Math 280 Problems for September 26

Pythagoras Level

Problem 1: Let f be a real function satisfying $f(x) + y = f(x + y)$ for all real x and y. Assume that $f(0)$ is a positive integer, and that $f(2) \mid f(5)$. Find $f(2008)$. Note: For integers m and n, the symbol $m \mid n$ means that m divides n.

Problem 2: Consider a set S with binary operation $\circledast$, i.e. for each $a, b \in S$, $a \circledast b \in S$. Assume $(a \circledast b) \circledast a = b$ for all $a, b \in S$. Prove that $a \circledast (b \circledast a) = b$ for all $a, b \in S$. Note: We do not know in general if $a \circledast b = b \circledast a$ (commutivity) or if $a \circledast (b \circledast c) = (a \circledast b) \circledast c$ (associativity).

Newton Level

Problem 3: Let f be a continuous function on $[0, 1]$, differentiable on $(0, 1)$, and such that $f(1) = 0$. Show that for some $c \in (0, 1)$,

$$\frac{f(c)}{c} = -f'(c).$$

Problem 4: Let $f : [0, 1) \to \mathbb{R}$ be a continuous, strictly increasing function, such that

$$(f(x))^3 = \int_0^x t(f(t))^2 dt$$

for every $x \geq 0$. Show that for every $x \geq 0$ we have $f(x) = \dfrac{x^2}{6}$.

Wiles Level

Problem 5: Let f be a real, positive and continuous function on the interval $[0, 1]$, twice differentiable on $(0, 1)$, and such that $2(f')^2 \geq f^2 + (f'')^2$ on $(0, 1)$. Suppose also that $f(0) = f'(0) = 1$. Find the maximum possible value for $f(1)$.

Problem 6: Suppose n fair 6-sided dice are rolled simultaneously. What is the expected value of the score on the highest valued die?