SAGE Activity (Determinants)

Log in to SAGE and create a new SAGE worksheet titled "Determinants".

1. Let's all make sure we know how to get SAGE to compute a determinant for us.

    (a) Make a random matrix by calling `A=random_matrix(QQ,8)`

    (b) Take a good look at `A` and thank god you don't have to compute a determinant by hand.

    (c) Compute the determinant using `A.determinant()`.

2. (a) Suppose you have two distinct points on the plane (x_1, y_1) and (x_2, y_2). The unique line through the points can be given by finding coefficients c_i that <u>are not all zero</u> such that

$$c_1 x_1 + c_2 y_1 + c_3 = 0$$
$$c_1 x_2 + c_2 y_2 + c_3 = 0$$

Furthermore, any general (x, y) on this line must also satisfy

$$c_1 x + c_2 y + c_3 = 0$$

Notice these three equations can be written in matrix form as:

$$\begin{pmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Since the vector of c_i's is not the zero vector, the 3×3 matrix must has nontrivial null space. Hence it is singular and has determinant 0. In other words,

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

This is called the determinant form of a line.

    (b) Use (2a) to find the equation of the line through the points $(3, 5)$ and $(-4, 1)$. Remember to make variables in SAGE use `x=var('x')`.

    (c) The same process in (2a) also works for circles. In this case, suppose (x_1, y_1), (x_2, y_2) and (x_3, y_3) are 3 distinct noncolinear points. Then there is a unique circle of the form

$$c_1(x^2 + y^2) + c_2 x + c_3 y + c_4 = 0$$

which passes through the points.

    (d) Redo the line a thinking in (2a) for this situation. Then give the determinant form of the circle through the points $(1, 2)$, $(-3, 7)$ and $(9, 5)$.

    (e) Have SAGE compute the determinant you made in part (2d) to find an equation of the circle

    (f) Complete the squares in you equation from (2e) to find the center and radius.

    (g) The same process in (2a) and (2c) can be done in higher dimensions as well. A sphere in 3-space has an equation

$$c_1(x^2 + y^2 + z^2) + c_2 x + c_3 y + c_4 z + c_5 = 0$$

    (h) How many points in three space do you need to determine a sphere?

    (i) Consider the points $(1, 1, 1)$, $(2, 4, 8)$, $(3, 9, 27)$, ... Find the determinant form of the sphere through these points.

    (j) Find the center and radius for the sphere.

3. One of the most well-known formulas involving determinants is "Cramer's Rule" (often taught in high school algebra courses).

    (a) Cramer's Rule: If $Ax = b$ represents a system of n equations in n unknowns and $\det(A) \neq 0$, then the unique solution is

$$x_i = \frac{\det(A_i)}{\det(A)}$$

were A_i is the matrix A where the ith column has been replaced with b.

(b) Use Cramer's Rule to solve

$$x_1 + \qquad +2x_3 \quad = \quad 6$$
$$-3x_1 + 4x_2 + 6x_3 \quad = \quad 30$$
$$-x_1 - 2x_2 + 3x_3 \quad = \quad 8$$

SAGE HOMEWORK

For each, give <u>one</u> cell of input that gives the desired output.

1. (5 points) Create a function that takes as input a positive whole number n and outputs the determinant of the $n \times n$ matrix:

$$\begin{pmatrix} 1 & i & 0 & & \cdots & & 0 \\ i & 1 & i & 0 & \cdots & \cdots & 0 \\ 0 & i & 1 & i & 0 & \cdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & i & 1 & i & 0 \\ 0 & \cdots & \cdots & 0 & i & 1 & i \\ 0 & & \cdots & & & 0 & i & 1 \end{pmatrix}$$

Demonstrate your function works on the values $n = 1$, $n = 2$, and $n = 13$.

2. (5 points) A parabola has a general equation

$$c_1 y + c_2 x^2 + c_3 x + c_4 = 0.$$

Give the determinant form and standard equation $(y = a(x - h)^2 + k)$ of the parabola through the points $(20, 7)$, $(-12, 5)$ and $(-16, 8)$.

3. (5 points) Solve for y in the following without solving for x, z or w:

$$4x + y + z + w \quad = \quad 6$$
$$3x + 7y - z + w \quad = \quad 1$$
$$7x + 3y - 5z + 8w \quad = \quad -3$$
$$x + y + z + 2w \quad = \quad 3$$

Bonus. (5 points) Prove that if $f_1(x)$, $f_2(x)$, $g_1(x)$, and $g_2(x)$ are differentiable functions then

$$\frac{d}{dx} \begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{vmatrix} = \begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) \\ g_1'(x) & g_2'(x) \end{vmatrix}.$$

SAGE Activity (Determinants)

Log in to SAGE and create a new SAGE worksheet titled "Eigenvectors".

1. (a) The following SAGE function was emailed out:
```
def Transform(A):
        precision=200
        CirclePoints=[vector(RR,[cos(i*2*pi/precision),sin(i*2*pi/precision)])
                for i in range(precision)]
        ACirclePoints=[A*CirclePoints[i] for i in range(precision)]
        CirclePlot=sum([list_plot([CirclePoints[i]], hue=i/precision, aspect_ratio=1)
                for i in range(precision)])
        ACirclePlot=sum([list_plot([ACirclePoints[i]], hue=i/precision, aspect_ratio=1)
                for i in range(precision)])
        return CirclePlot+ACirclePlot
```

This function will show the before and after of tranforming a rainbow colored unit circle according to the 2×2 matrix given as input.
Let's see the regular circle first by calling `Transform(identity_matrix(2))`

(b) Create matrix $F = \begin{pmatrix} 1 & 1/10 \\ 1/10 & 1 \end{pmatrix}$ and call `Transform(F)` to see what it does to the unit circle.

