lecture and paying attention
- Studying outside of class & asking questions
- Homework Days & ALEKS
- Dr. Errthum's enthusiasm & availability for help.
- Errthum was very approachable & easy to ask questions of

26: What do you think most contributed to any troubles you had in this class?

- The course was <u>too</u> fast. Homework was much harder than expected.
- Not doing my HW
- Long Wednesday classes
- Not enough examples or practice

27: Do you have any suggestions for the instructor that may be helpful for the next time this course is taught?

- Go a little slower & not have so many quizzes.
- Spend less time on earlier stuff & more on last chapter
- Don't do ALEKS

28: Do you have any advice you would give to students in the first week/day of this class in future semesters?

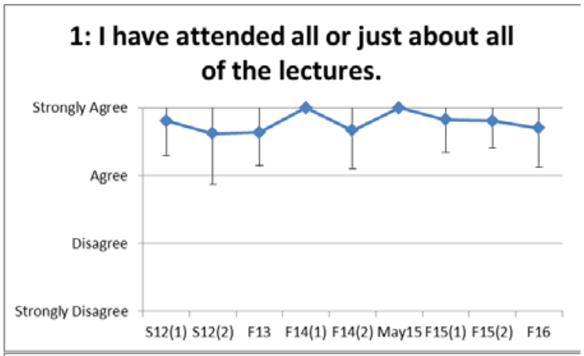
- Don't be afraid to ask for help.
- Go to class & keep up with homework
- Read the book
- Don't get behind

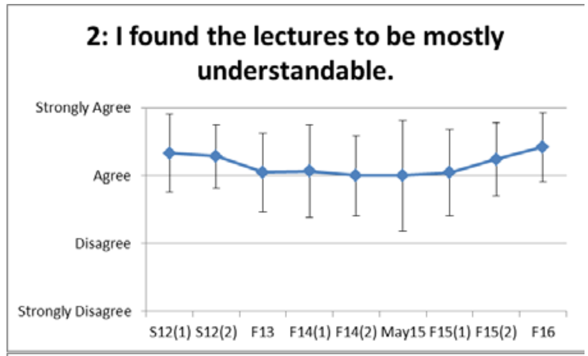
29: Any additional comments?

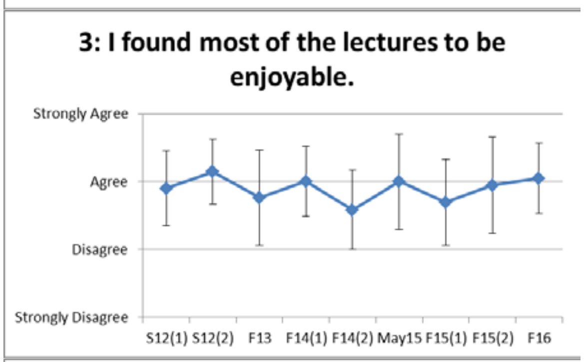
- Great instructor, useful info & good at tying together various elements of math.
- Made me really enjoy math.
- I loved your class. You made it fun to come to class.

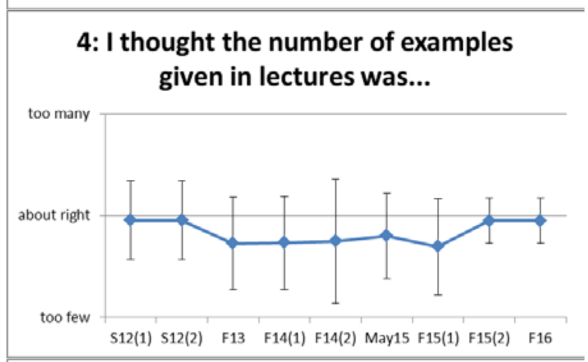
Math 140 – Applied Calculus

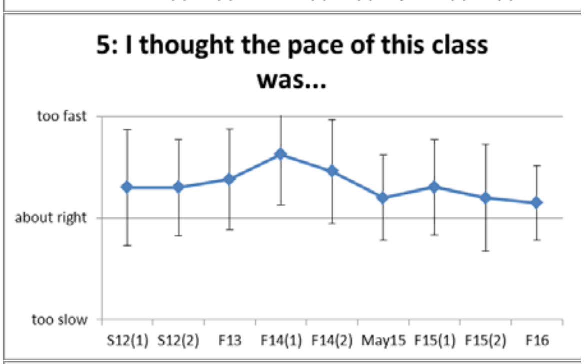
The charts below give the average (with standard deviation) for responses on course evaluations for Math140 – Applied Calculus. After the charts are representative replies to the free response questions.

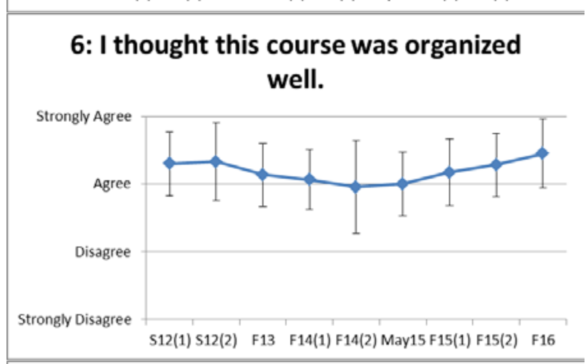


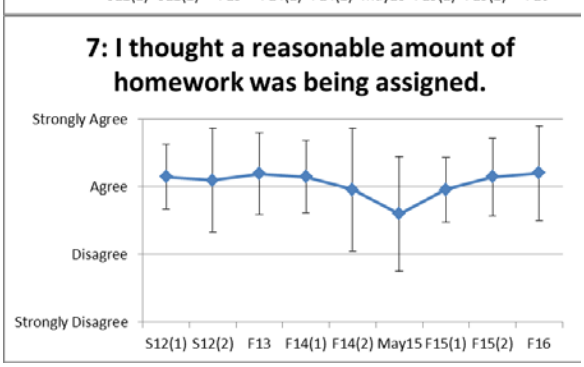


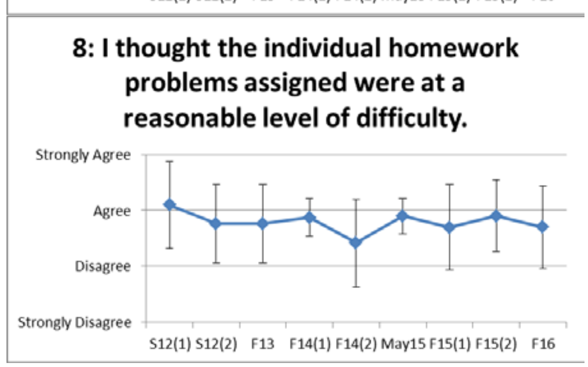


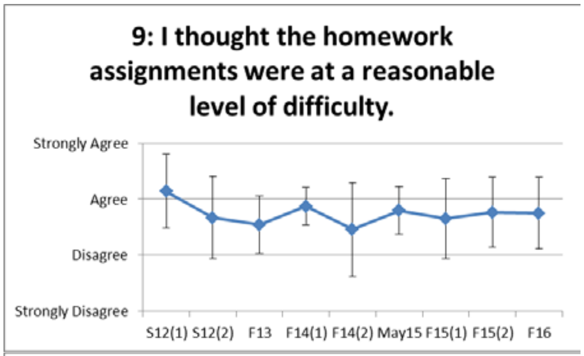


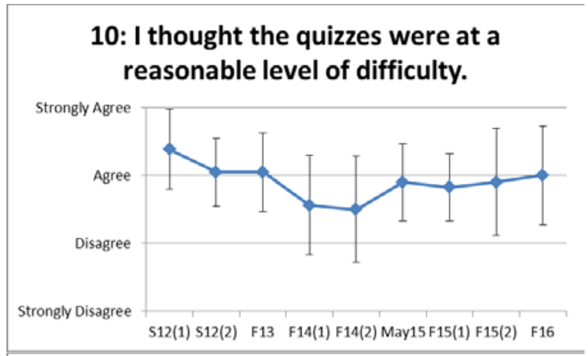


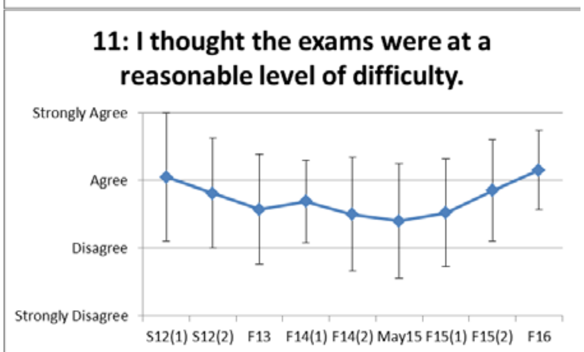


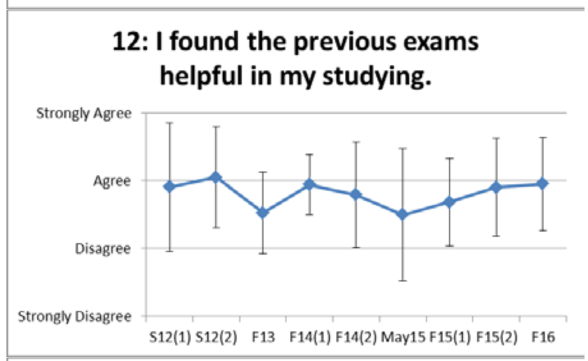


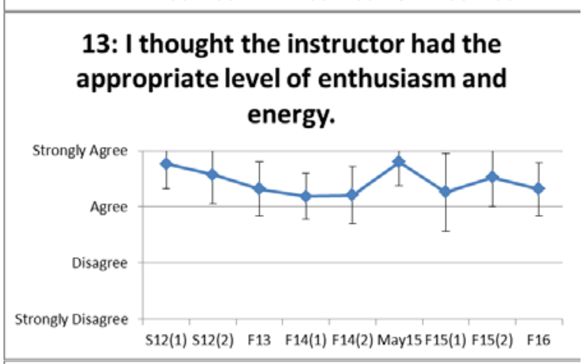


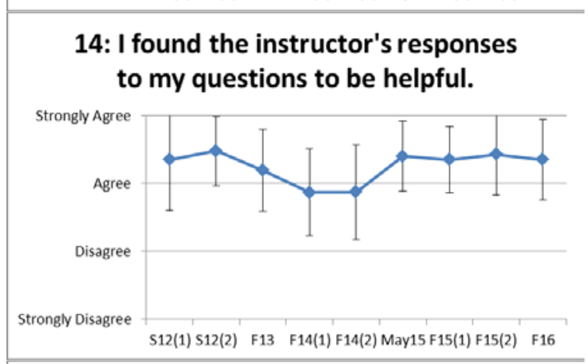

9: I thought the homework assignments were at a reasonable level of difficulty.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12(1) S12(2) F13 F14(1) F14(2) May15 F15(1) F15(2) F16


10: I thought the quizzes were at a reasonable level of difficulty.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12(1) S12(2) F13 F14(1) F14(2) May15 F15(1) F15(2) F16


11: I thought the exams were at a reasonable level of difficulty.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12(1) S12(2) F13 F14(1) F14(2) May15 F15(1) F15(2) F16


12: I found the previous exams helpful in my studying.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12(1) S12(2) F13 F14(1) F14(2) May15 F15(1) F15(2) F16


13: I thought the instructor had the appropriate level of enthusiasm and energy.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12(1) S12(2) F13 F14(1) F14(2) May15 F15(1) F15(2) F16


14: I found the instructor's responses to my questions to be helpful.
Strongly Agree
Agree
Disagree
Strongly Disagree
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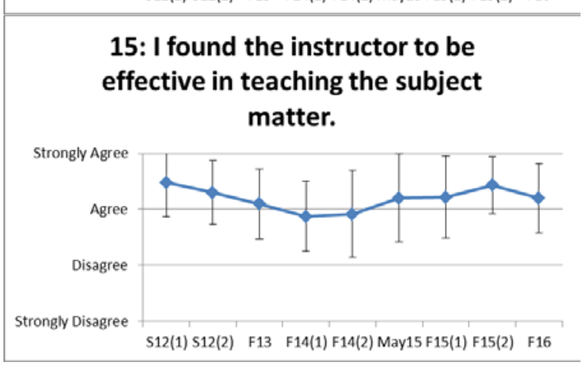
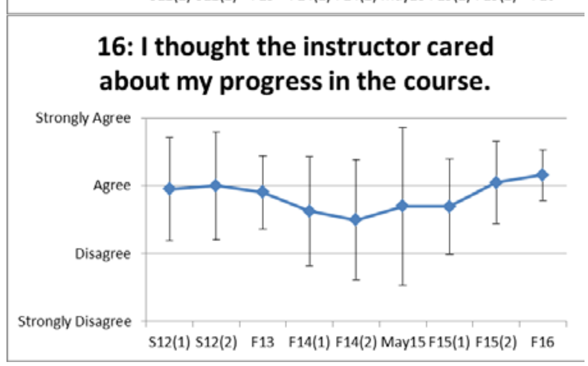
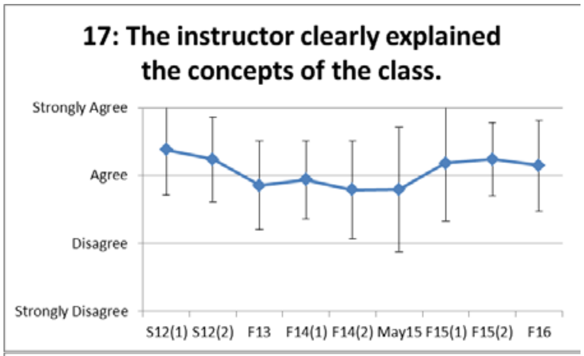
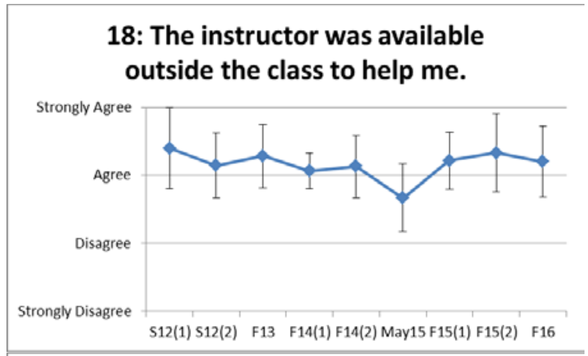
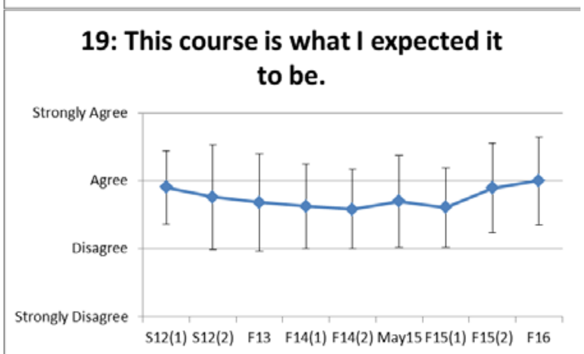
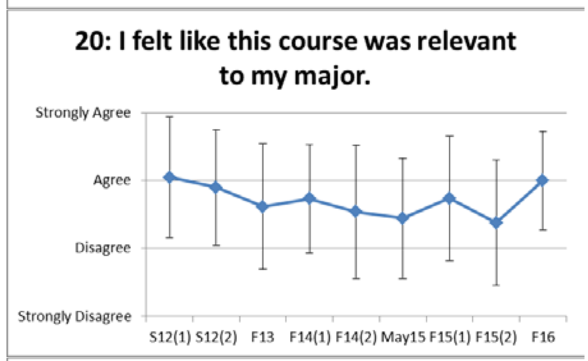

