

# Learning Your ABC

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## Examples

- $662415793599696251 = 239 \cdot 57301 \cdot 94873 \cdot 509833$
- $\frac{27008742384}{27680640625} = 2^4 \cdot 3^5 \cdot 5^{-6} \cdot 11^{-6} \cdot 13 \cdot 17^2 \cdot 43^2$

# Factorizations of Consecutive Numbers

$$2 = 2$$

$$3 = 3$$

$$4 = 2^2$$

$$5 = 5$$

$$6 = 2 \cdot 3$$

$$7 = 7$$

$$8 = 2^3$$

$$9 = 3^2$$

$$10 = 2 \cdot 5$$

$$11 = 11$$

$$12 = 2^2 \cdot 3$$

# Factorizations of Consecutive Numbers

$$22 = 2 \cdot 11$$

$$23 = 23$$

$$24 = 2^3 \cdot 3$$

$$25 = 5^2$$

$$26 = 2 \cdot 23$$

$$27 = 3^3$$

$$28 = 2^2 \cdot 7$$

$$29 = 29$$

$$30 = 2 \cdot 3 \cdot 5$$

$$31 = 31$$

$$32 = 2^5$$

# Factorizations of Consecutive Numbers

$$122 = 2 \cdot 61$$

$$123 = 3 \cdot 41$$

$$124 = 2^2 \cdot 31$$

$$125 = 5^3$$

$$126 = 2 \cdot 3^2 \cdot 7$$

$$127 = 127$$

$$128 = 2^7$$

$$129 = 3 \cdot 43$$

$$130 = 2 \cdot 5 \cdot 13$$

$$131 = 131$$

$$132 = 2^2 \cdot 3 \cdot 11$$

# Factorizations of Consecutive Numbers

$$55122 = 2 \cdot 3 \cdot 9187$$

$$55123 = 199 \cdot 277$$

$$55124 = 2^2 \cdot 13781$$

$$55125 = 3^2 \cdot 5^4 \cdot 7^2$$

$$55126 = 2 \cdot 43 \cdot 641$$

$$55127 = 55127$$

$$55128 = 2^3 \cdot 3 \cdot 2297$$

$$55129 = 29 \cdot 1901$$

$$55130 = 2 \cdot 5 \cdot 37 \cdot 149$$

$$55131 = 3 \cdot 17 \cdot 23 \cdot 47$$

$$55132 = 2^2 \cdot 7 \cdot 11 \cdot 179$$

# Factorizations of Consecutive Numbers

$$7796955122 = 2 \cdot 11 \cdot 354407051$$

$$7796955123 = 3^2 \cdot 17 \cdot 50960491$$

$$7796955124 = 2^2 \cdot 7 \cdot 12527 \cdot 22229$$

$$7796955125 = 5^3 \cdot 62375641$$

$$7796955126 = 2 \cdot 3 \cdot 1299492521$$

$$7796955127 = 13 \cdot 23 \cdot 3929 \cdot 6637$$

$$7796955128 = 2^3 \cdot 523 \cdot 1863517$$

$$7796955129 = 3 \cdot 37 \cdot 70242839$$

$$7796955130 = 2 \cdot 5 \cdot 2777 \cdot 280769$$

$$7796955131 = 7 \cdot 229 \cdot 1487 \cdot 3271$$

$$7796955132 = 2^2 \cdot 3^3 \cdot 2503 \cdot 28843$$

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- We call these numbers **smooth**.

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## Example

$$a + b = 2^{-3} \cdot 3^{-2} \cdot 7^{-6} \cdot 40949 \cdot 122698687$$

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*Typical Example:* (0.36287)

$$7 \cdot 5701 + 37 \cdot 1361 = 2^3 \cdot 3 \cdot 3761$$

# Measuring ABC triples

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*Example:*  $\alpha(2, 3^{10} 109, 23^5) = \frac{\ln(23^5)}{\ln(2 \cdot 3 \cdot 23 \cdot 109)} = 1.62991$

# Good ABC Triples

- Top three known ABC ratio (verified up to  $10^{20}$ ):

$(2, 3^{10}109, 23^5)$  with  $\alpha = 1.62991$

$(11^2, 3^25^67^3, 2^{21}3)$  with  $\alpha = 1.62599$

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- Largest known good ABC triples:

$(2^{24}5^547^5181^2, 13^{14}19 \cdot 103 \cdot 571^2 \cdot 4261, 7^{28}17 \cdot 37^2)$

with  $\alpha = 1.447420$  and 29 digits

$(5^917^223^437^243 \cdot 4817, 3^{14}11^861^2173^4, 2^{52}19^6127^2)$

with  $\alpha = 1.419184$  and 28 digits

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$$\begin{aligned}\frac{\ln(C)}{\ln(\text{rad}(ABC))} &= \alpha \\ \ln(C) &= \alpha \ln(\text{rad}(ABC)) = \ln((\text{rad}(ABC))^{\alpha}) \\ C &= (\text{rad}(ABC))^{\alpha}\end{aligned}$$

## ABC Conjecture (Oesterle and Masser, 1985)

For every  $\eta > 1$ , there exists only a finite number of ABC triples such that

$$C > (\text{rad}(ABC))^{\eta}$$

i.e. with  $\alpha(A, B, C) > \eta$ .

