MATH 120: Polar Form of Complex Numbers and Euler's Identity.

These notes modify the text (Section 8.3) to take advantage of Euler's Identity:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Example

1. $e^{i\pi/3} = \cos\pi/3 + i\sin\pi/3 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$.

2. $e^{i\pi} = \cos\pi + i\sin\pi = -1$.

3. $e^{i3\pi/2} = \cos 3\pi/2 + i\sin 3\pi/2 = -i$.

4. $e^{i8\pi} = \cos 8\pi + i\sin 8\pi = 1$.

Definition A complex number $z = a + bi$ has **polar form**:

$$z = re^{i\theta}$$

where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan\theta = b/a$.

Example 5

Write each complex number in polar form.

1. $1 + i$

2. $-1 + \sqrt{3}i$

3. $-4\sqrt{3} - 4i$

4. $3 + 4i$

Solution

1. The argument associated to a positive r is $\theta = \pi/4$. Then $r = \sqrt{1^2 + 1^2} = \sqrt{2}$. Thus $1 + i = \sqrt{2}e^{i\pi/4}$.

2. The argument associated to a positive r is $\theta = 2\pi/3$. Then $r = \sqrt{1+3} = 2$. Thus $-1 + \sqrt{3}i = 2e^{i2\pi/3}$.

3. The argument associated to a positive r is $\theta = 7\pi/6$. Then $r = \sqrt{48 + 16} = 8$. Thus $-4\sqrt{3} - 4i = 8e^{i7\pi/6}$.

4. The argument associated to a positive r is $\theta = \tan^{-1}(4/3)$. Then $r = \sqrt{3^2 + 5^2} = 5$. Thus $3 + 4i = 5e^{i\tan^{-1}(4/3)}$.

Theorem 0.1 (Multiplication and Division of Complex Numbers). *If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then*

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$

In other words, the usual rules of exponents apply.

Example 6

Let $z_1 = 2e^{i\pi/4}$ and $z_2 = 5e^{i\pi/3}$. Then $z_1 z_2 = 10e^{i7\pi/12}$ and $z_1/z_2 = \frac{2}{5}e^{i\pi/12}$.

Theorem 0.2 (De Moivre's Theorem). *If $z = re^{i\theta}$ then*

$$z^n = r^n e^{i\theta n}.$$

In other words, the usual rules of exponents apply.

Example 7

Find $\left(\frac{1}{2} + \frac{1}{2}i\right)^{10}$.

Solution

Converting to polar form gives $\frac{1}{2} + \frac{1}{2}i = \frac{1}{\sqrt{2}}e^{i\pi/4}$. So by rules of exponents

$$\left(\frac{1}{2} + \frac{1}{2}i\right)^{10} = \left(\frac{1}{\sqrt{2}}e^{i\pi/4}\right)^{10} = \frac{1}{2^5}e^{i10\pi/4} = \frac{1}{32}e^{i5\pi/2}.$$

Since the original question was posed in rectangular form, we should return to that form through Euler's Identity:

$$\frac{1}{32}e^{i5\pi/2} = \frac{1}{32}(\cos 5\pi/2 + i\sin 5\pi/2) = \frac{1}{32}i.$$

Example 9

Find the three cube roots of $2 + 2i$.

Solution

Let $z = 2 + 2i$. This can be written in polar form in the following three ways:

$$z \quad = \quad 2\sqrt{2}e^{i\pi/4} \quad = \quad 2\sqrt{2}e^{i9\pi/4} \quad = \quad 2\sqrt{2}e^{i17\pi/4}$$

Thus

$$
\begin{array}{lllllll}
z^{1/3} & = & (2\sqrt{2}e^{i\pi/4})^{1/3} & \text{or} & (2\sqrt{2}e^{i9\pi/4})^{1/3} & \text{or} & (2\sqrt{2}e^{i17\pi/4})^{1/3} \\
z^{1/3} & = & (2\sqrt{2})^{1/3}(e^{i\pi/4})^{1/3} & \text{or} & (2\sqrt{2})^{1/3}(e^{i9\pi/4})^{1/3} & \text{or} & (2\sqrt{2})^{1/3}(e^{i17\pi/4})^{1/3} \\
z^{1/3} & = & \sqrt{2}e^{i\pi/12} & \text{or} & \sqrt{2}e^{i3\pi/4} & \text{or} & \sqrt{2}e^{i17\pi/12}
\end{array}
$$

The answers in rectangular form are then

$$
\begin{array}{lll}
z^{1/3} & = & \sqrt{2}(\cos pi/12 + i\sin \pi/12) \approx 1.366 + 0.366i \\
& \text{or} & \sqrt{2}(\cos 3\pi/4 + i\sin 3\pi/4) = -1 + i \\
& \text{or} & \sqrt{2}(\cos 17\pi/12 + i\sin 17\pi/12) \approx -0.336 - 1.336i
\end{array}
$$

Exercises

Write the complex number in polar form with argument θ between 0 and 2π

30. $1 + \sqrt{3}i$

34. $-1 + i$

41. -20

42. $\sqrt{3} + i$

Compute the following.

70. $(1 - \sqrt{3})^5$

74. $(\sqrt{3} - i)^{-10}$

80. $(1 - i)^{-8}$

82. $\sqrt[3]{4\sqrt{3} + 4i}$

84. $\sqrt[5]{32}$

89. $\sqrt[4]{-1}$

Solve for all values of z.

92. $z^8 - i = 0$

94. $z^6 - 1 = 0$

Factor completely.

b1. $x^5 - 32$

b2. $x^4 + 1$