University of Waterloo

Fall 2012

PMATH 733 - Linear Representations of Finite Groups

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Last updated: December 14, 2012

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1 Introduction to Representation Theory

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1.1 Introduction

DEFINITION **1.** Finite Group. Let G be a set with a binary operation $(\cdot)$ satisfying the following properties

- $\forall f, g, h \in G, (f \cdot g) \cdot h = f \cdot (g \cdot h)$
- There exists an element e_l, called the left identity, such that for all $g \in G$. $e_l \cdot g = g$ and there exists $g_l^{-1} \in G$ such that $g_l^{-1} \cdot g = e_l$

Then $(G, \cdot)$ is called a group. If the cardinality of G is finite, we say G is a finite group. Otherwise, G is infinite.

REMARK **2.** You can prove that e_l and $e_l \cdot g = g \cdot e_l = g$ and $g_l^{-1} \cdot = g \cdot g_l^{-1} = e_l$.

DEFINITION **3.** We say a group G is Abelian if for all $g, h \in G$, $gh = hg$.

DEFINITION **4.** Let G be a group and $H \subset G$. H is a subgroup of G if $h_1 h_2^{-1} \in H$ for all $h_1, h_2 \in H$. If the only normal subgroup of G are $\{e\}, G$, we say G is a simple group. A subgroup H is called "normal" if forall $g \in G, h \in H, ghg^{-1} \in H$.

EXAMPLE **5.** Examples of Finite Groups

- Let n be a positive integer. The cyclic group $(\mathbb{Z}/n\mathbb{Z}, +)$ is a finite abelian group. $|C_n| = n$. The size of a group G is called the order of the group.
- Let n be a positive integer. The dihedral group D_n is the group of rotations and reflections in the plane which preserve a regular polygon with n vertices.

  $$D_n = \{1, r, \dots, r^{n-1}, s, sr, sr^2, \dots, sr^{n-1}\}$$

  Or we could write the group using generator relations

  $$D_n = \langle r, s \rangle / (r^n = 1, s^2 = 1, srs = r^{-1})$$

- S_n is the set of permutations of n elements, with $|S_n| = n!$.
- The quaternion group $\mathbb{H}$ is the group generated by i, j such that if $k := ij, m := i^2$. Then

  1. $i^4 = j^4 = k^4 = e$

2. $i^2 = j^2 = k^2 = m$
3. $ij = mji$
4. $\mathbb{H} = 8$

EXAMPLE **6.** Infinite Groups

1. $(\mathbb{Z}, +)$ is an infintie group
2. The group of rotations of the plane, preserving the origin, denoted by C_∞. We have C_∞ is isomorphic to $\mathbb{R}/2\pi\mathbb{Z} := S^1 :=$ the unit circle.
3. $D_\infty = C_\infty \cup \{sr_\alpha | r_\alpha \in C_\infty\}$ with $s^2 = 1$ and $sr_\alpha s = r_\alpha^{-1}$
4. Let F be a field. The set of all invertible $n \times n$ matrices of coefficients in F denoted by $GL_n(F)$ is a group. If F is a finite field, then $GL_n(F)$ is finite. Otherwise, $GL_n(F)$ is infinite.

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Definition **7.** Let G be a Hausdorff topological space with a group law. We say G is a topological group if the mapping $G \times G \to G$ with $(g, h) \to gh$ is continuous.

Remark **8.** If G is finite, we take the discrete topology on G, then G is a topological group.

Example **9.** Since $GL(n, F) \subseteq F^{n^2}$, if F is a topological space, then F^{n^2} has a topology. Therefore, we have a subspace topology on $GL(n, F)$ and with this topology, $GL(n, F)$ is a topological group.

Definition **10.** Let V be a complex Hilbert space. We use $GL(V)$ to denote the group of bounded operators on V with bounded inverse.

Definition **11.** Let G be a topological group and V be a non-trivial complex Hilbert space. A **(linear) representation** of G on V is a group of homomorphisms ρ, called the Group Action, from $G \to GL(V)$ such that the map

$$\begin{aligned} G \times V &\to V \\ (g, v) &\to \rho(g)v \end{aligned}$$

is continuous.

Remark **12.** The representation (ρ, v) map may not be a continuous mapping from $\rho : G \to GL(V)$ (For example, maybe if V is an infinite dimensional space, though this won't come up in our class). However, if G is finite and $dim(V) < \infty$, then ρ will be continuous.

Remark **13.** Concepts

1. if V is a finite dimensional vector space over $\mathbb{C}$, we can choose a basis $\{e_i\}_{i=1}^{n=dim(V)}$. If $(\rho, V) = \rho(V)$ is a representation of G on V

$$\rho(g)e_j = \sum_{i=1}^{n} a_{ij}(g)e_i$$

   Therefore, it induces a mapping (homomorphism) from $G \to GL(n, \mathbb{C})$ with $g \to (a_{ij}(g))$. This is called the **Matrix Form** of (ρ, V).

2. If (ρ, V) is a representation of G and $dim(V) < \infty$, we say that (ρ, V) is a finite dimensional representation with degree $dim(V)$.

3. Let G be a group, we define the **Group Algebra**

$$\mathbb{C}[G] := \{\sum_{g \in G} c_g g | c_g \in \mathbb{C}, \text{ almost all } c_g = 0\}$$

We can define the product on $\mathbb{C}[G]$ in the obvious way. If G is not abelian, $\mathbb{C}[G]$ is non-commutative. If (ρ, V) is a representation of G on V, then V can be viewed as a $\mathbb{C}[G]-$ Module with $(\sum c_g \cdot g) \cdot v := \sum c_g \rho(g) v$. If $|G| < \infty$ and $dim V < \infty$, we do not need to worry about the topologies. Therefore, (ρ, V) is a representation of G on V if and only if V is a $\mathbb{C}[G]-$ Module.

Example **14.** Examples of Representations

1. Let V be a one dimensional vector space over $\mathbb{C}$. 1_v is the identity linear transformation from V to V. The **trivial representation** of G is the homomorphism from $G \to GL(V) \cong \mathbb{C}^*$ defined by $\rho(g) := 1_v, \forall g \in G$.

2. Let G be a finite group of order n and $V := \mathbb{C}[G]$ $(dim(V) = n)$. We know that V has a basis $\{e_g\}_{g \in G}$. The **Regular Representation** (r, V) of G defined by

$$r : G \to V$$
$$r(s)e_t := e_{st}, \forall s, t \in G$$

Definition **15.** Let (ρ, V) and (τ, W) be two representations of a finite group G. A linear map $f : V \to W$ is called a homomorphism or **intertwining operator** between (ρ, V) and (τ, W) if $\forall g \in G, v \in V, f(\rho(g)v) = \tau(g)f(V)$. The homormorphism conditions is nothing but the diagram above commutes. If we view V and W as $\mathbb{C}[G]-$Modules (G- Modules via ρ, τ respectively. f is a hom. if and only if f is a G-Module homormophism from V to W. If f is an isomorphism we say that (ρ, V) and (τ, W) are isomorphic.

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Definition **16.** Let (ρ, V) and (τ, W) be representations of G. We define $(\rho \oplus \tau, V \oplus W)$ a representation of G defined by

$$(\rho \oplus \tau)(g)((v, w)) = (\rho(g)v, \tau(g)w)$$

Definition **17.** Let $\rho, W)$ be a representation of G on V and W be non-zero subspace of V. We say that W is a **subrepresentation** of (ρ, V) if W is "stable" under ρ-action, that is $\forall g \in G, w \in W, \rho(g)w \in W \Rightarrow (\rho|_W, W)$ is a representation of G.

Definition **18.** We say (ρ, V) is irreducible if (ρ, V) have only two different G-stable subspaces, namely, $\{0\}, V$. This is equivalent to saying the only subrepresentation of (ρ, V) is (ρ, V) itself.

Example **19.** All representations of degree 1 is irreducible.

Strategy: Find all the irreducible representations

Theorem **20.** *If G is finite, then all irreducible representations of G are finite dimensional.*

Proof. Let (ρ, V) be irreducible representation of G and let $e \in V$ be a non-zero vector. Let $W := \langle \rho(g)e \rangle_{g \in G}$. Then $dim(W) \leq |G| < \infty$. It is enough to show that W is G-Stable (G-Invariant). $w \in W$, $\exists C_g \in \mathbb{C}$ such that

$$w = \sum_{g \in G} C_g \rho(g) e$$

Then for all $h \in G$

$$\begin{aligned} \rho(h)w &= \rho(h) \left(\sum_{g \in G} C_g \rho(g) e \right) \\ &= \sum_{g \in G} C_g \rho(hg) e \in W \end{aligned}$$

Therefore, W is a finite dimensional subrepresentation of V and $W = V$. ∎

Remark **21.** If (ρ, V) is irreducible, $\forall v \neq 0 \in V$, $W = \langle \rho(g)v \rangle_{g \in G} = V$.

Example **22.** Find all irreducible representations of C_n
We have $C_n \cong \mathbb{Z}/n\mathbb{Z} = \{\bar{0}, \bar{1}, \ldots, n \bar{-} 1\}$. $\forall 0 \leq k \leq n-1$, we define a one -dimensional represnetation (X_k, V_k) as follows, $\forall k$, V_k is just a one dimensional vector spaces. So $\forall 0 \leq j \leq n-1, \forall v \in V_k$

$$X_k(\bar{j})v = e^{2\pi k \frac{j}{n}} v$$

Or we can X_k as a homomorphism

$$\begin{aligned} C_n &\to \mathbb{C}^* \\ \bar{j} &\to e^{2\pi\frac{kji}{n}} = \left(e^{\frac{2\pi ji}{n}}\right)^k \end{aligned}$$

$\{(X_k, V_k)\}_{k=0}^{n-1}$ is a set of irreducible representations of C_n.

Let (χ, V) be an irreducible representation of C_n, the set $\{\chi(g)\}_{g\in C_n} \subset GL(V)$ is a commutative subset. That is, for all $g_1, g_2 \in C_n$

$$\begin{aligned} \chi(g_1)\chi(g_2) &= \chi(g_2)\chi(g_1) \\ \chi(g_1g_2) &= \chi(g_2g_1) \\ (\chi(g_1))^n &= \chi(g_1^n) = \chi(1) = 1_V \end{aligned}$$

The minimal polynomial of $\chi(g_1)$ is a divisor of $x^n = 1$. Since $x^n - 1$ has no repeated roots, $\chi(g_1)$ is diagnalizable. By a theorem of linear algebra, $\{\chi(g)\}_{g\in G}$ has a common eigenvector v. That is, $\forall g \in G, \chi(g)v = \lambda_g v$ for some $\lambda_g \in \mathbb{C}$. Therefore, $W = \langle v\rangle$ is G-Invariant and $W = V$ and $dim(W) = dim(V) = 1$.

So $(\chi(\bar{1}))^n = 1_V$ implies that $\chi(\bar{1})$ is a root of unity of order n, thus

$$\chi(\bar{1}) = e^{(2\pi i)\frac{k}{n}}$$

for some $k \in \mathbb{Z}(0 \leq k \leq n-1)$. thus $\chi = \chi_k$.

Example **23.** Representations of D_n ($n \geq 2$ even)
$D_n = \langle r, s\rangle / \langle r^n = 1, s^2 = 1, srs = r^{-1}\rangle$. There are 4 1-dimensional representations

	r^k	sr^k
ψ_1	1	1
ψ_2	1	-1
ψ_3	$(-1)^k$	$(-1)^k$
ψ_4	$(-1)^k$	$(-1)^{k+1}$

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We know $C_n \triangleright D_n$ and $D_n/C_n \cong C_2$. Any representation of C_2 gives a representation of D_n by

$$D_n \to D_n/C_n \cong C_2 \to GL(V)$$

C_2 has 2 irredcible representations: namely the trivial one and the non-trivial one. However, a representation of C_n might not be extended to a representation of D_n.

For degree 2 representations, let $\xi = e^{\frac{2\pi i}{n}}$ and $h \in \mathbb{Z}$. We define a representation (ρ_n, V_n) of D_n on a two- dimensonal space V_n.

$$\rho_n(r^h) = \begin{pmatrix} \xi^{hk} & 0 \\ 0 & \xi^{-hk} \end{pmatrix}$$
$$\rho_n(sr^k) = \begin{pmatrix} 0 & \xi^{-hk} \\ \xi^{hk} & 0 \end{pmatrix}$$

We can check that (ρ_n, V_n) is a indeed a representation of D_n. Notice that ρ_n and ρ_{n-h} are isomorphic (Exercise: find an intertwining operator from ρ_n to ρ_{n-h})

The extreme cases $h = 0$ or $h = \frac{n}{2}$ are isomorphic to $\psi_1 \oplus \psi_2$ and $\psi_3 \oplus \psi_4$ respectively. To summarize, we get

$$\psi, \psi_2, \psi_3, \psi_4, \rho_1, \ldots, \rho_{\frac{n}{2}-1}$$

a set of irreducible representations of D_n.