# Consequences

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**Proof:** Suppose there was a solution, then let  $A = x^n$ ,  $B = y^n$ ,  $C = z^n$ .

Then  $\text{rad}(ABC) \leq xyz \leq z^3$ . Applying the conjecture gives  $z^n < (\text{rad}(ABC))^2 \leq (z^3)^2 = z^6$ . Hence  $n \leq 6$ .

The cases of  $3 \leq n \leq 6$  were proved in 1825 by Legendre and Dirichlet.

# More Consequences

## Corollary

If the ABC conjecture is true then the following are also proved:

- *The generalized Fermat equation*
- *Wieferich primes statement*
- *The Erdos-Woods conjecture*
- *Hall's conjecture*
- *The Erdos-Mollin-Walsh conjecture*
- *Brocard's Problem*
- *Szpiro's conjecture*
- *Mordell's conjecture*
- *Roth's theorem*
- *Dressler's conjecture*
- *Bounds for the order of the Tate-Shafarevich group*
- *Vojta's height conjecture*
- *Greenberg's conjecture*
- *The Schinzel-Tijdeman conjecture*
- *Lang's conjecture*

... and many more!



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## ABC Conjecture (Rephrased)

Given  $\epsilon > 0$ , there exists a constant  $K_\epsilon$  such that for every  $A, B, C$  coprime integers with  $A + B = C$ ,

$$\log C \leq K_\epsilon + (1 + \epsilon) \log R$$

where  $R = \text{rad}(ABC)$ .

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## Theorem (Gyory (2007))

Let  $A, B, C$  be coprime integers with  $A + B = C$ . Let  $t$  be the number of prime factors in  $R = \text{rad}(ABC)$ . Then

$$\log C < \frac{2^{10t+22}}{t^{t-4}} R (\log R)^t$$

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(But  $x + 1$  and  $x - 1$  are.)
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which is just like  $\ln(AB) = \ln(A) + \ln(B)$ .

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## PQR Theorem (Hurwitz, Stothers, Mason)

Let  $P, Q, R$  be nonconstant relatively-prime polynomials that satisfy  $P + Q = R$ , then

$$\deg(R) < \deg(\text{rad}(PQR)).$$

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- Example:*  $F = (x - 1)^2(x^2 + 1)^3$   
then  $F' = 2(x - 1)(x^2 + 1)^2(4x^2 - 3x + 1)$   
so  $\gcd(F, F') = (x - 1)(x^2 + 1)^2$   
and  $\frac{F}{\gcd(F, F')} = (x - 1)(x^2 + 1) = \text{rad}(F)$ .

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- There are “interesting” surfaces in algebraic geometry with “special” points that correspond to algebraic numbers.
- The corresponding algebraic numbers satisfy  $\alpha + \beta = \gamma$  and are usually smooth.
- I used some algorithms developed in my thesis to generate 350 of these examples and computed their algebraic ABC ratios.

