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A Circular Lemma

2002. *Proposed by Dorin Marghidanu, Colegiul National "A. I. Cuza," Corabia, Romania.*

Let a_1, a_2, . . . , a_n be positive real numbers. Prove that

$$\frac{a_1^2}{a_1+a_2}+\frac{a_2^2}{a_2+a_3}+\cdots+\frac{a_{n-1}^2}{a_{n-1}+a_n}+\frac{a_n^2}{a_n+a_1}\geq\frac{a_1+a_2+\cdots+a_n}{2}.$$

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We require two lemmas to prove the main result. Thie first is a basic technical result.

Lemma 1: For any nonnegative real numbers a_i and a_j,

$$\frac{a_i^2}{a_i+a_j}+\frac{a_j^2}{a_j+a_i}\geq\frac{a_i+a_j}{2}.$$

Proof: Consider

$$\begin{aligned}(a_i-a_j)^2 &\geq 0\\ 2a_i^2+2a_j^2 &\geq a_i^2+a_j^2+2a_ia_j\\ 2(a_i^2+a_j^2) &\geq (a_i+a_j)^2\\ \frac{a_i^2+a_j^2}{a_i+a_j} &\geq \frac{a_i+a_j}{2}.\end{aligned}$$

The next lemma is a curious identity that will allow us to shift our numerators.

Lemma 2: For any finite collection of nonnegative real numbers a_i,

$$\frac{a_1^2}{a_1+a_2}+\frac{a_2^2}{a_2+a_3}+\ldots+\frac{a_{n-1}^2}{a_{n-1}+a_n}+\frac{a_n^2}{a_n+a_1}=\frac{a_2^2}{a_1+a_2}+\frac{a_3^2}{a_2+a_3}+\ldots+\frac{a_n^2}{a_{n-1}+a_n}+\frac{a_1^2}{a_n+a_1}.$$

Proof: We proceed by induction. The base case ($n=2$) is trivial. Assume the inductive hypothesis for k:

$$\sum_{i=1}^{k-1}\frac{a_i^2}{a_i+a_{i+1}}+\frac{a_k^2}{a_k+a_1} = \sum_{i=2}^{k}\frac{a_i^2}{a_{i-1}+a_i}+\frac{a_1^2}{a_k+a_1}. \tag{1}$$

Then for the $k+1$ case, we have the expression

$$\sum_{i=1}^{k-1}\frac{a_i^2}{a_i+a_{i+1}}+\frac{a_k^2}{a_k+a_{k+1}}+\frac{a_{k+1}^2}{a_{k+1}+a_1}. \tag{2}$$

Solving for the summation in (1) and substituting into (2) yields

$$\begin{aligned}
(2) \quad &= \sum_{i=2}^{k} \frac{a_i^2}{a_{i-1}+a_i} + \frac{a_1^2}{a_k+a_1} - \frac{a_k^2}{a_k+a_1} + \frac{a_k^2}{a_k+a_{k+1}} + \frac{a_{k+1}^2}{a_{k+1}+a_1} \\
&= \sum_{i=2}^{k} \frac{a_i^2}{a_{i-1}+a_i} - \frac{(a_k+a_1)(a_k-a_1)}{a_k+a_1} + \frac{a_k^2}{a_k+a_{k+1}} + \frac{a_{k+1}^2}{a_{k+1}+a_1} \\
&= \sum_{i=2}^{k} \frac{a_i^2}{a_{i-1}+a_i} - \frac{a_k(a_k+a_{k+1})}{a_k+a_{k+1}} + \frac{a_k^2}{a_k+a_{k+1}} + \frac{a_1(a_{k+1}+a_1)}{a_{k+1}+a_1} + \frac{a_{k+1}^2}{a_{k+1}+a_1} \\
&= \sum_{i=2}^{k} \frac{a_i^2}{a_{i-1}+a_i} - \frac{a_k a_{k+1}}{a_k+a_{k+1}} + \frac{a_1 a_{k+1}}{a_{k+1}+a_1} + \frac{a_1^2}{a_{k+1}+a_1} + \frac{a_{k+1}^2}{a_{k+1}+a_1} \\
&= \sum_{i=2}^{k} \frac{a_i^2}{a_{i-1}+a_i} + \frac{(a_1 a_{k+1}+a_{k+1}^2)(a_k+a_{k+1}) - a_k a_{k+1}(a_{k+1}+a_1)}{(a_{k+1}+a_1)(a_k+a_{k+1})} + \frac{a_1^2}{a_{k+1}+a_1} \\
&= \sum_{i=2}^{k} \frac{a_i^2}{a_{i-1}+a_i} + \frac{a_{k+1}^2(a_{k+1}+a_1)}{(a_{k+1}+a_1)(a_k+a_{k+1})} + \frac{a_1^2}{a_{k+1}+a_1} \\
&= \sum_{i=2}^{k} \frac{a_i^2}{a_{i-1}+a_i} + \frac{a_{k+1}^2}{(a_k+a_{k+1})} + \frac{a_1^2}{a_{k+1}+a_1}.
\end{aligned}$$

Main Proposition For any finite collection of real numbers a_i,

$$\sum_{i=1}^{n-1} \frac{a_i^2}{a_i+a_{i+1}} + \frac{a_n^2}{a_n+a_1} \geq \sum_{i=1}^{n} \frac{a_i}{2}.$$

Proof: Consider the left hand side of the desired inequality

$$\begin{aligned}
\text{LHS} &= \sum_{i=1}^{n-1} \frac{a_i^2}{a_i+a_{i+1}} + \frac{a_n^2}{a_n+a_1} \\
&= \left(\sum_{i=1}^{n-1} \frac{a_i^2}{a_i+a_{i+1}} + \frac{a_{i+1}^2}{a_{i+1}+a_i}\right) - \sum_{i=1}^{n-1} \frac{a_{i+1}^2}{a_{i+1}+a_i} + \left(\frac{a_n^2}{a_n+a_1} + \frac{a_1^2}{a_n+a_1}\right) - \frac{a_1^2}{a_n+a_1} \\
&\geq \sum_{i=1}^{n-1} \frac{a_i+a_{i+1}}{2} + \frac{a_n+a_1}{2} - \left(\sum_{i=1}^{n-1} \frac{a_{i+1}^2}{a_{i+1}+a_i} + \frac{a_1^2}{a_n+a_1}\right) \text{ (by Lemma 1)} \\
&= \sum_{i=1}^{n} a_i - \text{LHS (by Lemma 2)}.
\end{aligned}$$

Now adding the LHS to both sides and dividing by 2 yields the desired result.