To show that $1 \leq h\frac{n}{2} - 1$, ρ_n is irreducible, we assume that there is a 1-dimensional G-invariant space $W = \langle V \rangle$. Thus, $\rho_n(r^k)v \in \langle v \rangle \Rightarrow v = [0,1]$ or $= [1,0]$. $\rho_h(s)v \in \langle v \rangle \Rightarrow v$ is an eigenvector $\rho_s(s)v = \lambda v$ of the matrix of

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

However, $[1,0]$ and $[0,1]$ are not eigenvectors of the above matrix. Thus, we have a contradiction and so ρ_n is irreducible.

Example **24.** D_n when n is odd

	r^k	sr^k
ψ_1	1	1
ψ_2	1	-1

The same linear transformations from the above example will work

Example **25.** $\{\psi_1, \psi_2, \rho_1\}$ is the full set of irreducible representations ofD_3.

Let (ρ, V) be an irreducible representations of D_3, we can consider (ρ, V) is a representation of $C_3 = \langle r \rangle / \langle r^3 = 1 \rangle$. Since $r^3 = 1$, V can be decomposed into the direct sum of eigenvectors of $]rho(r)$

$$V = \bigoplus_{i=1}^{m} V_i, dim(V) = m$$

Let $v_1 = \langle v_1 \rangle$ with $\rho(r)v_1 = \xi^{\alpha_1} v_1$ where $\xi = e^{\frac{2\pi i}{3}}, \alpha_1 \in \mathbb{Z}$. Then, we have

$$\begin{aligned} \rho(r)(\rho(s)v_1) &= \rho(rs)v_1 \\ &= \rho(sr^{-1})v_1 \\ &= \rho(s)(\rho(r^{-1}v_1) \\ &= \rho(s(\rho(r^2)v_1 \\ \rho(s)&[\xi^{2\alpha_i}v_1] \\ &= \xi^{2\alpha_i}(\rho(s)v_1) \end{aligned}$$

Thus $\langle v_1, \rho(S)v_1 \rangle = w$ is a Ds invariant subspace of (ρ, V).

Since (ρ, V) is irreduclbe, $V = W = \langle v_1, \rho(s)v_1 \rangle, \dim V = \dim W \leq 2$. Note that $ip\rho(s)v_1$ is an C_3 invariant suspace with eigenvalues $\xi^{2\alpha_i}$ respectiely to $\rho(v)$.

If $dim(W) = 1$ then $\xi^{2\alpha_1} = \xi^{\alpha_1} \Rightarrow \xi^{\alpha_1} = 1 \Rightarrow \alpha_1 = 0$ for a multiple of 3. That is, ρ is trivial on C_3 and it gives us ψ_1, ψ_2.

If $dim(W) = 2$, we have a basis of W, namely $\{v_1, \rho(s)v_1\}$. If we write down its matrix form, we will see that it is $\rho_s, V_n)$ for some $h = \alpha_1$. So we are done.

Remark **26.** The Grand goal of representation theory is to find all the irreducible representations of any group

1. Given a group, how many irreducible representations of the given one?
2. Is there any way to describe the set of irreducible representations?
3. How to find those representations

1.2 Direct Sum and Tensor Product

Let (ρ, V) and (τ, W) be two finite dimensional represnetations of a finite group G. Let $\{v_i\}$ and $\{w_i\}$ be bases of V and W respectively. We can define $(\rho \oplus \tau, V \oplus W)$ a representation of G. Let $V = \langle v_i \rangle, E = \langle w_j \rangle$.We have the matrix form for ρ and τ as follows

$$\begin{aligned} \rho(g)(v_i) &= \sum_k a_{k_i}(g)v_k \\ \tau(g)(w_j) &= \sum_l b_{lj}(g)w_l \end{aligned}$$

Then we define

$$\begin{aligned} p \otimes \tau(g)(v_i \otimes w_j) &:= \sum_{k,l} a_{k_i}(g)b_{l_j}(g)v_k \otimes w_l \\ p \oplus \tau(g)(v_i \oplus w_j) &:= \rho(g)v_i \oplus \tau(g)w_j \end{aligned}$$

Where $v_i \otimes w_j$ is a basis of $V \otimes W$.

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1.3 Decomposition of Representations

DEFINITION **27.** Let (ρ, V) be a representation of G. We say (ρ, V) is **unitary** if there exists a (G-) invariant inner product $\langle , \rangle$ on V such that $\langle , \rangle$ is positive and Hermitian and for all $g \in G$ and $w, v \in V$, $\langle \rho(g)v, \rho(g)w \rangle = \langle v, w \rangle$

REMARK **28.** A matrix is unitary if $A \cdot \bar{A}^t = I$. If (ρ, V) is a unitary representation with $\langle , \rangle$ a G-invariant inner product, then given an orthonormal basis, the matrix form $(a_i, (g))$ are all unitary.

LEMMA **29.** *Let (ρ, V) be a unitary representatio with an invariant inner product $\langle , \rangle$. W a subrepresentation, then, the orthogonal complement $W^\perp$ with respect to $\langle , \rangle$ is a subrepresentation of V and $V = W \oplus W^\perp$.*

Proof. We only need to show that $W^\perp$ is G-invariant. This happens if and only if $\forall g \in G, \tilde{w} \in W^\perp \Rightarrow \rho(\tilde{w}) \subseteq W^\perp \Leftrightarrow \forall g \in G, \tilde{w} \in W^\perp, \forall w \in W, \langle w, \rho(g)(\tilde{w}) \rangle = 0$.

$$\begin{aligned}
\langle w, \rho(g)(\tilde{w}) \rangle &= \left\langle \rho(g^{-1}w, \rho(g^{-1}\rho(g)\tilde{w} \right\rangle \\
&= \left\langle \rho(g^{-1}w, \tilde{w} \right\rangle \\
&= 0
\end{aligned}$$

Note: $\rho(g^{-1})w \in W, \tilde{w} \in W^\perp$ so the inner product is 0. Thus, $\langle w, \rho(g)\tilde{w} \rangle = 0$ if $w \in W, \tilde{w} \in W^\perp$ and we finish the proof. ∎

COROLLARY **30.** *If (ρ, V) is a unitary representation. (ρ, V) is a direct sum of irreducible representations of G.*

LEMMA **31.** *All representations of a finite group G is unitary.*

Proof. Let (ρ, V) be a representation of G and $\langle , \rangle$ an inner product on V. We define a new inner product $\langle , \rangle_G$ by $\forall v, w \in V$

$$\langle v, w \rangle_G := \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)v, \rho(g)w \rangle$$

We need to check whether our new inner product is G-invariant. $\forall h \in G.v, w \in V$

$$\begin{aligned}
\langle \rho(h)v, \rho(h)w \rangle_G &= \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)\rho(h)v, \rho(g)\rho(h)w \rangle \\
&= \frac{1}{|G|} \sum_{t \in G} \langle \rho(t)v, \rho(t)w \rangle = \langle v, w \rangle \\
t &:= gh
\end{aligned}$$

We used a change of variable above. Therefore, $\langle , \rangle_G$ is G-invariant. ∎

2 Character Theory

DEFINITION **32.** Let V be a vector space with a basis $\{e_i\}$ and A is a linear transformation from $V \to V$ with matrix form $\{a_{ij}\}$. Define the **Trace** of A denoted by $Tr(A)$ to be $Tr(A) := \sum a_i i$. We can consider Tr as a function from the set of linear transformations from V to V to the complex numbers. $Tr(A)$ is independent of the choice of basis.

DEFINITION **33.** Let (ρ, V) be a representation of G for $g \in G$. The **Character** $\chi_\rho(g)$ of (ρ, V) by

$$\chi_\rho(g) := Tr(\rho(g))$$

This gives us a function on G.

PROPOSITION **34.** *If χ is a character of the representation of (ρ, V) of V, then*

1. $\chi(e) = dim(V)$
2. $\chi(g^{-1}) = \chi(\bar{g})$
3. $\chi(st) = \chi(ts), \forall s, t \in G$

REMARK **35.** Let G be a group and f a function on G, we say f is a **class function** if $\forall s, t \in G, f(st) = f(ts)$ i.e. $f(sts^{-1}) = f(t)$.

Proof. 1. We know $\rho(e) = I_v$. Thus, $\chi(e) = Tr(I_v) = \sum 1 = dim(V)$.

2. Since G is finite, $\rho(g)$ has a finite order and it is diagonalizable. We can choose a basis such that the matrix form of $\rho(g)$ is diagonal with entries $\lambda_1, \ldots, \lambda_n$, where $n = dimV$ and $|\lambda_i| = 1$, that is, λ_i are roots of unity and so $\lambda_i^{-1} = \bar{\lambda}_i$. Then $\rho^{-1}(g)$ has the matrix form of a diagonal matrix with λ_i^{-1} on diagonal. Then

$$\chi(g^{-1}) = \sum \lambda_i^{-1} = \sum \bar{\lambda}_i = \chi(\bar{g})$$

∎

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PROPOSITION **36.** *Let (ρ, V) and (τ, W) be two representations of G and χ_ρ and χ_τ be characters of (ρ, V) and (τ, W) respectively. Then,*

$$\chi_{\rho \oplus \tau} = \chi_\rho + \chi_\tau$$
$$\chi_{\rho \otimes \tau} = \chi \cdot \chi_\tau$$

Proof. Let $\{v_i\}$ and $\{w_j\}$ be bases of V and W respectively. Then,

$$\rho(g)v_i = \sum a_{ki}v_k$$
$$\tau(g)w_j = \sum b_{lj}w_l$$

Given $g \in G$, $\rho \oplus \tau(v_i) = \sum a_{ki}v_k$, the coefficient of v_i is a_{ii} and $\rho \oplus \tau(w_j) = \sum b_{lj}w_l$, the coefficient of w_j is b_{jj}. Then,

$$\begin{aligned}\chi_{\rho\oplus\tau}(g) &= Tr(\rho \oplus \tau(g)) = \sum a_{ii} + \sum b_{jj}\\ &= \chi_\rho(g) + \chi_\tau(g)\\ \rho \oplus \tau(g) &= \begin{pmatrix} \rho(g) & 0 \\ 0 & \tau(g) \end{pmatrix}\end{aligned}$$

Now for the tensor product, $\rho \otimes \tau(g)(v_i \otimes w_j) = \sum a_{k_i}(g)b_{l+j}v_k \otimes w_l$, the coefficent of $v_i \otimes w_j$ is $a_{ii}b_{jj}$. Thus,

$$\begin{aligned}\chi_{\rho\otimes\tau}(g) &:= Tr(\rho \otimes \tau(g))\\ &= \sum_i \sum_j a_{ii}b_{jj}\\ &= \left(\sum_i a_{ii}\right)\left(\sum_j b_{jj}\right)\\ &= \chi_\rho(g) \cdot \chi_\tau(g)\end{aligned}$$

∎

LEMMA **37.** [SCHUR'S LEMMA] *Let (ρ, V) and (τ, W) be two irreducible representations of G and f be an intertwining operator from V to W.*

1. *If (ρ, V) and (τ, W) are not isomorphic, then $f = 0$*
2. *If $(\rho, V) = (\tau, W)$, then f is a scalar multiple of the identity (called homothety) and the scalar is called the ratio.*

Proof. If $f = 0$, then (i) and (ii) are true. Suppose, $f \neq 0$, let $ker(f) := V' \subsetneq V$. Then, $\forall v' \in V'$

$$f(\rho(g)(v')) = \tau(g)(f(v')) = \tau(g)(0_w) = 0_w$$

Thus, V' is G-invariant. Since V is irreducible, it must be the case that $V' = \{0_v\}$.

Let $Im(f) = W' \subseteq W$. $\forall w' \in W', \exists v \in V$ such that $f(v) = w'$. $\forall g \in G$

$$\tau(g)(w') = \tau(g)(f(v)) = f(\rho(g)(v)) \in W'$$

Thus, W' is G-invariant. Since W is irreducible, $W' = \{0_w\}$ or $W' = W$. Since $f \neq 0$, $W' = W$ and f is bijective, i.e. an isomorphism. If (ρ, V) and (τ, W) are not isomorphic,

$f \equiv 0$. This finishes the first case.