**Introduction**  
○

**Factorizations**  
○○○

**ABCs**  
○○○○

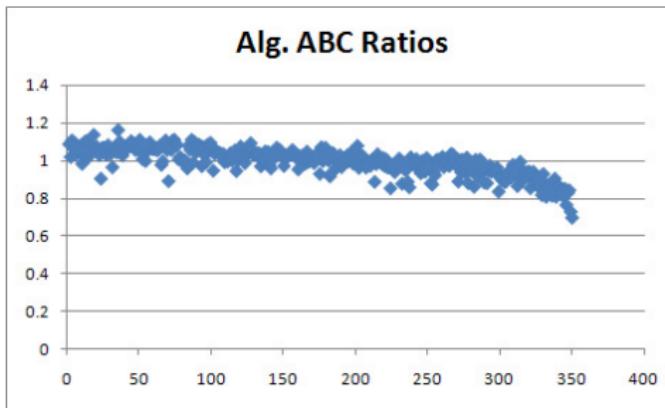
**ABC Conjecture**  
○○○○

**PQR**  
○○○

**My Work**  
○●○

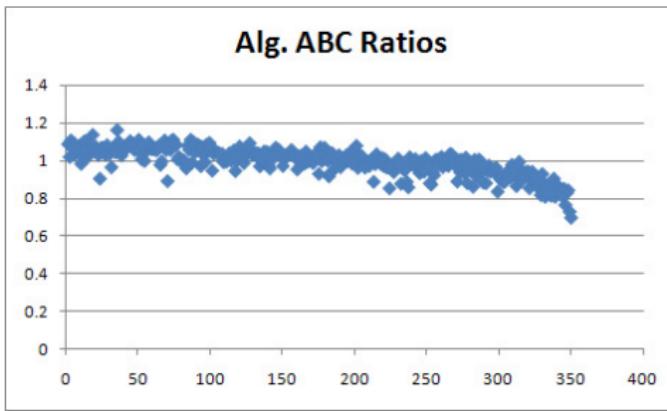
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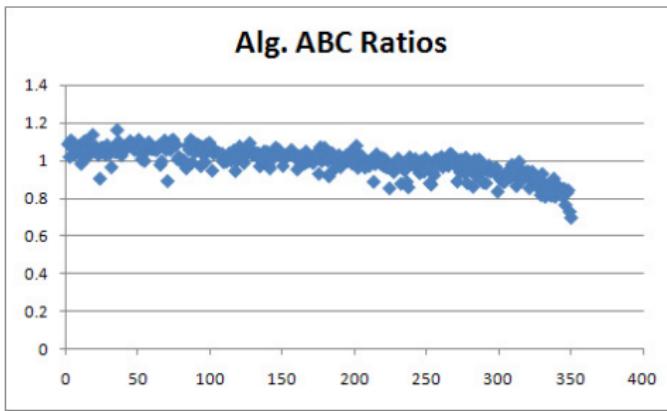
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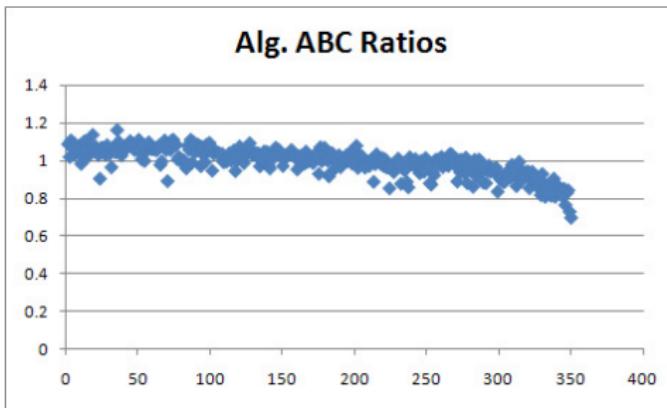
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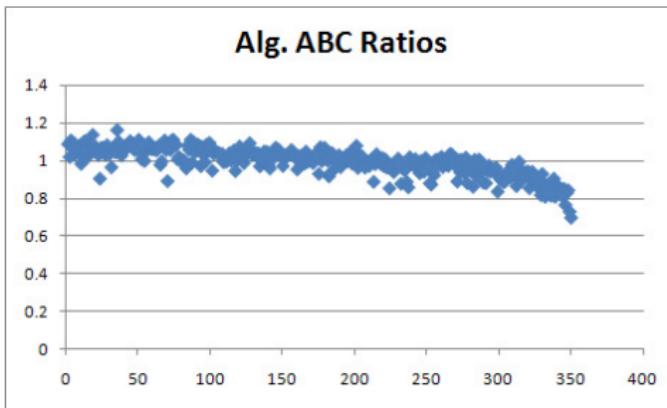
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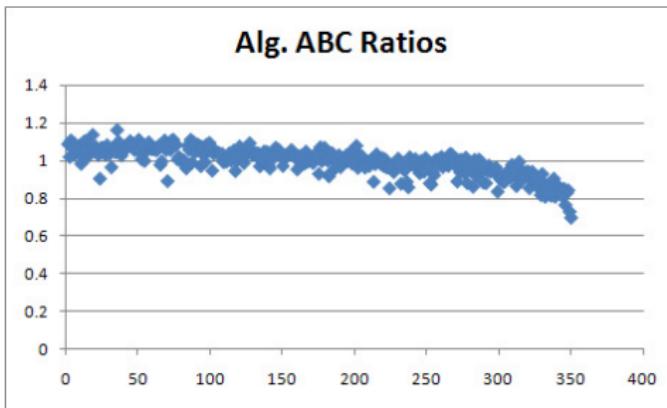
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- Data does follow a trend. Proof? No idea how to even begin.
- Failure? Well, yes, but no.

# The End?

Thanks!

More information:

The ABC Conjecture Home Page

<http://www.math.unicaen.fr/~nitaj/abc.html>