Math 280 Solutions for September 12

Pythagoras Level

Problem 1: [Nick's Math Puzzle #9]

$$\begin{aligned} 1/x + 1/y &= -1 & (1) \\ x^3 + y^3 &= 4 & (2) \end{aligned}$$

(1) implies $x + y = -xy$. (2) implies $(x+y)^3 - 3xy(x+y) = 4$. Hence $-(xy)^3 + 3(xy)^2 - 4 = 0$.

By inspection, $xy = -1$ is a solution of this cubic equation. Factorizing, we have $(xy+1)(xy-2)^2 = 0$. Hence $xy = -1$, $x + y = 1$, or $xy = 2$, $x + y = -2$.

If $xy = -1$ and $x + y = 1$, then x, y are roots of the quadratic equation $u^2 - u - 1 = 0$. (Consider the sum and product of the roots of $(u - A)(u - B) = u^2 - (A + B)u + AB = 0$.) Hence $u = (1 \pm \sqrt{5})/2$.

If $xy = 2$ and $x + y = -2$, then x, y are roots of $u^2 + 2u + 2 = 0$. This has complex roots: $u = -1 \pm i$.

Therefore the real solutions are $x = (1 \pm \sqrt{5})/2$, $y = (1 \mp \sqrt{5})/2$.

Problem 2: [Nick's Math Puzzle #11] Let p be the probability that student A wins. We consider the possible outcomes of the first two rolls. (Recall that each roll consists of the throw of two dice.) Consider the following mutually exclusive cases, which encompass all possibilities.

- If the first roll is a 12 (probability 1/36), A wins immediately.
- If the first roll is a 7 and the second roll is a 12 (probability 1/6 * 1/36 = 1/216), A wins immediately.
- If the first and second rolls are both 7 (probability 1/6 * 1/6 = 1/36), A cannot win. (That is, B wins immediately.)
- If the first roll is a 7 and the second roll is neither a 7 nor a 12 (probability 1/6 * 29/36 = 29/216), A wins with probability p.
- If the first roll is neither a 7 nor a 12 (probability 29/36), A wins with probability p.

Note that in the last two cases we are effectively back at square one; hence the probability that A subsequently wins is p.
Probability p is the weighted mean of all of the above possibilities.
Hence $p = 1/36 + 1/216 + (29/216)p + (29/36)p$.
Therefore $p = 7/13$.

Newton Level

Problem 3: [MAA-NCS 2006 #2] The value is $\ln \dfrac{(10)(101)(1002)(2011)}{(11)(102)(1003)}$. We can calculate it as follows:

$$\begin{aligned} \int_1^{2008} \frac{dx}{x + \lfloor \log_{10} x \rfloor} &= \int_1^{10} \frac{dx}{x} + \int_{10}^{100} \frac{dx}{x+1} + \int_{100}^{1000} \frac{dx}{x+2} + \int_{1000}^{2008} \frac{dx}{x+3} \\ &= \ln 10 + \ln \frac{101}{11} + \ln \frac{1002}{102} + \ln \frac{2011}{1003} \\ &= \ln \frac{(10)(101)(1002)(2011)}{(11)(102)(1003)} \end{aligned}$$

Problem 4: [MAA-NCS 2006 #3] The product is $\frac{2009}{2 \cdot 2008} = \frac{2009}{4016}$. Let

$$\Pi_n = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right).$$

Examination of some cases for small n suggest the following formula, which we will prove by induction for every integer $n \geq 2$:

$$\Pi_n = \frac{n+1}{2n}.$$

With $n = 2$ we have

$$1 - \frac{1}{2^2} = \frac{3}{4} = \frac{n+1}{2n}.$$

Suppose that

$$\Pi_k = \frac{k+1}{2k}.$$

Then

$$\begin{aligned}\Pi_{k+1} &= \Pi_k\left(1-\frac{1}{(k+1)^2}\right) = \left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right) \\ &= \frac{(k+1)(k+2)k}{2k(k+1)^2} = \frac{k+2}{2(k+1)},\end{aligned}$$

and by induction the claim is established.

Wiles Level

Problem 5: [Putnam 2006 B1] The "curve" $x^3+3xy+y^3-1=0$ is actually reducible, because the left side factors as

$$(x+y-1)(x^2-xy+y^2+x+y+1).$$

Moreover, the second factor is

$$\frac{1}{2}((x+1)^2+(y+1)^2+(x-y)^2),$$

so it only vanishes at $(-1,-1)$. Thus the curve in question consists of the single point $(-1,-1)$ together with the line $x+y=1$. To form a triangle with three points on this curve, one of its vertices must be $(-1,-1)$. The other two vertices lie on the line $x+y=1$, so the length of the altitude from $(-1,-1)$ is the distance from $(-1,-1)$ to $(1/2,1/2)$, or $3\sqrt{2}/2$. The area of an equilateral triangle of height h is $h^2\sqrt{3}/3$, so the desired area is $3\sqrt{3}/2$.

Remark: The factorization used above is a special case of the fact that

$$\begin{aligned}&x^3+y^3+z^3-3xyz \\ &= (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z),\end{aligned}$$

where ω denotes a primitive cube root of unity. That fact in turn follows from the evaluation of the determinant of the *circulant matrix*

$$\begin{pmatrix} x & y & z \\ z & x & y \\ y & z & x \end{pmatrix}$$

by reading off the eigenvalues of the eigenvectors $(1,\omega^i,\omega^{2i})$ for $i=0,1,2$.

Problem 6: [Putnam 2006 A2] Suppose on the contrary that the set B of values of n for which Bob has a winning strategy is finite; for convenience, we include $n=0$ in B, and write $B=\{b_1,\dots,b_m\}$. Then for every nonnegative integer n not in B, Alice must have some move on a heap of n stones leading to a position in which the second player wins. That is, every nonnegative integer not in B can be written as $b+p-1$ for some $b\in B$ and some prime p. However, there are numerous ways to show that this cannot happen.

First solution: Let t be any integer bigger than all of the $b\in B$. Then it is easy to write down t consecutive composite integers, e.g., $(t+1)!+2,\dots,(t+1)!+t+1$. Take $n=(t+1)!+t$; then for each $b\in B$, $n-b+1$ is one of the composite integers we just wrote down.

Second solution: Let $p_1,\dots,p_{2m}$ be any prime numbers; then by the Chinese remainder theorem, there exists a positive integer x such that

$$\begin{aligned} x-b_1 &\equiv -1 \pmod{p_1p_{m+1}} \\ &\dots \\ x-b_n &\equiv -1 \pmod{p_mp_{2m}}.\end{aligned}$$

For each $b\in B$, the unique integer p such that $x=b+p-1$ is divisible by at least two primes, and so cannot itself be prime.

Third solution: Put $b_1=0$, and take $n=(b_2-1)\cdots(b_m-1)$; then n is composite because $3,8\in B$, and for any nonzero $b\in B$, $n-b_i+1$ is divisible by but not equal to b_i-1. (One could also take $n=b_2\cdots b_m-1$, so that $n-b_i+1$ is divisible by b_i.)