Let λ be an eigenvalue of f. Thus, $f - \lambda \cdot I_v$ hss a non-trivial kernel. Moreover, $f - \lambda \cdot I_v$ is an intertwining operator since

$$\begin{aligned} \rho(g)(f - \lambda I_v) &= \rho(g) \circ f - \lambda \rho(g) \circ I_v \\ &= f \circ \rho(g) - \lambda I_v \circ \rho(g) \\ &= (f - \lambda I_v) \circ \rho(g) \end{aligned}$$

As before, $f - \lambda I_v \equiv 0$, i.e. $f = \lambda I_v$. ∎

Corollary **38.** *Keep the notations as above. Let h be a linear mapping from V to W, not necessarily an intertwining operator. Put*

$$h_G := \frac{1}{|G|} \sum_{t \in G} \tau(t)^{-1} \circ h \circ \rho(t)$$

1. *If (ρ, V) and (τ, W) are not isomorphic, then $h_G \equiv 0$*

2. *If $(\rho, V) = (\tau, W)$, the h_G is a homothety of ratio $\frac{1}{n} Tr(h), n = dim(V)$.*

Proof. We claim that h_G is an intertwining operator. $\forall g \in G, \tau(g) \circ h_G = h_G \circ \rho(g) \Rightarrow h_G = \tau^{-1}(g) \circ h_G \circ \rho(g)$ Thus,

$$\begin{aligned} \tau)g)^{-1} \circ h_G \circ \rho(g) &= \frac{1}{|G|} \sum_{t \in G} \tau(g)^{-1} \circ \tau(t)^{-1} \circ h \circ \rho(t) \circ \rho(g) \\ &= \frac{1}{|G|} \sum_{t \in G} \tau(tg)^{-1} \circ h \circ f(tg) \\ &= \frac{1}{|G|} \sum_{s \in G} \tau(S)^{-1} \circ h \circ \rho(s) \\ &= h_G \end{aligned}$$

Thus, h_G is an intertwining operator. By Schur's Lemma, (i) is done. Next,

$$\begin{aligned} Tr(h_G) &= \frac{1}{|G|} \sum_{t \in G} Tr(\rho(t)^{-1} \circ h \circ \rho(t)) \\ &= \frac{1}{|G|} \sum_{t \in G} Tr(h) = Tr(h) \end{aligned}$$

If $h_g = \lambda I_v$, $Tr(h_g) = \lambda dim(V) = \lambda n$. Thus the ratio λ is equal to $\frac{1}{n} Tr(h)$ ∎

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Example **39.** Let $G = C_{10} \cong \mathbb{Z}/10\mathbb{Z}$ and $h = 5 \cdot I_{\mathbb{C}}$. Suppose that χ_1 and χ_2 are two 1-dimensional representations of G on $\mathbb{C}$.

$$\chi_2(\bar{j}) := e^{\frac{2\pi i j}{10}}$$
$$\chi_i(\bar{j}) := e^{\frac{4\pi i}{10}}$$

$j = \{0, 1, \ldots, 0\}$. Then,

$$\begin{aligned} h_G &= \frac{1}{10}\sum_{j=0}^{9} e^{\frac{-14\pi i j}{10}} 5 e^{\frac{4\pi i j}{10}} \\ &= \frac{1}{2}\sum_{j=0}^{9} e^{\frac{-10\pi i j}{10}} = \frac{1}{2}\sum_{j=0}^{9}\left(e^{-\pi i j}\right) \\ &= \frac{1}{2}\sum_{j=0}^{9}(-1)^i = 0 \end{aligned}$$

Theorem **40.** *Let (ρ, V) and (τ, W) be 2 irreducible representations of G with matrix forms (a_{ij}) and (b_{kl}) respectively. Then,*

1. *For the case that ρ and τ are not isomorphic, we have $\frac{1}{|G|}\sum_{t\in G} a_{ij}(t^{-1})b_{kl}(t) = 0$*
2. *For the case $\rho = \tau$, we have*

$$\frac{1}{|G|}\sum_{t\in G} a_{ij}(t^{-1})b_{kl}(t) = \frac{1}{h}\delta_{il}\delta_{jk} = \begin{cases} \frac{1}{n} & i = l, j - k \\ 0 & \text{otherwise} \end{cases}$$

$n = \dim(V)$

Proof. Let h be a linear mapping from V to W with a matrix representation (χ_{rs}) and h_G with a matrix representation (y_{rs}).

$$\begin{aligned} y_{il} &= \frac{1}{|G|}\sum_{t\in G} a_{ij}(t^{-1})\chi_{jk} b_{kl}(t) \\ &= \left(\frac{1}{|G|}\sum_{t\in G} a_{ij}(t^{-1})b_{kl}(t)\right)\chi_{jk} \end{aligned}$$

In case (i)$y_{il} \equiv 0$, $\frac{1}{|G|}\sum_{t\in G} a_{ij}(t^{-1})b_{kl}(t) = 0$.

In case (ii), we have $h_G = \lambda I_v$, i.e.

$$\begin{aligned} y_{il} = \lambda, \delta_{il} &= \frac{1}{h}Tr(h)\cdot\delta_{il} \\ &= \delta_{il}\left(\frac{1}{n}\sum \delta_{jk}\chi_{jk}\right) \\ &= \left(\frac{1}{n}\sum \delta_{jk}\delta_{il}\right)\chi_{jk} \end{aligned}$$

Thus,

$$\frac{1}{|G|}\sum_{t\in G} a_{ij}(t^{-1})b_{kl}(t) = \frac{1}{n}\sum \delta_{jk}\delta_{il} = \begin{cases} \frac{1}{n} & j=k, i=l \\ 0 & \text{otherwise} \end{cases}$$

∎

REMARK **41.** 1. Suppose that the matrices $(a_{ij}(t)$ are unitary. It can be realized by a suitable choice of basis. We have $a_{ij}(t^{-1}) = \bar{a_{ji}(t)}, (A^{-1} = \bar{A}^t)$

2. If ϕ and ψ are two functions on G, put

$$(\phi|\psi) = \frac{1}{|G|}\sum_{t\in G}\phi(t)\bar{\psi(t)}$$

It is an inner product.

Let (ρ, V) and (τ, W) be two irreducible representations of G with matrix forms (a_{ij}) and b_{kl} with respectively orthonormal bases on V and W respectively. Then, $(a_{ij}|b_{kl}) = 0$ if $((\rho, V) \ncong (\tau, W)$ and $(a_{ik}|b_{kl}) = \frac{1}{n}\delta_{ik}\delta jl$

In conclusion, the functions coming from matrix forms are orthogonal (and therefore linearly independent).

THEOREM **42.** *Orthogonality of irreducible characters*
(1) If χ is a character of an irreducible representation, we have $(\chi|\chi) = 1$.

(2) If χ, ψ are characters of two non-isomorphic irreducible representations, we have $(\chi|\psi) = 0$.

Proof. Let (ρ, V) and (τ, W) be two irreducible representations of G with matrix form (a_{ij}) and (b_{kl}) with respect to orthonormal basis of V and W respectively. Let χ and ψ be characters of ρ and τ respectively.

$$\begin{aligned}
\chi(g) &:= \sum a_{ii}(g) \\
\psi(g) &:= \sum b_{kk}(g) \\
(\chi|\chi) &= \sum_{i,j}(a_{ii}|a_{jj}) = \sum_{i,j}\frac{1}{n}\delta_{ij} = 1 \\
(\chi|\psi) &= \sum (a_{ii}|b_{kk}) = 0
\end{aligned}$$

by the orthogonal relations. ∎

Monday, September 26

THEOREM **43.** *χ, ψ are characters of an irreducible representation. Then,*

1. $(\chi|\chi) = 1$

2. $(\chi|\psi) = 0$ *if the representations are not isomorphic and* $(\chi|\psi) = 1$ *is they are isomorphic.*

Remark **44.** Let (ρ, v) be a representation of G with a matrix form $(a_{ij}(g))$.

1. If we change the basis, then we get a new matrix form

$$\left(\sum_{k,l} T_{ik}^{-1} a_{kl}(g) T_{lj}\right)$$

   when T is the matrix of change of basis and T^{-1} is its inverse. Note that $\sum_{kl} T_{ik}^{-1} a_{kl}(g) T_{lj}$ is a linear combination of functions $a_{kl}(g)$. Let $F_\rho := \langle a_{kl}(g)\rangle \subseteq C(G)$, the set of "continuous" functions on G (For finite groups, all functions are continuous, so not really relevant, but matters for Lie Groups,etc). By observation, we have that F_ρ is independent of choices of V.

2. If (ρ, V) is irreducible, then $dim(F_\rho) = (\dim(V))^2$ because we can represent $C(G)$ as $(\chi|\psi)$, and so the elements are linearly independent because they are orthogonal.

3. If $(\rho, V) \cong (\tau, W)$m the characters of ρ and τ are the same.

Recall that every representation of G can be decomposed into a direct sum of irreducible representations (not necessarily unique). Fix a representation of G and $\{\chi_1, \ldots, \chi_n\}$ the set of all characters of irreducible representations of G (it might be infinite. Then $\chi_i(e_i) = n_i =$ dimension of the new representation.

Theorem **45.** *(**)Let V be a representaion of G with character ϕ. If $V = \bigoplus W_i$ where W_i irreducible with character of χ_{W_i}. Then if W is an irreducible representation of G with character χ, then the number of W_i that are isomorphic to W is equal to $(\phi|\chi)$. In particular, the number is independent of the decomposition and the decomposition is unique.*

Proof. We know that $\phi = \sum \chi_{w_i}$. Thus,

$$\begin{aligned}(\phi|\chi) &= (\sum \chi_{w_i}|\chi)\\ &- \sum(\chi_{W_i}|\chi)\\ &= \text{ number of } W_i \text{ isomorphic to } W\end{aligned}$$

Because by the previous theorem, the above inner product is 0 or 1 depending whether the representations are isomorphic. ∎

Friday, September 2012

Corollary **46.** *(**)Two representations are isomorphic if and only if their characters are the same.*

Example **47.** In class, we claim ρ_{n-h} and ρ_h are isomorphic (ρ_n is an irreducible representation of D_n). Where we had

$$\rho_h(r^k) = \begin{pmatrix} \xi^{hk} & 0 \\ 0 & \xi^{-hk} \end{pmatrix}$$

Where $\chi_{\rho_h}(r^) = \xi^{hk} + \xi^{hk}, \xi_{\rho_h}(sr^k) = 0, \xi_{\rho_h} = \chi_{\rho_{n-h}},$

Corollary **48.** *Let $\chi_1, \ldots, \chi_n$ are distinct irreducible characters of G and $W_1, \ldots, W_k$ denote its corresponding representation space, where an* **irreducible character** *means a function coming from character of irreducible representation. Each representation is isomorphic to*

$$V = m_1 W_1 \oplus \cdots \oplus m_k W_k$$

where m_i are integers (non-negative). Then $m_i := (\chi_v|\chi_i)$ where χ_v is the character of V and $\chi_v = \sum m_i \chi_i$. As a consequence,

$$\begin{aligned}(\chi_v|\chi_v) &= \left(\sum m_i \chi_i | \sum m_j \chi_j\right) \\ &= \sum_{i,j} m_i m_j (\chi_i|\chi_j) \\ &= \sum_{i,j} m_i m_j \delta_{ij} \sum_i m_i^2\end{aligned}$$

In particular, $(\chi_v|\chi_v)$ is a sum of squares.

Theorem **49.** *Let ϕ be a character of a representation V. Then, $(\phi|\phi) = 1$ if and only if V is irreducible.*

Proof. Let $V = \bigoplus m_i W_i$,W_i are distinct irreducible. $(\phi|\phi) = \sum_r m_i^2 = 1 \Leftrightarrow$ there is only one component and $m_1 = 1 \Leftrightarrow V$ is irreducible. ∎

Corollary **50.** *All one dimensional representations are irreducible*

Proof. Let (χ, V) be a one-dimensional irreducible representation of G and χ a character of (χ, V).

$$\begin{aligned}(\chi|\chi) &= \frac{1}{|G|}\sum_{g\in G} \chi(g)\bar{\chi}(g) \\ &= \frac{1}{|G|}\sum_{g\in G} 1 = 1\end{aligned}$$

∎

Definition **51.** Let G be a finite group of order n and $V = \mathbb{C}[G]$ (Note that $\mathbb{C}[G] \cong \langle g \rangle_{g \in G}$). Let (r, V) be the "regular" representation $r(h)g := hg$. Then,,

$$\begin{cases} |G| & \text{If } g = e \\ 0 & \text{If } g \neq e \end{cases}$$

Let χ be an irreducible character and χ_r be a character of regular representation. Then consider

$$\begin{aligned} (\chi_r|\chi) &= \frac{1}{|G|} \sum_{t \in G} \chi_r(t)\bar{\chi}(t) \\ &= \frac{1}{|G|} \chi_r(e)\bar{\chi}(e) = dim_\chi \end{aligned}$$

Theorem **52.** *(**) Let $\chi_1, \ldots, \chi_h$ be the set of full irreducible characters (it might be infinite). Then,*

$$\chi_r = \sum_{i=1}^{h} (dim_\chi)\, \chi_i$$

In particular, h is finite.

Corollary **53.** $|G| = \sum_{i=1}^{h} \left(dim^2_{\chi_i} \right)$.

Proof.

$$\begin{aligned} (\chi_r|\chi_r) &= \frac{1}{|G|} \sum_{t \in G} \chi_r(t)\bar{\chi}(t) \\ &= \frac{1}{|G|} |G|\,|G| = |G| \end{aligned}$$

On the other hand, $(\chi_r|\chi_r) = \sum_{i=1}^{h} (dim_{\chi_i})^2$. Therefore $|G| = \sum_{i=1}^{h} (dim \chi_i)^2$ ∎

Example **54.** Consider D_3 which has 2 one dimensional representations ψ_1, ψ_2 and 1 2 dimensional ρ_1.

$$|D_3| = 6 = 1^2 + 1^2 + 2^2 = 6$$

Monday, October 1

Remark **55.** Note that the set of all irreducible characters form an orthogonal system in the space of class functions on G.

Definition **56.** Let $C^n(G) = \{f : F \to \mathbb{C} | f(sts^{-1}) = f(t), s, t \in G\}$ denote the space of class functions on G and $\dim(C^n(G))$ is the number of conjugacy classes.

