Covariance between Fourier coefficients representing the time waveforms observed from an array of sensors*

W. S. Hodgkiss and L. W. Nolte

Department of Electrical Engineering, Duke University, Durham, North Carolina 27706
(Received 29 July 1975; revised 14 November 1975)

Typically, the assumption of a long observation interval is made to achieve uncorrelatedness between Fourier coefficients at different frequency indicies. An explicit relationship between observation interval length and covariance between the Fourier coefficients is obtained to provide insight as to how long the observation interval should be to approximate this desirable condition.

Subject Classification: [43]60.20.

INTRODUCTION

Representation of the band-limited random processes observed at the array element outputs by a vector of Fourier coefficients can lead to a significant degree of mathematical tractability in array processing problems. The chief benefit arises when the observables are jointly Gaussian and their covariance matrix is of block diagonal form. Typically, the assumption of a long observation interval T is made to achieve uncorrelatedness between Fourier coefficients at different frequency indices.[1–6] Utilizing the notation of Appendix A, the limiting condition under which the Fourier coefficients become uncorrelated has been expressed by Papoulis[7]

$$\lim_{T\to\infty} E[z_k(n)z_k(m)^*]=0\ , \quad n\neq m\ , \tag{1}$$

where "*" denotes conjugate transpose.

A practical question arises as to how long T must be in order that these Fourier coefficients be approximately uncorrelated. This is also an important question when implementing array processors sequentially where the total observation length is broken into incremental time intervals. In these cases, uncorrelatedness between Fourier coefficients at different frequency indices is desired in each incremental interval. In addition, zero covariance between Fourier coefficients arising from adjacent observation intervals is desirable since it can lead to a convenient implementation of the sequential array processor.[8] In this paper, equations will be presented which express the covariance between the Fourier coefficients explicitly as a function of observation period length.

I. COVARIANCE ARISING FROM SCALAR RANDOM PROCESS

Let $z(t)$ be a sample function from the zero mean stationary random process observed at the output of a single array element. The Fourier coefficients for this time wave form will be as defined in Appendix A:

$$z(n)=\left(\frac{1}{T}\right)^{1/2}\int_{-T/2}^{T/2} z(t)\exp(-jn\omega_0 t)\,dt\ , \tag{2}$$

where $\omega_0=2\pi/T$. The expression in (1) indicates that the Fourier coefficients at different frequency indices become uncorrelated as the observation interval T increases. Unfortunately, little in insight is gained as to how fast this occurs. The derivation in Ref. 8 due to Blachman is most beneficial to this respect.[9] Summarizing those results,

$$E[z(n)z(m)^*]=\int_{-\infty}^{\infty} N(x\omega_0)\,\mathrm{sinc}(x-n)\,\mathrm{sinc}(x-m)\,dx\ , \tag{3}$$

where $N(x\omega_0)$ is the power spectral density function of the random process and $\mathrm{sinc}(x)=\sin(\pi x)/(\pi x)$.

Note the orthonormal property of $\mathrm{sinc}(x)$

$$\int_{-\infty}^{\infty}\mathrm{sinc}(x-n)\,\mathrm{sinc}(x-m)\,dx=\delta_{nm}\ , \tag{4}$$

where

$$\delta_{nm}=\begin{cases}1\ , & n=m\ ,\\ 0\ , & n\neq m\ .\end{cases}$$

The variance and covariance of the Fourier coefficients now easily can be pictured as areas under a curve. The integrand in (3) is illustrated graphically in Fig. 1 for two specific cases of the following expressions:

(i) Variance:

$$E[z(n)z(n)^*]=\int_{-\infty}^{\infty} N(x\omega_0)\,\mathrm{sinc}^2(x-n)\,dx\ . \tag{5}$$

(ii) Covariance:

$$E[z(n)z(m)^*]=\int_{-\infty}^{\infty} N(x\omega_0)\,\mathrm{sinc}(x-n)\,\mathrm{sinc}(x-m)\,dx\ . \tag{6}$$

Two conclusions may be drawn:

(i) Variance: As long as $N(x\omega_0)$ is approximately constant within $2\omega_0$ or $3\omega_0$ either side of $n\omega_0$, then

$$E[z(n)z(n)^*]\simeq N(n\omega_0)\ . \tag{7}$$

(ii) Covariance: As long as $N(x\omega_0)$ is approximately constant over the interval where the product $\mathrm{sinc}(x-n)\times\mathrm{sinc}(x-m)$ has appreciable value, then

$$E[z(n)z(m)^*]\simeq 0\ , \quad n\neq m\ . \tag{8}$$

Essentially, increasing the observation interval leads to a smoothing of the power spectral density function when it is written as a function of $x\omega_0$ (as $T\to\infty$, $\omega_0=2\pi/T\to 0$). Thus, for a given random process, Eqs. (7) and (8) may be considered valid if the observation