15: I found the instructor to be effective in teaching the subject matter.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12(1) S12(2) F13 F14(1) F14(2) May15 F15(1) F15(2) F16

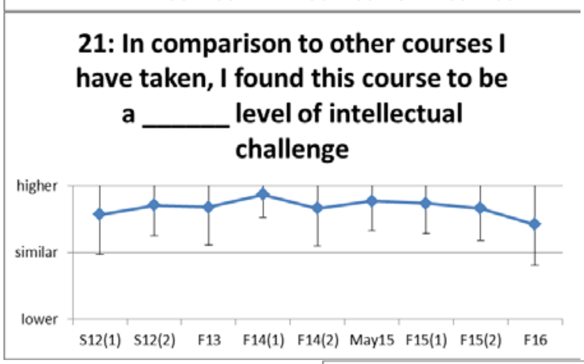

16: I thought the instructor cared about my progress in the course.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12(1) S12(2) F13 F14(1) F14(2) May15 F15(1) F15(2) F16

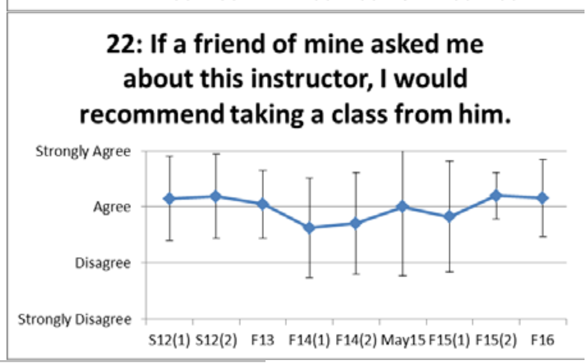


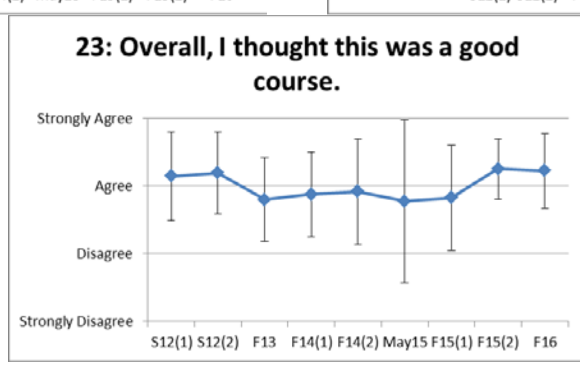












Representative Comments (Spring 2012, both sections)

26: What do you think most contributed to your success in this class?

- Coming to class especially helped and all the examples we did
- Studying for each quiz/test using class notes
- Practice tests

- The MAC
- Attending class, taking notes, doing homework and having previous calculus knowledge
- Professor Errthum because he made the subject easy to understand and was very effective at teaching the course.
- Reviewing past quizzes & tests before exams and Project 3 really helped me understand what I was doing w/ material I had just learned
- Homework system
- Projects/homework

27: What do you think most contributed to any troubles you had in this class?

- Not studying enough.
- It moved a bit fast.
- Projects, just struggles putting the work together on paper & finding the right way to get the answers.
- His higher level of teaching
- The examples in class were nothing like the homework, tests or quizzes
- Difficult material
- Online homework!
- Not asking enough questions
- Projects

28: Do you have any suggestions for the instructor that may be helpful for the next time this course is taught?

- Do examples during the lecture that are harder/more complex
- Slow down, make examples similar to homework
- Explain the projects more at the beginning of the year. I felt almost blind going into the first one.
- Make outlines of his notes that can be filled in during lecture
- Drop WileyPlus
- Have projects due closer to the chapter they are relevant to. Make homework more similar to the inclass examples

29: Do you have any advice you would give at the beginning of the semester to another student taking this course in the future?

- Attend each lecture because the material is explained well & class notes are key.
- Make sure you come to class & ask questions if you don't understand
- Take two precalc classes, and use the tutors
- Attend all lectures, study from lecture notes, not necessarily book/online system
- Don't get behind, go to class only if you need to.
- If you don't understand something get help ASAP since different parts of the class are repeated in depth and with different concepts later on in the course.

- Find groups to work w/ on projects & don’t be scared to go in & ask questions.
- Do the homework.
- My only advice would be show up for every class, like I did because Errthum’s examples really help in order to understand this course
- Attend lectures, work hard on projects
- Pay attention to the in-class examples & ask questions. Also start the projects early.
- Don’t be afraid to ask questions.

30: Any additional comments?

- Awesome professor
- Nope.
- Good job
- Errthum is a great teacher because he understands the subject very well and gives examples of problems in class that really helps to understand the material
- Dr. Errthum was a nice guy & means well, but the class was difficult for me.
- It was a tough class, but I learned a lot and feel good about the skills I learned in this class.
- I enjoyed this class.

Representative Comments (Fall 2013)

24: What do you think most contributed to your success in this class?

- The homework and the textbook examples
- Spending time studying
- Studying the notes from lecture and tutoring
- Multiple examples in lectures
- Going to the tutor center to get help.

25: What do you think most contributed to any troubles you had in this class?

- Homework examples were not covered in class; exams very unfamiliar to quizzes
- The projects were tough
- The fact that math is not my best subject and no calc background
- A lack of explanation from the instructor.

26: Do you have any suggestions for the instructor that may be helpful for the next time this course is taught?

- Please do give examples that will come up in the homework
- Sometimes you get going really fast, there was some homework that was hard to understand

- Don’t use examples on tests we have never seen in class
- Projects that are easier to understand
- Keep up the good work, we can tell you love math and it shows in your instructing

27: Do you have any advice you would give to students in the first week/day of this class in future semesters?

- Study / ask questions / do no procrastinate
- Start projects early!
- Don’t be lazy
- Go to every class
- Do the homework
- If you are having problems, see a tutor!
- Go to all lectures; he is a great teacher; the HW is deceivingly easier
- Put in the work, put in the time. You need to!

28: Any additional comments?

- Decently hard class
- Thanks for being enthusiastic – makes a class I didn’t want to take much more bearable
- Great class, wish I did better
- I enjoyed the class. It tests my knowledge and intelligence and few classes have done that to me.
- Course is a good intro to calculus

Representative Comments (Fall 2014, both sections)

24: What do you think most contributed to your success in this class?

- Doing practice exams
- Exam reclaims; exam examples
- Studying a lot
- Partial credit on tests
- Lecture notes
- Classwork was very important. Most of material was learnt from class.
- Going into his office hours for help.
- Lectures & friends who met in groups
- Reviewing the hand-outs
- Online homework

25: What do you think most contributed to any troubles you had in this class?

- Difficulty jump between samples and homework and the quizzes
- Sometimes examples were done in too fast of a pace.
- Quizzes, hw was hard
- The homework, it was irrelevant to quizzes/exams.
- WileyPlus
- The test material wasn't really covered in class
- Pace of class. It was too much to cover with only 2 classes a week.

26: Do you have any suggestions for the instructor that may be helpful for the next time this course is taught?

- Make quizzes a bit shorter
- Slowly explain each step for examples
- Do more in class examples
- Slower pace, less content
- Make the lectures harder or the exams easier
- Keep printed notes!
- The quizzes are too hard for the few questions that it has & too little time to finish quizzes
- Explain the basics

27: Do you have any advice you would give to students in the first week/day of this class in future semesters?

- Stay on top of things. Ask for help during office hours… it helps!
- Do practice problems in book. Review exams & quizzes
- Attend all sessions to understand & participate in class.
- Study every day, and do homework right away.
- Do homework on time
- Go to every class
- Pay attention
- Take your time getting to know the first few chapters & review them often.

28: Any additional comments?

- Good course, tough freshman course.
- This was awesome class. Finally I got numerical & mathematical views of theories I learn in Econ & Fin classes
- How did you memorize everyone's names?
- I enjoyed taking your class ☺
- The handouts were actually very helpful to organize notes

Representative Comments (May 2015)

24: What do you think most contributed to your success in this class?

- Group studies
- Paying attention/always showing up/doing the hw
- The teacher did a very good job of explaining
- The way how you broke up concepts really helped me learn the material.
- Reading from the book & reviewing the lecture notes!
- The organization of the material. I thought it was easier tying everything together in a 3 week course. I think a reg. term class would have been more difficult
- I like how he printed notes for the class. It made it easier to study. I also liked how he had study guides and posted the solution

25: What do you think most contributed to any troubles you had in this class?

- Too fast paced, overlapping material
- Just the fact that is was a May term class & had to be fast-paced
- Time constraints and overload of HW
- No tutors available
- A lot of the homework questions didn't correlate with the notes in class

26: Do you have any suggestions for the instructor that may be helpful for the next time this course is taught?

- Less chapters
- No, I thought that it was very well taught and he has a lot of energy
- Not quite as many HW problems
- Homework that reflects quiz & exam!
- It was good, I liked having the option to start the exams a little early so we weren't rushed.

27: Do you have any advice you would give to students in the first week/day of this class in future semesters?

- Do you really need this course?
- Always do the hw
- Plan accordingly
- Do the homework, ask questions, and go to class.
- Work hard!!

28: Any additional comments?

- Great professor
- Good job and I had a good 3 weeks

- Thank you for a wonderful class.
- You can't have a Calc class & no tutor!!!
- College Algebra should NOT be a pre req for this class. Wasn't helpful at all.

Representative Comments (Fall 2015, both sections)

24: What do you think most contributed to your success in this class?

- Tutoring, really focusing on this class, a lot of studying
- Going to class
- The notes
- Studying examples
- Taking Calculus 1 before was a big help
- Examples in class were effective and clear in its methods. Concepts were thoroughly explained before going into the content
- Studying and reviewing the quizzes/exams as well looking over the problems that were done in class
- The way the class was taught by a great professor
- Studying/Reviewing notes before tests
- Good effort and homework
- WIleyPlus
- Developing study methods that allowed me to recognize problems and how to solve them
- The instructor made it fun to listen to lecture

25: What do you think most contributed to any troubles you had in this class?

- Reviewing a lot
- Homework
- The exams were difficult
- Not understanding concepts
- I was instructed to take this class before taking precalculus
- You went too fast sometimes and skipped through some steps when teaching a concept
- If I wasn't focused the whole time, it is easy to get lost because so much material is covered in one class period
- Homework was hard sometimes
- Not studying enough
- I didn't take PreCalc in High School
- Definitely missing any classes
- Falling behind on homework/not studying enough

26: Do you have any suggestions for the instructor that may be helpful for the next time this course is taught?

- Test scores are commonly low, maybe better prepare
- Show some harder examples to prepare better for quizzes and exams
- Give less hw or make exams more generally conceptual
- Make the homework more like the examples in notes/book
- Keep doing your thing
- Nope
- Simplify the content of the questions
- Explain multiple times
- More time on quizzes
- More examples in class, force students to learn calculator better

27: Do you have any advice you would give to students in the first week/day of this class in future semesters?

- Take this course seriously, it’s a lot of work but pays off if you put the adequate time in
- Go to class
- Always do all homework, its easy pts
- Study double what you think you should, know concepts inside and out
- Study hard and get help from tutors and the professor
- Take notes and study them
- Quickly switch to another professor. If you can’t, be ready to go to the MAC <u>at least</u> once a week.
- Know concepts from past units, they can apply to things in units to come.
- Focus in lecture & study outside of class
- Do the homework it helps your grade
- For tests, don’t study homework, study notes
- Don’t worry about bad quiz grades, they are a lesson in itself
- Go to a tutor right away
- Come to every class
- Don’t underestimate the course

28: Any additional comments?

- Really enjoyed class
- Very good professor
- I think a class average of 55% on Exam 2 (before the reclaim) was ridiculous. A class average should not be that low.
- You are great at teaching the subject, but perhaps expect too much out of many students who simply are required to take this course.

- I loved Prof. Errthum as a teacher! ☺
- I really enjoyed your teaching style and I had a lot of fun in this class.
- No.

Representative Comments (Fall 2016)

24: What do you think most contributed to your success in this class?

- The teacher
- The pace, helpful examples
- Attending lecture
- Calculator
- Going to tutoring!
- The pre-made notes available online to print
- The homework and the reclaims on exams
- Taking a calc class in high school.
- Review days

25: What do you think most contributed to any troubles you had in this class?

- Homework
- Not studying enough
- Quizzes and exams, more examples
- The large amount of material in a short amount of time

26: Do you have any suggestions for the instructor that may be helpful for the next time this course is taught?

- Better practice exams. I felt like they were too easy but then the test questions where more complicated.
- Notes relevant to homework.
- Keep doing what you’re doing
- Slow down

27: Do you have any advice you would give to students in the first week/day of this class in future semesters?

- Study hard
- Do your homework
- Attend lecture
- Get a tutor
- Make sure to print out the notes
- Be on top of your stuff
- Do homework on the same day you learn the topic

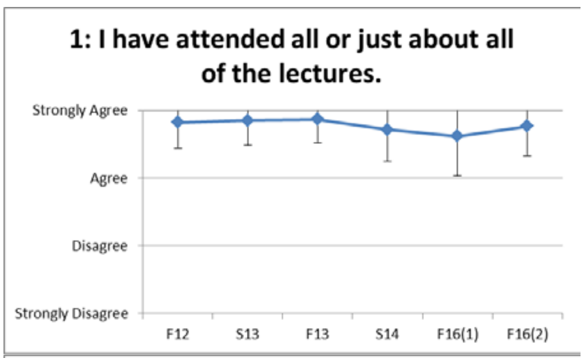
- Stay on top of notes

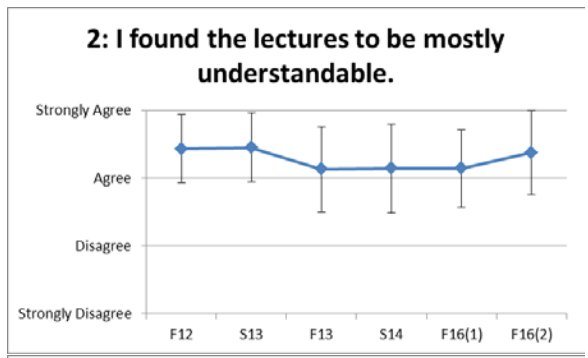
28: Any additional comments?

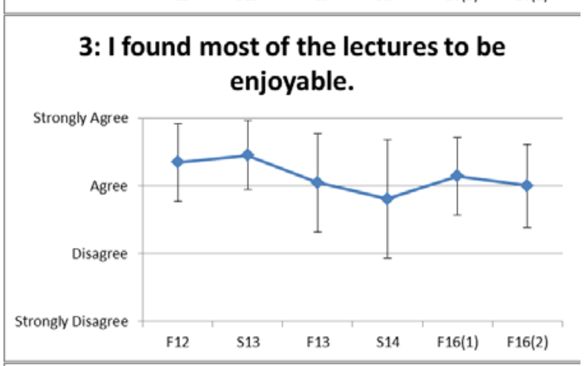
- I really enjoyed this course. Thanks Professor Errthum.
- Really happy with this course. As an econ major this helped me A LOT.