PROPOSITION **57.** *Let f be a class function on G and (ρ, V) a representation of G. Let $\rho_f : V \to V$ defined as $\rho_f := \sum_{t\in G} f(t)\rho(t)$. If (ρ, V) is an irreducible representation of degree n with character χ, then ρ_f is a homothety of the ratio λ_i,*

$$\lambda = \frac{|G|}{n}(f|\bar{\chi})$$

Proof. We would like to show that ρ_f is an intertwining operator. $\forall s \in G$

$$\begin{aligned}
\rho(s)^{-1} \circ \rho_f \circ \rho(s) &= \rho(s)^{-1}\left(\sum_{t\in G} f(t)\rho(t)\right) \circ \rho(s) \\
&= \sum_{t\in G}(f(t)\rho(s)^{-1} \circ \rho(t) \circ \rho(s)) \\
&= \sum_{t\in G} f(t)\rho(s^{-1}ts) \\
&= \sum_{u=s^{-1}ts} f(sus^{-1})\rho(t) \\
&= \sum_{u\in G} f(u)\rho(u) = \rho_f
\end{aligned}$$

Thus, ρ_f is a homothety.

$$\begin{aligned}
\lambda &= \frac{1}{n}Tr(\rho_f) \\
&= \frac{1}{n}\sum_{t\in G} f(t)Tr(\rho(t)) \\
&= \frac{1}{n}\sum_{t\in G} f(t)\chi(t) \\
&= \frac{|G|}{n}(f|\bar{\chi})
\end{aligned}$$

∎

THEOREM **58.** *Let $C^h(G)$ be the space of class functions on G. Then, the full set $\{\chi_1, \dots, \chi_k\}$ of irreducible characters is an orthonormal basis of $C^h(G)$. In particular, the number of irreducible representations of G is the same as the number of conjugacy classes.*

Proof. Let $f \in C^h(G)$ and $(f|\chi_i) = 0$ for all $1 \le i \le k$. It is enough to show that $f \equiv 0$. Note that there is no harm in assuming that $(f|\bar{\chi}_i) \equiv 0$ for all $1 \le i \le k$. For each representation ρ of G, put $\rho_f := \sum_{t\in G} f(t)\rho(t)$. The previous proposition shows that ρ_f is 0 if ρ is irreducible. From the direct sum of decomposition, we conclude that ρ_f is 0 for all representations. In particular, let $\rho = r$, where r is the regular representation, i.e. $r_f \equiv 0$.

$$\begin{aligned} 0 = r_f(e_l) &= \sum_{t \in G} f(t) r(t)(e_l) \\ &= \sum_{t \in G} f(t) e_{tl} \\ &= \sum_{t \in G} f(t) e_t \Rightarrow \\ f(t) &= 0 \end{aligned}$$

For all $t \in G$. It finishes the proof. ∎

Corollary **59.** *A group is abelian if and only if all irreducible representations are 1-dimensional.*

Proof. G is abelian $\Leftrightarrow$ the number of conjugacy classes is equal to $|G|$. By the previous theorem, the number of irreducible representation is $|G|$. We want to show this is equivalent to all the irreducible representations being 1-dimensional. Recall that $|G| = \sum_{i=1}^{k} m_i^2$ where m_i is the dimension of the irreducible representation, so $m_i = 1$ and this concludes the proof. ∎

Corollary **60.** *Let A be an abelian subgroup of G, then each irreducible representation of G has degree less than or equal $\frac{|G|}{|A|}$, i.e. the index of A in G.*

Proof. Let (ρ, V) be an irreducible representation of G. We can restrict the G-action to A-action and make (ρ, V) a representation of A. Since A is abelian, all the irreducible representations are 1-dimensional. Let $V = \bigoplus_{i=1}^{l} V_i$ where V_i are irreducible representations of A and $\text{Dim}(V) = 1$. Let $\{[s]\}$ be a collection of the representatives of the coset G/A. Define $W = \langle \rho(s) V_i \rangle_{s \in G/A}$. More precisely if $V_1 = \langle v \rangle$, then $W = \langle \rho(s) v \rangle$. It is enough to show that W is G-invariant. Let $t \in G$, $\forall w \in W, w = \sum C_s \rho(s) v$,

$$\begin{aligned} \rho(t) w &= \sum_{s \in G/A} C_s \rho(t) \rho(s) v \\ &= \sum_{s \in G/A} C_s \rho(s_t) \rho(g_{st}) v \\ &= \sum_{s_t \in G/A} C_s \rho(s_t) C_{st} v \\ &= \sum_{s_t \in G/A} C_s C_{st} \rho(s_t) v \in W \end{aligned}$$

where $\{s_k\}$ is a permutation of $\{[s]\}$ and $a_{st} \in A$ $(ts = s_t a_{st})$. We have showed that W is G-invariant and therefore $V = W$ and $\text{Dim } V = \text{Dim W} \leq \frac{|G|}{n}$ ∎

Wednesday, October 3

EXAMPLE **61.** Suppose $G = S_3 \cong D_3$. $G = \{(1), (12), (23), (13), (123), (213)\}$. There are 3 conjugacy classes. Recall, that if $H \lhd G$ can be lifted to an irreducible representation of G by

$$\tilde{\rho} : G \to G/H \to GL(V)$$

$A_3 = \{(1), (123), (132)\} \lhd S_3 \wedge [S_3 : A_3] = 2$ thus $S_3/A_3 \cong C_2$. Let

$$\begin{aligned}
\psi_0 : C_2 &\to \mathbb{C}^* \\
[i] &\to 1 \\
\psi_0 : C_2 &\to \mathbb{C}^* \\
[i] &\to [1]^i \\
\theta_0 : S_2 &\to \mathbb{C}^* \\
\sigma &\to 1 \\
\theta_1 : S_3 &\to \mathbb{C}^* \\
\sigma &\to sgn(\sigma)
\end{aligned}$$

are induced by ψ_0 and ψ_1. By the theorem we proved we know that there exists at least one irreducible representation of degree greater than 1, say k. $|G| = 6 \geq 1^2 + 1^2 + k^2 \Rightarrow k = 2$ and there exists exactly one representation of degree 2. Let θ be the character of this representation. In fact, we can find its character

$$\chi_r = \chi_{\theta_0} + \chi_{\theta_1} + 2\theta$$

We can determine the value now by looking at the value of the characters on each of the elements. A natural question how can we construct this n-dimensional irreducible represen-

Table 1: Value of the Characters

	(1)	(12)	(123)
χ_{θ_0}	1	1	1
χ_{θ_1}	1	-1	1
θ	2	0	-1
χ_r	6	0	0

tation? S_3 can act of $\mathbb{C}^3$ by permuting the basis vectors $\{1, 1, 1\}$. By observation, $e_1 + e_2 + e_3$ is a G-invariant vector with eigenvalue 1. $v^\perp = \{w \in \mathbb{C}^3 | \langle v, w \rangle = 0\}$ is a G-invariant subspace. In fact, $v^\perp = \{(x, y, z) \in \mathbb{C}^3 | x + y + z = 0\}$. This is the 3-dimensional irreducible representation of S_3.

Product Groups

DEFINITION **62.** Let G_1, G_2 be two groups. We define $G_1 \times G_2$, the product of G_1 and G_2 by $(s_1, t_1) \cdot (s_2, t_2) = (s_1 s_2, t_1 t_2)$ for all $s_1, s_2 \in G$ and $t_1, t_2 \in G_2$

Remark **63.** Notes about product groups

1. G_1 and G_2 are finite groups, then $|G_1 \times G_2| = |G_1|\,|G_2|$.
2. If $H_1, H_2 \lhd G$ with $H_1H_2 = G$ and $H_1 \cap H_2 = \{e\}$ and for all h_1, h_2 we have $h_1h_2 = h_2h_1$ then $G \cong H_1 \times H_2$.

Definition **64.** Let (ρ_1, V) and (ρ_2, V) be representations of G_1 and G_2, respectively. Define a representation $\rho_1 \otimes \rho_2$ of $G_1 \times G_2$ on $V_1 \otimes V_2$ by setting $\rho_1 \otimes \rho_2(s_1, s_2)(v_1 \otimes v_2) := \rho_1(s_1)v_1 \otimes \rho_2(s_2)v_2$ where $s_1 \in G_1, s_2 \in G_2, v_1 \in V_1, v_2 \in V_2$, we can check that it is a representation of $G_1 \times G_2$.

Remark **65.** Notes on product

1. If $G_1 = G_2$ then $\rho_1 \otimes \rho_2$ has two different meanings. It can be viewed as a representation of G by $\rho_1 \otimes \rho_2(s)(v_1 \otimes v_2) := \rho_1(s)v_1 \otimes \rho_2(s)v_2, \forall s \in G, v_1 \in V, v_2 \in V_2$. On the other hand it can be viewed as a representation of $G \times G$ as we just defined.
2. In general, if we have a group homomorphism $f : H \to K$ and (tau, W) be a representation of K. Then, we can define a representation on H by $H \to K \to GL(W)$ by $\tau \circ f$. If we set $H = G, K = G \times G$ and $f : G \to G \times G$ with a diagonal mapping $G \to (g, g)$. Then, the first meaning is induced from the second one by f.

Friday, October 5

Remark **66.** If χ_i is a the character of ρ_i respectively, the character of $\rho_1 \otimes \rho_2$ is given $\chi(s_1, s_2) = \chi_1(s_1)\chi_2(s_2)$.

Theorem **67.** *Keeping our notations. We have*

1. *If ρ_1 and ρ_2 are irreducible, $\rho_1 \otimes \rho_2$ is irreducible*
2. *Each irreducible representation of $G_1 \times G_2$ is isomorphic to a representation $\rho_1 \otimes \rho_2$ when ρ_1 and ρ_2 are irreducible for G_1 and G_2 respectively.*

Proof. If ρ_1 and ρ_2 are irreducible, we have

$$\frac{1}{|G_1|}\sum_{s_1\in G_1}\left|\chi_1(s_1)^2\right| = 1$$

$$\frac{1}{|G_2|}\sum_{s_1\in G_2}\left|\chi_2(s_2)^2\right| = 1$$

$$\begin{aligned}\frac{1}{|G_1\times G_2|}\sum_{s\in G_1,s_2\in G_2}|\chi(s_1,s_2)|^2 &= \frac{1}{|G_1||G_2|}\sum_{s_1\in G_1,s_2\in G_2}|\chi_1(s_1)|^2|\chi_2(s_2)|^2 \\ &= \left(\frac{1}{|G|}\sum_{s_1\in G_1}|\chi_1(s_1)^2\right)\left(\frac{1}{|G^2|}\sum_{s\in G_2}|\chi_2(s_2)|^2\right) = 1\end{aligned}$$

Let $V_1,...,V_l$ (reap. $W_1,...,W_k$) be all irreducible representations of G_1 (reap. G_2) with degree $n_1,...,n_l$ (reap. $m_1,...,m_k$). Then $\sum_i n_i^2 = |G_1|, \sum_j m_j^2 = |G_2|$

$$\begin{aligned}\sum_{i,j}(dimV_i\otimes W_j)^2 &= \sum_{i,j}(n_i,m_j)^2 \\ &= \sum_{i,j}n_i^2m_j^2 \\ &= (\sum_i n_i^2)(\sum_j m_j^2) \\ &= |G_1||G_2| = |G_1\times G_2|\end{aligned}$$

Thus, $\{V_i\otimes W_j\}$ is the full set of irreducible representations of $G_1\times G_2$.

∎

3 Induced Representations

Let G be a finite group, H a subgroup of G. We have a system of representatives $\{r_\sigma\}_{\sigma\in G/H} \subset G$ such that the disjoint union of $r_\sigma H = G$. $\forall t \in G$, we can write uniquely $t = rh$ where $r \in \{r_\sigma\}, h \in H$.

Definition **68.** Let (ρ, V) be a representation of G and H a subgroup of G and W be an invariant subspace of V. i.e. $\forall h \in H, w \in W, hw \in W$ $(\rho(H)W \subseteq W)$. Let $s \in G, \rho(S)W$ depends only on the left coset of H since $s = rh$. $r \in \{r_\sigma\}, h \in H, \rho(s)W = \rho(r)\rho(h)W = \rho(r)W$.

Let $W_\sigma = \rho(r_\sigma)W \subseteq V$. We say (ρ, V) is **induced** by (θ, W) where θ is the restriction of $\rho|_H$ on W, i.e. $\theta(h)W = \rho(h)W, h \in H, w \in W'$ if $V = \bigoplus_{\sigma\in G/H} W_\gamma$. We also say that (ρ, V) is the induced representation of (θ, W).

