Math 280 Problems for October 5

Pythagoras Level

Problem 1: A $3 \times 3 \times 3$ cube is assembled from 27 $1 \times 1 \times 1$ cubes all of whose faces are white. We paint all of the faces of the large cube black, and then disassemble it. A blindfolded man reassembles the large cube from the 27 little cubes. What is the probability that all the faces of the reassembled cube are completely black?

[Blakers 2011 #4] Of the 27 little cubes, one is completely white, six are black on one face, twelve are black on two (edge-sharing) faces and eight are black on three (corner-sharing) faces. When the large cube is reassembled, each of the little cubes has 27 possible positions and 24 possible orientations. Therefore, the total number of possible large cubes is

$$27! \times 24^{27}.$$

Suppose that all of the faces of the reassembled large cube are completely black. There is 1! possible way to position the white cube, 6! possible ways to position the cubes with one black face, 12! for the cubes with two black faces, and 8! for the cubes with three black faces. There are also 24 ways to orient the white cube, 4 ways to orient each of the cubes with one black face, 2 ways to orient each of the cubes with two black faces, and 3 ways to orient each of the cubes with three black faces. Therefore, the total number of possible large, completely black cubes is

$$1! \times 6! \times 12! \times 8! \times 24^1 \times 4^6 \times 2^{12} \times 3^8.$$

Therefore, the probability that the reassembled cube is completely black is

$$\frac{6! \times 12! \times 8! \times 24 \times 4^6 \times 2^{12} \times 3^8}{27! \times 24^{27}} = \frac{6! \times 12! \times 8!}{27! \times 24^{18}}.$$

Problem 2: A polynomial of degree 2011 with real coefficients is such that $P(n) = \dfrac{n}{n+1}$ for all integers $n \in \{0, 1, 2, \ldots, 2011\}$. What is the value of $P(2012)$?

[Blakers 2011 #10] Let $Q(x) = (x+1)P(x) - x$. Then $Q(x)$ is a polynomial of degree 2012 over $\mathbb{R}$ and for $n \in \{0, 1, 2, \ldots, 2011\}$, $Q(n) = 0$, so that $Q(x)$ has zeroes $0, 1, 2, \ldots, 2011$. Since $Q(x)$ is of degree 2012 and has 2012 distinct zeros, it can have no other zeros. Hence,

$$Q(x) = k \cdot \prod_{i=0}^{2011} (x - i)$$

for some real constant k. Now,

$$\begin{aligned} 1 &= Q(-1) \\ &= k \prod_{i=0}^{2011} (-1 - i) \\ &= k \cdot (-1)^{2012} \cdot \prod_{i=0}^{2011} (i + 1) \\ &= k \cdot 2012!. \end{aligned}$$

Hence $k = \frac{1}{2012!}$. Thus

$$\begin{aligned} (2012+1)P(2012) - 2012 = Q(2012) &= \frac{1}{2012!} \prod_{i=0}^{2011} (2012 - i) \\ &= \frac{1}{2012!} \cdot 2012! = 1 \\ P(2012) &= \frac{1 + 2012}{2013} = 1. \end{aligned}$$

Newton Level

Problem 3: Find all polynomials $P(x)$ such that $P(2x) = P'(x) \cdot P''(x)$ for all $x \in \mathbb{R}$.

[Blakers 2011 #1] Let $P(x)$ be a real polynomial such that $P(2x) = P'(x) \cdot P''(x)$ for all $x \in \mathbb{R}$. The zero polynomial is one such polynomial $P(x)$. Suppose that $P(x)$ is not the zero polynomial and let n be the degree of $P(x)$. Then the degree of $P(2x)$ is n. If n is equal to 0 or 1, then $P'(x) \cdot P''(x)$ will be the zero polynomial, and so will not be equal to $P(2x)$. If $n \geq 2$, then the degree of $P'(x) \cdot P''(x)$ will be $(n-1)+(n-2) = 2n-3$. Therefore, $P(2x) = P'(x) \cdot P''(x)$ implies that $n = 2n-3$ and hence $n = 3$. So, $P(x)$ is a cubic polynomial of the form $a + bx + cx^2 + dx^3$, with $d \neq 0$. Therefore,

$$\begin{aligned} P(2x) &= a + 2bx + 4cx^2 + 8dx^3 \\ P'(x) \cdot P''(x) &= (b + 2cx + 3dx^2)(2c + 6dx) \\ &= 2bc + (4c^2 + 6bd)x + 18cdx^2 + 18d^2x^3 \end{aligned}$$

Thus

$$\begin{aligned} a &= 2bc \\ 2b &= 4c^2 + 6bd \\ 4c &= 18cd \\ 8d &= 18d^2. \end{aligned}$$

Solving this set of simultaneous equations, first we find $d = 4/9$ (recall $d \neq 0$), from which we see that successively $c = 0$, $b = 0$ and $a = 0$, giving us the solution $P(x) = \frac{4}{9}x^3$.

Problem 4: Let $y = x^{1/x}$ for $x > 0$. Find the intervals on which $y(x)$ is monotonic, and on each such interval, find its range.

[Blakers 2009 #1] Since $x > 0$ implies $y > 0$, we can take logs on both sides to conclude

$$\ln y = \frac{1}{x} \ln x$$

Differentiating with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2}(1 - \ln x)$$

Since y and $x^2 > 0$, the only crtical point is at $lnx = 1$, $x = e$. Furthermore, dy/dx is positive for $0 < x < e$ and negative for $e < x < \infty$. Hence $y(x)$ is increasing on $(0, e]$ with range $(0, e^{1/e}]$ and decreasing on $[e, \infty)$ with range $[e^{1/e}, 1)$.

Wiles Level

Problem 5: A vector $\vec{v} = (x, y, z) \in \mathbb{R}^3$ is integral if each component is an integer. Prove that if $\vec{u}$, $\vec{v}$, and $\vec{w}$ are mutually orthogonal integral vectors with the same length L, then L is an integer.

[Blakers 2011 #6] Given three mutually orthogonal integral vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ with the same length $L \neq 0$, they define a parallelepiped that is actually a cube. (Note that we can ignore the trivial case $L = 0$, since $0 \in \mathbb{Z}$, and so there is nothing to prove in this case.) The volume of the cube defined by $\vec{u}$, $\vec{v}$, $\vec{w}$ is $L^3 = |\vec{u} \cdot (\vec{v} \times \vec{w})|$ and so L^3 is an integer. Furthermore, $L^2 = \vec{u} \cdot \vec{u}$ and so L^2 is an integer. Therefore, $L = \frac{L^3}{L^2}$ is rational, and since L^2 is an integer, L must also be an integer.

Problem 6: Consider a binary operation $*$ on a set S, that is, for all $a, b \in S$, $a * b$ is in S. Prove that if for all $a, b \in S$, $(a * b) * a = b$, then for all $a, b \in S$, $a * (b * a) = b$. (Obviously you cannot assume $*$ is associative.)

[Blakers 2010 #2] Since $(a * b) * a = b$ for all $a, b \in S$, it follows that (for all $a, b \in S$)

$$(b * a) * b = a \tag{1}$$

$$(c * b) * c = b \qquad \text{where } c = b * a. \tag{2}$$

So for all $a, b \in S$,

$$\begin{aligned} a * (b * a) &= ((b * a) * b) * (b * a) && \text{by (1)} \\ &= b, && \text{by (2).} \end{aligned}$$