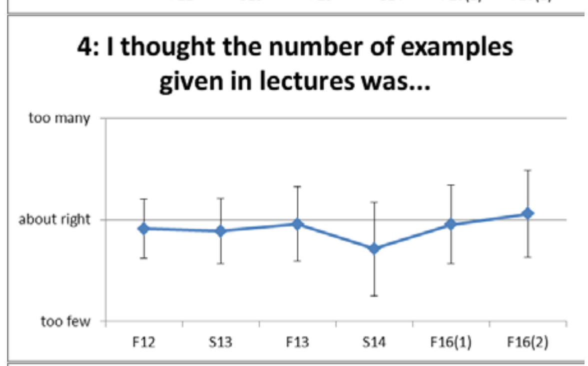
Math 160/212 – Calculus I

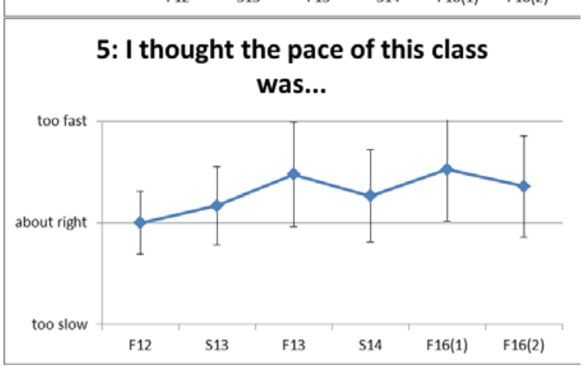
The charts below give the average (with standard deviation) for responses on course evaluations for Math160/212 –Calculus I. After the charts are representative replies to the free response questions.

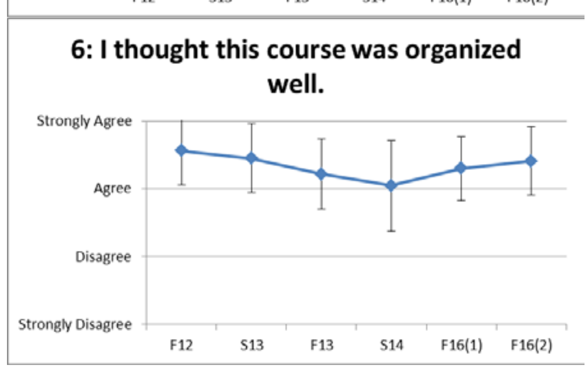


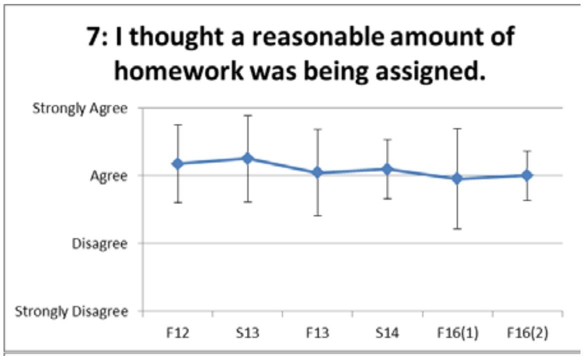


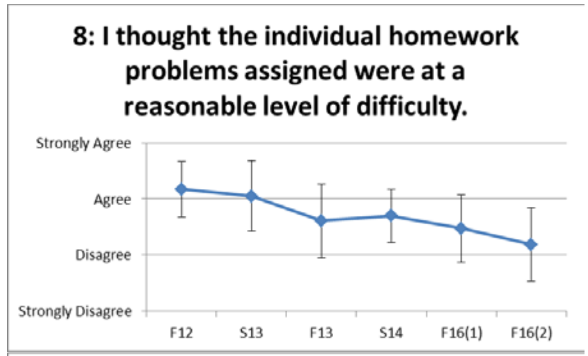


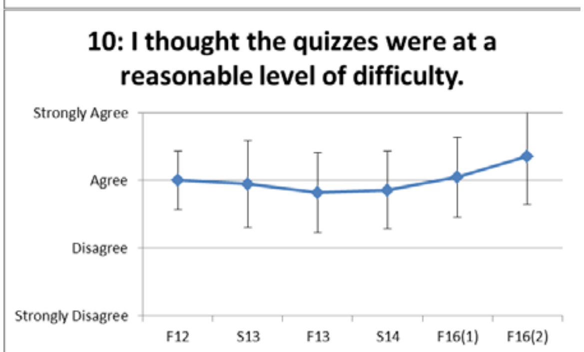


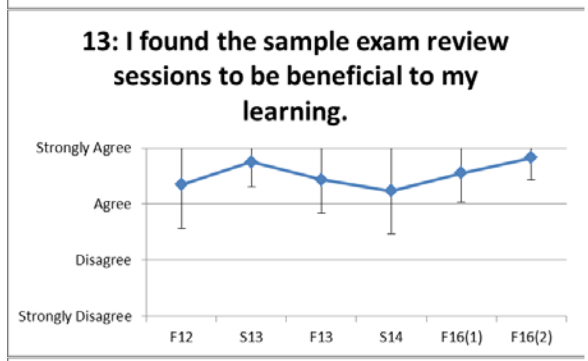


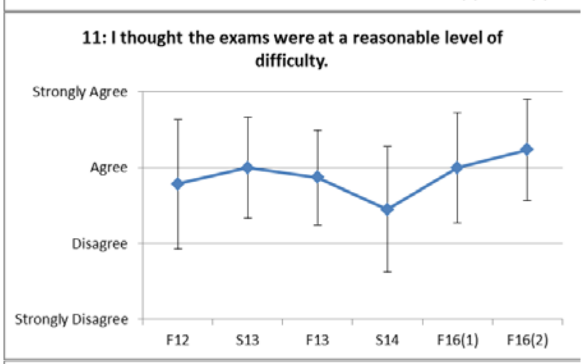


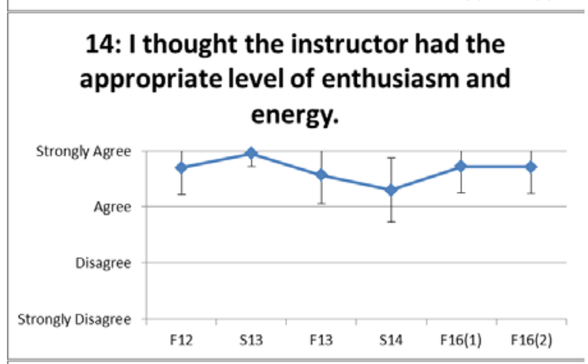

7: I thought a reasonable amount of homework was being assigned.
Strongly Agree
Agree
Disagree
Strongly Disagree
F12
S13
F13
S14
F16(1)
F16(2)

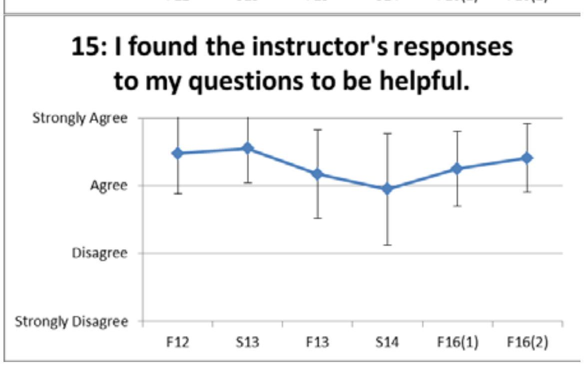

8: I thought the individual homework problems assigned were at a reasonable level of difficulty.
Strongly Agree
Agree
Disagree
Strongly Disagree
F12
S13
F13
S14
F16(1)
F16(2)

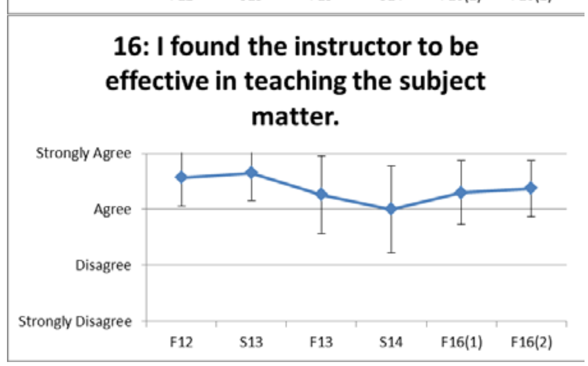

10: I thought the quizzes were at a reasonable level of difficulty.
Strongly Agree
Agree
Disagree
Strongly Disagree
F12
S13
F13
S14
F16(1)
F16(2)

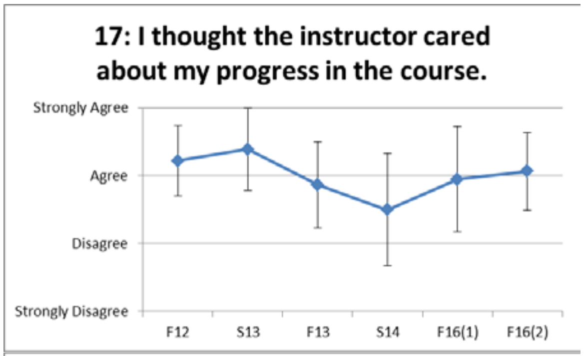

13: I found the sample exam review sessions to be beneficial to my learning.
Strongly Agree
Agree
Disagree
Strongly Disagree
F12
S13
F13
S14
F16(1)
F16(2)

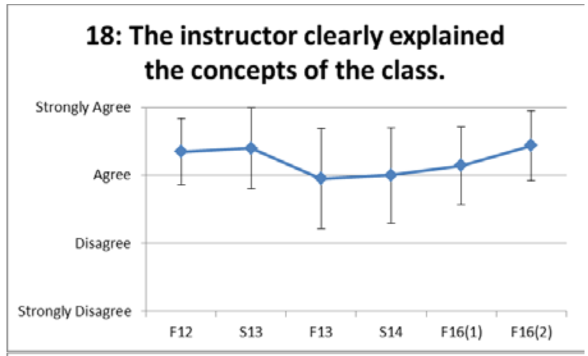

11: I thought the exams were at a reasonable level of difficulty.
Strongly Agree
Agree
Disagree
Strongly Disagree
F12
S13
F13
S14
F16(1)
F16(2)

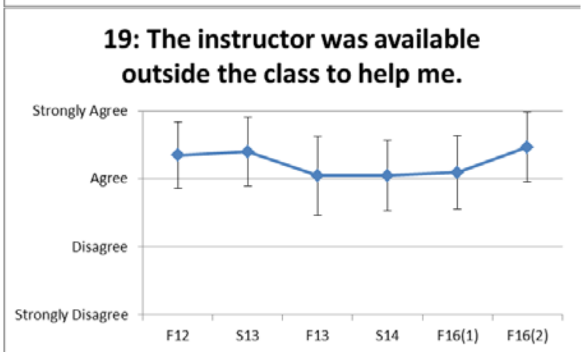

14: I thought the instructor had the appropriate level of enthusiasm and energy.
Strongly Agree
Agree
Disagree
Strongly Disagree
F12
S13
F13
S14
F16(1)
F16(2)


15: I found the instructor's responses to my questions to be helpful.
Strongly Agree
Agree
Disagree
Strongly Disagree
F12
S13
F13
S14
F16(1)
F16(2)

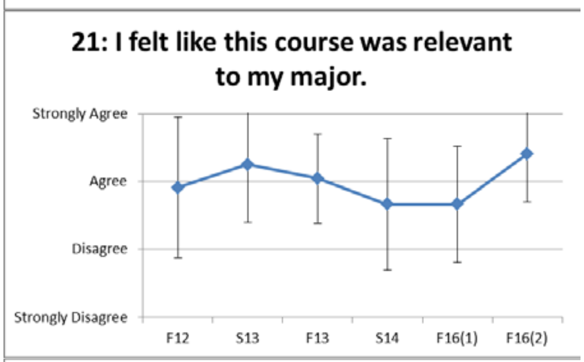

16: I found the instructor to be effective in teaching the subject matter.
Strongly Agree
Agree
Disagree
Strongly Disagree
F12
S13
F13
S14
F16(1)
F16(2)

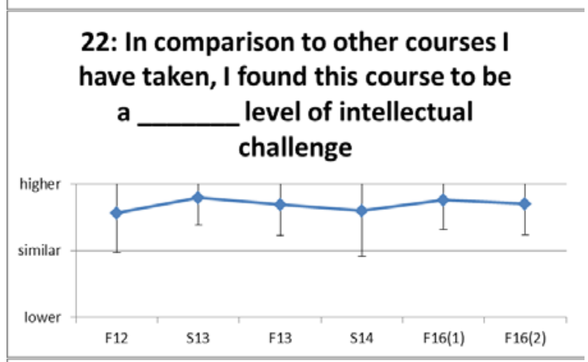

17: I thought the instructor cared about my progress in the course.
Strongly Agree
Agree
Disagree
Strongly Disagree
F12
S13
F13
S14
F16(1)
F16(2)

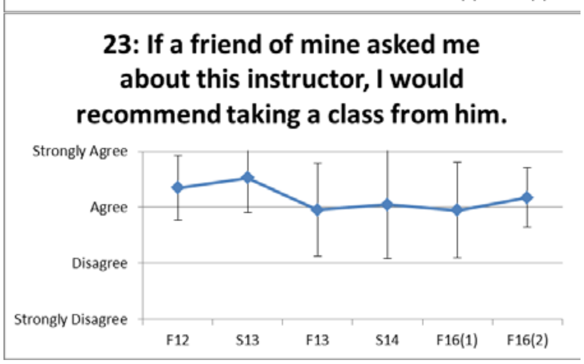

18: The instructor clearly explained the concepts of the class.
Strongly Agree
Agree
Disagree
Strongly Disagree
F12
S13
F13
S14
F16(1)
F16(2)

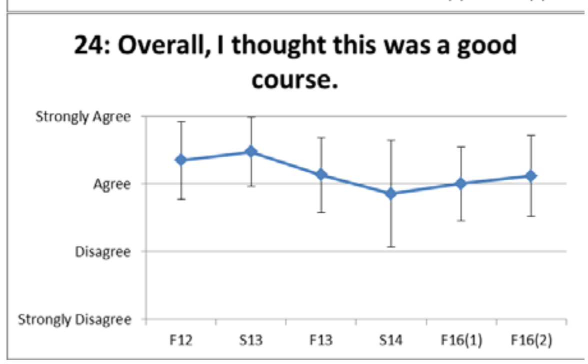

19: The instructor was available outside the class to help me.
Strongly Agree
Agree
Disagree
Strongly Disagree
F12
S13
F13
S14
F16(1)
F16(2)

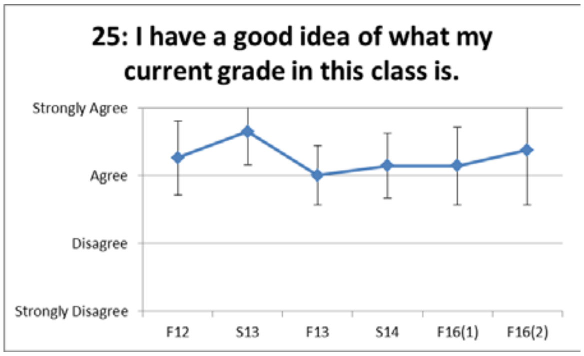

20: This course is what I expected it to be.
Strongly Agree
Agree
Disagree
Strongly Disagree
F12
S13
F13
S14
F16(1)
F16(2)

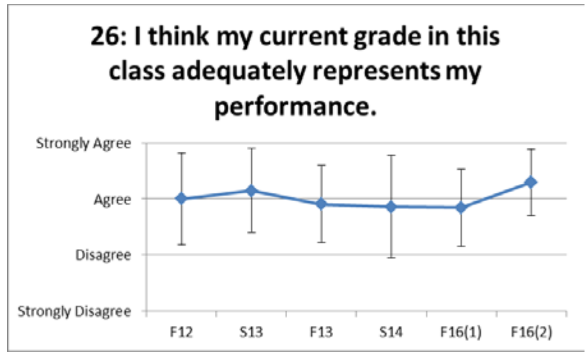

21: I felt like this course was relevant to my major.
Strongly Agree
Agree
Disagree
Strongly Disagree
F12
S13
F13
S14
F16(1)
F16(2)


22: In comparison to other courses I have taken, I found this course to be a _______ level of intellectual challenge
higher
similar
lower
F12
S13
F13
S14
F16(1)
F16(2)


23: If a friend of mine asked me about this instructor, I would recommend taking a class from him.
Strongly Agree
Agree
Disagree
Strongly Disagree
F12
S13
F13
S14
F16(1)
F16(2)


24: Overall, I thought this was a good course.
Strongly Agree
Agree
Disagree
Strongly Disagree
F12
S13
F13
S14
F16(1)
F16(2)





Representative Comments (Fall 2012)

25: What do you think most contributed to your success in this class?

