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American Mathematical Monthly, Vol 122, No 7, August-September 2015

11853. Find
$$\sum_{n=1}^{\infty} \frac{1}{\sinh 2^n}.$$

Solution, by Eric Errthum, Winona State University, Winona, MN

By inspection the partial sums are found to be given by
$$\sum_{i=1}^{n} \frac{1}{\sinh 2^n} = \frac{2(e^{2^{n+1}} - e^2)}{(e^{2^{n+1}} - 1)(e^2 - 1)}$$
since
$$\begin{aligned}
\frac{2(e^{2^{n+1}} - e^2)}{(e^{2^{n+1}} - 1)(e^2 - 1)} + \frac{2}{e^{2^{n+1}} - e^{-2^{n+1}}} &= \frac{2(e^{2^{n+1}} - e^2)}{(e^{2^{n+1}} - 1)(e^2 - 1)} + \frac{2e^{2^{n+1}}}{(e^{2^{n+1}} - 1)(e^{2^{n+1}} + 1)} \\
&= \frac{2(e^{2^{n+1}} - e^2)(e^{2^{n+1}} + 1) + 2e^{2^{n+1}}(e^2 - 1)}{(e^{2^{n+1}} - 1)(e^{2^{n+1}} + 1)(e^2 - 1)} \\
&= \frac{2(e^{2^{n+2}} - e^{2^{n+1}+2} + e^{2^{n+1}} - e^2) + 2(e^{2^{n+1}+2} - e^{2^{n+1}})}{(e^{2^{n+1}} - 1)(e^{2^{n+1}} + 1)(e^2 - 1)} \\
&= \frac{2(e^{2^{n+2}} - e^2)}{(e^{2^{n+2}} - 1)(e^2 - 1)}.
\end{aligned}$$
Hence the limit of the partial sums gives
$$\sum_{i=1}^{\infty} \frac{1}{\sinh 2^n} = \frac{2}{e^2 - 1}.$$