Theorem **69.** *Given a representation (θ, W) of a subgroup H of G there exists a unique representation (ρ, V) of G denoted by $Ind_H^G\theta$ or $Ind_H^G W$ induced by (θ, W) (up to isomorphism).*

Wednesday, October 10

Proof. We first prove uniqueness. Let $R = \{r_\sigma\}_{\sigma \in \sigma/H}$ be a set of representatives. By definition, $V = \bigoplus_{\sigma \in G/H} W_\sigma, W_\sigma := \rho(r_\sigma)W$. Thus, each element v of V has a unique expression

$$v = \sum_{\sigma \in G/H} \rho(r_\sigma) w_\sigma$$

Given $g \in G$, $g \cdot r_\sigma = r_{g \cdot \sigma} \cdot h_{g \cdot \sigma}$ where $r_{g \cdot \sigma} \in R, h_{g \cdot \sigma} \in A$.

$$\begin{aligned}
\rho(g)(\rho(r_\sigma)w_\sigma) &= \rho(g \cdot r_\sigma)w_\sigma \\
&= \rho(r_{g\cdot\sigma})\rho(h_{g\cdot\sigma})w_\sigma \\
&= \rho(r_\sigma \cdot \sigma)(\theta(h_{g\cdot\sigma})w_\sigma)
\end{aligned}$$

This expression is only dependent on θ and H. Therefore, (ρ, V) is unique.

Next, we show existence, Define a representation (ρ, V) by $V = \bigoplus_{\sigma \in G/H} W_\sigma$ where $W_\sigma \cong W$. $\rho : G \to GL(V)$, for all $g \in G, v = \sum_{\sigma \in G/H} w_\sigma$ with $w_\sigma \in W_\sigma$, for all $\sigma \in G/H$.

$$\rho(g)v := \sum_{\sigma \in G/H} \theta(h_{g\cdot\sigma}) w_{g\cdot\sigma}$$

To show that ρ is well defined, we must verify that it is a homomorphism. That is, $\rho(g)'\rho(g)w_\sigma = \rho(gg')w_\sigma$, for $g, g' \in G, \sigma \in G/H$. We have

$$\begin{aligned}
g \cdot r_\sigma &= r_{g\sigma}h_{g\sigma} \\
g' r_{g\sigma} &= r_{g'(g\sigma)}h_{g'(g\sigma)} \\
(g'g)r_\sigma &= g'(gr_\sigma) \\
&= g'(r_{g\sigma}h_{g\sigma}) \\
&= r_{g'(g\sigma)}(h_{g'(g\sigma)}h_{g\sigma}) \\
\rho(g')(\rho(g)w_\sigma) &= \rho(g')(\theta(h_{g\sigma})w_{g\sigma}) \\
&= \theta(h_{g'(g\sigma)}\theta(h_{g\sigma})w_{g'(g\sigma)} \\
\rho(g;g)w_\sigma &= \theta(h_{g'(g\sigma)}h_{g\sigma})w_{g'(g\sigma)} \\
&= \theta(h_{g'(g\sigma)})\theta(h_{g\sigma})w_{g'(g\sigma)}
\end{aligned}$$

∎

Remark **70.** A few remarks on induced representations

1. If $\rho_1 = Ind_r^G\theta_1, \rho_2 = Ind_H^G\theta_2$, then $\rho_1 \oplus \rho_2 = Ind_H^G(\theta_1 \oplus \theta_2) = Ind_H^G\theta_1 \oplus Ind_\theta^G\theta_2$.

   Proof. If V_i, W_i are representation spaces of ρ_i, θ_i respectively, then $W_1 \oplus W_2 \subseteq V_1 \oplus V_2$ and $\rho_i(r_\sigma)W_i$ are distinct sums. By our theorem, $V_1 \oplus V_2 = Ind_H^G W_1 \oplus W_2$.

2. If $\rho_1 = Ind_H^G$ and ρ_2 a representation of G, then $(Ind_H^G\theta) \otimes \rho_2 = Ind_H^G(\theta \otimes Res_H^G\rho_2)$ where $Res_H^G\rho_2$ is the representation of H by forgetting the other part of the G-action.
3. Let H be a subgroup of G and K be a subgroup of H. Given a representation (θ, W) of K, $Ind_H^G(Ind_K^H\theta) = Ind_K^G\theta$.
4. $Ind_H^G W \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$.

THEOREM **71.** *Let H be a subgroup of G and $|H|$ is the order of H, $R = \{r_\sigma\}_{\sigma\in G/H}$ a system of representatives. Suppose that (ρ, V) is induced by a representation (θ, W) of H and let χ_ρ and χ_θ be the corresponding characters of G and H. For all $g \in G$,*

$$\chi_\rho(g) = \sum_{r\in R, r^{-1}gr\in H} \chi_\theta(r^{-1}gr) = \frac{1}{|H|} \sum_{s\in G, s^{-1}gs\in H} \chi_\theta(s^{-1}gs)$$

Proof. $V = \oplus_{\sigma\in G/H} W_\sigma, W_\sigma = \rho(r_\sigma)W$. We know that $\rho(g)$ permutes W_σ. $\chi_(g)|_{W\sigma}$ is not 0, if and only if $g\sigma = \sigma \Leftrightarrow gr_\sigma = r_\sigma h_{g\sigma} \Leftrightarrow r_\sigma^{-1}gr_\sigma = h_{g\sigma} \in H$. So we get

$$Tr_{W_\sigma}(\rho(g)|_{W_\sigma}) = Tr_W\theta(r_\sigma^{-1}gr_\sigma) = X_\theta(r_\theta^{-1}gr_\sigma)$$

∎

Friday, October 12

This finishes the first equality. For the second one, $\forall s \in r_\sigma H$, i.e. $s = r_\sigma h$ $X_\theta(s^{-1}gs) = X_\theta(h^{-1}r_\sigma^{-1}gr_\sigma h) = Tr_W[\theta(h^{-1})\theta(r^{-1}\sigma gr_\sigma)\theta(h)] = Tr_W(\theta(r_\sigma^{-1}gr_\sigma)) = X_\theta(r_\sigma^{-1}gr_\sigma)$. Thus, $g\sigma = \sigma \Leftrightarrow gr_\sigma = r_\sigma h_{g\sigma} \Leftrightarrow r_\sigma^{-1}gr_\sigma = h_{g\sigma} \in H$.

PROPOSITION **72.** *Let H be a subgroup of G, (θ, W) a representation of H, (τ, V) a representation of G. Then, any H-module homomorphism $\varphi : W \to V$ extends uniquely to a G-module homomorphism. $\tilde{\varphi} : Ind_H^G W \to V$.*

$$Hom_H(W, Res_H^F V) \cong Hom_G(Ind_H^G W, V)$$

In particular, this universal property uniquely determines $Ind_H^G W$ up to isomorphism.

Proof. Let $(\rho, V) = Ind_H^G\theta$ and $R = \{r_\sigma\}_{\sigma\in G/H}$ a system of representatives of G/H. $V = \bigoplus_{\sigma\in G/H} W_\sigma, W_\sigma = \rho(r_\sigma)W$. We define $\tilde{\varphi}$ on W_σ as follows: $\forall\sigma \in G/H$

$$W_\sigma(\rho(r_\sigma)^{-1} \to W(\varphi) \to V(\tau(r_\sigma)) \to V$$

Which is independent of the representative of r_σ for r, since γ is H-invariant, ∎

THEOREM **73.** [FROBENIUS RECIPROCITY] *Let H be a subgroup of G, (θ, W) a representation of H, (τ, V) a representation of G. For a representation (ϵ, M), we use χ_ϵ or χ_M to denote the character of (ϵ, M).*

$$(\chi_{Ind_H^G W}|\chi_V)_G = (\chi_W|\chi_{Res_H^G V})_H$$

Proof. Since the inner product is linear w.r.t direct sum. it is enough to show that θ and τ are irreducible. The left hand side is the number of times U appears in $Ind_H^G W$ which is equal to $dim_{\mathbb{C}}(Hom_G(Ind_H^G W, U))$. Similarly, the right hand side is the number of times W appearing in $Res_H^G V$ which is equal to $dim_{\mathbb{C}}(Hom_H(W, Res_H^G V))$. Since $Hom_G(Ind_H^G W, V) = Hom_G(W, Res_H^G, V)$, we have that they have the same dimension over $\mathbb{C}$. ∎

Example **74.** Suppose we have $G = D_n$ where n is even and $n \geq 2$. Recall that $D_n = \langle r, s \rangle / \langle r^n = 1, s^2 = 1, srs = r^{-1} \rangle$.

Table 2: 1 - Dimensional representations

	r^k	sr^k
ψ_1	1	1
ψ_2	1	(-1)
ψ_3	$(-1)^k$	$(-1)^k$
ψ_4	$(-1)^k$	$(-1)^{k+1}$

We have ζ, ρ_h as defined earlier. We have

$$
\begin{aligned}
1^2 + 1^2 + 1^2 + 1^2 + (\frac{n}{2} - 1)2^2 &= 4 + (\frac{n}{2} - 1) \cdot 4 \\
&= 2n = |D_n|
\end{aligned}
$$

Thus, $\{\psi_1, \psi_2, \psi_3, \psi_4, \rho_h\}_{1 \leq h \leq \frac{n}{2} - 1}$ is the full set of irreducible representations of D_n. $C_n \lhd D_n$ so any representation of C_n can produce a representation of D_n with twice dimensions. All irreducible representations of C_n are $\{\chi_h\}$, $\chi_h(r^k) = \zeta^{hk}$. $Ind_{C_n}^{D_n}$ is a two-dimensional representation. In fact, $Ind_{C_n}^{D_n} \cong \rho_h$.

(χ_h, V) is a 1-dimensional representation of C_n. $Ind_{C_n}^{D_n} \chi_h = V \bigoplus \rho(s)V = \langle v, \rho(s)v \rangle$ where v is a basis of V.

$$
\begin{aligned}
\rho_h(r^k)v &= \chi_h(r^k v) = \zeta^{hk} v \\
\rho_h(r^k)(\rho_h(s)v) &= \rho_h(sr^{-k})v = \rho_h(s)\zeta^{-hk} v = \zeta^{-hk}\rho_h(s)v) \\
r^k s &= sr^k
\end{aligned}
$$

The matrix form of $\rho_h(r^k)$ is $\begin{pmatrix} \zeta^{hk} & 0 \\ 0 & \zeta^{-hk} \end{pmatrix}$.

Monday, October 15

Example **75.** Let $G \cong S_4$. We have $|S_4| = 4! = 24$. $H = \{(1), (12)(34), (13)(24), (14)(23)\}$. Then, $H \lhd S_4$, $S_3 \subseteq S_4$ and $H \rtimes S_3 \cong S_4$ (semi-direct), $S_4/H \cong S_3 \cong D_3$. We have 2 1-dimensional representations of S_4 and one 2-dimensional representation of S_4. S_4 acts on

$\mathbb{C}^4$ by permuting the basis elements $\{e_1, e_2, e_3, e_4\}$, then $e_1 + e_2 + e_3 + e_4$ is an S_4- invariant subspace

$$\{x + y + z + w = 0 | x, y, z, w \in \mathbb{C}^4\}$$

is a S_4-invariant subspace of $\mathbb{C}4$. This gives us a 3-dimensional irreducible representation of S_4.

	(1)	(12)	(12)(34)	(123)	(1234)
x_0	1	1	1	1	1
ϵ	1	-1	1	1	-1
θ	2	0	2	-1	0
ψ	3	1	-1	0	-1
$\epsilon \otimes \psi$	3	-1	-1	0	1
r_G	24	0	0	0	0

Remark **76.** Let H, K be two subgroups of G and $\rho : H \to GL(W)$ be a representation of H. $V = Ind_H^G W$. We would like to know $Res_K^G V$. First of all, we choose a set of representatives S for $K\backslash G/H$, that is, $G = \bigcup_{s \in S} KsH$ (disjoint union), so $s \sim s' \Leftrightarrow \exists k \in K, h \in H$ such that $ksh = s'$. $\forall s \in S$ define $H_s := sHs^{-1} \cap K \subseteq K$. We set $\rho^s(x) := \rho(s^{-1}xs), x \in H_s$ and obtain a representation $\rho_s : H_s \to GL(W)$, we denote this representation by W_s.

Proposition **77.** *The representation $Res_K^G(Ind_H^G W)$ is isomorphic to the direct sum of the representation $Ind_{H_s}^K W_s$ for $s \in S \cong K\backslash G/H$.*

Proof. We know that V is a direct sum of the image $\rho(x)W$ for $x \in G/H$. Let $s \in S$ and $V(s)$ be the space of V generated by the image of $\rho(x)W$, where $x \in KsH$. By definition, $V(s)$ is K-invariant. We just need to show that $V(s) \cong Ind_{H_s}^K W_s$. We only need to check that $W_s \subseteq V(s)$. In fact, $\rho(s)W$ is H_s isomorphic to W_s given by $s : W_s \to \rho(s)W$. ∎

Remark **78.** In particular, if $H = K$, we still use $H_s = sHs^{-1} \cap H$. The representation of ρ of H, define a $Res_s(\rho)$ by restriction to H_s. This might be different than W_s.