- S2I and practice tests
- Help from prof., homework days
- Paying close attention
- The lectures made a lot of sense and the time we spent working through examples was very helpful.

26: What do you think most contributed to any troubles you had in this class?

- My pre-calc knowledge
- The exams
- Waiting to do the homework instead of right away

27: Do you have any suggestions that may be helpful for the next time this course is taught?

- Go a little easier on tests/quizzes
- A little slower pace during lectures
- More review
- Suggest reading the chapter
- Make sample exams more like the exam
- I really like the Ch. 3 method of teaching

28: Do you have any advice to give to students in the first week of this class in future semesters?

- Really know your pre-calc
- Pay attention, do the homework, ask questions when you don't understand
- Do the homework. Always!
- Do homework right away and go to S2I

29: Any additional comments?

- Great class!

- ☺ I enjoyed the class
- Overall good class, it challenged me

Representative Comments (Spring 2013)

25: What do you think most contributed to your success in this class?

- SI was very helpful outside of class. Also very helpful to have practice exams.
- Learned a lot of things and I love math now!
- I really liked that many types of examples were done in class. I learn better when I am shown how to do something
- Attending all the lectures and doing the work assigned to me

26: What do you think most contributed to any troubles you had in this class?

- Not doing the homework completely and precisely hurt me the most.
- At times I thought the homework assigned was harder than any of things learned in class. At times, I needed to use a tutor/google to understand what I had to do
- The pace at which we did new material
- The test's being difficult also the quizzes

27: Do you have any suggestions that may be helpful for the next time this course is taught?

- Honestly…not really. I really enjoyed every aspect of class.
- Push for more attendance in S2I sessions. They were very helpful
- Have harder examples explained during lecture because homeowork problems were considerably hard

28: Do you have any advice to give to students in the first week of this class in future semesters?

- Do all homework, go to ALL lectures, and go to S2I Then you will do great!
- Know your pre-calc and make sure your algebra skills are sound. Don't panic, calc isn't that difficult
- Stay ahead and make sure you understand the homework

29: Any additional comments?

- Best professor I've had so far
- Great teacher. Not boring, keeps the lectures moving and not boring
- Aleks was kinda pointless. (The exam portion anyway). Maybe in the future just make it finishing the pie chart.

Representative Comments (Fall 2013)

24: What do you think most contributed to your success in this class?

- The homework and the textbook examples
- Spending time studying
- Studying the notes from lecture and tutoring
- Multiple examples in lectures
- Going to the tutor center to get help.

25: What do you think most contributed to any troubles you had in this class?

- Homework examples were not covered in class; exams very unfamiliar to quizzes
- The projects were tough
- The fact that math is not my best subject and no calc background
- A lack of explanation from the instructor.

26: Do you have any suggestions for the instructor that may be helpful for the next time this course is taught?

- Please do give examples that will come up in the homework
- Sometimes you get going really fast, there was some homework that was hard to understand
- Don't use examples on tests we have never seen in class
- Projects that are easier to understand
- Keep up the good work, we can tell you love math and it shows in your instructing

27: Do you have any advice you would give to students in the first week/day of this class in future semesters?

- Study / ask questions / do no procrastinate
- Start projects early!
- Don't be lazy
- Go to every class
- Do the homework
- If you are having problems, see a tutor!
- Go to all lectures; he is a great teacher; the HW is deceivingly easier
- Put in the work, put in the time. You need to!

28: Any additional comments?

- Decently hard class
- Thanks for being enthusiastic – makes a class I didn't want to take much more bearable
- Great class, wish I did better
- I enjoyed the class. It tests my knowledge and intelligence and few classes have done that to me.
- Course is a good intro to calculus

Representative Comments (Spring 2014)

27: What do you think most contributed to your success in this class?

- Talking to the instructor
- The MAC. Getting one on one help there saved my butt.
- Homework Day and Sample exam
- S2I and MAC sessions
- Coming to lectures and doing the homework

28: What do you think most contributed to any troubles you had in this class?

- The exams were much harder than the sample exams

29: Do you have any suggestions that may be helpful for the next time this course is taught?

- Harder examples in class to better prepare for HW & tests
- Have easier problems
- Try to slow down a bit
- Not as many homework days

30: Do you have any advice to give to students in the first week of this class in future semesters?

- Attend ALL lectures
- Go to the MAC!
- Do all of the homework and go to S2I for help doing it.
- Go to class and ask questions!

31: Any additional comments?

- Sometimes the homework assigned was a little too much (amount-wise)
- Most student only need Calc I, so have easier problems
- Professor Errthum is fantastic! He knows what he's talking about and makes class fun.
- Loved the enthusiasm in the instructor.
- Loved this class!
- I thought the connections and explanations of the concepts were very good and not only focused on procedures.

Representative Comments (Fall 2016, both sections)

27: What do you think most contributed to your success in this class?

- External help & notes/examples
- Help from other members of the class.
- The sample tests
- The reviews before the exam
- Instructor, online studying
- Taking Calc 1 and 2 in high school
- Tutoring
- The reclaims on exams
- Study groups and Wolfram Alpha
- Group exam review, pre-exam practice exams
- Example almost exclusively
- Example problems, quiz and exam study sheets
- Getting the homework done, sometimes I would leave lecture with questions and then as I proceeded through homework I figured it out.
- The weekly quizzes to keep me studying & reviewing material each week, & my background knowledge from calc in high school
- Attempting homework problems and being able to see if they were correct or incorrect right away

28: What do you think most contributed to any troubles you had in this class?

- Too few examples on the class
- Not having <u>any</u> pre-experience with this material – I never took any other calculus class before this. My ACT scores are what got me into this course, not preexisting knowledge.
- The concepts
- I feel WebWork was really annoying
- Poor time management
- Algebra-laden questions could be confusing despite understanding the calculus concept being evaluated
- Way too fast, Concepts were introduced and sprinted through. I don't know that t can be done better with the time available but there it is.
- Speed and expectations
- Homework was very hard
- Sometimes not getting as much help, or procrastinating
- Not attending class
- Applying concepts to harder questions
- Exams are a little hard

29: Do you have any suggestions that may be helpful for the next time this course is taught?

- WebWork should be avoided

- Maybe not as many quizzes
- Encourage study groups
- Slow down
- Make homework problems more similar to what quiz problems are
- Maybe slow down for students who cannot keep up with the examples
- I liked the white board days

30: Do you have any advice to give to students in the first week of this class in future semesters?

- Try solving homework yourself, it helps in the exam
- Do the homework in groups
- Make sure you take notes & attend class, practice the material often
- Better be ready
- Come to all classes
- Study for the quizzes and do the homework
- Try hard
- Notes!
- Only take this class if you think numerically, take a different calc class if you think of math spatially
- Always do your homework, don't miss class ever
- Attend, most of the people who dropped this course didn't ever come!
- Be ready to study. You're going to have too.
- Ask questions
- Focus on algebra and the precalc material
- Don't get behind, do your homework the day it's assigned & go into office hours or tutoring center if you need help.
- Take your notes carefully
- Make sure to pay close attention because you don't want to get too off track

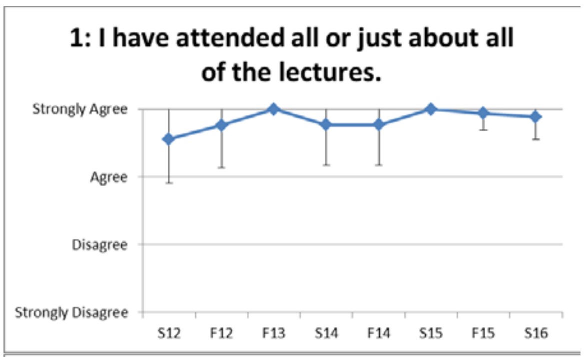
31: Any additional comments?

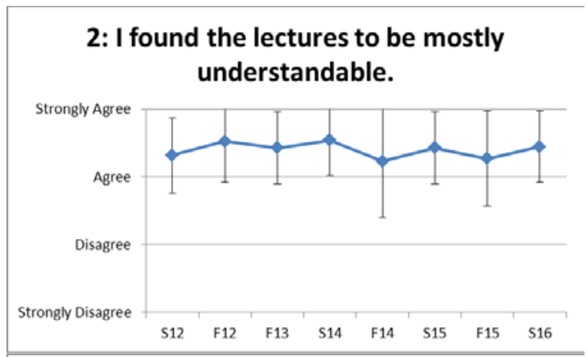
- Yikes
- It was a good class
- Good god math is hard.
- It was nice to have a professor who didn't bother getting bogged down in numbers, means that I liked the "no calculator" approach
- Good lectures, homework is hard, quizzes are fine
- Good teacher, always enthusiastic, makes waking up at 8am fun.
- Great professor, just a little too fast for me.
- Thanks!!

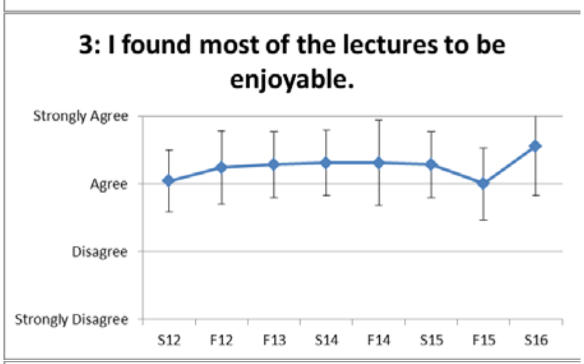
- I think the homework is a little too difficult if you're attempting to do it w/o looking it up on the internet
- I enjoyed the class
- Enjoyable class!
- Thank you for your effort in enthusiastically teaching this class.
- Great energy & enthusiasm!

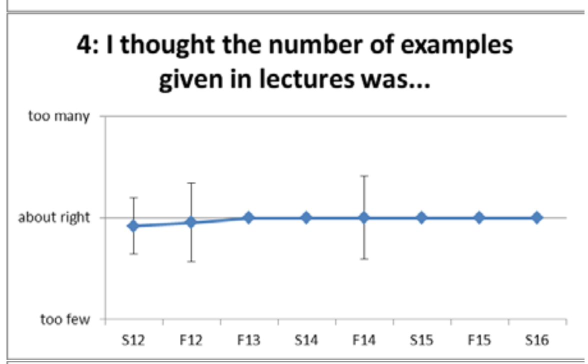
Math 165/213 – Calculus II

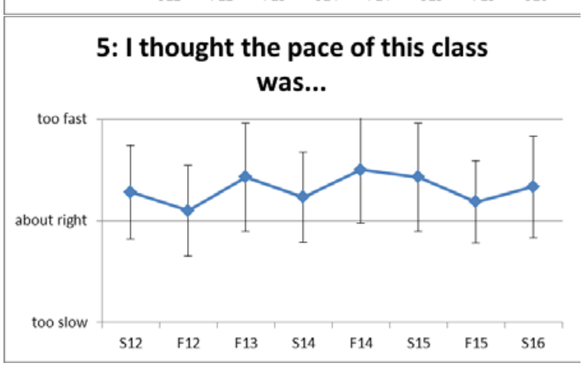
The charts below give the average (with standard deviation) for responses on course evaluations for Math165/213 –Calculus II. After the charts are representative replies to the free response questions.

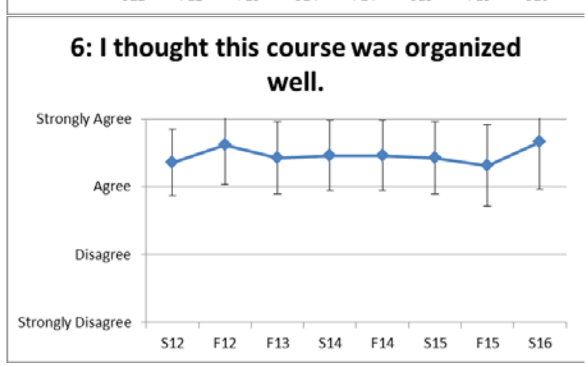


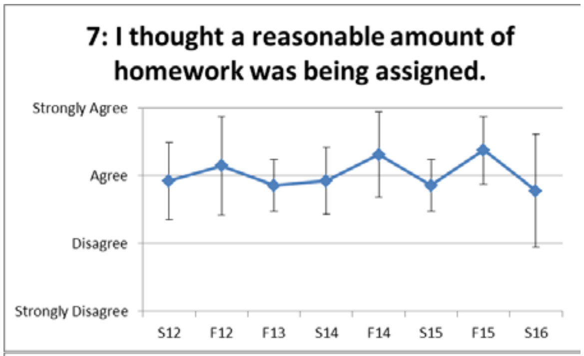


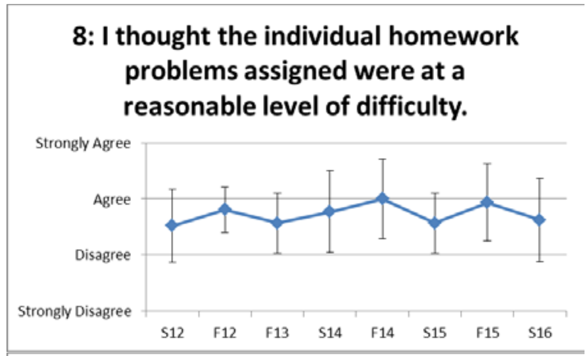


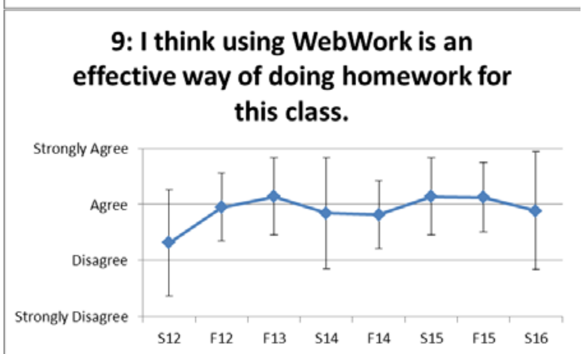


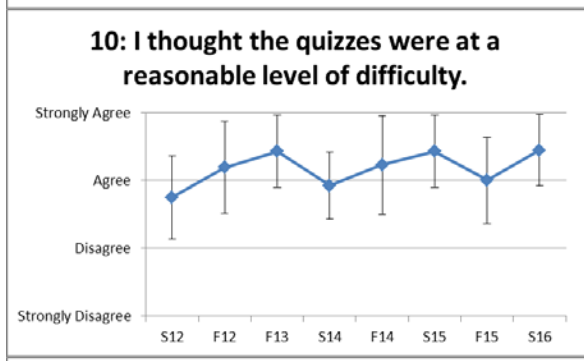


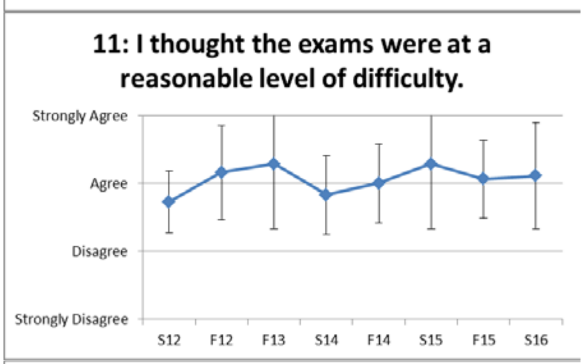


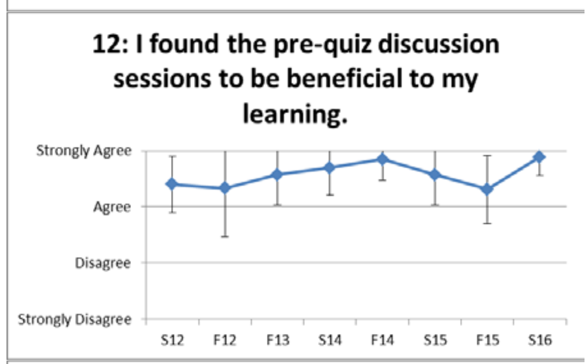

7: I thought a reasonable amount of homework was being assigned.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12 F12 F13 S14 F14 S15 F15 S16

