Saving and Breaking Entanglement

Project Outside the Course Scope

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Motivation and Big Picture

Entanglement and Noise

- **Entanglement** is a fundamental resource for quantum computing and communication.
- **Quantum channels** model the noise and open-system dynamics that degrade entanglement.

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Core Question

Which quantum channels **preserve** entanglement under repeated application, and which **destroy** it?

Breaking Entanglement

Separable States

$$\mathcal{S}_{AB} = \left\{ \sum_i P_i \otimes Q_i \,\middle|\, P_i, Q_i \geq 0 \ \forall i \right\} \subset M_{d_1}(\mathbb{C}) \otimes M_{d_2}(\mathbb{C})$$

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Entanglement-Breaking Channel (EB)

A channel Φ is **entanglement-breaking** if

$$(\Phi \otimes \mathrm{id})(\rho_{AB}) \in \mathcal{S}_{AB} \quad \forall \rho_{AB}$$

Breaking Entanglement

Entanglement-Breaking Index

The **entanglement-breaking index** $n(\Phi)$ is the smallest N such that Φ^N is EB, or $n(\Phi) = \infty$ if no such N exists.

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Inclusion Relations

$$\mathrm{EB} \subset \mathrm{EEB}$$

Saving Entanglement

Entanglement-Saving (ES) Channel

Φ is **entanglement-saving** if Φ^n is not entanglement-breaking for any $n \in \mathbb{N}$ (i.e $n(\Phi) = \infty$).

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Asymptotically Entanglement-Saving (AES)

Φ is **AES** if none of the limit points of $(\Phi^n)_{n\in\mathbb{N}}$ are entanglement-breaking.

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Universal Entanglement-Preserving (UEP)

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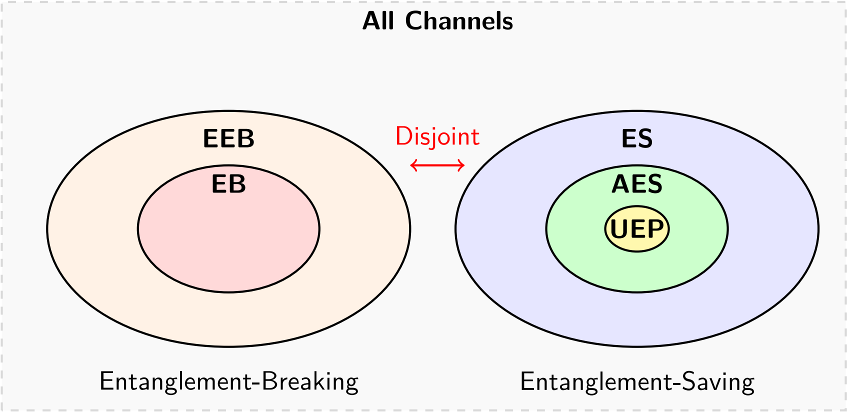
$$\text{UEP} \subset \text{AES} \subset \text{ES}$$

The Landscape of Quantum Channels



Main Result — UEP Channels are Unitaries

Theorem (Lami & Giovannetti, Thm 17)

The only universal entanglement-preserving (UEP) channels are unitary evolutions:

$$\Phi(\rho) = U\rho U^\dagger$$

for some unitary U.

Wigner's Theorem

Theorem (Wigner). Let $T : \mathcal{H} \to \mathcal{H}$ (not necessarily linear) on a Hilbert space $\mathcal{H}$. Suppose

$$|\langle T(x)|T(y)\rangle| = |\langle x|y\rangle| \quad \forall\, x, y \in \mathcal{H}.$$

Then there exists a real function $\varphi : \mathcal{H} \to \mathbb{R}$ and an isometry or anti-isometry $V : \mathcal{H} \to \mathcal{H}$ such that

$$T(x) = e^{i\varphi(x)} Vx.$$

In particular, if $\mathcal{H}$ is finite-dimensional, V is unitary or anti-unitary.

UEP Channels are Unitary Evolutions: Steps 1 & 2

Step 1: Strict Positivity of $\Phi(\mathbb{1})$

Suppose $\exists\, |\alpha\rangle$ such that $\langle\alpha|\Phi(\mathbb{1})|\alpha\rangle = 0$.

$$\Rightarrow \; \Phi^{\dagger}(|\alpha\rangle\langle\alpha|) = 0 \Rightarrow \; 0 \in \sigma(\Phi)$$

This leads to a contradiction with the UEP property (see Lemma 16 in Lami).