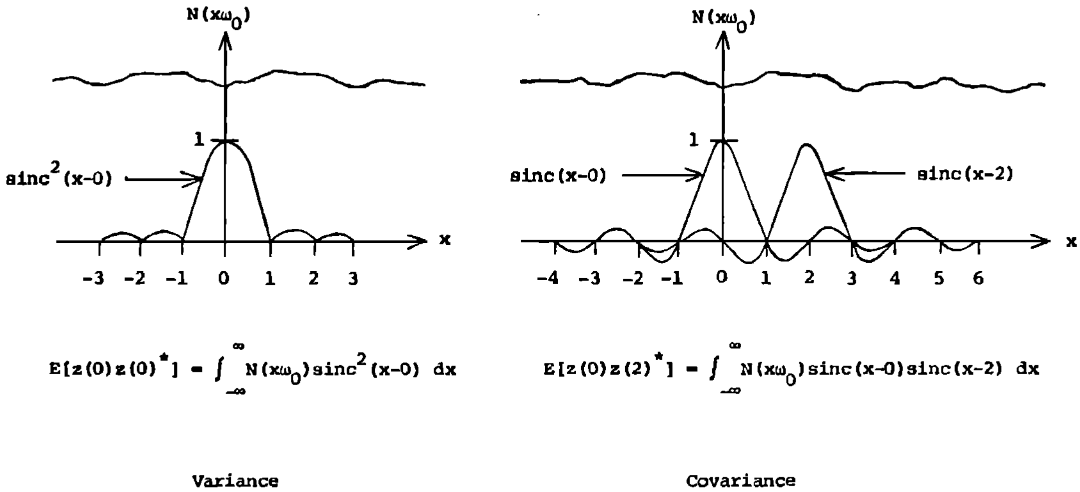


FIG. 1. Variance and covariance of the Fourier coefficients.

interval is chosen long enough so that the power spectrum $N(x\omega_0)$ is relatively smooth with respect to increments in ω_0.

A particular power spectrum often assumed is that of bandlimited white Gaussian noise

$$N(\omega) = \begin{cases} \frac{1}{2}N_0\,, & |\omega| < 2\pi W\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{9}$$

where W is the bandwidth in Hertz. It is interesting to observe how the elements of the covariance matrix associated with such a spectrum change as T is allowed to increase. Appendix B contains a set of five such matrices which represent successive doubling of the observation length.[10] The first matrix is for an observation length such that $W = 0.5\omega_0/2\pi$ or $T = 0.5/W$; the last matrix is for an observation length such that $W = 8\omega_0/2\pi$ or $T = 8/W$. As T increases, the matrices become progressively more diagonal in form. Roughly, a condition on the observation length can be established on the basis of the last matrix. With reference to the unit height spectrum, for at least 80% of the Fourier coefficients ($0 \le n \le N$) to have the following properties:

$$\begin{aligned} &0.95 \le E[z(n)z(n)^*]\,, \\ &|E[z(n)z(m)^*]| \le 0.02\,, \end{aligned} \tag{10}$$

then $2WT > 16$. Note that this is on the order of the usually assumed condition arising in the uniformly spaced time samples approach to random process representation (i.e., $2WT \gg 1$).

II. COVARIANCE ARISING FROM VECTOR RANDOM PROCESS

Section I dealt with the covariance between Fourier coefficients representing the time waveform observed at a single element. Presumably, the observation length will be chosen so that Eqs. (7) and (8) may be assumed valid. Now, let $z(t)$ be a vector of sample functions from the zero mean stationary vector random process observed as the collection of outputs from all the array elements. The Fourier coefficients for these time waveforms will be defined as in Appendix A:

$$z_k(n) = \left(\frac{1}{T}\right)^{1/2} \int_{-T/2}^{T/2} z_k(t) \exp(-jn\omega_0 t)\, dt\,, \tag{11}$$

where $\omega_0 = 2\pi/T$ and k is the array element index. Since the relationship between the coefficients representing a single element's output has already been discussed, only the covariance between pairs of Fourier coefficients arising from two different elements need be considered here.

