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PSET 3, Mathematical Foundations 01

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Let λ be the eigen value & x be the eigen vector

$$A x = \lambda x$$
$$(A - \lambda I) x = 0$$
$$|A - \lambda I| = 0$$

$$\therefore \quad A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 4 & 5-\lambda & 6 \\ 7 & 8 & 9-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (1-\lambda)[(\lambda-5)(\lambda-9) - 48] - 2[4(9-\lambda) - 42] + 3[32 - 7(5-\lambda)]$$
$$= (1-\lambda)(\lambda^2 - 14\lambda + 45 - 48) - 2(-4\lambda - 6) + 3(-3 + 7\lambda)$$
$$= \lambda^2 - 14\lambda - 3 - \lambda^3 + 14\lambda^2 + 3\lambda + 8\lambda + 12 - 9 + \cancel{7\lambda}\, 21\lambda$$
$$= -\lambda^3 + 15\lambda^2 + 18\lambda$$

$$\therefore \quad \lambda^3 - 15\lambda^2 - 18\lambda = 0$$
$$\lambda(\lambda^2 - 15\lambda - 18) = 0$$

$$\boxed{\lambda_1 = 0 \; ; \quad \lambda_2 = \frac{15 + 3\sqrt{33}}{2} \; ; \quad \lambda_2 = \frac{15 - 3\sqrt{33}}{2}}$$

For finding eigenvectors, $\begin{bmatrix} 1-\lambda & 2 & 3 \\ 4 & 5-\lambda & 6 \\ 7 & 8 & 9-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\lambda_1 = 0 : \quad \left.\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 4x_1 + 5x_2 + 6x_3 &= 0 \\ 7x_1 + 8x_2 + 9x_3 &= 0 \end{aligned}\right\} \text{Infinite solutions} \; : \; v_1 = [1, -2, 1]^T$$

$$\lambda_2 = \frac{15 + 3\sqrt{33}}{2} : \quad (-13 - 3\sqrt{33})x_1 + 4x_2 + 6x_3 = 0$$
$$8x_1 + (-5 - 3\sqrt{33})x_2 + 12x_3 = 0$$
$$14x_1 + 16x_2 + (3 - 3\sqrt{33})x_3 = 0$$

$$v_2 = \left[\frac{3\sqrt{33} - 11}{22}, \; \frac{9 + 3\sqrt{33}}{44}, \; 1\right]^T$$

$$\lambda_3 = \frac{15 - 3\sqrt{33}}{2} : \quad (-13 + 3\sqrt{33})x_1 + 4x_2 + 6x_3 = 0$$
$$8x_1 + (-5 + 3\sqrt{33})x_2 + 12x_3 = 0$$
$$14x_1 + 16x_2 + (3 + 3\sqrt{33})x_3 = 0$$

$$v_3 = \left[\frac{-3\sqrt{33} - 11}{22}, \; \frac{9 - 3\sqrt{33}}{44}, \; 1\right]^T$$

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$$\boxed{v_1 = [1,-2,1]^T \; ; \; v_2 = \left[\frac{3\sqrt{33}-11}{22}, \frac{3\sqrt{33}+9}{44}, 1\right]^T \; ; \; v_3 = \left[\frac{-3\sqrt{33}-11}{22}, \frac{-3\sqrt{33}+9}{44}, 1\right]^T}$$

Trace = Sum of diagonal elements

$= 1 + 5 + 9$

$= 15$

$=$ Sum of eigen values

Determinant = Product of eigenvalues = 0

$= 1(45-48) - 2(\cancel{45}36-42) + 3(32-35)$

$= 0$

Rank :

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank of a matrix is number of linearly independent rows.

$\therefore$ Rank = 2

Also, Rank = Number of non-zero eigenvalues

Trace = Sum of eigenvalues
Determinant = Product of eigenvalues
Rank = Number of non-zero eigenvalues.

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PSET 03, Mathematical Foundations 02

$y_i = Ax_i$

1) Dimensions of $y_i = p \times 1$

Dimensions of $x_i = q \times 1$

$\therefore$ Dimensions of $A = \boxed{p \times q}$

2) $y_1 = Ax_1$ and $y_2 = Ax_2$

~~Eucli~~ Let $x_1 = [a, b]^T$ & $x_2 = [c, d]^T$

Then, Euclidean distance $= ||x_1 - x_2|| = \sqrt{(a-c)^2 + (b-d)^2}$

Now, consider $(x_1 - x_2)^T(x_1 - x_2)$ whose dimension is 1×1

$$(x_1 - x_2)^T(x_1 - x_2) = [(a-c), (b-d)] \begin{bmatrix} a-c \\ b-d \end{bmatrix}$$

$$= [(a-c)^2 + (b-d)^2]$$

Now, we want, $||x_1 - x_2|| = ||y_1 - y_2|| = \sqrt{(a-c)^2 + (b-d)^2}$

$$\therefore (x_1 - x_2)^T(x_1 - x_2) = (y_1 - y_2)^T(y_1 - y_2)$$

$$(x_1 - x_2)^T(x_1 - x_2) = (A(x_1 - x_2))^T(A(x_1 - x_2))$$

$$(x_1 - x_2)^T(x_1 - x_2) = (x_1 - x_2)^T A^T A (x_1 - x_2)$$

$\Rightarrow A^T A = I$ ~~only if x_1 & x_2 are invertible~~.