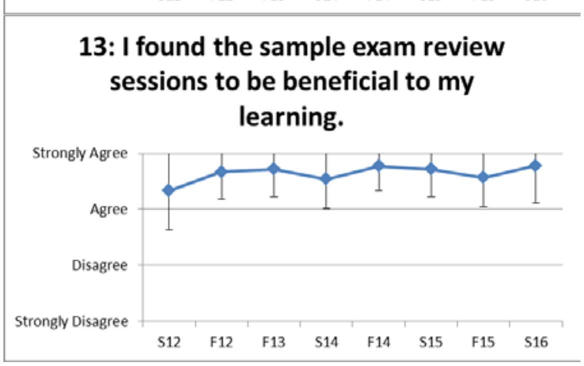

8: I thought the individual homework problems assigned were at a reasonable level of difficulty.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12 F12 F13 S14 F14 S15 F15 S16

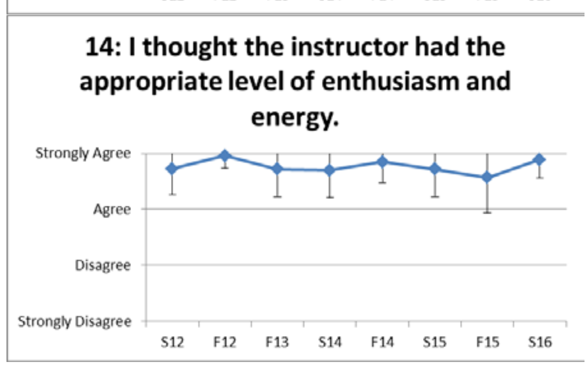

9: I think using WebWork is an effective way of doing homework for this class.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12 F12 F13 S14 F14 S15 F15 S16

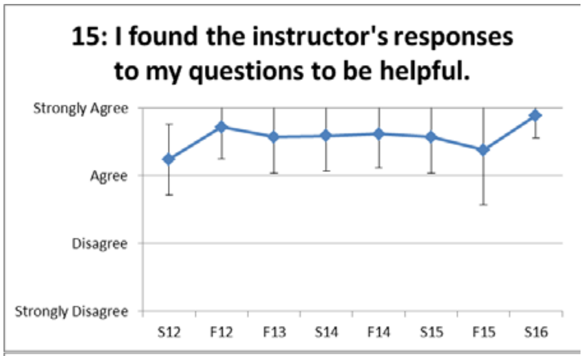

10: I thought the quizzes were at a reasonable level of difficulty.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12 F12 F13 S14 F14 S15 F15 S16

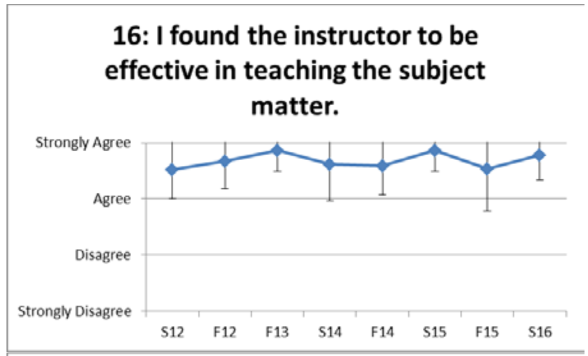

11: I thought the exams were at a reasonable level of difficulty.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12 F12 F13 S14 F14 S15 F15 S16

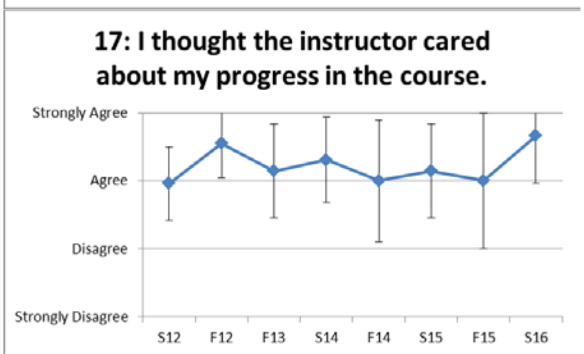

12: I found the pre-quiz discussion sessions to be beneficial to my learning.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12 F12 F13 S14 F14 S15 F15 S16

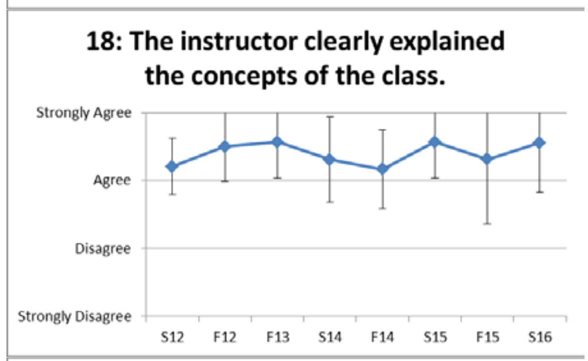

13: I found the sample exam review sessions to be beneficial to my learning.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12 F12 F13 S14 F14 S15 F15 S16

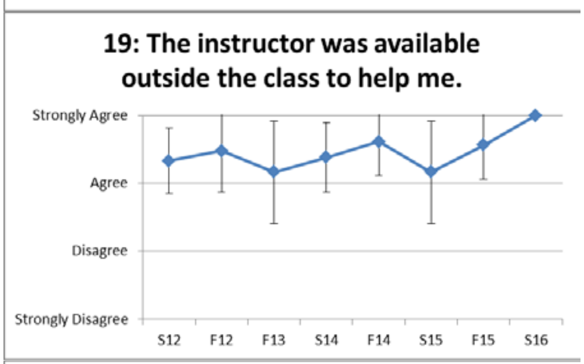

14: I thought the instructor had the appropriate level of enthusiasm and energy.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12 F12 F13 S14 F14 S15 F15 S16


15: I found the instructor's responses to my questions to be helpful.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12 F12 F13 S14 F14 S15 F15 S16

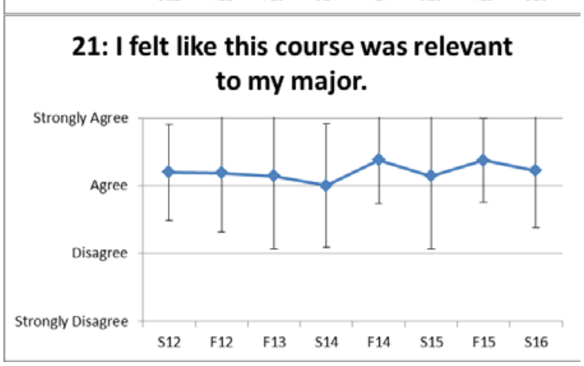

16: I found the instructor to be effective in teaching the subject matter.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12 F12 F13 S14 F14 S15 F15 S16

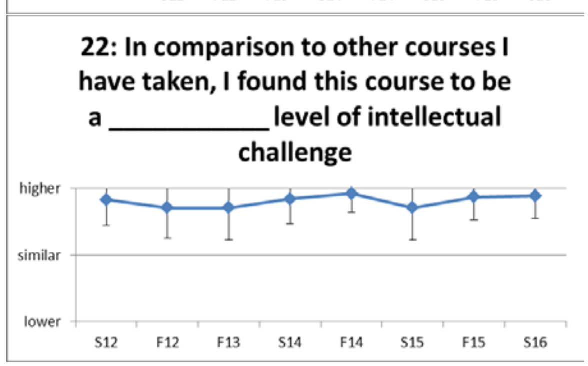

17: I thought the instructor cared about my progress in the course.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12 F12 F13 S14 F14 S15 F15 S16

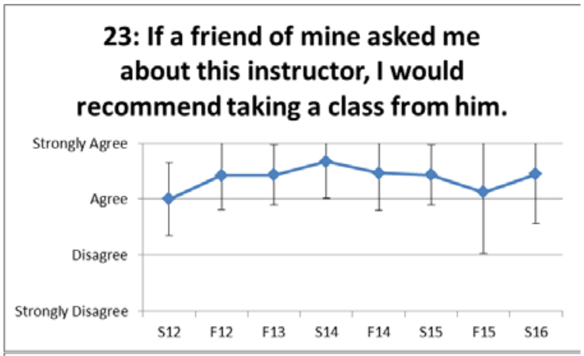

18: The instructor clearly explained the concepts of the class.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12 F12 F13 S14 F14 S15 F15 S16

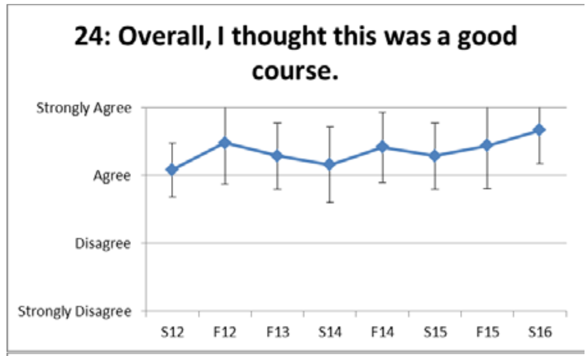

19: The instructor was available outside the class to help me.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12 F12 F13 S14 F14 S15 F15 S16

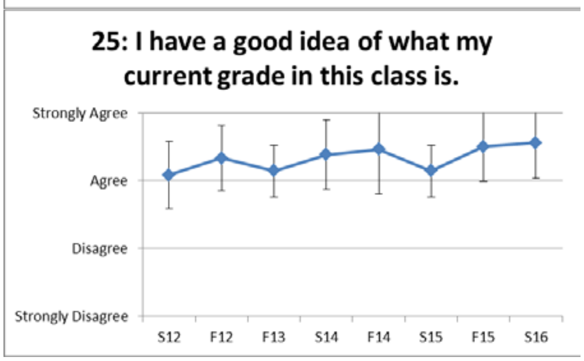

20: This course is what I expected it to be.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12 F12 F13 S14 F14 S15 F15 S16

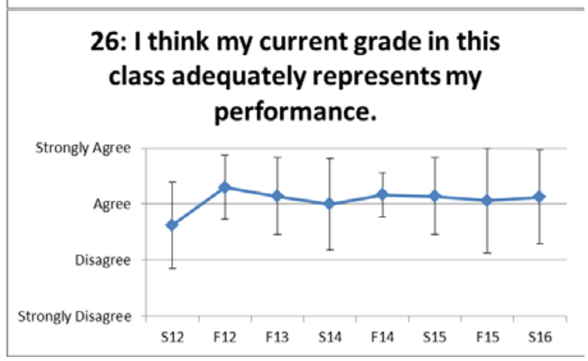

21: I felt like this course was relevant to my major.
Strongly Agree
Agree
Disagree
Strongly Disagree
S12 F12 F13 S14 F14 S15 F15 S16


22: In comparison to other courses I have taken, I found this course to be a ___________ level of intellectual challenge
higher
similar
lower
S12 F12 F13 S14 F14 S15 F15 S16









Representative Comments (Spring 2012)

27: What do you think most contributed to your success in this class?

- I loved Webassign. I liked how it gave practice problem and show me how to do the problem. I also like all the example that were given in class.
- going to lecture, the sample exams
- I really like your teaching style. Examples we learn in class directly relate to what we need to know. I was not surprised for tests and quizzes. I knew what I needed to know for them.
- Hard work, lots of study time
- Number of examples worked & good lectures

28: What do you think most contributed to any troubles you had in this class?

- My own distractions that kept me off task.
- Webassign
- Fast paced
- Not studying or reading the book
- Not having a good foundation in calc I
- Differences between instructor’s presentation of material and the book.

29: Do you have any suggestions that may be helpful for the next time this course is taught?

- Just use harder examples in class
- Different method for homework

- Quizs not so hard
- More extra credit
- Teach with calculators & more on paper homework instead of webassign
- No. Fantastic if you teach it.
- Slow down lectures

30: Do you have any advice to give to students in the first week of this class in future semesters?

- Take good notes and work hard
- Do those PSP's for review
- Keep up with the homework
- Do not skip lecture!
- Study and go over things a lot
- Prepare for notes and stay on task
- Don't take the course lightly
- Review old Calc 1 stuff & basic algebra
- If you have a question, don't be afraid to ask.
- Good luck! ☺

31: Any additional comments?

- ☺
- Overall Calc II was hard, but the concepts were hard no matter what. He did a great job teaching.
- You're a really good teacher! ☺
- I do NOT like Webassign. Most problems come with answer input & PSPs.
- Great prof, but hard grader.
- Great! ☺
- Thank you.

Representative Comments (Fall 2012)

27: What do you think most contributed to your success in this class?

- Studying
- Going to class
- Using class notes on WebAssign
- The lectures and taking good notes
- The review sessions before exams and quizzes

28: What do you think most contributed to any troubles you had in this class?

- Not doing the WebAssign
- Pace of the class was a bit fast

- I didn't have as strong algebra skills as I should have

29: Do you have any suggestions that may be helpful for the next time this course is taught?

- In class examples closer to test questions
- More review in the beginning maybe. Especially for fall semester when we have been out of class for a while.
- Slow down during series a little

30: Do you have any advice to give to students in the first week of this class in future semesters?

- Don't save the homework for the last minute
- Have fun and do the work
- Pay Attention
- It's a hard class, no way around it. Do well on the quizzes. They prepare you for tests.
- Absolutely, 100%, do NOT skip any lectures of this class because it will probably put you very far behind.

31: Any additional comments?