Clearly, if the random processes observed at all the array elements are independent of one another, their respective collections of Fourier coefficients also will be independent. Such will be the case when independent sensor noise is considered. A different situation arises when the noise field contains an additive directional noise component. Now, a portion of the random process observed at one element simply will be a time delayed version of that observed at another. Given that the observation length has been chosen long enough so that (7) and (8) may be assumed valid, then any two Fourier coefficients at different frequency indices will be approximately uncorrelated. However, coefficients at the same frequency index will be related as shown in Ref. 8. Summarizing those results,

$$\begin{aligned} E[z_l(n)z_k(n)^*] = \exp(+jn\omega_0\tau_{lk}) \int_{-\infty}^{\infty} D(x\omega_0) \\ \times \operatorname{sinc}^2(x-n) \exp[-j(x-n)(-\tau_{lk}/T)2\pi]\, dx\,, \end{aligned} \tag{12}$$

where τ_{lk} is the time delay between the lth and kth elements of the directional noise component $d(t)$ and $D(x\omega_0)$ is its power spectral density function. Note that if the reception situation is such that plane waves are incident across a uniformly spaced linear array, then $\tau_{lk} = (k-l)\tau_n$, where τ_n is the time delay of $d(t)$ between adjacent elements. As before, the covariance of the Fourier coefficients can be pictured as the area under a curve. The real part of the integrand in (12) is illustrated graphically in Fig. 2 for the case $\tau_{lk}/T = \frac{1}{4}$. Two conclusions may be drawn:

(i) As long as $D(x\omega_0)$ is approximately constant with-

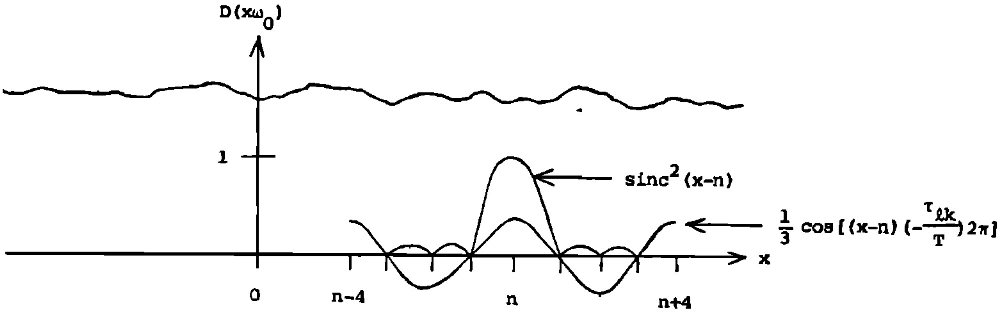
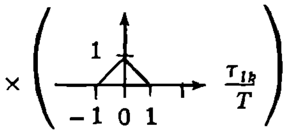


FIG. 2. Real part of the integrand in Eq. (12): $\tau_{lk}/T=\frac{1}{4}$.

in $2\omega_0$ or $3\omega_0$ either side of $n\omega_0$, then

$$E[z_l(n)z_k(n)^*]\simeq\exp(-jn\omega_0\tau_{lk})\,D(n\omega_0)$$



(13)

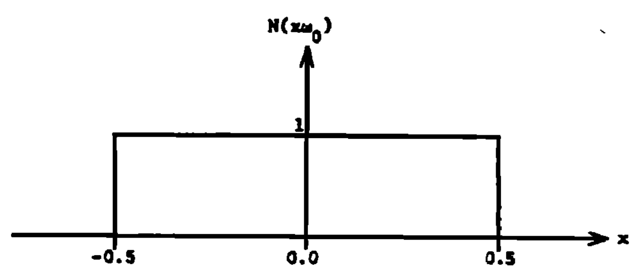
(ii) As long as $D(x\omega_0)$ is approximately constant over the interval where the product $\mathrm{sinc}(x-n)\,\mathrm{sinc}(x-m)$ has appreciable value, then