$\therefore \boxed{A^T A = I \text{ \& } A \text{ must be square matrix}}$

3) $A^T A = I \Rightarrow (A^T A)^T = I^T = I \Rightarrow AA^T = I$

$\Rightarrow$ A must be a square matrix if we want the the Euclidean distance to remain same.

a) $p = 2, q = 2 \Rightarrow$ A will be a square matrix

Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow AA^T = I$

Let $x_1 = [1, 2]^T$ & $x_2 = [1, -1]^T$

$\therefore y_1 = [2, -1]^T$ & $y_2 = [-1, -1]^T$

$||x_1 - x_2|| = 3$ & $||y_1 - y_2|| = 3 \Rightarrow$ Euclidean distance remains same.

b) $q = 2, p = 1$ and c) $q = 4, p = 2$

Here its not possible since dimensions of A are (1×2) & (2×4)

A is not a square matrix $\Rightarrow$ Not feasible

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Pset 3, Mathematical Foundations - 03

① $w_1 x_1 + w_2 x_2 + w_3 = 0 \Rightarrow x_2 = \frac{-w_1 x_1 - w_3}{w_2}$

$\mu_1 = \frac{\Sigma x_1}{N} \qquad \mu_2 = \frac{\Sigma x_2}{N}$

$\therefore \frac{\Sigma x_2}{N} = \frac{1}{N} \Sigma \left(\frac{-w_1 x_1 - w_3}{w_2} \right) = \frac{-\frac{w_1}{N} \Sigma x_1 - \frac{\Sigma}{N} w_3}{w_2} = \frac{-w_1 \mu_1 - w_3}{w_2}$

$\therefore \mu_2 = \frac{-w_1 \mu_1 - w_3}{w_2}$

$x_2 - \mu_2 = \frac{w_1}{w_2} (\mu_1 - x_1) = \frac{-w_1}{w_2} (x_1 - \mu_1)$

$$A' = \begin{bmatrix} x_1' - \mu_1 \\ x_2' - \mu_2 \end{bmatrix} [x_1' - \mu_1 \ , \ x_2' - \mu_2]$$

$$= \begin{bmatrix} (x_1' - \mu_1)^2 & (x_1' - \mu_1)(x_2' - \mu_2) \\ (x_1' - \mu_1)(x_2' - \mu_2) & (x_2' - \mu_2)^2 \end{bmatrix}$$

$$= \begin{bmatrix} (x_1' - \mu_1)^2 & \frac{-w_1}{w_2}(x_1' - \mu_1)^2 \\ \frac{-w_1}{w_2}(x_1' - \mu_1)^2 & \frac{w_1^2}{w_2^2}(x_1' - \mu_1)^2 \end{bmatrix}$$

$$= \begin{bmatrix} (x_1' - \mu_1)^2 & \frac{-w_1}{w_2}(x_1' - \mu_1)^2 \\ 0 & 0 \end{bmatrix}$$

Number of non-zero eigen values = Rank of matrix = 1

$$A = \frac{1}{N} \sum_{i=1}^{N} [x_i - \mu]^T [x_i - \mu] = \frac{1}{N} \Sigma \begin{bmatrix} (x_1^i - \mu_1)^2 & \frac{-w_1}{w_2}(x_1^i - \mu_1)^2 \\ \frac{-w_1}{w_2}(x_1^i - \mu_1)^2 & \frac{w_1^2}{w_2^2}(x_1^i - \mu_1)^2 \end{bmatrix}$$

$$= \frac{1}{N} \begin{bmatrix} \Sigma (x_1^i - \mu_1)^2 & \frac{-w_1}{w_2} \Sigma (x_1^i - \mu_1)^2 \\ \frac{-w_1}{w_2} \Sigma (x_1^i - \mu_1)^2 & \frac{w_1^2}{w_2^2} \Sigma (x_1^i - \mu_1)^2 \end{bmatrix}$$

$$= \frac{1}{N} \begin{bmatrix} \Sigma (x_1^i - \mu_1)^2 & \frac{-w_1}{w_2} \Sigma (x_1^i - \mu_1)^2 \\ 0 & 0 \end{bmatrix}$$

Number of non-zero eigen values = 1

② Slope of original line $= \frac{-w_1}{w_2}$

$\therefore$ Slope of perpendicular line $= \frac{w_2}{w_1}$

Line passes through $\mu = [\mu_1, \mu_2]^T$

$y = mx + c$

$\therefore x_2 = \frac{w_2}{w_1} x_1 + c \Rightarrow c = \mu_2 - \frac{w_2}{w_1} \mu_1$