- Thank you!
- Congrats. You made me semi-enjoy math!

Representative Comments (Fall 2013)

27: What do you think most contributed to your success in this class?

- Staying on top of the homework, coming to class everyday, and weekly quizzes made me stay prepared throughout the course.
- The homework and his examples were very helpful
- How well the professor taught and the amount of work assigned was great but there were many chances to get points in the class.
- Trying to understand the math behind everything
- Sample exams
- Having already taken the course in high school
- Errthum's in-depth coverage of everything as well as pointing out where common mistakes happen

28: What do you think most contributed to any troubles you had in this class?

- The course requires a lot of time to learn some of the topics, sometimes I was busy
- The pace is faster than any I have ever taken, but I enjoyed the challenge
- Not doing the work

- Not going in for extra help on material that I did not understand
- Remembering previous Calc I or pre-calc knowledge
- Lots of concepts that I didn't fully understand, making little mistakes that turned into big problems

29: Do you have any suggestions that may be helpful for the next time this course is taught?

- Do the same thing!
- Some day there were an excessive number of examples
- Really describe how to do Taylor series and what is happening each step.
- Hammer out the geometry more.
- Spend a little extra time on the section involving work => lots of variables
- No WebAssign
- A little more time between quizzes
- Give more difficult examples in class

30: Do you have any advice to give to students in the first week of this class in future semesters?

- Stay on top of the homework, do the PSP's, quizzes are your best friend, good luck with Taylor series
- If you want to increase your math knowledge, stay in this class.
- Pace yourself on the homework. Don't just do it all before the quizzes.
- Take Errthum and learn the geometry.
- Take advantage of the E.C. opportunities & show up to class.
- Make sure to know Calc I

31: Any additional comments?

- Loved the class! Thanks so much!
- Examples in-class and teaching style made it easier to learn.
- Ditch WebAssign and drop Stewart

Representative Comments (Spring 2014)

27: What do you think most contributed to your success in this class?

- Examples while in lecture
- Studying and asking questions
- Homework review days before quizzes
- Doing the homework, attending <u>all</u> lectures, sample exams
- The lectures were presented in a way that I understood and enjoyed.

28: What do you think most contributed to any troubles you had in this class?

- The quizzes, not fully understanding, not having all the formulas memorized
- Procrastinating on webassign and cramming for exams
- The pace, it was a little too fast for me
- Not knowing pre-calc stuff
- WebAssign is a pain in the ass. Sometimes too little time was spent on more difficult subjects.

29: Do you have any suggestions that may be helpful for the next time this course is taught?

- Maybe for exams allow a note card or small note sheet for the formulas
- Have some paper homework that is due at times throughout the semester.

30: Do you have any advice to give to students in the first week of this class in future semesters?

- Do the PSP's & study for the exams & quizzes. Also ask any questions may have. Do it before the night before
- Study series, understand series
- Study for the final continuously throughout the course
- Attend <u>every</u> lecture, do webassign on time, pace yourself, don't cram
- Don't skip class, <u>study!</u>

31: Any additional comments?

- Good, challenging course. Frustrating at times but a great intellectual challenge.
- Professor is a great teacher just an extremely hard course

Representative Comments (Fall 2014)

27: What do you think most contributed to your success in this class?

- Attending all of the lectures
- In-class review
- Doing the homework and spending a lot of time outside of class on the material.
- WebAssign
- Seeing the professor during office hours and studying with classmates

28: What do you think most contributed to any troubles you had in this class?

- Distractions and being tired/not focused in class
- In-class → out of/after class retention
- It was very high paced and the problems on the quiz/tests weren't very similar to those on webassign

- Failure to do homework
- Textbook
- Difficult material being taught that I had never seen before.

29: Do you have any suggestions that may be helpful for the next time this course is taught?

- Keep doing what your doing
- Maybe make the webassign questions more relevant to the in-class material
- A little less time spent on the easy stuff, a little more time spent on the complicated stuff
- A few more examples in class even just basic simple examples in order to get the basics down before trying harder problems
- Errthum should write his own book, with his own notation & chapter order, especially the notation thing

30: Do you have any advice to give to students in the first week of this class in future semesters?

- Memorize formulas that are given in lecture. Don't wait to do PSP and other homework.
- Your going to bomb your first couple quizzes until you figure out you need to study your notes over and over.
- Sit back & absorb the material at your pace w/ prof's guidance
- Do the homework & extra practice problems
- Read the book before lecture
- Do <u>NOT</u> miss any lecture days. You will fall behind fast.

31: Any additional comments?

- You did a great job teaching this class.
- Being a freshman, this course has been an eye opener and I believe it will prepare me very well for what is ahead of me.
- You are a great guy. The class was a nice challenge. Your tests and quizzes are incredibly hard and degrading considering I put hours of studying into preparing for every test.
- I really liked you as a professor for this course. I liked how you were able to crack some jokes.
- Thank you, It was a pleasure

Representative Comments (Spring 2015)

27: What do you think most contributed to your success in this class?

- Going to all of the lectures and completing all the homework thoroughly
- Plenty of in class examples & enough time to do homework and ask questions before quizzes & exams
- Instructor's enthusiasm & willingness to answer my dumb questions

28: What do you think most contributed to any troubles you had in this class?

- Inadequate studying or preparation. Also could have been not paying attention in class or not paid attention to the homework
- The pace that you go from example to example
- The test questions were sometimes different than the example problems

29: Do you have any suggestions that may be helpful for the next time this course is taught?

- Maybe spend a little time going over most frequently missed questions after each quiz or exam
- Make tests worth less of your grade
- More group activities
- Not really
- Split the homework into smaller assignments
- Don't explain too fast because not all the international students can get the idea when you are explaining so fast.

30: Do you have any advice to give to students in the first week of this class in future semesters?

- Listen to the lectures and you will do fine.
- Stick with the class, the stuff you learn changes so you always have a chance to pick up your grade
- Do homework the day you learn that subject instead of right before it's due
- Make sure to come to class and to take good notes!
- Ask questions even if you think they're dumb; find a way to be interested & personally invested in course material; study, study, study.

31: Any additional comments?

- Overall, great course!
- Thanks!!
- Easily the best math course I've ever taken. Will be back for more of Dr. Errthum's classes

Representative Comments (Fall 2015)

27: What do you think most contributed to your success in this class?

- The bonus questions offered on every test/quiz
- Asking questions, studying
- Doing homework problems
- In class review for tests/quizzes

28: What do you think most contributed to any troubles you had in this class?

- The WebAssign problems were sometimes more challenging than things we learned in class
- Time management, 1st "adult math class"
- Not studying enough. Need to do more problems
- I was slow at understanding some of the concepts in class
- Professor going to fast or wont help answer refreshing questions
- The instructor's view of having students prove themselves to him.
- Bad Pre-Calc/Average algebra background mixed with different instructor.

29: Do you have any suggestions that may be helpful for the next time this course is taught?

- None. Very well taught
- More real world problems
- No not really, I really enjoyed the lectures
- Perhaps a brief geometry review during the first week
- Slow down & explain

30: Do you have any advice to give to students in the first week of this class in future semesters?

- Do the homework because it helps you review, Go to class
- Keep up with homework. DO NOT miss lectures. STUDY!
- Give yourself enough time to work out all the homework problems on webassign. Don't try to do them all at once
- Try your best
- Be prepared

31: Any additional comments?

- Great job, love the enthusiasm
- I really did enjoy this class.
- This was a fun class to be a part of. ☺

Representative Comments (Spring 2016)

27: What do you think most contributed to your success in this class?

- Attending lecture / help on hw outside of class
- Consistently doing the homework and making sure I actually understood the "why" of problems.
- Asking questions
- Taking notes, studying, participating in class

28: What do you think most contributed to any troubles you had in this class?

- It requires a lot of time/practice to understand the concepts
- Not doing the homework
- Not studying, not being comfortable enough with algebra/precalc
- Webworks took too long, usually spent 3-4 hours per assignment

29: Do you have any suggestions that may be helpful for the next time this course is taught?

- No, really enjoyed it
- Harder examples in class
- I have to say sometimes the nature of your problems seems so daunting sometimes students just give up

30: Do you have any advice to give to students in the first week of this class in future semesters?

- Take notes / <u>attend lecture</u>
- Get in the habit of doing the homework
- Stick with it, tough course but manageable if participating
- Study more
- Make sure they are very comfortable,/brushed up on prerequisite material

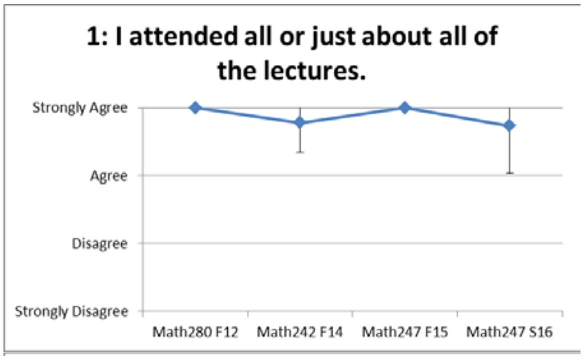
31: Any additional comments?

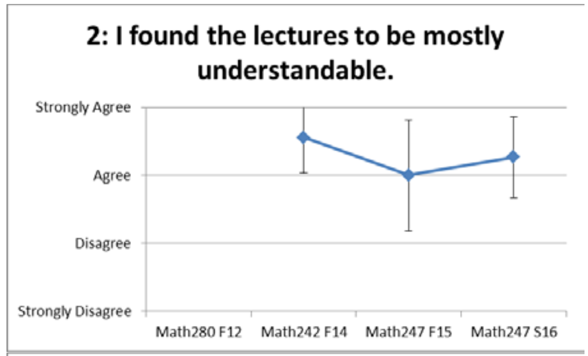
[no replies]

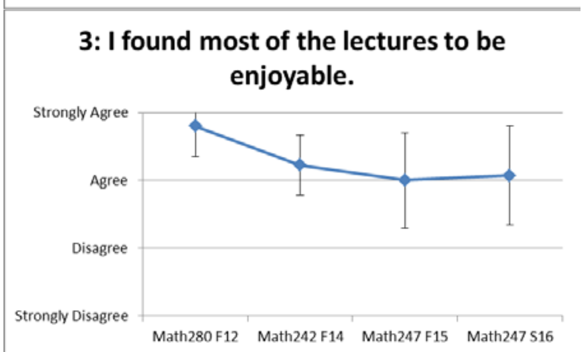
Other 200-level Math courses

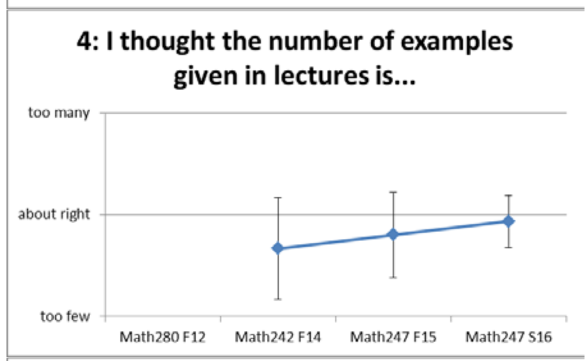
The charts below give the average (with standard deviation) for responses on course evaluations for 200-level Math courses I've taught beside Calculus I and II. After the charts are representative replies to the free response questions.

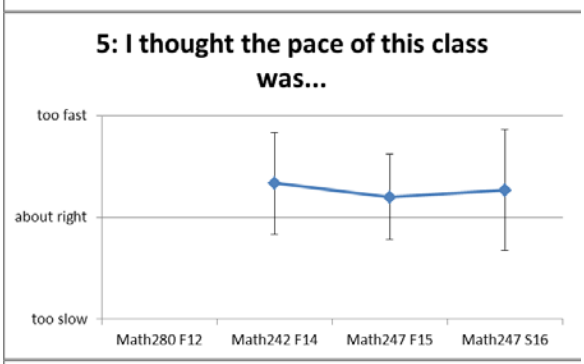
Note: As these evaluations correspond to a variety of courses, there is a larger variation in responses and the numbering of evaluation questions is inconsistent and contains gaps since some questions were specific to the course.

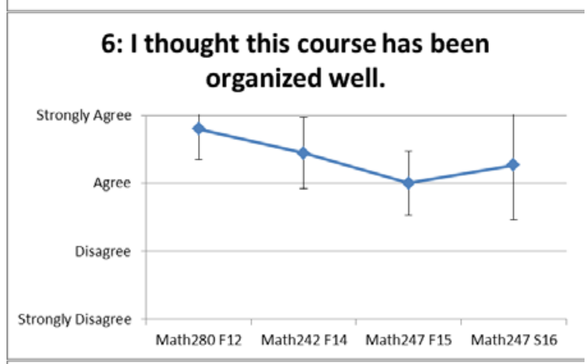

1: I attended all or just about all of the lectures.
Strongly Agree
Agree
Disagree
Strongly Disagree
Math280 F12
Math242 F14
Math247 F15
Math247 S16

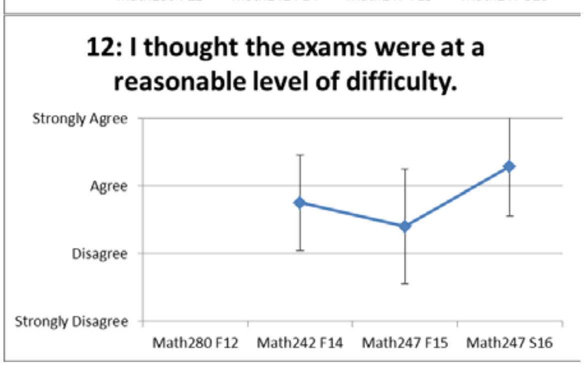

2: I found the lectures to be mostly understandable.
Strongly Agree
Agree
Disagree
Strongly Disagree
Math280 F12
Math242 F14
Math247 F15
Math247 S16

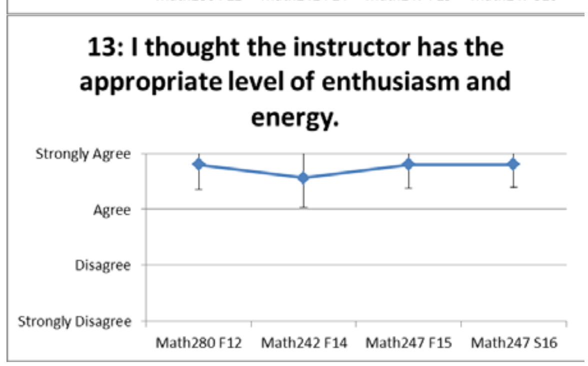

3: I found most of the lectures to be enjoyable.
Strongly Agree
Agree
Disagree
Strongly Disagree
Math280 F12
Math242 F14
Math247 F15
Math247 S16

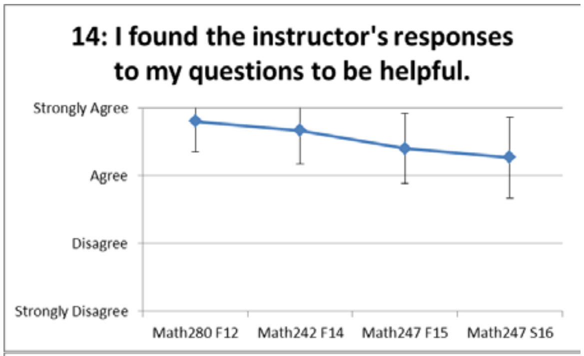

4: I thought the number of examples given in lectures is...
too many
about right
too few
Math280 F12
Math242 F14
Math247 F15
Math247 S16

