20171104 - PSet 6 - Q2

BG cell ; $N(\mu_1, \sigma_1^2)$

FG cell ; $N(\mu_2, \sigma_2^2)$ $(\mu_1 < \mu_2)$

a) $\sigma_1 = \sigma_2 = \sigma$

$P(BG) = P(FG)$

$$\therefore \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(\frac{-1}{2}\frac{(\theta^* - \mu_1)^2}{\sigma^2}\right)} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(\frac{-1}{2}\frac{(\theta^* - \mu_2)^2}{\sigma_2^2}\right)}$$

$$(\theta^* - \mu_1)^2 = (\theta^* - \mu_2)^2$$

$$\theta^2 - 2\mu_1\theta^* + \mu_1^2 = \theta^2 - 2\mu_2\theta^* + \mu_2^2$$

$$\therefore \boxed{\theta^* = \frac{\mu_1 + \mu_2}{2}}$$

b) Let $P(BG) = \lambda P(FG)$

$$\therefore \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{\left(\frac{-1}{2}\left(\frac{\theta - \mu_1}{\sigma_1}\right)^2\right)} = \frac{\lambda}{\sqrt{2\pi}\,\sigma_2} e^{\left(\frac{-1}{2}\left(\frac{\theta - \mu_2}{\sigma_2}\right)^2\right)}$$

Now, $\theta = \frac{\mu_1 + \mu_2}{2}$

$$\therefore \frac{1}{\sigma_1} e^{\left(\frac{-1}{2}\left(\frac{\mu_1 - \mu_2}{\sigma_1}\right)^2\right)} = \frac{\lambda}{\sigma_2} e^{\left(\frac{-1}{2}\left(\frac{\mu_1 - \mu_2}{\sigma_2}\right)^2\right)}$$

$$\ln\left(\frac{\sigma_2}{\lambda\sigma_1}\right) = \frac{1}{2}\left(\frac{\mu_1 - \mu_2}{\sigma_1}\right)^2 - \frac{1}{2}\left(\frac{\mu_1 - \mu_2}{\sigma_2}\right)^2$$

$$\boxed{\ln\left(\frac{\sigma_2}{\lambda\sigma_1}\right) = \frac{1}{2}(\mu_2 - \mu_1)^2\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right)}$$

c) $$\frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(\frac{-1}{2}\left(\frac{\theta - \mu_1}{\sigma}\right)^2\right)} = \frac{4}{\sqrt{2\pi\sigma^2}} e^{\left(\frac{-1}{2}\left(\frac{\theta - \mu_2}{\sigma}\right)^2\right)}$$

$$\frac{-1}{2}\left(\frac{\theta - \mu_1}{\sigma}\right)^2 = \ln 4 - \frac{1}{2}\left(\frac{\theta - \mu_2}{\sigma}\right)^2$$

$$\sigma^2 \ln 16 + (\mu_1^2 - \mu_2^2) = 2\theta(\mu_1 - \mu_2)$$

$$2\theta = \frac{\sigma^2 \ln 16}{\mu_1 - \mu_2} + (\mu_1 + \mu_2) = 300 - \frac{\sigma^2 \ln 16}{100}$$

$$\boxed{\theta = 150 - \frac{\sigma^2 \ln 4}{100}}$$