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This leads to a contradiction with the UEP property (see Lemma 16 in Lami).

Step 2: Reduction to the Unital Case

Let $A := \Phi(\mathbb{1})^{1/2}$.
Define:

$$\Psi(X) := A^{-1}\Phi(X)A^{-1}$$

Then Ψ is UEP, CP, and

$$\Psi(\mathbb{1}) = \mathbb{1}$$

Step 3: Preservation of Non-Invertibility

Statement

If $\rho \geq 0$ and $\det \rho = 0$, then $\det \Psi(\rho) = 0$.

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Proposition 15

$$\rho_A \otimes \rho_B \in \partial \mathcal{S}_{AB} \iff \det \rho_A \cdot \det \rho_B = 0$$

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Sketch

- $\rho \otimes \frac{\mathbb{1}}{d} \in \partial \mathcal{S}_{AB}$ (Prop. 15)
- $\exists\, (\sigma_\epsilon)_{\epsilon > 0} \notin \mathcal{S}_{AB}$, $\sigma_\epsilon \to \rho \otimes \frac{\mathbb{1}}{d}$
- Ψ UEP $\Rightarrow (\Psi \otimes \mathrm{id})(\sigma_\epsilon) \notin \mathcal{S}_{AB}$
- Ψ continuous $\Rightarrow \lim_{\epsilon \to 0} (\Psi \otimes \mathrm{id})(\sigma_\epsilon) = \Psi(\rho) \otimes \frac{\mathbb{1}}{d}$
- $\Psi(\rho) \otimes \frac{\mathbb{1}}{d} \in \partial \mathcal{S}_{AB} \Rightarrow \det \Psi(\rho) = 0$ (Prop. 15)

Step 4: Preservation for Hermitian Matrices

Statement

If $X = X^\dagger$ and $\det X = 0$, then $\det \Psi(X) = 0$.

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Sketch

- $X^2 \geq 0$, $\det(X^2) = 0$
- Step 3 $\Rightarrow \det \Psi(X^2) = 0$
- $\exists\, |\eta\rangle : \langle\eta|\Psi(X^2)|\eta\rangle = 0$
- Kadison's inequality:

$$0 = \langle\eta|\Psi(X^2)|\eta\rangle = \langle\eta|\Psi(X^\dagger X)|\eta\rangle \geq (\Psi(X)|\eta\rangle)^\dagger(\Psi(X)|\eta\rangle)$$

- $\Rightarrow \Psi(X)|\eta\rangle = 0 \Rightarrow \det \Psi(X) = 0$

Step 5: Preservation of the Spectrum

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If $X = X^\dagger$, then $\sigma(\Psi(X)) = \sigma(X)$.

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Sketch

- $\lambda \in \sigma(X) \overset{\text{Step 4}}{\Rightarrow} \det \Psi(X - \lambda \mathbb{1}) = 0$
- Ψ linear and unital $\Rightarrow$ $\lambda \in \sigma(\Psi(X))$
- For X Hermitian and non-degenerate, $\sigma(X) \subseteq \sigma(\Psi(X))$ and both have d elements, so $\sigma(X) = \sigma(\Psi(X))$
- For any $X = X^\dagger$, take $(X_\epsilon)_\epsilon$ non-degenerate, $X_\epsilon \to X$
- By continuity (Weyl's theorem):

$$\sigma(\Psi(X)) = \lim_{\epsilon \to 0} \sigma(\Psi(X_\epsilon)) = \lim_{\epsilon \to 0} \sigma(X_\epsilon) = \sigma(X)$$

Step 6: Pure States and Inner Products

Statement

Ψ sends pure states to pure states and preserves $|\langle\alpha|\beta\rangle|$.

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Sketch

- For $|\alpha\rangle$, the spectrum is $\sigma(|\alpha\rangle\langle\alpha|) = \{1, 0, \ldots, 0\}$
- Spectrum preservation $\Rightarrow \Psi(|\alpha\rangle\langle\alpha|) = |\alpha'\rangle\langle\alpha'|$
- For $|\alpha\rangle, |\beta\rangle$:

$$\sigma(|\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta|) = \{1 + |\langle\alpha|\beta\rangle|,\ 1 - |\langle\alpha|\beta\rangle|,\ 0, \ldots, 0\}$$

- Spectrum preservation:

$$\sigma(\Psi(|\alpha\rangle\langle\alpha|) + \Psi(|\beta\rangle\langle\beta|)) = \{1 + |\langle\alpha'|\beta'\rangle|,\ 1 - |\langle\alpha'|\beta'\rangle|,\ 0, \ldots, 0\}$$

- $\Rightarrow\ |\langle\alpha|\beta\rangle| = |\langle\alpha'|\beta'\rangle|$

Step 7: Conclusion via Wigner's Theorem

Statement

Ψ is conjugation by a unitary: $\Psi(X) = UXU^\dagger$ for all X.

