Math 280 Solutions for October 31

Pythagoras Level

Problem 1: [2008 Illinois MAA #1] Let $A < B < C$. The largest sum of the numbers on five zombies is $5C$ and the smallest sum is $5A$. Let x be the sum of the five zombies that will balance the five C zombies and y the sum of the five cards which balance the five A zombies. Then $x + 5C = 0$ and $x \geq 5A$ so $5A + 5C \leq x + 5C = 0$. This means $A + C \leq 0$. Similarly, $y + 5A = 0$ and $5C \geq y$ so $5A + 5C \geq y + 5A = 0$. Hence, $A + C \geq 0$. It follows that $A + C = 0$ and thus, A and C are negatives of each other.

The second largest sum is $4C + B$. Let z be the sum of the numbers on the five zombies which balance the five cards four C and one B. If $z = 5A$, then $0 = 4C + B + z = B + A$ and $B = -A = C$, which is impossible since the three numbers are different. Hence z is at least as large as the second smallest sum which means $4A + B \leq z$.

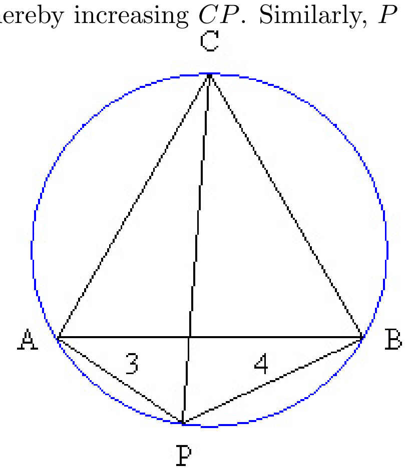
In a similar way, if w is the sum of the five zombies which balance the five zombies four A and one B, then $w \leq 4C + B$. Hence,

$$0 = (4A + B) + w \leq (4A + B) + (4C + B) \leq z + (4C + B) = 0.$$

Thus, $B = 0$. Therefore, the numbers are -2008, 0 and 2008.

Problem 2: [Nick's Math Puzzles #139] Let the point P denote the horde, A denote Alpha, B denote Beta, and C denote Gamma. Clearly P must be on the opposite side of AB to C, for otherwise we could reflect P in AB, thereby increasing CP, while keeping AP and BP the same.

Also, P must be on the same side of AC as B, for otherwise we could reflect P in AC, and then extend AB so that $BP = 4$, thereby increasing CP. Similarly, P must be on the same side of AB as C.



Hence quadrilateral $APBC$ is convex, and with diagonals AB and CP, so that we may apply Ptolemy's Inequality, which states that: $AB \cdot CP \leq AP \cdot BC + BP \cdot AC$, with equality if, and only if, $APBC$ is cyclic.

Since $AB = BC = AC$, we get CP less than or equal to $AP + BP = 7$, with equality if P lies on the arc AB of the (unique) circumcircle of $\triangle ABC$.

It is clear that equality can occur, as, for any side length, AP/BP increases continuously from 0 without limit as P moves anticlockwise along the arc AB (omitting the end point B.) Hence at some point AP/BP will reach the value $3/4$.

Therefore, the maximum possible distance of the horde from Gamma is 7 miles.

Newton Level

Problem 3: [2008 Illinois MAA #5] The quadratic equation $x^2 + px + q = 0$ has two real solutions when $p^2 - 4q > 0$ or $q < p^2/4$. Therefore, if $A = \{(p, q) : \; q < p^2/4, \; -1 \leq q, p \leq 1\}$, then the desired probability is

$$\frac{\text{area}(A)}{\text{area}(S)}.$$

The denominator is just 4. The numerator is the area of the region below the curve $q = p^2/4$ and above the line $q = -1$ on the interval $[-1, 1]$. This is given by

$$\int_{-1}^{1} p^2/4 - (-1) \; dp = 2 \int_{0}^{1} p^2/4 + 1 \; dp = 2 \left[p^3/12 + p\right]_0^1 = \frac{13}{6}.$$

So the probability is $13/24$.

Problem 4: [2007 Illinois MAA #2] The required area is the value of the integral

$$\int_0^{\pi} \left(\int_x^{\pi} g(t)\ dt \right)\ dx.$$

This integral is taken over the region $R = \{(t, x) :\ x \le t \le \pi, 0 \le x \le \pi\}$. Therefore the region R can also be described as $R = \{(t, x) :\ 0 \le x \le t, 0 \le t \le \pi\}$. Hence the desired integral is

$$\int_0^{\pi} \left(\int_0^{t} g(t)\ dx \right)\ dt = \int_0^{\pi} tg(t)\ dt = \int_0^{\pi} \sin t\ dt = -\cos t|_0^{\pi} = 2.$$

Wiles Level

Problem 5: [2008 Illinois MAA #3] There is only one triple of numbers which satisfy these equations. Note that $x^3 = 100 + y^2$, $y^3 = 100 + z^2$, and $z^3 = 100 + x^2$. In particular, all unknowns, x, y, and z are positive. The given equations imply

$$x^3 - y^3 = y^2 - z^2$$

$$y^3 - z^3 = z^2 - x^2$$

Factoring each of these gives

$$(x - y)(x^2 + xy + y^2) = (y - z)(y + z)$$
$$(y - z)(y^2 + yz + z^2) = (z - x)(z + x)$$

If $x > y$, then from the first of these equations we get $y > z$, since all unknowns are positive. Hence, from the second equation $z > x$. Combining these inequalities gives $x > y > z > x$, which is impossible. A similar contradiction arises from the assumption that $x < y$. Therefore, $x = y$. It follows that $x = y = z$. Therefore, we need to find the positive values of x such that $x^3 = 100 + x^2$ or $x^2(x - 1) = 100$. The positive integral solutions to this equation, if there are any, must be less than 10 and a divisor of 100. By examination, one such value is 5. Since $x^3 - x^2 - 100 = (x - 5)(x^2 + 4x + 20)$ and $x^2 + 4x + 20 = 0$ has no real roots, the only triple of real numbers which satisfies the given system is $(5, 5, 5)$.

Problem 6: [1995 Putnam B-4] The infinite continued fraction is defined as the limit of the sequence $L_0 = 2207, L_{n+1} = 2207 - 1/L_n$. Notice that the sequence is strictly decreasing (by induction) and thus indeed has a limit L, which satisfies $L = 2207 - 1/L$, or rewriting, $L^2 - 2207L + 1 = 0$. Moreover, we want the greater of the two roots.

Now how to compute the eighth root of L? Notice that if x satisfies the quadratic $x^2 - ax + 1 = 0$, then we have

$$\begin{aligned} 0 &= (x^2 - ax + 1)(x^2 + ax + 1) \\ &= x^4 - (a^2 - 2)x^2 + 1. \end{aligned}$$

Clearly, then, the positive square roots of the quadratic $x^2 - bx + 1$ satisfy the quadratic $x^2 - (b^2 + 2)^{1/2}x + 1 = 0$. Thus we compute that $L^{1/2}$ is the greater root of $x^2 - 47x + 1 = 0$, $L^{1/4}$ is the greater root of $x^2 - 7x + 1 = 0$, and $L^{1/8}$ is the greater root of $x^2 - 3x + 1 = 0$, otherwise known as $(3 + \sqrt{5})/2$.