Line : $w_2 x_1 - w_1 x_2 + (w_1 \mu_2 - w_2 \mu_1) = 0$

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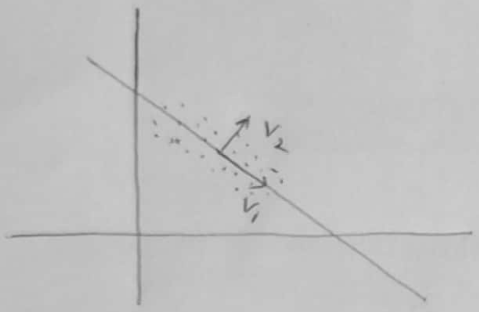
Now, proceeding in a similar way as part ①,

number of non-zero ~~values~~ eigenvalues of $B' = 1$

number of non-zero eigenvalues of $B = 1$

③ $\Sigma = \frac{1}{N}\sum_{i=1}^{N}[x_i - \mu]^T[x_i - \mu]$



Covariance matrix gives the spread of data

From image, we see that spread is mainly spread along two prependicular axes

∴ Eigenvectors of Σ will be-

- along the line $L \Rightarrow v_1$

- perpendicular to the line $L \Rightarrow v_2$

∴ Σ has two non-zero eigen values

The eigenvalue gives the amount of spread along the corresponding eigenvectors

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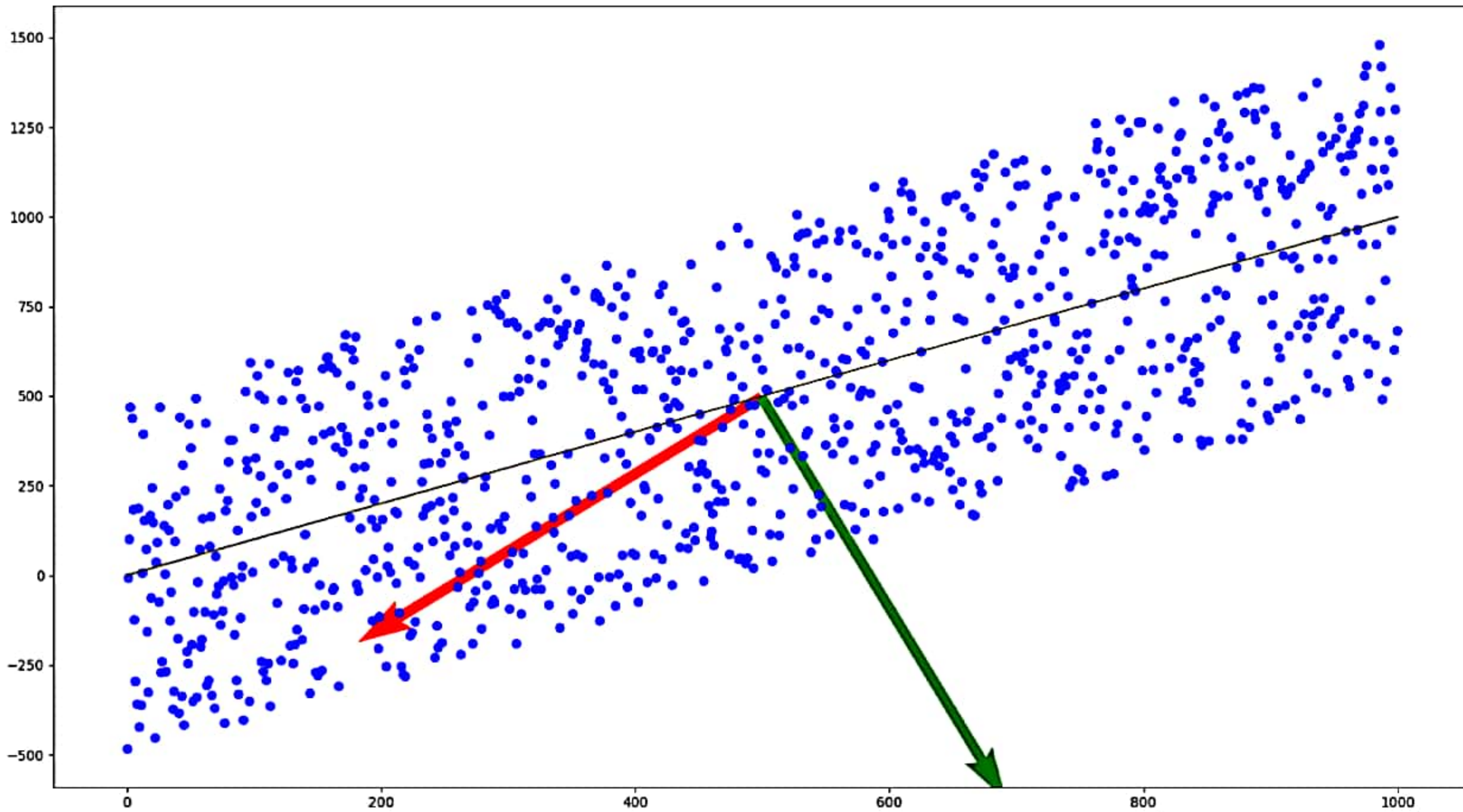

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