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Sketch

- By Step 6: Ψ preserves $|\langle\alpha|\beta\rangle|$
- Wigner's Theorem $\Rightarrow |\alpha'\rangle = e^{i\varphi(\alpha)}U|\alpha\rangle$
- $\Psi(|\alpha\rangle\langle\alpha|) = U|\alpha\rangle\langle\alpha|U^\dagger$
- By linearity, $\Psi(X) = UXU^\dagger$ for all X
- Returning to Φ:
  $$\Phi(X) = AUXU^\dagger A$$
  where $A = \Phi(\mathbb{1})^{1/2}$
- Trace preservation $\Rightarrow$ $A = \mathbb{1}$
- $\Rightarrow$ $\Phi(X) = UXU^\dagger$

Theorem 21: Characterization of ES Channels (Nonzero Determinant)

Statement

Let $\phi \in \mathbf{CPt}_d$ be a quantum channel with $a_\phi(0) < 2(d-1)$ (in particular, $\det \phi \neq 0$ suffices). The following are equivalent:

1. ϕ is entanglement-saving (ES)
2. ϕ has a semipositive fixed point, or $|\sigma_P(\phi)| \geq 2$
3. $\exists\, 1 \leq n \leq d$ such that ϕ^n has a semipositive fixed point

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3. $\exists\, 1 \leq n \leq d$ such that ϕ^n has a semipositive fixed point

Notation

$\sigma_P(\phi) = \{\lambda \in \sigma(\phi) \mid |\lambda| = 1\}$ (peripheral spectrum)

Open Question: Why Do ES Channels Have Measure Zero?

Claim from Lami & Giovannetti

"The set of ES channels has measure zero in the space of all quantum channels."

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Open Questions for Discussion

- **Why exactly** do these spectral constraints lead to measure zero?
- What does this mean physically for the "probability" of encountering ES channels?
- Can anyone provide intuition for this measure-theoretic argument?

PPT Squared Conjecture

Definitions

A linear map $T : M_{d_1} \to M_{d_2}$ is called

- **entanglement breaking** if for any positive matrix $X \in (M_{d_2} \otimes M_{d_1})^+$, the matrix $(\mathrm{id}_{d_2} \otimes T)(X)$ is separable.
- **completely copositive** if $\vartheta_{d_2} \circ T$ is completely positive, where $\vartheta_d : M_d \to M_d$ denotes the matrix transposition map.

PPT Squared Conjecture - Version 1

If a linear map $T : M_d \to M_d$ is both completely positive and completely copositive, then its square $T \circ T$ is entanglement breaking.

Motivation: Quantum Key Distribution

A) Composition of different linear maps

PPT Squared Conjecture - Version 2

For any pair of linear maps $T_1 : M_{d_1} \to M_{d_2}$ and $T_2 : M_{d_2} \to M_{d_3}$ that are both completely positive and completely copositive, the composition $T_2 \circ T_1$ is entanglement breaking.

$2) \Rightarrow 1)$:

Obvious.

$1) \Rightarrow 2)$:

Assumption: $\exists$ linear maps $T_1 : M_{d_1} \to M_{d_2}$ and $T_2 : M_{d_2} \to M_{d_3}$ and both CP and PPT, such that the composition $T_2 \circ T_1$ is <u>not</u> EB.

A) Composition of different linear maps

1) $\Rightarrow$ 2):

Step 1: Isometric Embedding

Let $V_1 : \mathbb{C}^{d_1} \to \mathbb{C}^d$, $V_2 : \mathbb{C}^{d_2} \to \mathbb{C}^d$ denote the canonical isometries into the first d_i coordinates of $\mathbb{C}^d$ and $d = \max(d_1, d_2, d_3)$. Define the linear maps $\tilde{T}_1 : M_d \to M_d$ and $\tilde{T}_2 : M_d \to M_d$ as

$$\begin{aligned} \tilde{T}_1(X) &= T_1(V_1^\dagger X V_1) \oplus 0_{(d-d_2)} \\ \tilde{T}_2(X) &= T_2(V_2^\dagger X V_2) \oplus 0_{(d-d_3)} \qquad X \in M_d \end{aligned}$$