Proposition **79.** [Mackey's Irreducibility Criterion] *In order to make $V = Ind_H^G W$ irreducible, it is necessary and sufficient that the following two conditions be satisfied.*

1. *W is irreducible*
2. *$\forall s \in H\backslash G/H$, two representations (ρ^s, W_s) and $Res_s(\rho)$ are disjoint, i.e. ρ^s and $Res_s(\rho)$ have no common irreducible components. H is the same as $(\chi_{Res_s(\rho)} | \chi_{\rho(s)})_{H_s} = 0$ for all $s \in H\backslash G/H$.*

Proof. V is irreducible if and only if

$$
\begin{aligned}
(\chi_v|\chi_v)_G &= 1 \Leftrightarrow \\
(\chi_{Ind_H^G W}|\chi_{Ind_H^G W})_G &= 1 \Leftrightarrow \\
(\chi_w|\chi_{Res_H^G(Ind_H^G W)})_H &= 1 \\
Res_H^G(Ind_H^G w) &= \bigoplus_{s\in H\backslash G/H} Ind_{H^s}^H \rho^s
\end{aligned}
$$

$$
\begin{aligned}
1 = (\chi_v|\chi_v) &= \sum_{s\in H\backslash G/H} d_s \\
d_s &= (\chi_w|\chi_{Ind_{H^s}^H}\rho^s)_H = (\chi_{Res_s(\rho)}|\chi_{\rho^s})_{H_s}
\end{aligned}
$$

If $s = e$ then $d_s = d_e = 1$. Thus, the sum is equal to 1 if and only if $d_s \equiv 0$ for $s \neq e$. This is exactly the second condition. ∎

COROLLARY **80.** *Suppose that $H \lhd G$, $Ind_H^G W$ is irreducible if and only if W is irreducible and ρ is not isomorphic to and of ρ^s for all $s \notin H$.*

4 Module Theory

Wednesday, October 17

DEFINITION **81.** Let R be a ring with identity 1 (not necessarily commutative). A (left) R-Module is an abelian group $(M, +, 0)$ together with a left action of R on M by $R \times M \to M$ with $(r, m) \to r \cdot m$ such that for all $r, s \in R$ and $m, n \in M$

1. $r(m + n) = rm + rn$
2. $r(sm) = (rs)m$
3. $(r + s)m = rm + sm$
4. $1m = m$

We sometimes write ${}_RM$ to specify that M is a left module. We can define the right module in a similar fashion.

EXAMPLE **82.** ${}_RR$ for any ring with identity.

DEFINITION **83.** Recall that a (left) **Ideal** of a ring R is a subset if R such that $\forall r \in R, i \in I \Rightarrow r \cdot i \in I$ and also closed under addition. Similarly, we can define right ideals. If a subset I is both left and right ideals, we say I is an ideal of R. Any left ideal I of R gives an R-module.

DEFINITION **84.** If M, N are R-modules (always means left) then $\phi : M \to N$ is said to be a (module) homomorphism if

1. φ is a group homomorphism
2. $\forall r \in R$ and $m \in M$, we have $\phi(r \cdot m) = r \cdot \varphi(m)$.

DEFINITION **85.** Let N be a subset of an R-Module M. We say that N is a submodule of M if N is a subgroup and $R \cdot N = \{r \cdot n | r \in R, n \in N\} \subseteq N$.

DEFINITION **86.** A quotient module M/N, where N is a submodule, of M is a quotient group M/N with the R-action $r(m + N) = r \cdot m + N$. It is well defined.

THEOREM **87.** [FIRST ISOMORPHISM THEOREM] *Let R be a ring and $\varphi : M \to N$ is a module homomorphism. Then,*

1. *$\varphi(M)$ is a submodule, $ker(\varphi)$ is a submodule of M*
2. *$\varphi(M) \cong M/ker(\varphi)$*

Proof. Obvious. ∎

THEOREM **88.** [SECOND ISOMORPHISM THEOREM] *Let R be a ring and $B, C \subseteq A$ be R-Modules. Then*

$$(B+C)/B \cong C/(B \cap C)$$

where $B + C := \{b + c | b \in B, c \in C\} \subseteq A$.

Proof. Obvious. ∎

THEOREM **89.** *Let R be a ring and $C \subsetneq A$ be R-Modules. The sub-modules of A/C corresponds to submodules $C \subseteq B \subseteq A$ via $B \Leftrightarrow B/C$. Furthermore,*

$$\frac{A/C}{B/C} \cong A/B$$

THEOREM **90.** *Let R a ring and suppose that $0 \leq A_0 \leq A_1 \leq \cdots \leq A_n = M$ and $0 = B_0 \leq \cdots \leq B_m = M$ are two chains of R-Modules. Then, both chains can be refined so that they have the same length and the same factors)possibly in different order).*

Proof. Let $A_{i,j} := A_i + (A_{i+1} \cap B_j)$ where $0 \leq i \leq n$ and $0 \leq j \leq n$. Let $B_{i,j} := B_j + (A_i \cap B_{j+1})$. Then, by the previous theorem, $A_{i,j+1}/A_{i,j} \cong B_{i+j}/B_{i,j}$. $A_i = A_{0,i}$ and $B_j = B_{0,j}$. Thus, $\{A_{i,j}\}, \{B_{i,j}\}$ are the refinements which we are looking for. ∎

DEFINITION **91.** A module M is called irreducible (simple) if M has exactly two different submodules 0 and M. A **Composition Series** for a module M is a chain of submodules $0 \subsetneq A_1 \subsetneq A_2 \cdots \subsetneq A_n = M$ and A_i/A_{i-1} is irreducible. A_i/A_{i-1} are called the factors of M.

THEOREM **92.** [JORDAN - HOLDER] *If M has a composition series, then any two composition series have the same length and the same factors (up to isomorphism).*

Proof. By the previous theorem, two composition series share refinements with the same factors. ∎

REMARK **93.** The length of M and factors do not uniquely determine M.

EXAMPLE **94.** Consider $S_4 \cong H \rtimes S_3$ has the same factor as $H \times S_3$.
$H = \{(1), (12)(34), (13)(24), (14)(23)\}$

Friday, October 19

PROPOSITION **95.** *An R-Module M is irreducible if and only if M is isomorphic to R/A, when A is a maximal left ideal.*

Proof. $\Leftarrow M \cong R/A$, then the submodule of M corresponding to the left ideals containing A. Since A is maximal, M is irreducible.

$\Rightarrow \varphi :_R R \to M$ by $r \to r \cdot a$ when we fix a non-zero element $a \in M$, $\varphi \neq 0$. $\varphi(M)$ is a submodule of M, thus, $\varphi(R) = M$. $\ker \varphi$ is a left ideal of R and $R \cong R/ker\varphi$ and by the converse, $\ker \varphi$ must be maximal. ∎

Definition **96.** A module M is Noetherian (Artinian) if every non-empty set of submodules has a maximal element.

Definition **97.** The **ascending chain condition** abbreviated as ACC, says that if $\{A_n\}_{n=1}^{\infty}$ is a sequence of submodules with $A_n \subseteq A_{n+1}$ for all $n \geq 1$, then there is an N such that $A_n = A_{n+1}$ for all $n \geq N$ (similarly can define descending chain).

Proposition **98.** *A module is Noetherian (Artinian) if an donly if it satisfies ACC.*

Definition **99.** A module M is called finitely generated if $M = \langle b_1, \ldots, b_n \rangle = \sum_{i=1}^{n} Rb_i$.

Proposition **100.** *A module is Noetherian if and only if every submodule is finitely generated.*

Proposition **101.** *Let A be a module and $B \subseteq A$ be a submodule. A is Artinian (Noetherian) if and only if $B, A/B$ are Artinian (Noetherian).*

Corollary **102.** *A finite product of modules $M_1 \times M_2 \times \cdots \times M_k$ of modules is Artinian (Noetherian) if and only if each M_i is Artinian (Noetherian).*

Proof. By induction on k and $(A \times B)/B \cong A$. ∎

Corollary **103.** *A module M has a composition series if and only if M is Artinian and Noetherian.*

Proof. Suppose that M has a composition series $0 \leq M_0 \subsetneq \cdots \subsetneq M_r = M$, we know M_{i+1}/M_i is simple (irreducible) and therefore, it is both Artinian and Noetherian. By the previous proposition, $M = M_k$ is both Artinian and Noetherian.

Conversely, let C be a maximal chain of submodules. Since M is artinian and C is bounded below 0. Similarly, since M is Noetherian, C is bounded above by M. Therefore, C has a finite length. ∎

Radicals

DEFINITION **104.** Let R be a ring. For an R-module M, define $\text{Ann}(M) := \{r \in R | r \cdot m = 0, \forall m \in M\}$. The Jacobson radicals (radical) of R is

$$J(R) := \bigcap_{M \text{ irreducible R-Module}} \text{Ann}(M)$$

PROPOSITION **105.** *Let R be a ring and M an R-Module. Then $Ann(M)$ is an ideal of R. As a consequence, $J(R)$ is an ideal of R.*

THEOREM **106.** [HOMEWORK] *The following are equivalent for $J(R)$.*

1. $\bigcap_{M \text{ irreducible}} Ann(M)$
2. $\bigcap_{A \text{ maximal left ideals}} A$
3. $\{a \in R | \forall r \in R, \exists u \in R, a(1 - ra) = 1\}$
4. *The largest proper ideal J of R such that $1-a \in R^* = \{r \in R | \exists r', r'' \in R$ such that $r'r = r''r = 1\}$*

Furthermore, the right analogues of 1,2,3 are also equivalent to $J(R)$.

DEFINITION **107.** A ring R is a **semiprimitive** if $J(R) = 0$. Note that some books use "semisimple" to denote this property.

THEOREM **108.** *$R/J(R)$ is semiprimitive*

Proof. By corresponding theorem (third isomorphism theorem),

$$\begin{aligned} J(R/J(R)) &= \bigcap_{M \text{ Maximal}} M \\ &= \bigcap_{N \text{ maximal in R}} N/J(R) \\ &= \left(\bigcap_{N \text{ maximal}} \right) /J(R) \\ &= J(R)/J(R) \\ &= 0 \end{aligned}$$

∎

Monday, October 22

Definition **109.** A left ideal I is **nil** if for all $a \in I$ if a is nilpotent. That is, there exists $k \in \mathbb{Z}$ with $a^k = 0$. A left ideal is **nilpotent** if there exists $b \in \mathbb{Z}$ such that $I^k = 0$, where I^k is the ideal generated by $\{a_1, \dots, a_k | a_i \in I\}$.

Proposition **110.** *If I is left ideal then $I \subseteq J(R)$*

Proof. Let $a \in I, \forall r \in R$, we would like to show $1 - ra$ has a left inverse. Since $ra \in I$, there exists $k \in \mathbb{N}$ such that $(ra)^k = 0$. Hence,

$$(1 - ra)(1 + ra + \cdots + (ra)^{k-1}) = 0 \Rightarrow \\ 1 - ra \in R^*$$

Note the right hand of the product is $(1 - (ra)^k)$ and we are done. ∎

Theorem **111.** *Let A be $\mathbb{K}$-algebra such that $\dim_{\mathbb{K}}(A) < |\mathbb{K}|$, when $\mathbb{K}$ is a field. Then, $J(A)$ is nil.*

Proof. Let $a \in J(A)$ so that for all $\lambda \in \mathbb{K}, 1 - \lambda a$ is invertible in A. Then the set $\{(1 - \lambda a)^{-1} | \lambda \in \mathbb{K}\}$ must be linearly dependent since $\dim_{\mathbb{K}}(A) < |\mathbb{K}|$. Thus, there exists $\lambda_0 = 0, \lambda_1, \dots, \lambda_n \in \mathbb{K}$ and $c_0, \dots, c_n \in \mathbb{K}$ not all zero, such that

$$0 = \sum_{i=0}^{n} c_i (1 - \lambda_i)^{-1} = \left[\prod_{i=0}^{n} (1 - \lambda_i a)^{-1} \right] \sum_{i=0}^{n} c_i \prod_{j \neq i} (1 - \lambda_j a)$$

Let $p(x) = \sum_{i=0}^{n} c_i \prod_{i \neq j} (1 - \lambda_j x)$ and $p(a) = 0$. Since $\prod_{i=0}^{n} (1 - \lambda_i a)^{-1}$ is invertible. We would show that $p(x) \neq 0$.

1. $c_0 \neq 0$. Then, the coefficeint of x^n is $c_0 \prod_{i=1}^{n} (-\lambda_j) \neq 0$
2. If $c_0 = 0$, suppose that $c_i \neq 0, i > 0$, then $p(\lambda_i^{-1}) = c_i \prod_{i \neq j} (1 - \lambda_j \lambda_i^{-1}) \neq 0$.

Hence, $p(x) \neq 0$.

$$0 = p(a) = a^k (b_k + b_{k+1} a + \cdots + b_n a^{n-k}$$

for some $0 \leq k \leq m$ and $b_k \neq 0$. However, the right hand of the product is invertible which implies that $a^k = 0$ which means that $J(A)$ is nil. ∎

Theorem **112.** *If G is any group, then $\mathbb{C}[G]$ is semiprimitive, i.e. $J(\mathbb{C}[G]) = 0$.*

Proof. Define an involution on $\mathbb{C}[G]$ by

$$x^* = \left(\sum_{g \in G} x_g g \right)^* := \sum_{g \in G} \bar{x}_g g^{-1}$$

Clearly $(x^*)^* = x, (\alpha x)^* = \bar{\alpha} x^*, (x+g)^* = x^* + y^*, (xy)^* = y^* x^*$. Suppose that G is countable then $\mathbb{C}[G]$ is a $\mathbb{C}$-algebra of dimension $|G| < |\mathbb{C}|$. Thus, $J(\mathbb{C}[G])$ is nil.