(c) For each of the following matrices,

   - indicate how many eigenvectors there are
   - determine whether the eigenvalues are positive or negative
   - determine whether the magnitude of the eigenvalues is greater than, less than, or approximately equal to 1
   - give a description of what will happen to the circle if you repeatedly transform by that matrix

   i. $A = \begin{pmatrix} 3 & 2 \\ 1 & -3 \end{pmatrix}$

   ii. $B = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}$

   iii. $C = \begin{pmatrix} -2 & 1 \\ 1/2 & -1 \end{pmatrix}$

   iv. $D = \begin{pmatrix} -1/4 & 1/2 \\ 1/8 & -1/4 \end{pmatrix}$

   v. $E = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ (Note: If you can't tell what happened, modify the precision in your function to `precision=60` and try again.)

2. Let's all make sure we know how to get SAGE to compute eigenvalues and eigen vectors for us.

   (a) Make the matrix
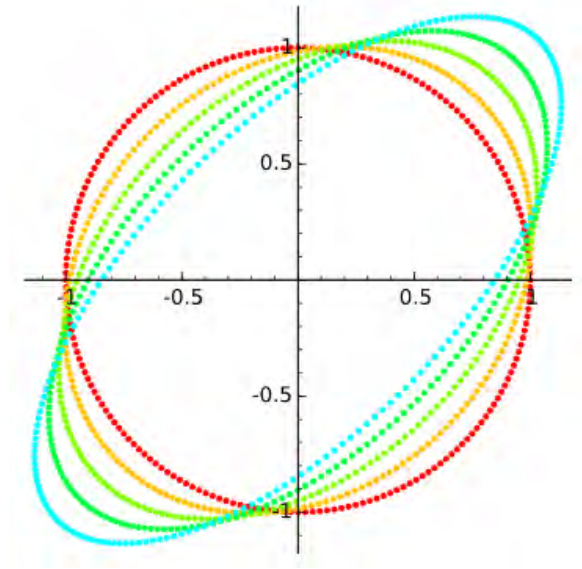 $$A = \begin{pmatrix} -2 & 0 & 1 \\ -2 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}$$

   (b) Compute the characteristic polynomial with `A.charpoly()`

   (c) Compute the eigenvalues using `A.eigenvalues()`

   (d) Compute the eigenvectors using `MyEigenInfo=A.eigenvectors_right(); MyEigenInfo`

   (e) That looks like a mess: it's a list of lists. The lists contain [eigenvalue λ, eigenvectors for λ, $\gamma_A(\lambda)$].

   (f) You can access just the eigenvectors by digging into the list with `[MyEigenInfo[i][1][0] for i in [0,1,2]]`

   (g) Make a matrix S with the columns of eigenvectors for A and check that $S^{-1}AS$ gives a diagonal matrix of the eigenvalues.

   (h) Determine $A^5 - 3A^2 + 2$ with no more than 2 matrix multiplications.

   (i) Use the idea of a Taylor Series to compute the matrices e^A and $\arctan(A)$.

   (j) Recall in (b) you found the characteristic polynomial $p_A(x)$. Now compute $p_A(A)$.

   (k) Prove that what you observed in (j) will always be true.

   (l) Give a conjecture (stating all the necessary conditions and then the conclusion) to describe in general what you found in (j). Check with me before going on.

   (m) What does this mean we can say about the set of matrices $\{I, A, A^2, A^3, ..., A^n\}$?

SAGE HOMEWORK
For each, give one cell of input that gives the desired output.

1. (5 points) Compute the eigenvalues and eigenvectors for the matrices in 1(b).

2. (5 points) Modify the code above to create a function that takes as input a matrix A and positive integer n and gives as output the tranformations of A, A^2, A^3, ... A^n on a solid colored circle but with each power of A takes on evenly spaced out colors in the hue range 0 to 1/2. Demonstrate your function on the first 5 powers of the matrix C from part 1(b).

   For example, with the matrix $F = \begin{pmatrix} 1 & 1/10 \\ 1/10 & 1 \end{pmatrix}$ up to the 4th power, we would get output of:

3. (5 points) Create a function that takes as input a list of 2 eigenvalues and a list of two linearly independent eigenvectors and outputs a matrix with those properties. Demonstrate your function on the inputs:
   `[5,7], [(1,2), (-3,4)]`
   and
   `[I,-I], [(1,1),(1,-1)]`

Bonus. (5 points) Prove that if A is diagonalizable, then $\sin^2(A) + \cos^2(A) = I$.

Markov Chains: Thomas Train Game Analysis

1. WARM-UP: Determine the probability of reaching home on the 7th turn if you are the yellow train.

2. TASK: Rank order the trains from most likely to win first to least likely to win using a 50% threshold.

3. CHALLENGE: Approximate (to 3 decimals) the expected number of turns for each train to reach home. (Recall: Expected value is given by $\sum_n n \cdot P(n)$.)