A) Composition of different linear maps

1) $\Rightarrow$ 2):

Step 2: Switch Map

Define the switch map $T : M_d \otimes M_2 \to M_d \otimes M_2$ as

$$\begin{aligned} T(X) = {} & \tilde{T}_1\big((\mathbb{I}_d \otimes \langle 1|)X(\mathbb{I}_d \otimes |1\rangle)\big) \otimes |2\rangle \langle 2| \\ & + \tilde{T}_2\big((\mathbb{I}_d \otimes \langle 2|)X(\mathbb{I}_d \otimes |2\rangle)\big) \otimes |1\rangle \langle 1| \end{aligned}$$

for any $X \in M_d \otimes M_2$.

Note: T is still completely positive and completely copositive.

A) Composition of different linear maps

$1) \Rightarrow 2)$:

Step 3: Putting it all together

Result in Paper: $T_2 \circ T_1(Y) = T \circ T\left(V_1 Y V_1^\dagger \otimes |1\rangle\langle 1|\right)$

Our Result: $T_2 \circ T_1(Y) \oplus 0_{(d-d_3)} \otimes |1\rangle\langle 1| = T \circ T\left(V_1 Y V_1^\dagger \otimes |1\rangle\langle 1|\right)$

By assumption, this channel is not entanglement breaking. Therefore, $T \circ T$ cannot be entanglement breaking either. $\square$

B) Connection to local entanglement annihilation

Definition (2-locally entanglement annihilating maps)

A linear map $T : M_{d_1} \to M_{d_2}$ is called *2-locally entanglement annihilating* if the image $(T \otimes T)(X)$ is separable for any positive matrix $X \geq 0$.

PPT squared conjecture - Version 3

For any pair of linear maps $T_1 : M_{d_1} \to M_{d_2}$ and $T_2 : M_{d_3} \to M_{d_4}$, both CP and PPT, the image $(T_1 \otimes T_2)(X)$ is separable for any positive matrix $X \in (M_{d_1} \otimes M_{d_3})_+$.

Tricks with maximally entangled state (MES)

Notation: *(Unnormalized) MES:* $|\Omega_d\rangle = \sum_{i=1}^{d} |i\rangle \otimes |i\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$

Corresponding matrix: $\omega_d := |\Omega_d\rangle \langle\Omega_d|$

Lemma $\mathbb{I}.1$

Any vector $|\psi\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ can be written as

$$|\psi\rangle = (\mathbb{I}_{d_1} \otimes A) |\Omega_{d_1}\rangle = (B \otimes \mathbb{I}_{d_2}) |\Omega_{d_2}\rangle ,$$

with linear maps $A : \mathbb{C}^{d_1} \to \mathbb{C}^{d_2}$ and $B : \mathbb{C}^{d_2} \to \mathbb{C}^{d_1}$.

Lemma $\mathbb{I}.2$

For any map $L : M_{d_1} \to M_{d_2}$ that is Hermiticity-preserving (i.e., it maps Hermitian matrices to Hermitian matrices), we have

$$(\mathrm{id}_{d_1} \otimes L)(\omega_{d_1}) = (\vartheta_{d_1} \circ L^* \circ \vartheta_{d_2} \otimes \mathrm{id}_{d_2})(\omega_{d_2}).$$

B) Connection to local entanglement annihilation

$2) \Rightarrow 3)$:

Note: Suffices to check that $(T_1 \otimes T_2)(|\psi\rangle\langle\psi|)$ is separable for any pure state $|\psi\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_3}$.

Rewrite expression using MES-tricks

$$(T_1 \otimes T_2)(|\psi\rangle\langle\psi|) = [\mathrm{id}_{d_2} \otimes (T_2 \circ \mathrm{Ad}_A \circ \vartheta_{d_1} \circ T_1^* \circ \vartheta_{d_2})]\,(|\Omega_{d_2}\rangle\langle\Omega_{d_2}|).$$

Note: $\mathrm{Ad}_A \circ \vartheta_{d_1} \circ T_1^* \circ \vartheta_{d_2} : M_{d_2} \to M_{d_3}$ is still CP and PPT.

By assumption, their composition is entanglement breaking, and therefore the expression $(T_1 \otimes T_2)(|\psi\rangle\langle\psi|)$ is separable.