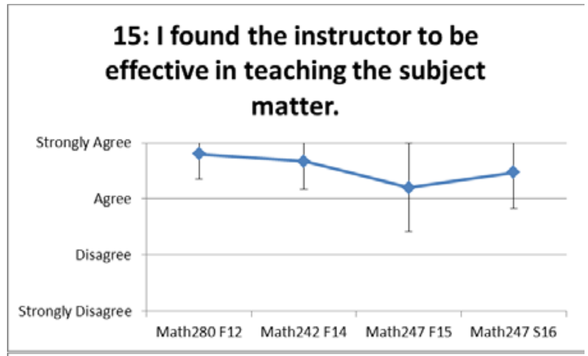

5: I thought the pace of this class was...
too fast
about right
too slow
Math280 F12
Math242 F14
Math247 F15
Math247 S16

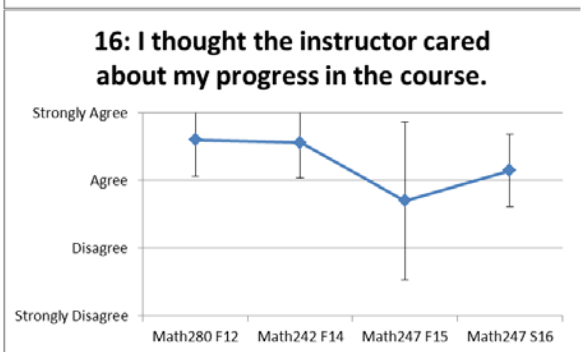

6: I thought this course has been organized well.
Strongly Agree
Agree
Disagree
Strongly Disagree
Math280 F12
Math242 F14
Math247 F15
Math247 S16

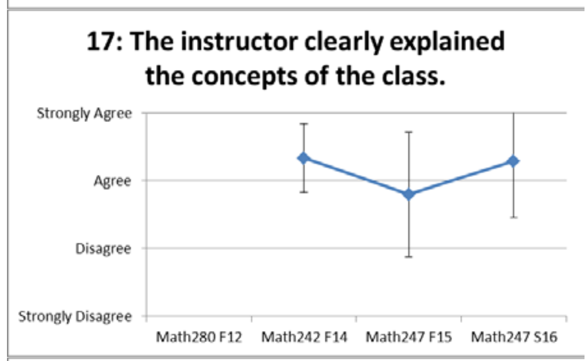

12: I thought the exams were at a reasonable level of difficulty.
Strongly Agree
Agree
Disagree
Strongly Disagree
Math280 F12
Math242 F14
Math247 F15
Math247 S16

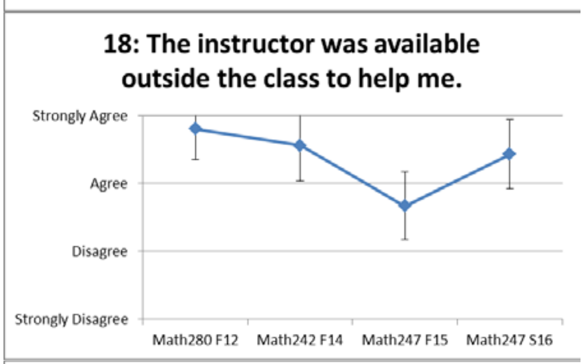

13: I thought the instructor has the appropriate level of enthusiasm and energy.
Strongly Agree
Agree
Disagree
Strongly Disagree
Math280 F12
Math242 F14
Math247 F15
Math247 S16

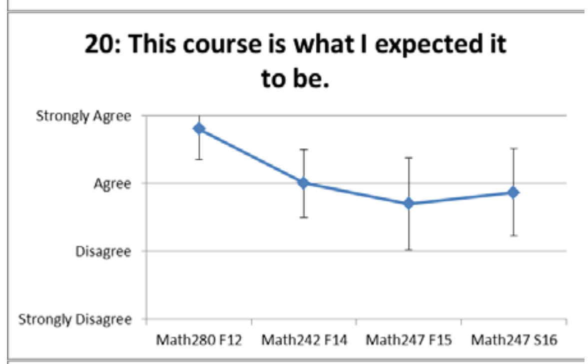

14: I found the instructor's responses to my questions to be helpful.
Strongly Agree
Agree
Disagree
Strongly Disagree
Math280 F12
Math242 F14
Math247 F15
Math247 S16

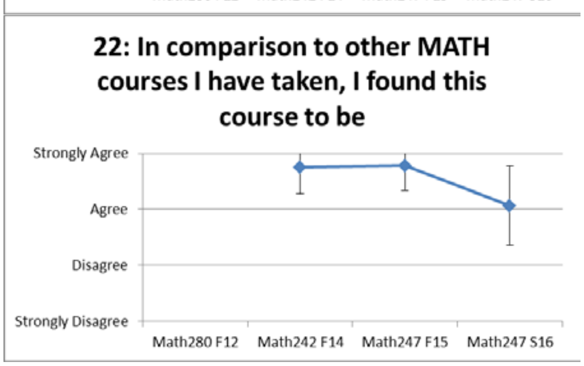

15: I found the instructor to be effective in teaching the subject matter.
Strongly Agree
Agree
Disagree
Strongly Disagree
Math280 F12
Math242 F14
Math247 F15
Math247 S16

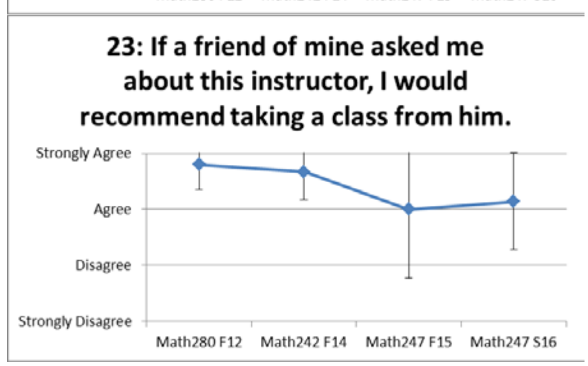

16: I thought the instructor cared about my progress in the course.
Strongly Agree
Agree
Disagree
Strongly Disagree
Math280 F12
Math242 F14
Math247 F15
Math247 S16

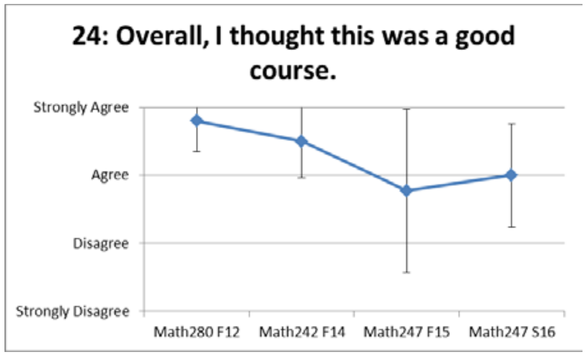

17: The instructor clearly explained the concepts of the class.
Strongly Agree
Agree
Disagree
Strongly Disagree
Math280 F12
Math242 F14
Math247 F15
Math247 S16


18: The instructor was available outside the class to help me.
Strongly Agree
Agree
Disagree
Strongly Disagree
Math280 F12
Math242 F14
Math247 F15
Math247 S16


20: This course is what I expected it to be.
Strongly Agree
Agree
Disagree
Strongly Disagree
Math280 F12
Math242 F14
Math247 F15
Math247 S16


22: In comparison to other MATH courses I have taken, I found this course to be
Strongly Agree
Agree
Disagree
Strongly Disagree
Math280 F12
Math242 F14
Math247 F15
Math247 S16


23: If a friend of mine asked me about this instructor, I would recommend taking a class from him.
Strongly Agree
Agree
Disagree
Strongly Disagree
Math280 F12
Math242 F14
Math247 F15
Math247 S16



Representative Comments (Math242, Fall 2014)

27: What do you think most contributed to your success in this class?

- Good lectures
- I read all of the assigned homework and studied a lot before each test
- The professor; online sources other than the book
- The help I received during office hours

28: What do you think most contributed to any troubles you had in this class?

- The book, not always clear when explaining
- The pace was slightly too fast and I didn't always have enough time to soak in the information

29: Do you have any suggestions that may be helpful for the next time this course is taught?

- I think you did a very good job. Maybe just assign the practice problems rather than solely recommending them.
- Slow down and give more examples as will be seen on the tests.

30: Do you have any advice to give to students in the first week of this class in future semesters?

- Do homeworks and come to lectures; do not drop the class without trying your best.
- Read all the readings and do the activities early, not last minute.
- Be prepared to work hard & spend time in this course.

31: Any additional comments?

- Great professor!
- Thanks for always helping when questions were asked!
- I appreciate your willingness to help outside of class.

Representative Comments (Math247, Fall 2015)

25: What do you think most contributed to your success in this class?

- Working hard and going to every class. I wouldn't go as far to call it success though. Unless not failing is a success. My goal was an A and I got a B so I failed.
- Taking notes
- Attend class. Do homework regularly
- Study group
- Being able to go into office hours & oral hw days
- Having prior knowledge
- Office hours
- All the examples we did in class

26: What do you think most contributed to any troubles you had in this class?

- Sometimes the lectures were confusing & he would just move on
- Written homework
- Not studying enough

27: Do you have any suggestions that may be helpful for the next time this course is taught?

- Written homework needs to not be impossible.
- Go slower and not so much homework. Even amount of oral & written, not a lot of both.
- Actually care about your students. Do your own assignments.
- Not really, the class seemed to be the issue

28: What advice would you give to another student taking this course next time?

- Be prepared. Its Math247 but it felt like a Math 400 level course. Only reason I recommend it is because while Errthum can be kinda rough, the dude is hilarious. It's a love hate relationship. You get killed in the homeworks, but his humor was fantastic to me.
- Get a tutor if you don't know anything. Join a study group. Figure out Errthum's office hours.
- Read the textbook ahead of classes
- Don't procrastinate on homework
- Go in for help if you need it, don't just complain
- Go to his office hours. It may be difficult stuff, but he always helps when you see him in his office.

29: Any additional comments?

- See ya in Number Theory
- Have a good break!

Representative Comments (Math247, Spring 2016)

25: What do you think most contributed to your success in this class?

- Having oral, written, and review days to study
- Keeping up with all the homework and visiting office hours
- Attending lectures and doing the readings
- Oral homework days, even though the written homework was usually difficult, it definitely solidified the concepts.
- The combination of the website and the lectures

26: What do you think most contributed to any troubles you had in this class?

- The lecture was very fast.
- Not studying hard enough for exams
- The written homework
- I thought the exams were too long.
- Simply being a different "set" of math than I usually deal with.

27: Do you have any suggestions that may be helpful for the next time this course is taught?

- Some of the later topics should be taken at a slower pace
- I think maybe less frequent hw days. I think the course sometimes felt rushed b/c we sacrifice approximately 1 day a week for hw days.

28: What advice would you give to another student taking this course next time?

- Come to every class, information is important
- The written homework is going to seem like a pain, but work through it & don't be afrain to go to office hours for help!
- Do the readings before class; Don't think the class is easy because you might be wrong; Practice problems at home
- Do the online work – you know what to pay attention to, and it makes life easier

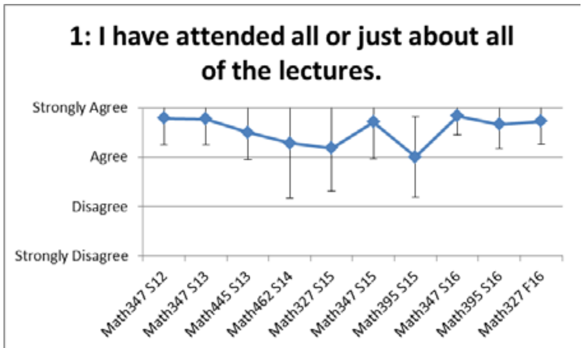
29: Any additional comments?

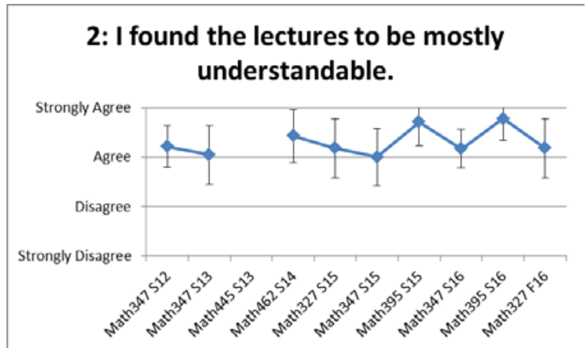
- Enjoyed class
- Generally, great class
- Cut out ebook
- Professor is very helpful. Makes the content easy to understand in class.
- Although this class wasn't closely related to my major it helped me understand a broader range of math concepts & improved my critical thinking.

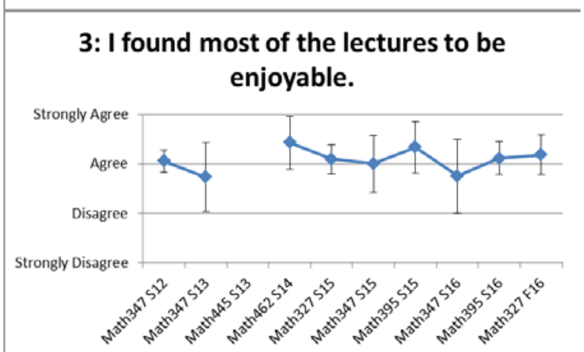
300- and 400-level Math courses

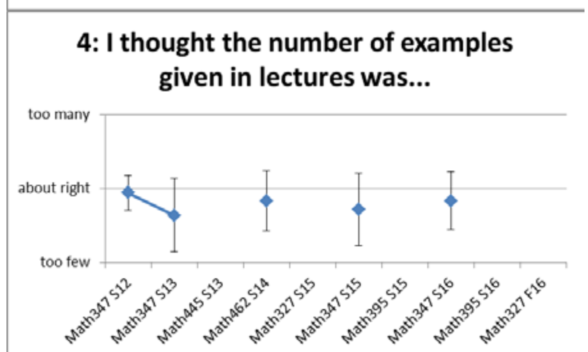
The charts below give the average (with standard deviation) for responses on course evaluations for 300- and 400-level Math courses. After the charts are representative replies to the free response questions.

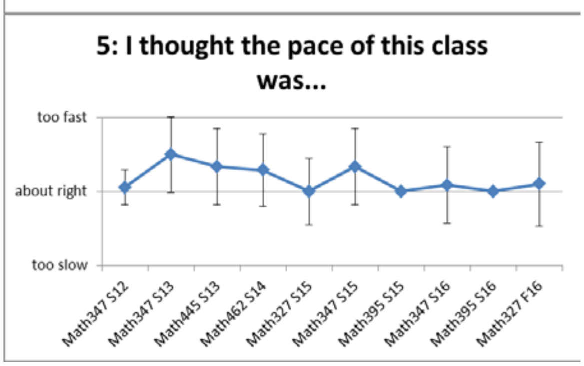
Note: As these evaluations correspond to a variety of courses, there is a larger variation in responses and the numbering of evaluation questions is inconsistent and contains gaps since some questions were specific to the course.

