Math 280 Solutions for November 14

Pythagoras Level

Problem 1: [MAA-NCS 2006 #6] From $\sqrt[3]{r} + \dfrac{1}{\sqrt[3]{r}} = 3$, we have upon cubing

$$r + 3\sqrt[3]{r} + \frac{3}{\sqrt[3]{r}} + \frac{1}{r} = 27,$$

so

$$r + \frac{1}{r} = 27 - 3\left(\sqrt[3]{r} + \frac{1}{\sqrt[3]{r}}\right) = 27 - 9 = 18.$$

Cubing once more gives

$$r^3 + 3\left(r + \frac{1}{r}\right) + \frac{1}{r^3}$$

so

$$r^3 + \frac{1}{r^3} = 18^3 - 3 \cdot 18 = 5778.$$

Problem 2: [MAA-NCS 2003 #1] From the given information

$$\pi(r+t)^2 = 2\pi r^2.$$

So

$$r^2 + 2rt + t^2 = 2r^2.$$

Rearranging terms yeilds

$$2t^2 = r^2 - 2rt + t^2 = (r-t)^2.$$

So

$$|r - t| = \sqrt{2}\ t.$$

If $t > r$ then

$$2\pi r^2 = \pi(r+t)^2 > \pi(2r)^2 = 4\pi r^2. \Rightarrow\Leftarrow$$

So $t < r$ and $r - t = |r - t| = \sqrt{2}\ t$, so $r = (1+\sqrt{2})t$.

Newton Level

Problem 3: [MAA-NCS 2000 #10] Consider

$$\sum_{k=n}^{n^2} \frac{1}{k} = \int_n^{n^2} \frac{1}{\lfloor x \rfloor}\ dx > \int_n^{n^2} \frac{1}{x}\ dx = \ln(n^2) - \ln(n) = \ln(n).$$

And $\ln(n) > 2008$ when $n > r = e^{2008}$.

Problem 4: [MAA-NCS 2002 #4] Write the integrand as $e^{-x^2} e^{f(x)}\ dx$ and make the substitution $u = f(x)$, $du = f'(x)\ dx$. By the fundamental theorem of calculus, $f'(x) = e^{-x^2}$, so now our integrand is $e^u\ du$. To find the limits we need to evaluate $f(0) = 0$ and

$$\lim_{x\to\infty} f(x) = \int_0^\infty e^{-x^2}\ dx = \frac{\sqrt{\pi}}{2}.$$

So

$$\int_0^\infty e^{-x^2+f(x)}\ dx = \int_0^{\frac{\sqrt{\pi}}{2}} e^u\ du = e^{\frac{\sqrt{\pi}}{2}} - 1.$$

Wiles Level

Problem 5: [MAA-NCS 2004 #6] Let P represent the permutation executed by the shuffler. As P^2 in cycle notation is

$$(A, 3, 10, 6, 9, 8, J, 7, 4, 2, K, 5, Q),$$

a single cycle, it follows that P is a single cycle, and therefore that $P^{13} = I$. Then $P = P^{14} = (P^2)^7$, which one computes from P to be the cycle

$$(A, 7, 3, 4, 10, 2, 6, K, 9, 5, 8, Q, J).$$

This cycle applied to the cards in their original order puts them in the order

$$7, 6, 4, 10, 8, K, 3, Q, 5, 2, A, J, 9.$$

Problem 6: [MAA-NCS 2001 #4] We may assume without loss of generality that $r > 1$, and then $0 < \frac{1}{r} < 1$. Then $F(\frac{1}{r}) = \frac{1}{r}$ and $F(r) = r - \lfloor r \rfloor$. If $F(r) + F(\frac{1}{r}) = 1$, then

$$r + \frac{1}{r} = F(r) + \lfloor r \rfloor + F\left(\frac{1}{r}\right) = 1 + \lfloor r \rfloor = n$$

For some integer $n \geq 2$. Solving for r we obtain

$$r = \frac{n + \sqrt{n^2 - 4}}{2},$$

and we see that $n > 2$ (since $n = 2$, make $r = 1$). To find a specific solution, choose a value of $n \geq 3$. For example, $n = 4$ yeilds $r = 2 + \sqrt{3}$. And indeed $F\left(\frac{1}{2+\sqrt{3}}\right) = 2 - \sqrt{3}$ and $F(2 + \sqrt{3}) = \sqrt{3} - 1$.