Math 280 Solutions for September 19

Pythagoras Level

Problem 1: [MAA/NCS 2006 #4] The problem is equivalent to counting the positive integers which in base 3 which have seven or fewer digits, where each digit is 0 or 1, and at least two 1s are present. There are 27 seven-digit strings of 0s and 1s, of which 7 have exactly one 1 and 1 has no 1s. Thus the count is 27 - 7 - 1 = 120.

Problem 2: [MAA/NCS 2005 #2] The only solution is $\left(\frac{32}{5}, \frac{-14}{5}\right)$. If $x < 0$ the first equation implies $y = 10$. But when $y = 10$, the second equation implies $x = 12$, so there are no solutions with $x < 0$. If $y \geq 0$, the second equation reduces to $x = 12$. But with $x = 12$ in the first equation we get $y < 0$, so there are no solutions with $y \geq 0$. This leaves only the fourth quadrant. If $x \geq 0$ and $y < 0$ the equations simplify to $2x + y = 10$ and $x - 2y = 12$, with the unique solution $x = \frac{32}{5}$, $y = \frac{-14}{5}$.

Newton Level

Problem 3: [Nick's Math Puzzles #15] Factorizing numerator and denominator, we have

$$\begin{aligned} k^3 - 1 &= (k-1)(k^2 + k + 1) \\ k^3 + 1 &= (k+1)(k^2 - k + 1) \end{aligned}$$

Note that $k^2 - k + 1 = (k-1)^2 + (k-1) + 1$, and so $k^3 + 1 = [(k-1) + 2][(k-1)^2 + (k-1) + 1]$, allowing cancellation of the quadratic factor across successive terms, and of the linear factor across "next but one" terms.

We can now calculate P_n, the partial product of the first $n - 1$ terms.

$$\begin{aligned} P(n) &= \frac{7}{9} \times \frac{26}{28} \times \frac{63}{65} \times ... \times \frac{n^3 - 1}{n^3 + 1} \\ &= \left(\frac{1}{3} \times \frac{7}{3}\right) \times \left(\frac{2}{4} \times \frac{13}{7}\right) \times \left(\frac{3}{5} \times \frac{21}{13}\right) \times ... \times \left(\frac{n-1}{n+1} \times \frac{n^2 + n + 1}{n^2 - n + 1}\right) \\ &= \frac{2}{3} \times \left(\frac{n^2 + n + 1}{n(n+1)}\right) \\ &= \frac{2}{3} \times \left(1 + \frac{1}{n(n+1)}\right) \end{aligned}$$

As n tends to infinity, P_n tends to 2/3. Hence, the infinite product, P, converges to 2/3.

Problem 4: [2000 Putnam B-2] Since $\gcd(m, n)$ is an integer linear combination of m and n, it follows that

$$\frac{gcd(m,n)}{n} \binom{n}{m}$$

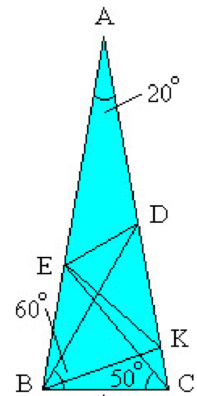
is an integer linear combination of the integers

$$\frac{m}{n} \binom{n}{m} = \binom{n-1}{m-1} \text{ and } \frac{n}{n} \binom{n}{m} = \binom{n}{m}$$

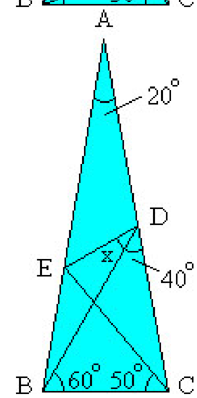
and hence is itself an integer.