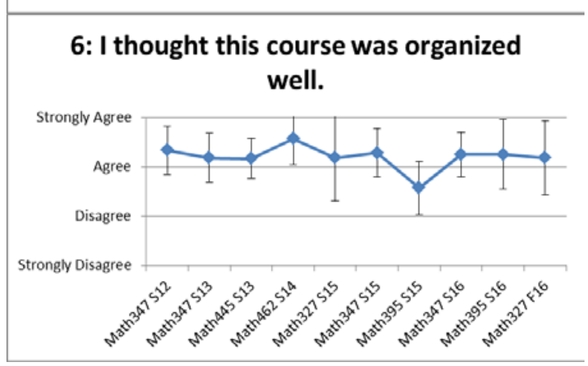




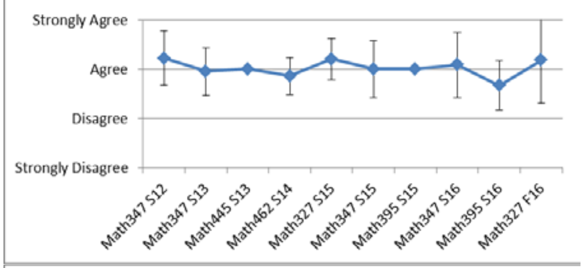




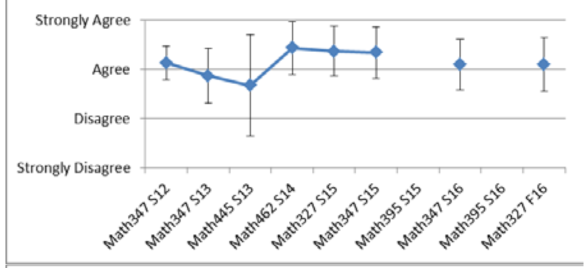




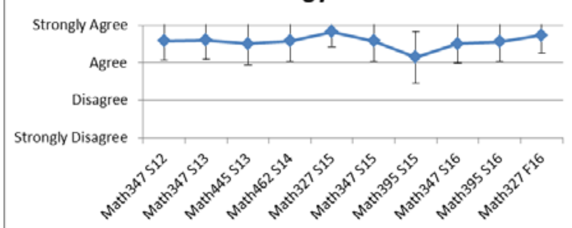
7: I thought a reasonable amount of homework was being assigned.



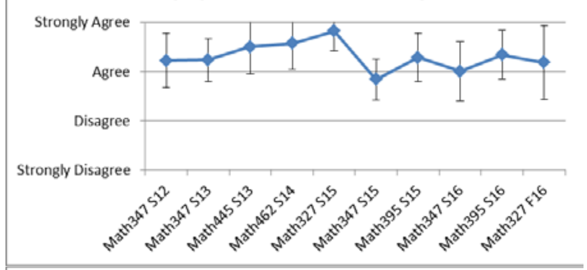
11: I thought the exams were at a reasonable level of difficulty.



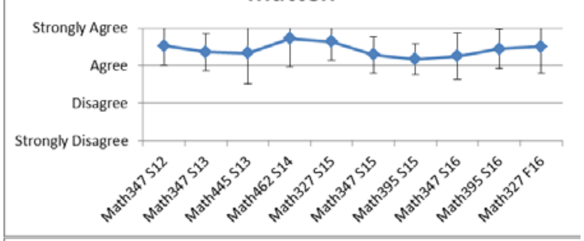
13: I thought the instructor had the appropriate level of enthusiasm and energy.



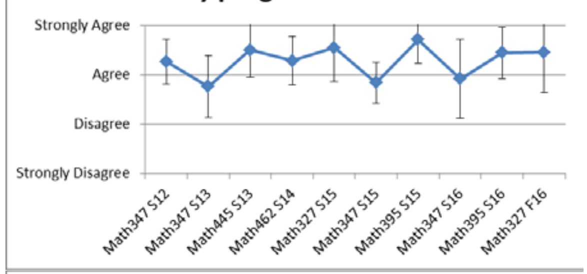
14: I found the instructor's responses to my questions to be helpful.



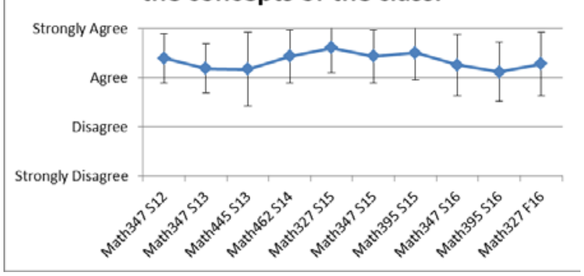
15: I found the instructor to be effective in teaching the subject matter.



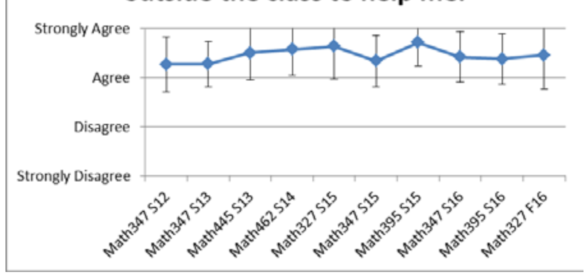
16: I thought the instructor cared about my progress in the course.

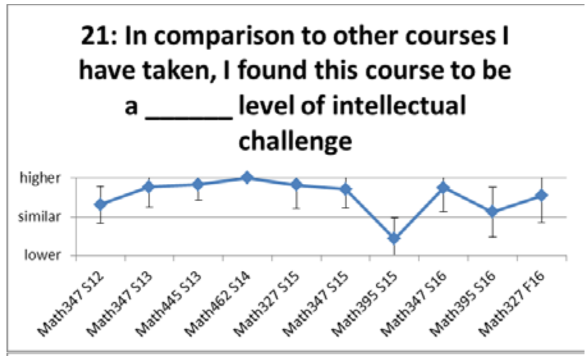


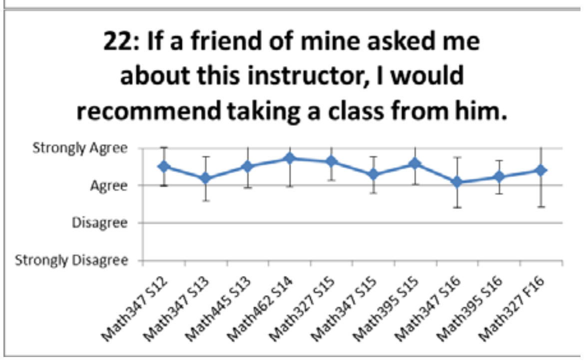
17: The instructor clearly explained the concepts of the class.

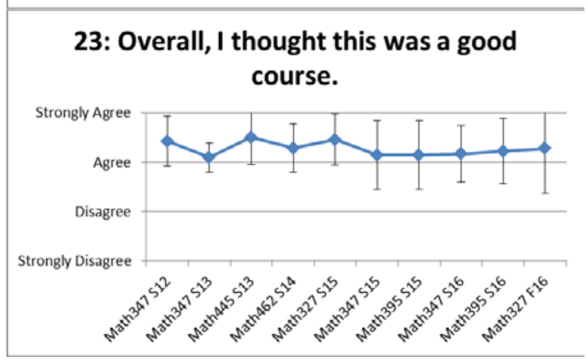


18: The instructor was available outside the class to help me.









Representative Comments (Math347, Spring 2012)

31: What do you think most contributed to your success in this class?

- I think the homework (both oral & written) were very beneficial. This helped me to be able to understand the problems before the exam.
- Seeing examples and doing oral homework.
- Doing the HW & studying in a group.
- Attending class and working with others on homework and studying

32: What do you think most contributed to any troubles you had in this class?

- Understanding of proofs/methods of proofs, Picking the correct formula/equation
- The textbook was difficult to understand.
- Didn't ask questions when I had trouble
- Not studying enough
- Not enough example in text

33: Do you have any suggestions that may be helpful for the next time this course is taught?

- I think doing more formal proofs in class would help (example proofs)
- Help more w/ proofs, or teach more about them when we're assigned them.
- Hard to do on T/TH, more class days
- I liked this course, no suggestions

34: What advice would you give to another student taking this course next time?

- Make sure you stay on task because it will help you through the semester. Come to class!
- Emphasize the importance of asking for help and coming to office hours.
- I recommend working with others and attend class every day.
- Study in groups, do your stuff on time & don't wait till the last moment.
- Don't procrastinate with this class! Start making flashcards of important definitions/theorems.
- It gets a lot easier as the semester goes on. The first two weeks were the worst for me.
- Keep up with the homework.
- Lectures are beneficial. Stay on top of HW and study notes.

35: Any additional comments?

- Great Job! I think you made this class one of my favorites!
- Make a 3 day a week class for 50 min. Hard to concentrate and focus for an hour and a half.
- Even though this class was very difficult, it was still fun & entertaining to take!

Representative Comments (Math347, Spring 2013)

31: What do you think most contributed to your success in this class?

- Coming to lecture, getting help during office hours, practice exams
- The oral homework and quizzes allowed me to understand and study many concepts
- being able to talk to Errthum outside of class usually in his office
- Working in groups and doing written homework

32: What do you think most contributed to any troubles you had in this class?

- very fast paced lecture
- not understanding everything and studying incorrectly
- Written hw was too difficult
- Not having definitions or formulas memorized.
- Challenging content and I believe I could have put in more effort at the start of the semester

33: Do you have any suggestions that may be helpful for the next time this course is taught?

- more midterms
- Provide feedback on written homework, I have no idea how to do the ones that I got wrong
- Slow down the pace to give feedback

34: What advice would you give to another student taking this course next time?

- Come to class and do homework.
- Go to class! work on HW individually as much as you can & then meet w/ peers for help.
- Don't slack off

35: Any additional comments?

- Great enthusiasm and I would have enjoyed the class but the material/concepts were very confusing at times.

Representative Comments (Math445, Spring 2013, first 9 weeks of course covering for other faculty member)

23: What do you think most contributed to your success in this class?

- My motivation to work & study outside of class
- Instructor
- Already knowing much of the material
- Homework presentations

24: What do you think most contributed to any troubles you had in this class?

- Lack of instruction on material, Didn't learn much from watching others presentations on material
- Pace
- Moving the same speed on old material as newer, more difficult material

25: Do you have any suggestions that may be helpful for the next time this course is taught?

- This material is quite difficult to self-teach… perhaps more lecture
- Redo the exams
- Quick exercises/warm ups were mostly lame/boring

26: What advice would you give to another student taking this course next time?

- Keep up with course work
- Study
- Take Number Theory 1st

27: Any additional comments?

[no replies]

Representative Comments (Math462, Spring 2014)

25: What do you think most contributed to your success in this class?

- Asking questions during office hours
- The quizzes helped a lot and also the activities. Both showed me what I know and don't know.
- Overall your enthusiasm and energy with the topic.

- Attending most of the lectures

26: What do you think most contributed to any troubles you had in this class?

- Me not showing up to class
- The book was difficult to understand at times.
- Mornings.
- The topic itself, not necessarily how it was present but abstracting from certain theorems, etc.
- Sleeping through a couple of lectures

27: Do you have any suggestions that may be helpful for the next time this course is taught?

- Not so fast pace
- The format was great.
- If anything try to incorporate more examples whether from the homework or questions from the book, but that's about it.
- Finding a way to motivate class participation

28: What advice would you give to another student taking this course next time?

- Study, study, study
- Keep up with the work.
- Really stay focused on the material and conceptualize it.
- Go to class

29: Any additional comments?

- I liked how Chapter 7 was interspersed.

Representative Comments (Math347, Spring 2015)

31: What do you think most contributed to your success in this class?

- The lectures
- Oral homework days. Completing the homework.
- The instructor is very enthusiastic
- Doing & seeing examples in class

32: What do you think most contributed to any troubles you had in this class?

- Felt like I couldn't talk to you outside of class
- The written homework
- Amount of homework & didn't relate to what we went over in class

33: Do you have any suggestions that may be helpful for the next time this course is taught?

- Have more proofs so it's not so scary
- Slow down a bit
- Predict tough content and arrange appropriate time for it.

34: What advice would you give to another student taking this course next time?

- Build a relationship with the professor
- Start homework as soon as you get it
- Go to the oral HW days!
- Read the chapters
- Come to class as much as you can

35: Any additional comments?

- Easily my favorite math course I've ever taken.

Representative Comments (Math395, Spring 2015, co-taught with Chris Malone)

19: What do you think most contributed to your success in this class?

- Writing formally. Relaxed pace.
- The class was relaxed, and small size so everyone communicates well
- Asking questions and getting assignments done.
- Helpful professors and good group of students (small class)
- The good pacing and constructive criticism on assignments

20: What do you think most contributed to any troubles you had in this class?

- Given examples of what wanted done.
- No troubles
- procrastinating

21: Do you have any suggestions that may be helpful for the next time this course is taught?

- Section on grad school
- If someone is Math/Stat maybe have them get more of a hint of both sides
- Be more organized and on the same page about assignment dates.
- Maybe more options of seminars since noon on Wednesday doesn't always work – online options
- More emphasis on researching than just writing a paper about a past assignment

22: Do you have any advice to give to students in the first week of this class in future semesters?

- Be comfortable with speaking w/ your peers.
- Have an idea of easy topics you know well that you could potentially present and/or write a paper on
- For presentations pick a topic that you fully understand.
- Brainstorm speech & paper topics early!!

23: Any additional comments?

- Good teamwork
- Field trip!!!
- Thanks for the neat-o class!
- Students should know this class exists sooner.

Representative Comments (Math347, Spring 2016)

31: What do you think most contributed to your success in this class?

- Studying
- Oral homework days
- Reading the book & other outside sources
- Going to lecture & asking for help outside of class

32: What do you think most contributed to any troubles you had in this class?

- Poor attendance
- Moving fast in lectures sometimes
- Not being able to complete homework assignments
- Not fully understanding the material and quizzes and exams being something different then what we learned.

33: Do you have any suggestions that may be helpful for the next time this course is taught?

- Assign more homework, especially oral ones
- Spend more time on the last chapter
- If oral is eliminated, many more days for lecture.
- None, thought the course was great

34: What advice would you give to another student taking this course next time?

- Don’t read the book. It gets somewhat confusing most of the time.

- Don’t be afraid to be wrong
- Start your homework right away & ask questions
- Just stick with it. The first few weeks are <u>rough</u> but it gets a lot more fun.
- Start out strong. Don’t let yourself get behind. He gives you a calendar on when to work on something. Do it! Keep up, once behind you will stay behind.

35: Any additional comments?

- Put in more stuffs!
- More applications to real world, why do we care? Maybe emphasize more on other number systems & norms as well.
- I found this class to be very challenging and although I struggled through a good amount of it, I found that I learned quite a bit.

Representative Comments (Math395, Spring 2016, co-taught with Chris Malone)

19: What do you think most contributed to your success in this class?

- Feedback on our speeches
- The 3 presentations really helped improve skills, especially getting to watch ourselves

20: What do you think most contributed to any troubles you had in this class?

- 8am
- Not timing my speeches when practicing

21: Do you have any suggestions that may be helpful for the next time this course is taught?

- Organize dates on the online calendar in a more timely manner
- More time on building resume and feedback on it.
- Be more organized.

22: Do you have any advice to give to students in the first week of this class in future semesters?

- Listen, ask questions, don’t skip classes.
- This class is very helpful. Paying attention will pay off in the future.
- Don’t procrastinate the poster or paper.

23: Any additional comments?

[no replies]