B) Connection to local entanglement annihilation

$3) \Rightarrow 1)$:

Choose a choose a map $T : M_d \to M_d$ that is CP and PPT.

$$\begin{aligned} T_1(X) &= (\vartheta_d \circ T^* \circ \vartheta_d)(X) \\ T_2(X) &= T(X) \end{aligned}$$

for any $X \in M_d$. Using the previous tricks for the MES again

Rewrite expression using MES-tricks again

$$(T_1 \otimes T_2)(\omega_d) = ((\vartheta_d \circ T^* \circ \vartheta_d) \otimes T)\,(\omega_d) = (\mathrm{id}_d \otimes T^2)(\omega_d) = \mathcal{C}_{T^2}$$

By assumption, $(\mathrm{id}_d \otimes T^2)(\omega_d)$ is separable and therefore, T^2 is entanglement breaking. $\square$

C) Decomposability of certain positive maps

Definition (Decomposable maps)

A positive map $P : M_{d_1} \to M_{d_2}$ is called *decomposable* if $P = T_1 + \vartheta_{d_2} \circ T_2$, for some CP maps $T_1, T_2 : M_{d_1} \to M_{d_2}$.

PPT squared conjecture – Version 4

For any completely positive and completely copositive map $T : M_{d_1} \to M_{d_2}$ and any positive map $P : M_{d_2} \to M_{d_3}$, the composition $P \circ T : M_{d_1} \to M_{d_3}$ is decomposable.

Important Results

Lemma $\mathbb{III}.1$ (Horodeckis)

A linear map $T : M_{d_1} \to M_{d_2}$ is *entanglement breaking* if and only if for any positive map $P : M_{d_2} \to M_{d_1}$, the composition $P \circ T : M_{d_1} \to M_{d_1}$ is completely positive.

Lemma $\mathbb{III}.2$ (Størmer)

A linear map $T : M_{d_1} \to M_{d_2}$ is *decomposable* if and only if $(\mathrm{id}_{d_A} \otimes T)(X) \geq 0$ for any $X \in (M_{d_A} \otimes M_{d_B})^+$ with $X^\Gamma \geq 0$ where Γ denotes the partial transpose.

C) Decomposability of certain positive maps

$1) \Rightarrow 4):$

Consider a map $T : M_{d_1} \to M_{d_2}$ that is completely positive and completely copositive, and a map $P : M_{d_2} \to M_{d_3}$ that is positive.

C) Decomposability of certain positive maps

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Consider a map $T : M_{d_1} \to M_{d_2}$ that is completely positive and completely copositive, and a map $P : M_{d_2} \to M_{d_3}$ that is positive.

For any completely positive and completely copositive map $S : M_{d_3} \to M_{d_1}$, the composition $T \circ S : M_{d_3} \to M_{d_2}$ is entanglement breaking by assumption.

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By **Lemma** $\mathbb{III}.1$, the composition $P \circ T \circ S$ is completely positive.

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Consider a map $T : M_{d_1} \to M_{d_2}$ that is completely positive and completely copositive, and a map $P : M_{d_2} \to M_{d_3}$ that is positive.

For any completely positive and completely copositive map $S : M_{d_3} \to M_{d_1}$, the composition $T \circ S : M_{d_3} \to M_{d_2}$ is entanglement breaking by assumption.

By **Lemma** $\mathbb{III}.1$, the composition $P \circ T \circ S$ is completely positive.

Since this holds for any completely positive and completely copositive map S, **Lemma** $\mathbb{III}.2$ shows that $P \circ T$ must be decomposable.

C) Decomposability of certain positive maps

$4) \Rightarrow 1)$:

Assume the maps $T : M_{d_1} \to M_{d_2}$ and $S : M_{d_3} \to M_{d_1}$ are CP and PPT, but $T \circ S : M_{d_3} \to M_{d_2}$ is <u>not</u> EB.

C) Decomposability of certain positive maps

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Assume the maps $T : M_{d_1} \to M_{d_2}$ and $S : M_{d_3} \to M_{d_1}$ are CP and PPT, but $T \circ S : M_{d_3} \to M_{d_2}$ is <u>not</u> EB.

By **Lemma** $\mathbb{III}.1$, $\exists$ a positive map $P : M_{d_2} \to M_{d_3}$ s.t. $P \circ T \circ S$ is <u>not</u> completely positive.