Wiles Level

Problem 5: [Nick's Math Puzzles #22] Solution by Construction: Mark K on AC such that $\angle KBC = 20°$. Draw KB and KE. $\angle BEC = \angle ECB$, and so $\triangle BEC$ is isosceles with $BE = BC$. $\angle BKC = \angle BCK$, and so $\triangle BKC$ is isosceles with $BK = BC$. Therefore $BE = BK$. $\angle EBK = 60°$, and so $\triangle EBK$ is equilateral. $\angle BDK = \angle DBK = 40°$ and so $\triangle BDK$ is isosceles, with $KD = KB = KE$. So $\triangle KDE$ is isosceles, with $\angle EKD = 40°$, since $\angle EKC = 140°$. Therefore $\angle EDK = 70°$, yielding $\angle EDB = 30°$.



Solution by Trigonometry: Let $\angle EDB = x$. Then $\angle BED = 160° - x$, and $\angle BDC = 40°$. Applying the law of sines (also known as the sine rule) to: $\triangle BED$, $BE/\sin x = BD/\sin(160° - x)$. Applying the law to $\triangle BDC$, $BC/\sin 40° = BD/\sin 80°$. Therefore $BD = BE \cdot \sin(160° - x)/\sin x = BC \cdot \sin 80°/\sin 40°$. Then $\angle BEC = \angle ECB$, and so $\triangle BEC$ is isosceles with $BE = BC$. Hence $\sin(160° - x)/\sin x = \sin 80°/\sin 40°$. Then $\sin(160° - x) = \sin(20° + x)$, (since $\sin a = \sin(180° - a)$), and $\sin 80° = 2\sin 40° \cos 40°$, (since $\sin 2a = 2\sin a \cos a$.) Therefore $\sin(20° + x) = 2\cos 40° \sin x = \sin(x + 40°) + \sin(x - 40°)$, (since $\sin a \cos b = [\sin(a + b) + \sin(a - b)]/2$.) Then $\sin(20° + x) - \sin(x - 40°) = 2\cos(x - 10°)\sin 30° = \sin(x + 80°)$, (since $\sin a = \cos(90° - a)$.) Hence $\sin(x + 40°) = \sin(x + 80°)$. If $x < 180°$, the only solution is $x + 80° = 180° - (x + 40°)$, (since $\sin a = \sin(180° - a)$.) Hence $x = 30°$. Therefore $\angle EDB = 30°$.



Problem 6: [Putnam 2001 B-3] Since $(k - 1/2)^2 = k^2 - k + 1/4$ and $(k + 1/2)^2 = k^2 + k + 1/4$, we have that $\langle n \rangle = k$ if and only if $k^2 - k + 1 \le n \le k^2 + k$. Hence

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n} &= \sum_{k=1}^{\infty} \sum_{n, \langle n \rangle = k} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n} \\
&= \sum_{k=1}^{\infty} \sum_{n=k^2-k+1}^{k^2+k} \frac{2^k + 2^{-k}}{2^n} \\
&= \sum_{k=1}^{\infty} (2^k + 2^{-k})(2^{-k^2+k} - 2^{-k^2-k}) \\
&= \sum_{k=1}^{\infty} (2^{-k(k-2)} - 2^{-k(k+2)}) \\
&= \sum_{k=1}^{\infty} 2^{-k(k-2)} - \sum_{k=3}^{\infty} 2^{-k(k-2)} \\
&= 3.
\end{aligned}
$$

Alternate solution: rewrite the sum as $\sum_{n=1}^{\infty} 2^{-(n+\langle n \rangle)} + \sum_{n=1}^{\infty} 2^{-(n-\langle n \rangle)}$. Note that $\langle n \rangle \ne \langle n+1 \rangle$ if and only if $n = m^2 + m$ for some m. Thus $n + \langle n \rangle$ and $n - \langle n \rangle$ each increase by 1 except at $n = m^2 + m$, where the former skips from $m^2 + 2m$ to $m^2 + 2m + 2$ and the latter repeats the value m^2. Thus the sums are

$$\sum_{n=1}^{\infty} 2^{-n} - \sum_{m=1}^{\infty} 2^{-m^2} + \sum_{n=0}^{\infty} 2^{-n} + \sum_{m=1}^{\infty} 2^{-m^2} = 2 + 1 = 3.$$