$$E[z_l(n)z_k(m)^*]\simeq 0\ ,\quad n\neq m\ . \tag{14}$$

Thus, for a given additive directional noise component,

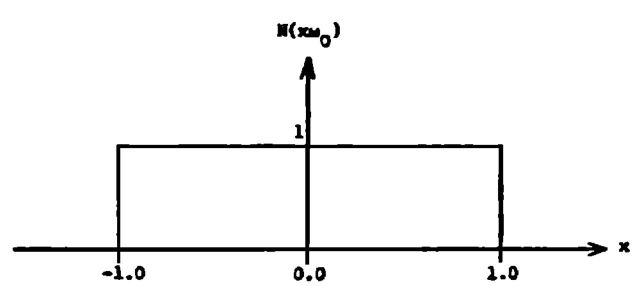
n \ m	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
-6	.00	.00	.00	.00	.00	-.01	.00	.01	.00	.00	.00	.00	.00
-5	.00	.00	.00	.00	-.01	.01	.00	-.01	.01	.00	.00	.00	.00
-4	.00	.00	.00	.00	.01	-.02	.00	.01	-.01	.00	.00	.00	.00
-3	.00	.00	.00	.01	-.01	.02	.01	-.02	.01	-.01	.00	.00	.00
-2	.00	-.01	.01	-.01	.01	-.03	-.01	.03	-.01	.01	-.01	.01	.00
-1	-.01	.01	-.02	.02	-.03	.08	.06	-.06	.03	-.02	.01	-.01	.01
0	.00	.00	.00	.01	-.01	.06	.77	.06	-.01	.01	.00	.00	.00
1	.01	-.01	.01	-.02	.03	-.06	.06	.08	-.03	.02	-.02	.01	-.01
2	.00	.01	-.01	.01	-.01	.03	-.01	-.03	.01	-.01	.01	-.01	.00
3	.00	.00	.00	-.01	.01	-.02	.01	.02	-.01	.01	.00	.00	.00
4	.00	.00	.00	.00	-.01	.01	.00	-.02	.01	.00	.00	.00	.00
5	.00	.00	.00	.00	.01	-.01	.00	.01	-.01	.00	.00	.00	.00
6	.00	.00	.00	.00	.00	.01	.00	-.01	.00	.00	.00	.00	.00

FIG. 3. Covariance matrix No. 1.



n \ m	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
-6	.00	.00	.00	.00	.01	-.03	.00	.02	-.01	.00	.00	.00	.00
-5	00.	.00	.00	.00	-.01	.03	.00	-.03	.01	.00	.00	.00	.00
-4	.00	.00	.00	-.01	.01	-.04	-.01	.04	-.01	.00	.00	.00	.00
-3	.00	.00	-.01	.01	-.02	.06	.01	-.05	.02	.00	.00	.00	.00
-2	.01	-.01	.01	-.02	.03	-.10	-.03	.07	-.03	.02	-.01	.01	-.01
-1	-.03	.03	-.04	.06	-.10	.47	.16	-.16	.07	-.05	.04	-.03	.02
0	.00	.00	-.01	.01	-.03	.16	.90	.16	-.03	.01	-.01	.00	.00
1	.02	-.03	.04	-.05	.07	-.16	.16	.47	-.10	.06	-.04	.03	-.03
2	-.01	.01	-.01	.02	-.03	.07	-.03	-.10	.03	-.02	.01	-.01	.01
3	.00	.00	.00	.00	.02	-.05	.01	.06	-.02	.01	-.01	.00	.00
4	.00	.00	.00	.00	-.01	.04	-.01	-.04	.01	-.01	.00	.00	.00
5	.00	.00	.00	.00	.01	-.03	.00	.03	-.01	.00	.00	.00	.00
6	.00	.00	.00	.00	-.01	.02	.00	-.03	.01	.00	.00	.00	.00

FIG. 4. Covariance matrix No. 2.



(13) and (14) may be considered valid if the observation interval is chosen long enough so that the power spectrum $D(x\omega_0)$ is relatively smooth with respect to increments in ω_0.

III. COVARIANCE BETWEEN ADJACENT OBSERVATION VECTORS

When the total observation period is broken into several incremental periods each of length T, a sequence of observation vectors $\vec{z}^i$ will result. The real random processes observed at the output of each array element will be assumed zero mean and stationary. Assuming T is chosen long enough so that (7) and (8) may be considered valid, then two Fourier coefficients at different frequency indices and in adjacent observation vectors will be approximately uncorrelated. Coefficients at the same frequency index are related as discussed in Ref. 8. Summarizing those results for the case of independent sensor noise

$$E\left[z_k^i(n) z_k^{i+1}(n)^*\right] = \int_{-\infty}^{\infty} N(x\omega_0)\,\mathrm{sinc}^2(x-n) \times \exp\left[-j(x-n)2\pi\right] dx\,, \tag{15}$$

where $\vec{z}^i$ and $\vec{z}^{i+1}$ are adjacent observation vectors and $N(x\omega_0)$ is the power spectral density function of the independent noise. And, for an additive directional noise component

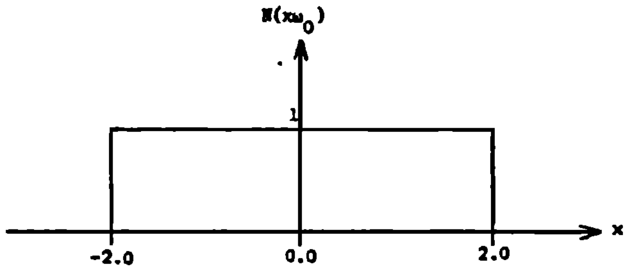
$$E\left[z_l^i(n) z_k^{i+1}(n)^*\right] = \exp(-jn\omega_0\tau_{lk}) \int_{\infty}^{\infty} D(x\omega_0)\,\mathrm{sinc}^2(x-n) \times \exp\left[-j(x-n)(1-\tau_{lk}/T)2\pi\right] dx\,, \tag{16}$$

where $D(x\omega_0)$ is the power spectral density function of the directional noise. Two conclusions may be drawn:

(i) As long as $N(x\omega_0)$ is approximately constant with-

n \ m	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
-6	.00	.00	.00	.00	.04	.00	-.01	.01	-.03	.00	.00	.00	.00
-5	.00	.00	.00	.02	-.05	.00	.01	-.02	.03	.00	.00	.00	.00
-4	.00	.00	.00	-.03	.07	.00	-.01	.02	-.04	.00	.00	.00	.00
-3	.00	.02	-.03	.04	-.11	-.01	.03	-.03	.05	.00	.00	.00	.00
-2	.04	-.05	.07	-.11	.49	.14	-.10	.08	-.10	.05	-.04	.03	-.03
-1	.00	.00	.00	-.01	.14	.93	.05	.05	.08	-.03	.02	-.02	.01
0	-.01	.01	-.01	.03	-.10	.05	.95	.05	-.10	.03	-.01	.01	-.01
1	.01	-.02	.02	-.03	.08	.05	.05	.93	.14	-.01	.00	.00	.00
2	-.03	.03	-.04	.05	-.10	.08	-.10	.14	.49	-.11	.07	-.05	.04
3	.00	.00	.00	.00	.05	-.03	.03	-.01	-.11	.04	-.03	.02	.00
4	.00	.00	.00	.00	-.04	.02	-.01	.00	.07	-.03	.00	.00	.00
5	.00	.00	.00	.00	.03	-.02	.01	.00	-.05	.02	.00	.00	.00
6	.00	.00	.00	.00	-.03	.01	-.01	.00	.04	.00	.00	.00	.00

FIG. 5. Covariance matrix No. 3.

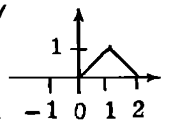


in $2\omega_0$ or $3\omega_0$ either side of $n\omega_0$, then for independent noise

$$E[z_k^i(n) z_k^{i+1}(n)^*] \simeq 0 \,, \tag{17}$$

(ii) As long as $D(x\omega_0)$ is approximately constant within $2\omega_0$ or $3\omega_0$ either side of $n\omega_0$, then for an additive direction noise component

$$E[z_l^i(n) z_k^{i+1}(n)^*] \simeq \exp(-jn\omega_0 \tau_{lk}) \cdot D(n\omega_0)$$

$$\cdot \Bigg($$  $$\frac{\tau_{lk}}{T} \Bigg) \,. \tag{18}$$

Thus, (17) and (18) may be considered valid if the incremental observation length T is chosen long enough so that the power spectra $N(x\omega_0)$ and $D(x\omega_0)$ are relatively smooth with respect to increments in ω_0. Under these conditions plus the additional requirement that $\tau_{lk} \ll T$, Fourier coefficients within adjacent observation vectors will have a covariance of approximately zero.

The covariance expressions discussed in this paper are summarized in Appendix C. Both the exact expressions and those conditioned on a sufficiently long incremental observation length are given.

IV. SUMMARY

The representation of the band-limited random processes observed at the array element outputs by a vector of Fourier coefficients can greatly simplify the mathematics of the likelihood ratio, especially when the noise field contains an additive source in the farfield. Typically, the assumption is made that the observation period is long enough for the Fourier coefficients