C) Decomposability of certain positive maps

$4) \Rightarrow 1)$:

Assume the maps $T : M_{d_1} \to M_{d_2}$ and $S : M_{d_3} \to M_{d_1}$ are CP and PPT, but $T \circ S : M_{d_3} \to M_{d_2}$ is not EB.

By **Lemma** $\mathbb{III}.1$, $\exists$ a positive map $P : M_{d_2} \to M_{d_3}$ s.t. $P \circ T \circ S$ is not completely positive.

By **Lemma** $\mathbb{III}.2$, $P \circ T$ cannot be decomposable. $\square$

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Backup Slides

Additional material for questions and discussion

Preliminaries for Theorem 21

Spectral Properties (Theorem 6, Lami)

Let $\phi \in \mathbf{CPt}_d$.

- All eigenvalues λ satisfy $|\lambda| \leq 1$.
- $1 \in \sigma(\phi)$, and ϕ has at least one positive fixed point.

Preliminaries for Theorem 21

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Let $\phi \in \mathbf{CPt}_d$.

- All eigenvalues λ satisfy $|\lambda| \leq 1$.
- $1 \in \sigma(\phi)$, and ϕ has at least one positive fixed point.

Supporting Results

Corollary 14: If $|\sigma_P(\phi)| \geq 2$, $\exists\, 1 \leq n \leq d$ such that $1 \in \sigma_P(\phi^n)$ with multiplicity > 1.

Lemma 20: If $1 \in \sigma(\phi)$ with multiplicity > 1, then ϕ admits a semipositive fixed point.

Corollary 19: If ϕ is entanglement-breaking and has a semipositive fixed point, then $\dim \ker \phi \geq 2(d-1)$ and $\det \phi = 0$.

Proof of Theorem 21: $(1) \Rightarrow (2)$

Statement

If ϕ is ES, then ϕ has a semipositive fixed point or $|\sigma_P(\phi)| \geq 2$.

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Sketch

- Suppose by contradiction: ϕ has a strictly positive fixed point $\rho_0 > 0$ and $\sigma_P(\phi) = \{1\}$
- Then $\lim_{n\to\infty} \phi^n = D_{\rho_0}$, where $D_{\rho_0}(X) = \rho_0 \operatorname{Tr} X$
- Choi–Jamiolkowski: $R_{\phi^n} \to R_{D_{\rho_0}} = \rho_0 \otimes \frac{\mathbb{1}}{d}$
- By Proposition 15, $R_{D_{\rho_0}}$ is interior to the separable set
- $\Rightarrow \phi$ is not ES (contradiction)

Proof of Theorem 21: $(2) \Rightarrow (3)$

Statement

If ϕ has a semipositive fixed point or $|\sigma_P(\phi)| \geq 2$, then $\exists\, 1 \leq n \leq d$ such that ϕ^n has a semipositive fixed point.

Proof of Theorem 21: $(2) \Rightarrow (3)$

Statement

If ϕ has a semipositive fixed point or $|\sigma_P(\phi)| \geq 2$, then $\exists\, 1 \leq n \leq d$ such that ϕ^n has a semipositive fixed point.

Sketch

- If ϕ has a semipositive fixed point, done.
- Else, $|\sigma_P(\phi)| \geq 2$
- **Corollary 14:** $\exists\, 1 \leq n \leq d$ such that $1 \in \sigma_P(\phi^n)$ with multiplicity > 1
- **Lemma 20:** ϕ^n has a semipositive fixed point

Proof of Theorem 21: $(3) \Rightarrow (1)$

Statement

If $\exists\, 1 \leq n \leq d$ such that ϕ^n has a semipositive fixed point and $a_\phi(0) < 2(d-1)$ (or $\det \phi \neq 0$), then ϕ is ES.

Proof of Theorem 21: $(3) \Rightarrow (1)$

Statement

If $\exists\, 1 \leq n \leq d$ such that ϕ^n has a semipositive fixed point and $a_\phi(0) < 2(d-1)$ (or $\det \phi \neq 0$), then ϕ is ES.

Sketch

- Suppose, for contradiction, that ϕ is not ES: $\exists\, N$ such that ϕ^N is entanglement-breaking and has a semipositive fixed point.
- **Corollary 19:** ϕ^N entanglement-breaking with semipositive fixed point $\Rightarrow \dim \ker \phi^N \geq 2(d-1) \Rightarrow \det \phi^N = 0$
- But $\det \phi \neq 0 \Rightarrow \det \phi^N \neq 0$ (contradiction)
- $\Rightarrow \phi$ is ES