Math 280 Solutions for September 26

Pythagoras Level

Problem 1: [Iowa MAA 2008 #3] When $x = 0$, the functional relationship implies that

$$f(y) = f(0 + y) = y + f(0)$$

so f is a linear function. By the divisibility condition we have $2+f(0) \mid 5+f(0)$ if and only if $2+f(0) \mid (5+f(0))-(2+f(0)) = 3$ so we must have $f(0) = 1$. Therefore $f(y) = y + 1$ and $f(2008) = 2009$.

Alternatively, one can get a linear function by setting $y = -x$ in the functional equation. This time the equation takes form $f(x) - x = f(0)$, and the rest of the solution is the same as above.

Problem 2: [Putnam 2001 A-1] The hypothesis implies $((b \circledast a) \circledast b) \circledast (b \circledast a) = b$ for all $a, b \in S$ (by replacing a by $b \circledast a$), and hence $a \circledast (b \circledast a) = b$ for all $a, b \in S$ (using $(b \circledast a) \circledast b = a$).

Newton Level

Problem 3: [Iowa MAA 2008 #7] Consider the function $g(x) = xf(x)$. Since $f(1) = 0$, it follows that $g(0) = g(1) = 0$, and thus the function g satisfies the assumptions of the Rolle's Theorem on the interval $[0, 1]$. Therefore $g'(c) = 0$ at some number $0 < c < 1$. So

$$0 = g'(c) = f(c) + cf'(c), \text{ and } \frac{f(c)}{c} = -f'(c).$$

Problem 4: [Iowa MAA 2008 #4] Take the derivative of each side to obtain, for every $x \geq 0$,

$$3(f(x))^2 f'(x) = x(f(x))^2, \text{ that is, } (f(x))^2(3f'(x) - x) = 0.$$

Hence for all $x > 0$ we have either $f(x) = 0$ or $3f'(x) - x = 0$. Since $f(0)$ is expressed in terms of a definite integral from 0 to 0, therefore $f(0) = 0$. Because $f(x)$ is strictly increasing, it follows that $f(x) > 0$ for all $x > 0$. Thus $3f'(x) - x = 0$ for all $x > 0$, and $f'(x) = x/3$ for all $x \geq 0$. It follows that $f(x) = x^2/6$, for all $x \geq 0$.

Wiles Level

Problem 5: [Iowa MAA 2008 #10] Let $g(x) = \frac{f'(x)}{f(x)}$. The inequality $2(f'(x))^2 \geq (f(x))^2 + (f''(x))^2 \geq 2f(x)f''(x)$ implies that $g'(x) \leq 0$ and therefore $g(x)$ is non-increasing on the interval $[0, 1]$. Hence

$$\ln f(1) = \int_0^1 \frac{f'(x)}{f(x)} dx \leq g(0) \cdot 1 = 1,$$

which implies that $f(1) \leq e$. The function $f(x) = e^x$ satisfies the conditions of the problem, with $f(1) = e$.

Problem 6: [Nick's Math Puzzles #50] We seek the expected value of the highest individual score when n dice are thrown. We first find $p_n(k)$, the probability that the highest score is k.

There are k^n ways in which n dice can each show k or less. For the highest score to equal k, we must subtract those cases for which each die shows less than k; these number $(k-1)^n$. So, k is the highest score in $k^n - (k-1)^n$ cases out of 6^n. In other words, $p_n(k)$, the probability that the highest individual score is k, is $(k^n - (k-1)^n)/6^n$.

The expected value, $E(n)$, of the highest score is the sum, from $k = 1$ to 6, of $kp_n(k)$. Hence $E(n) = [6(6^n - 5^n) + 5(5^n - 4^n) + 4(4^n - 3^n) + 3(3^n - 2^n) + 2(2^n - 1^n) + 1(1^n - 0^n)]/6^n = 6 - (1^n + 2^n + 3^n + 4^n + 5^n)/6^n$.