n \ m	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
-9	.00	.00	.00	.00	.00	-.04	.01	.00	.00	.01	-.01	.01	-.01	.02	.00	.00	.00	.00	.00
-8	.00	.00	.00	.00	.00	.04	-.01	.00	.00	-.01	.01	-.01	.01	-.02	.00	.00	.00	.00	.00
-7	.00	.00	.00	.00	.02	-.05	.01	.00	-.01	.01	-.01	.01	-.02	.03	.00	.00	.00	.00	.00
-6	.00	.00	.00	.00	-.03	.07	-.01	-.01	.01	-.01	.02	-.02	.02	-.03	.00	.00	.00	.00	.00
-5	.00	.00	.02	-.03	.04	-.12	-.01	.02	-.02	.02	-.02	.02	-.03	.04	.00	.00	.00	.00	.00
-4	-.04	.04	-.05	.07	-.12	.49	.13	-.09	.07	-.06	.05	-.05	.05	-.06	.04	-.03	.03	-.02	.02
-3	.01	-.01	.01	-.01	-.01	.13	.94	.04	-.04	.03	-.03	.03	-.03	.05	-.03	.02	-.02	.01	-.01
-2	.00	.00	.00	-.01	.02	-.09	.04	.97	.03	-.03	.03	-.03	.03	-.05	.02	-.02	.01	-.01	.01
-1	.00	.00	-.01	.01	-.02	.07	-.04	.03	.97	.03	-.03	.03	-.03	.05	-.02	.02	-.01	.01	-.01
0	.01	-.01	.01	-.01	.02	-.06	.03	-.03	.03	.97	.03	-.03	.03	-.06	.02	-.01	.01	-.01	.01
1	-.01	.01	-.01	.02	-.02	.05	-.03	.03	-.03	.03	.97	.03	-.04	.07	-.02	.01	-.01	.00	.00
2	.01	-.01	.01	-.02	.02	-.05	.03	-.03	.03	-.03	.03	.97	.04	-.09	.02	-.01	.00	.00	.00
3	-.01	.01	-.02	.02	-.03	.05	-.03	.03	-.03	.03	-.04	.04	.94	.13	-.01	-.01	.01	-.01	.01
4	0.2	-.02	.03	-.03	.04	-.06	.05	-.05	.05	-.06	.07	-.09	.13	.49	-.12	.07	-.05	.04	-.04
5	.00	.00	.00	.00	.00	.04	-.03	.02	-.02	.02	-.02	.02	-.01	-.12	.04	-.03	.02	.00	.00
6	.00	.00	.00	.00	.00	-.03	.02	-.02	.02	-.01	.01	-.01	-.01	.07	-.03	.00	.00	.00	.00
7	.00	.00	.00	.00	.00	.03	-.02	.01	-.01	.01	-.01	.00	.01	-.05	.02	.00	.00	.00	.00
8	.00	.00	.00	.00	.00	-.02	.01	-.01	.01	-.01	.00	.00	-.01	.04	.00	.00	.00	.00	.00
9	.00	.00	.00	.00	.00	.02	-.01	.01	-.01	.01	.00	.00	.01	-.04	.00	.00	.00	.00	.00

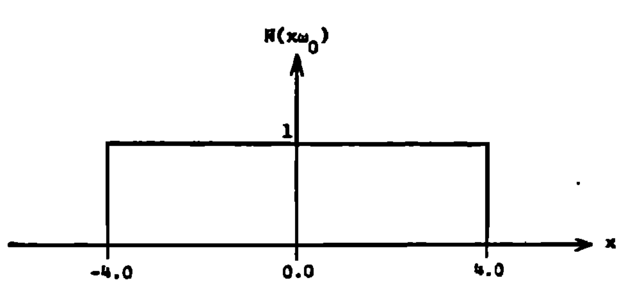


FIG. 6. Covariance matrix No. 4.

at different frequency indices to be uncorrelated. This paper has examined the question as to how long T has to be for such an assumption to be considered valid for practical purposes.

APPENDIX A: RANDOM PROCESS REPRESENTATION

The optimal array processors mentioned in this paper observe a vector of real-time waveforms on the

n \ m	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
-9	.05	-.12	.00	.01	-.02	.02	-.02	.02	-.02	.02	-.02	.02	.00	.02	-.02	.00	-.02	.02	.00
-8	-.12	.49	.13	-.08	.06	-.05	.04	-.04	.04	-.03	.03	-.03	.03	-.03	.03	-.03	.03	-.03	.02
-7	.00	.01	.95	.04	-.03	.03	-.03	.02	-.02	.02	.00	.02	-.02	.02	-.02	.02	-.02	.03	-.02
-6	.01	-.08	.04	.97	.02	-.02	.02	-.02	.02	-.02	.02	-.02	.02	-.02	.02	-.02	.02	-.03	.00
-5	-.02	.06	-.03	.02	.98	.02	-.02	.02	-.02	.01	-.01	.01	-.01	.01	-.01	.02	-.02	.03	-.02
-4	.02	-.05	.03	-.02	.02	.98	.02	-.01	.01	-.01	.01	-.01	.01	-.01	.01	-.02	.02	-.03	.02
-3	-.02	.04	-.03	.02	-.02	.02	.98	.01	-.01	.01	-.01	.01	-.01	.01	-.01	.02	-.02	.03	.00
-2	.02	-.04	.02	-.02	.02	-.01	.01	.99	.01	-.01	.01	-.01	.01	-.01	.01	-.02	.02	-.03	.02
-1	-.02	.04	-.02	.02	-.02	.01	-.01	.01	.99	.01	-.01	.01	-.01	.01	-.01	.02	.00	.03	-.02
0	.02	-.03	.02	-.02	.01	-.01	.01	-.01	.01	.99	.01	-.01	.01	-.01	.01	-.02	.02	-.03	.02
1	-.02	.03	.00	.02	-.01	.01	-.01	.01	-.01	.01	.99	.01	-.01	.01	-.02	.02	-.02	.04	-.02
2	.02	-.03	.02	-.02	.01	-.01	.01	-.01	.01	-.01	.01	.99	.01	-.01	.02	-.02	.02	-.04	.02
3	.00	.03	-.02	.02	-.01	.01	-.01	.01	-.01	.01	-.01	.01	.98	.02	-.02	.02	-.03	.04	-.02
4	.02	-.03	.02	-.02	.01	-.01	.01	-.01	.01	-.01	.01	-.01	.02	.98	.02	-.02	.03	-.05	.02
5	-.02	.03	-.02	.02	-.01	.01	-.01	.01	-.01	.01	-.02	.02	-.02	.02	.98	.02	-.03	.06	-.02
6	.00	-.03	.02	-.02	.02	-.02	.02	-.02	.02	-.02	.02	-.02	.02	-.02	.02	.97	.04	-.08	.01
7	-.02	.03	-.02	.02	-.02	.02	-.02	.02	.00	.02	-.02	.02	-.03	.03	-.03	.04	.95	.13	.00
8	.02	-.03	.03	-.03	.03	-.03	.03	-.03	.03	-.03	.04	-.04	.04	-.05	.06	-.08	.01	.49	-.12
9	.02	.02	.02	.00	-.02	.02	.00	.02	-.02	.02	-.02	.02	-.02	.02	-.02	.01	.00	-.12	.05