Let $x \in J(\mathbb{C}[G])$ and suppose $x \neq 0$.

$$y := x^* x = \sum_{h \in G} \left(\sum g \in G (\bar{x}_g x_{gh}) h \right)$$

In particular, $y_e = \sum_{y \in G} |x_g|^2 > 0$, so $y \neq 0$.

$$y^* = (x^* x)^* = x^* (x^*)^* = x^* x = y$$

Thus, $y^2 = y^* y \neq 0$. Repeat this construction we get $y^{2^k} \neq 0$ for all $k \geq 1, k \in \mathbb{Z}$. Also, $y^{2^k} \in J(\mathbb{C}[G])$. H is a contraction since $J(\mathbb{C}[G])$ is nil. So the countable case is done.

Next suppose that G is any group and $x \in J(\mathbb{C}[G])$. $H := \langle \{g \in G | x_g \neq 0\} \rangle$ is countable. We have $x \in \mathbb{C}[H]$. Our goal is to show that $x \in J(\mathbb{C}[G])$ for all $r \in \mathbb{C}[H]$, $(1-rx)^{-1} \in \mathbb{C}[G]$. Let $1 - rx = \sum_{h \in H} a_h h$.

$$(1 - rx)^{-1} = \sum_{g \in G} b_g g$$
$$b = \sum_{g \in H} b_g g \in \mathbb{C}[H]$$

Since $(1 - rx)(1 - rx)^{-1} = 1$.

$$1 = \sum_{g \in G} a_g b_{g^{-1}} = \sum_{h \in H} a_h b_{h-1}$$

For any non-identity element $k \in H$

$$0 = \sum_{g \in G} a_g b_{g^{-1}k} = \sum_{h \in G} a_h b_{h^{-1}k}$$

Thus, $(1-rx)b' = 1$. Since the inverse is unique, $b' = (1-rx)^{-1} \in \mathbb{C}[H]$ and we are done. ∎

Wednesday, October 24

Definition 113. A ring R is called Artinian if ${}_R R$ is a left Artinian R-moudle. In other words, R is Artinian if and only any collection of left ideals has a minimal element.

Example 114. If R is a finite dimensional $\mathbb{K}$ - algebra for a field $\mathbb{K}$. Then, R is artinian. The length of R is less or equal to $\dim_{\mathbb{K}}(R)$.

Theorem 115. *If A is artinian, then $J(A)$ is nilpotent.*

Proof. Let $J = J(A)$, $J \supset J^2 \supset J^3 \ldots$, then there exists $N \in \mathbb{N}$ such that $J^n = J^{n+1}$ for all $n \geq N$. Let $B = J^N$, So

$$BJ = J^N J = J^{N+1} = J^N = B = B^2$$

If $B = 0$, we are done. If not, let S be the set of left ideals I such that $BI \neq 0$. S is non-empty since $J, B \in S$. Since A is artinian, S has a minimal element I_0. There exits $x \in I_0$ such that $Bx \neq 0$. $B(Bx) = B^2x = Bx \neq 0$. Thus, $Bx \in S$. By minimality, $Bx = I_0$, i.e. there exists $b \in B$ such that $bx = x$. This implies $(1-b)x = 0 \Rightarrow x = 0$ Since $1 - b \in J$ is invertible. ∎

Corollary **116.** *If A is artinian then $J(A)$ is the unique largest nilpotent ideal and every nilpotent ideal is contained in $J(A)$.*

Lemma **117.** [Schur's Lemma] *If M is an irreducible left R-module, then $End_R(M)$ is a division ring.*

Proof. Let $\rho \in End_R(M)$, $\varphi \neq 0$, then $\varphi(M)$ is a a submodule of M and $\varphi(M) \neq 0$. Thus, $M = \varphi(M)$ since M is irreducible. Similarly, $\ker \varphi$ is a submodule of M. Since $\ker \varphi$ is a proper submodule of M, $\ker \varphi = 0$. Thus, φ is an isomorphism and it is invertible. ∎

Theorem **118.** *Let M be a left ideal of a ring R. Then,*

1. *If $M^2 \neq 0$, $End_R(M)$ is a division ring, then $M = Re$ for some $e = e^2 \in R$ and $End_R(M) \cong eR^{op}e$ where R^{op} is the opposite ring of R, i.e. $r_1, r_2 \in R.r_1 * r_2 := r_2 r_1$.*
2. *If M is a minimal left ideal and $M^2 \neq 0$ then $M = Re$ for some $e = e^2 \in R$.*
3. *If R has no non-zero nilpotent ideal and $M = Re$ for some $e = e^2 \in R$, then M is minimal if and only if eRe is a division ring.*

Proof. Since $M^2 \neq 0$ there exists $a \in M$ such that $Ma \neq 0$. Define $\rho_a : M \to M$ by $\rho_a(x) := xa$. Let $e := \rho_a^{-1}(a)$ (If $M = Re, e^2e$, then $m \in M, me = -m$).

$$\begin{aligned}
ea &= \rho_a(e) = a \\
ea = e(ea) &= e^2 a \Rightarrow \\
(e - e^2)a &= 0 \Rightarrow \\
\rho(a)(e - e^2) &= 0 \\
e - e^2 &= 0 \Rightarrow \\
e &= e^2
\end{aligned}$$

$M = \rho_a(M) = M_a \supseteq R_a \supseteq M_a$. This(? $M = r$ nd we get that $re = M$?erased). $\forall c \in M, c = re = re^2 = ce$ for some $r \in R$. Suppose that $\rho \in End_R(M)$. Let $b = \rho(e)$

$$v = \rho(e^2) = e\rho(3) = eb = ebe$$

Thus, $b \in eMe \leq eRe$.

$$\rho(x) = \rho(xe) = x\rho(e) = xh$$

Then $\rho = \rho_b$. Conversel, for all $b \in eRe$, $\rho_b \in End_S(M)$.

$$\rho_b \circ \rho_c = \rho_{cb} = \rho_{b*c}$$

in R^{op}. Thus, $End_R(M) \cong eR^{op}e$.

(ii) Suppose that M is minimal and therefore M is irreducible by (i) we are done.

(iii) If M is minimal, by (ii) we are done. Assume eRE is a division rng and $0 \neq N \leq M$. If $eN = 0$, then $N^2 \leq MN = ReN = 0$. It contradicts our assumption. Thus, $eN \neq 0$. Take $n \in N, en \neq 0 mene = en \neq 0$. Thus, there exsits $r \in R$ such that $eren = (ere)(ene) = e$. Since $eren \in N$, this implies $e \in N \Rightarrow M \subseteq N \Rightarrow M = N$. Thus, M is minimal. ∎

Monday, October 29

THEOREM **119.** *If R is artinian and semiprimitive then $R = \bigoplus A_i$, A_i is simple and artinian.*

THEOREM **120.** [ARTIN - WEDDERBURN] *If R is Artinian and simple, then there exists a division ring F such that $R \cong M_n(D)$ for some $n \in \mathbb{N}$. Moreover, n and D are unique.*

COROLLARY **121.** [GENERALIZATION OF ARTIN-WEDDERBURN] *If R is Artinian and semi-primitive (semi-simple) then $R \cong \bigoplus M_{n,i}(D_i)$ for some division rings D_i and $n_i \in \mathbb{N}$. Moreover, n_i, D_i are unique.*

DEFINITION **122.** A ring R is called primitive if there exists a faithful irreducible R-module M, i.e. $\text{Ann}(M) = 0$. An ideal A of R is called primitive if $A = \text{Ann}(R)$ for some irreducible R-module M.

REMARK **123.**

- R/A is primitive since M is a faithful irreducible R/A-Module.
- If R is primitive, then R is semi-primitive. Since $J(R) = \bigcap \text{Ann}(N) = 0 \subseteq \text{Ann}(M)$

PROPOSITION **124.** *If R is Artinian and simple, then R is primitive.*

Proof. Since R is Artinian, there exists a minimal left ideal. A is irreducible by minimality. $\text{Ann}(A)$ is an ideal of R. Since R is simple, $\text{Ann}(A) = 0$ and A is faithful. ∎

REMARK **125.** Let R be primitive and M a faithful irreducible R-module. $D = \text{End}_R(M)$ is a division algebra. M is also a D-module by $\varphi \in D$. $\varphi x = \varphi(x)$. Moreover, $\forall r \in R$, $\varphi(rm(= r\varphi(m)$. Thus, we get $R \to \text{End}_R(M)$ by $r \to \varphi_r : M \to M$ by $r \cdot m \to 1m$. Since M is faithful, $R \to \text{End}_D(M)$ is injective. What we need now is to show that this map is surjective.

DEFINITION **126.** An R-module is called **semisimple** if every submodule $N \subseteq M$ is a direct summand, i.e., there exists $N' \subseteq M$ such that $M = N \oplus N'$.

Proposition **127.** *If M is semi-simple, then every submodule and every quotient module is semisimple.*

Lemma **128.** *If M is a non-zero semi-simple R module, then M has an irreducible submodule.*

Proof. Take $0 \neq m \in M$. Let S be the set of all submodules that are contained in M but do not contain m. S is non-empty since $0 \in S$. By Zorn's Lemma, S has a maximal element, N_0. $R_m = N_0 \oplus N'$ for some N'. Then, N' is irreducible. If not, there exists N'', $0 \neq N'' \subsetneq N'$ and $N_0 \oplus N'' \supsetneq N_0$. By maximality, $R_m \subseteq N_0 \oplus N''$ (one line missing here, a contradiction is thrown) ∎

Theorem **129.** *Let M be an R-module. The following are equivalent*

1. *M is semisimple*
2. *M is a direct sum of irreducible modules*
3. *M is equal to the sum of all irreducible modules.*

Proof. $(i) \to (ii)$. Let M_1 be the maximal submodule of M such that M_1 is a direct sum of irreducible modules. Since M is semisimple, there exists M_1' such that $M = M_1 \oplus M_1'$. IF M_1' is not zero, by lemma, there exists an irreducible submodule $N \subseteq M_i$. $M_1 \oplus N \supseteq$. It contradicts the maximality of M.
$(ii) \to (iii)$ Trivial.

$(iii) \to (i)$. Let N be a submodule of M. $S' :=$ the collections of irreducible submodules such that $\sum_{L \in S} L$ is a direct sum and $L \cap N = 0$. If $N \neq M$, S' is non-empty by our condition. S' has a maximal element, say S_0 and $N' = \oplus_{L \in S_0} L$. If $N \oplus N' \subseteq M$ there exist $N \oplus N'$ and $L' \cap N \oplus N' = 0$, since L' is irreducible. Take $N'' - N' \oplus L'$ and $N'' \cap N = -$ and n'' is a direct sum of irreducible modules $N'' \supsetneq N'$. It contradicts the maximality of N. ∎

Definition **130.** A ring R is semi-simple if ${}_RR$ is semi-simple, i.e. $\forall I$ left ideal of R there exists J (left ideal of R suchthat $R = I \oplus J$.

Corollary **131.** *If R is semi-simple, then every R-module is semi-simple.*

Proof. $M = \sum_{m \in M} R_m$, thus, it is sufficient to show that R_m is semi-simple. $R_m \cong R/N, N = \{r \in R, rm = d\}$, R is semi-simple and therefore $R/N \cong R_m$ is semi-simple. Thus, R_m is the direct sum of irreducible modules and we are done. ∎

Wednesday, October 31

Remark **132.** [Correction] In A3Q7, the division algebra must be finite dimensional over F.

Lemma **133.** *Let M be semi-simple over R, $D = End_R(M)$ and $f \in End_D(M)$. Let $m \in M$, there exists an element $r \in R$ such that $r \cdot m = f(m)$*

Proof. Since M is semi-simple, there exists an R-submodule N such that $M = R_m \oplus N$. Let $\pi : M \to R_m$ be the projection. This is an R-homomorphism and $\pi \in \mathrm{End}_E(M) = D, f(m) = f(\pi(m)) = \pi(f(m)) \in R_m$ (since π is identity on R_m since its a projection) ■

Theorem **134.** [Density Theorem] *Let M be semi-simple over R and $D = End_R(M)$. Let $f \in End_D(M)$, for any $m_1, \ldots, m_k$ there exists $r \in R$ such that $rm = f(m_i)$.*

Corollary **135.** *If M is finitely generated, then the image is onto. Moreover, if M is faithful irreducible and finitely generated, then $R \cong End_D(M) \cong M_n(D)$ where $n = \dim_D(M)$.*

Proof of Density Theorem. We may assume that M is irreducible. Define

$$f^(k) : M^k \to M^k$$
$$f^{(k)}(x_1, \ldots, x_k) \to (f(x_1), \ldots f(x_k))$$

$D = \mathrm{End}_R(M^k)$, ten D' is none other than the ring of matrices with coefficients in $D = \mathrm{End}_R(M)$. Thus, $f^{(k)}$ commutes with D' since f commutes with D and therefore $f^{(k)} \in \mathrm{End}_{D'}(M^k)$.