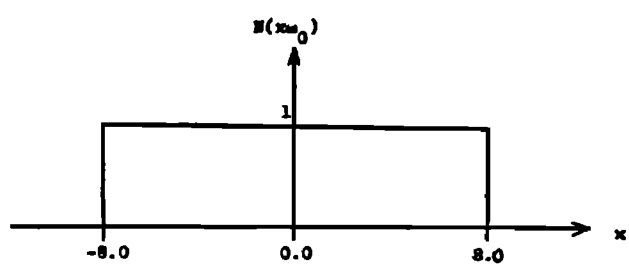


FIG. 7. Covariance matrix No. 5.

interval $(-\frac{1}{2}T, \frac{1}{2}T)$

$$\vec{z}(t)=[z_0(t), \ldots, z_{K-1}(t)]^T , \quad \text{(A1)}$$

where the subscript denotes the array element. Assuming the random processes are zero mean and band limited to $N\omega_0/2\pi$ Hz, let

$$z_k(n) = \left(\frac{1}{T}\right)^{1/2} \int_{-T/2}^{T/2} z_k(t) \exp(-jn\omega_0 t)\, dt \;, \tag{A2}$$

$$\vec{z}(n) = [z_0(n), \ldots, z_{K-1}(n)]^T \;, \tag{A3}$$

and

$$\vec{z} = [z(0)^T, \ldots, z(N)^T]^T \;. \tag{A4}$$

In this way, the time waveforms observed at the K elements are mapped into a $K \times (N+1)$ dimensional vector. Note that the Fourier coefficients for a single frequency index n and all K elements are grouped together.

APPENDIX B: COVARIANCE MATRICES FOR BAND-LIMITED WHITE GAUSSIAN NOISE

This section contains a set of five covariance matrices (Figs. 3–7) which represent successive doubling of the observation period length.[10] The Fourier coefficients representing a sample function from the scalar random process are related by the expression

$$E[z(n)z(m)^*] = \int_{-\infty}^{\infty} N(x\omega_0)\,\mathrm{sinc}(x-n) \times \mathrm{sinc}(x-m)\, dx \;, \tag{B1}$$

where $\omega_0 = 2\pi/T$, $N(x\omega_0)$ is the power spectral density function of the random process, and $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$. The spectrum considered is that of unit height white Gaussian noise band-limited to W Hz. In the first covariance matrix, the observation length is such that $W = 0.5\ \omega_0/2\pi$ or $T = 0.5W$. In the fifth covariance matrix, the observation length is sixteen times longer (i.e., $W = 8\omega_0/2\pi$ or $T = 8/W$). Note how the covariance matrices become progressively more diagonal in form as T is increased.

APPENDIX C: COVARIANCE BETWEEN FOURIER COEFFICIENTS

Let

$$z_k(t) = n_k(t) + d_0(t - \tau_{0K}) \;,$$

where $z_k(t)$ is the output observed at the kth element, $n_k(t)$ is the independent noise component at the kth element, $d_0(t)$ is the directional noise component at the 0th element, τ_{lk} is the time delay of directional component between the lth and kth elements.

1. Covariance within a single observation vector $\vec{z}^i$

$$E[z_l^i(n) z_k^i(m)^*] = \exp[-jm\omega_0 \tau_{lk}] \times \int_{-\infty}^{\infty} [N(x\omega_0)\cdot\delta_{lk} + D(x\omega_0)]\,\mathrm{sinc}(x-n) \times \mathrm{sinc}(x-m) \exp[-j(x-m)(-\tau_{lk}/T)2\pi]\, dx \;, \tag{C1}$$

where

$$\delta_{lk} = \begin{cases} 1, & l = k\;, \\ 0, & l \neq k\;, \end{cases}$$

$N(x\omega_0)$ is the power spectral density function of the independent noise component, and $D(x\omega_0)$ is the power spectral density function of the directional noise component.

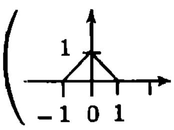
2. Covariance between adjacent observation vectors $\vec{z}^i$ and $\vec{z}^{i+1}$

$$E[z_l^i(n) z_k^{i+1}(m)^*] = \exp[-jm\omega_0(T - \tau_{lk})] \times \int_{-\infty}^{\infty} [N(x\omega_0)\cdot\delta_{lk} + D(x\omega_0)]\cdot\mathrm{sinc}(x-n) \times \mathrm{sinc}(x-m) \exp[-j(x-m)(1-\tau_{lk}/T)2\pi]\, dx \;. \tag{C2}$$

Now, assume that the incremental observation period is chosen long enough so that the power spectra $N(x\omega_0)$ and $D(x\omega_0)$ are relatively smooth with respect to increments in ω_0.

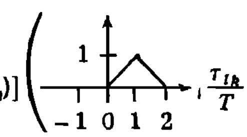
3. Covariance within a single observation vector $\vec{z}^i$

$$E[z_l^i(n) z_k^i(m)^*] \simeq \exp(-jm\omega_0 \tau_{lk}) \cdot [N(n\omega_0) \times \delta_{lk} + D(n\omega_0)] \left(\right) \cdot \delta_{nm} \;. \tag{C3}$$



4. Covariance between adjacent observation vectors $\vec{z}^i$ and $\vec{z}^{i+1}$

$$E[z_l^i(n) z_k^{i+1}(m)^*] \simeq [-jm\omega_0(T - \tau_{lk})] \times [0 + D(n\omega_0)] \left(\right) \cdot \delta_{nm} \;. \tag{C4}$$



*This work was supported by the Office of Naval Research (Code 222). The authors were visiting the Department of Electrical Engineering, University of Washington at Seattle and Colorado State University at Fort Collins, during the year 1974–1975.

[1] F. Bryn, "Optimum Signal Processing of Three-Dimensional Arrays Operating on Gaussian Signals and Noise," J. Acoust. Soc. Am. **34**, 289–297 (1962).

[2] H. Cox, "Interrelated Problems in Estimation and Detection, Parts I and II," Proceedings of the NATO Advanced Study Institute on Signal Processing with Emphasis on Underwater Acoustics, Enschede, The Netherlands, 12–23 Aug., 1968.

[3] J. B. Lewis and P. M. Schultheiss, "Optimum and Conventional Detection Using a Linear Array," J. Acoust. Soc. Am. **49**, 1083–1091 (1971).

[4] W. J. Bangs and P. M. Schultheiss, "Space-Time Processing for Optimal Parameter Estimation," Proceedings of the NATO Advanced Study Institute on Signal Processing with Emphasis on Underwater Acoustics, Loughborough, U. K., 21 Aug.–1 Sept., 1972; also appeared in *Signal Processing*, edited by J. W. R. Griffiths, P. L. Stocklin, and C. Van Schooneveld (Academic, New York, 1973).

[5] W. R. Hahn and S. A. Tretter, "Optimum Processing for Delay-Vector Estimation in Passive Signal Arrays," IEEE Trans. Inf. Theory IT-19, 608–614 (1973).

[6] S. L. Adams and L. W. Nolte, "Bayes Optimum Array Detection of Targets of Known Location," J. Acoust. Soc. Am. 58, 656–659 (1975).

[7] A. N. Papoulis, *Probability, Random Variables, and Stochas-*

tic Processes (McGraw-Hill, New York, 1965).

[8]W. S. Hodgkiss, "The Sequential Implementation of Array Processors when there is Directional Uncertainty," Technical Report No. 10, Adaptive Signal Detection Laboratory, Duke University, Durham, NC, 1975.

[9]N. M. Blachman, "On Fourier Series for Gaussian Noise," Inf. Control 1, 56–63 (1957).

[10]R. A. Marshall, "Correlation of Fourier Coefficients in the Covariance Matrix," Senior Honors paper (Duke University, 1973) (unpublished).