Since M is simple, M^k is semi-simple and we apply lemma to $f^{(k)}$ on M^k. It finishes the proof, there exists $r \in R$ such that $rm = f(m_i)$ for $1 \leq i \leq k$. ■

Theorem **136.** [Artin -Wedderburn] *If R is simple and Artinian, then there exists a division ring D such that $R \cong M_n(D)$.*

Proof. There exists a faithful irreducible R-module M and $D = \mathrm{End}_R(M)$. Assume that we have an infinite linearly independent sets $\{v_1, \ldots, v_k, \ldots\}$ in M over D.

$$L_k = \{r \in R | rv_i = 0, 1 \leq i \leq k\}$$

We have $L_{k+1} \supsetneq L_k$ by the density theorem. Since R is Artinian, it is impossible. ■

Lemma **137.** *Let D be a division ring and $R = M_n(D)$, then every irreducible R-module is isomorphic to ${}_RD^n$. Therefore, $M_n(D)$ has a unique class of irreducible modules.*

Proof. Let M be an irreducible R-module. If $0 \neq m \in M, M = R_m \neq R/N$ where $N = \{r \in R | rm = 0\}$. Since R is Artinian and semi-primitive, there exists $e = e^2 \in R$ such that $N = Re$.

$$M \cong R/N = R/Re \cong R(1-e)$$

(Recall $R = Re \oplus R(1-e)$). Since M is irreducible, $R(1-e)$ is irreducible and therefore it is a minimal left ideal. R acts transitively on $RE_{1 \cdot 1} \cong_R D^n$. Since $RE_{1 \cdot 1}$ is an irreducible R-module. $\mathrm{Ann}(RE_{1 \cdot 1} = 0$ since R is simple. In particular, $(1 \cdot e)E_{1 \cdot 1} \neq 0$. Pick $r \in R$ such that $f_r E_{1 \cdot 1} \neq 0$. Define $\varphi : R(1-e)$ by $x(1-e) \to x(1-e)rE_{1 \cdot 1}$. φ is a homomorphism and $\ker \varphi \supsetneq Rf$ and therefore $\ker \varphi = 0$. Similarly, φ is onto and thus φ is an isomorphism. ■

REMARK **138.** Let G be a finite group. $\mathbb{C}[G]$ is Artinian and semi-primitive. Therefore, we have

THEOREM **139.** *Let G be a finite group. Then $\mathbb{C}[G] \cong \bigoplus_{n_i} M_{n_i}(\mathbb{C})$. (proof by Artin Wedderburn).*

REMARK **140.** A_n, SN, GL_2 state the irreducible representation. Study assignment questions which we got wrong. Finally, a question from class.

November 2

Proof. Let $\{(\rho_i, V_i)\}$ be the set of all irreducible representations of G (i.e. V_i are irreducible $\mathbb{C}[G]$-Module via ρ_i, $\dim V_i = n$. Each ρ_i gives a homomorphism from $G \to GL(V_i)$ and extended to

$$\mathbb{C}[G] \to \mathrm{End}_{\mathbb{C}}(V_i) \cong M_{n_{\mathbb{C}}}\mathbb{C}$$

We can combine them together and get

$$\mathbb{C}[G] \to \bigoplus_{i=1}^{l} M_{n_i}(\mathbb{C})$$

where $l = \#$ of irreducible representations of G, since $|G| = \sum n_i^2$. If ρ is surjective and then it is injective and therefore it is an isomorphism. The surjective comes from the orthogonal relations. This finishes the proof. ∎

PROPOSITION **141.** [FOURIER INVERSION] *Let $\{u_i\}_{i \in Irr(G)}$ be an element of $\prod_{i \in Irr(G)} M_{n_i}(e)$ and $u = \sum_{g \in G} u_s s$ such that $\hat{\rho}_i(u) = u_i$. Then,*

$$u_s = \frac{1}{|G|} \sum_{i \in Irr(G)} n_i Tr(\rho_i(s^{-1}) u_i 0$$

PROPOSITION **142.** *Let $u = \sum u_s s, v = \sum v_s s,\ u, v \in \mathbb{C}[G]$*

$$\langle u, v \rangle = \sum_{s \in G} u_{s^{-1}} v_s$$

$$\langle u, v \rangle = \frac{1}{|G|} \sum_{i \in Ind(G)} n_i Tr(\hat{\rho}(uv))$$

THEOREM **143.** *Let (ρ, V) be an irreducible representation of G, then* $\dim V || G|$.

PROPOSITION **144.** *For any irreducible representation (ρ_i, V_i) , the set of all irreducible representations of G, the homomorphism $\hat{\rho}_i$ maps the center of $\mathbb{C}[G]$, denoted $Z(\mathbb{C}[G])$ into the set of homotheties of V_i and defines an algebra homomorphism.*

$$w_i : Z(\mathbb{C}[G]) \to \mathbb{C}$$

If $u = \sum_{s \in G} u_s s \in Z(\mathbb{C}[G])$

$$\begin{aligned} w_i(u) &= \frac{1}{n_i} Tr_{V_i}(\hat{\rho}(u)) \\ &= \frac{1}{n_i} \sum_{s \in G} u_i \chi_i(s) \end{aligned}$$

Remark **145.** $Z(\mathbb{C}[G]) = \langle u_c := \sum_{s \in G} s_c \rangle$ (conjugacy classes).

$$\dim Z(\mathbb{C}[G]) = \#\text{ of irreducible representations}$$

$\{u_c\}$ generates a subalgebra of $\mathbb{C}([G])$. Moreover, u_{c_1}, u_{c_2} is an integral linear combination of $\{u_c\}$. Thus, $\mathbb{Z}\langle u_c \rangle$ is a finite generated $\mathbb{Z}$-algebra inside $\mathbb{C}[G]$.

Definition **146.** Let R be a commutative ring of characteristic 0 (i.e $\mathbb{Z}$ is a subring). For $x \in R$, we say X is integral over $\mathbb{Z}$, if there exists $a_1, \ldots, a_n \in \mathbb{Z}$ such that

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

Remark **147.**

1. Let $R = \mathbb{C}$, the set of all elements which are integral over $\mathbb{Z}$ is called the set of algebraic integers.
2. The roots of unity are algebraic integers since $x^n = 1$.
3. Any algebraic integer in $\mathbb{Q}$ is in $\mathbb{Z}$ (By Gauss's Lemma, let $x \in \mathbb{Q}$ be an algebraic integer, there exists $a, b \in \mathbb{Z}$ such that $ax + b = 0$. Since x is integral over $\mathbb{Z}$. $a = 1$ and $x = -b \in \mathbb{Z}$.
4. $\forall g \in G$ and any (ρ, V) a representation of V, then $\chi_\rho(g)$ is an algebraic integer.
5. Algebraic integers form a ring in $\mathbb{C}$.

Proposition **148.** *Let $x \in R$ a commutative ring of characteristic 0. The following are equivalent*

1. *x is integral over $\mathbb{Z}$*
2. *$\mathbb{Z}[X]$ is finitely generated as a $\mathbb{Z}$-module.*
3. *There exists finitely generated sub $\mathbb{Z}$-Module of R which contains $\mathbb{Z}[X]$.*

Corollary **149.** *If R is finitely generated $\mathbb{Z}$ module, each element of R is integral over $\mathbb{Z}$. In particular, all $\mathbb{Z}(\langle u_I \rangle_c)$ are integral over $\mathbb{Z}$.*

Corollary **150.** *Let ρ be an irreducible representation of G of degree n with character χ. IF u is an element in $Z(\mathbb{C}[G])$ such that $u = \sum_{c,conj} a_c u_c$ and a_i are algebraic integers. Then, the number*

$$\frac{1}{n} \sum_{c} \sum_{s \in G, s \in c} a_c \chi(s)$$

is integral over $\mathbb{Z}$

Proof. Indeed, this number is just the image of $\omega : \mathbb{Z}[\mathbb{C}[G]) \to \mathbb{C}$. ∎

Theorem **151.** $\dim V||G|$

Proof. $u = \sum_{s \in G} \chi(s^{-1}) s \in Z(\mathbb{C}[G])$. Then, by the corollary,

$$\begin{aligned} \frac{1}{n} \sum_{s \in G} \chi(s^{-1}) \chi(s) & \\ &= \frac{|G|}{n} \langle \chi | \chi \rangle \\ &= \frac{|G|}{n} \in \mathbb{Q} \end{aligned}$$

Hence $\frac{|G|}{n} \in \mathbb{Z}$ and so $n||G|$. ∎

Wednesday - November 7, Burnside Thm

Theorem **152.** [Thm] *Let G be a group such that $|G| = p^a q^b$ where p, q are distinct primes, $a, b \geq 0$. Then G is solvable. s*

Lemma **153.** *If $N \rhd G$ and N is solvable, as well as G/N, then G is solvable.*

Proposition **154.** *p is prime then $\frac{1}{p}$ is not an algebraic integer. (proof on last friday)*

Proposition **155.** *Let $\lambda_1, \ldots, \lambda_n$ be the nth roots of unity. Let $a = \frac{1}{n} \sum \lambda_i$. If a is integral over $\mathbb{Z}m$ then either $a = 0$ or $\lambda_1 = \cdots = \lambda_n = a$.*

Proof. F is the splitting field of a over $\mathbb{Q}$, $f(x)$ is the minimal polynomial. $\sigma \in Gal(F/\mathbb{Q})$, $\sigma(\lambda_i)$ is a root of unity.

$$\begin{aligned} |\sigma(a)| &\leq \frac{1}{n} \sum |\sigma \lambda_i)| = 1 \\ N(a) &= \prod_{\sigma \in Gal(F/Q)} \sigma(a) \leq 1 \end{aligned}$$

Hence, the constant term of $f \in \{-1, 0, 1\}$. $|\sigma(a)| = 1 \Rightarrow \lambda_1 = \cdots = \lambda_n = a$.

∎

PROPOSITION **156.** *G a finite group. $S \in G$, $C(S) = p^r, r > 0$. Then, there exists an irreducible character χ such that $\chi(s) \neq 0, \chi(1) \ncong 0(\mod p)$. Further, $\rho(s)$ is a homothety.*

Proof.

$$\sum_{\chi} \chi(1)\chi(s) = 0$$
$$1 + \sum_{\chi n.t.} \chi(1)\chi(s) = 0$$
$$1 + \sum_{\chi, n.t} p^{u(\chi)} q_\chi \chi(s) = 0$$
$$u = \sum_{g \in C(S)} gu(g)$$
$$\frac{1}{n}\sum_{g \in G} u(g)\chi(G) = \frac{c(s)}{n}\chi(s)$$
$$n = \chi(1) \ncong 0(\mod p)$$

So there exists $k, l \in \mathbb{Z}$ such that $c(s)k + ln = 1$. Then,

$$\frac{kc(s)}{n}\chi(s) + l\chi(s) = \frac{1}{n}\chi(s)$$

Hence, $\chi(s) = \lambda$ and $\lambda^n = 1$. Thus, $\rho(s)$ is a homothety and $\ker(\rho)$ is not trivial. ∎

PROPOSITION **157.** *Let G be as in the theorem. Then, there exists $s \in G$ such that $c(S) \ncong 0(\mod q)$ $s \neq 1$, $|G| = p^a q^b = 1 + \sum c(g)$*

PROPOSITION **158.** *G as in the theorem, then G contains a non-trivial normal subgroup.*

of Theorem. By prop 5, G contains a normal subgroup N. Then, use induction, N, G/N is solvable. Lemma implies that G is Solvable. ∎

Let G be such that $|G| = pq$, $n_q \cong 1(\mod q)$, then $n_q | p$, $nq = 1$ so $p < q$.

EXAMPLE **159.**

Monday, November 11

Representations of A_n - Frank Ban

DEFINITION **160.**
$$A_n = \{\pi \in S_n : sgn\pi \cong 0 \mod 2\}$$
This yields that $[S_n : A_n] = 2$.

Definition **161.** Let V be a representation of G. Define $V' = V \otimes U'$ where U' maps H to 1 and $\bar{H}$ to 1.

Definition **162.** W a representation of H, then $\bar{W}$ by $\rho_{\bar{W}}(h) = \rho_w(tht^{-1})$ where $t \notin H$.

Proposition **163.** *V an irreducible representation of G, $W = Res_H^G V$, one of the following holds*

1. *$V \cong V'$, $W = W' \oplus W''$ such that W', W'' irreducible conjugate and not isomorphic.*

$$Ind_H^G W' = Ind_H^G W'' = V$$

2. *V not isomorphic to V', W is irreducible and self conjugate $Ind_H^G W = V \oplus V'$, each irreducible representation of H arises in one of these ways.*

Proof.

$$\begin{aligned}|G| &= \sum_{g\in G} |\chi(g)|^2 \\ 2|H| &= \sum_{h\in H} |\chi(h)|^2 + \sum_{t\in H} |\chi(t)|^2 \\ &= |H| \langle \chi|_H, \chi|_H\rangle + \sum_{t\notin H} |\chi(t)|^2\end{aligned}$$

If $\langle \chi|_H, \chi|_H\rangle = 2$ then $\chi(t) = 0$ for all $t \notin H$ then $\chi_{v'} = \chi_v$, then $V \cong V'$ which implies $W = W' \oplus W''$.

If $\langle \chi_H, \chi_H\rangle = 1$ then W is irreducible and so $\sum_{t\notin H} |\chi(t)|^2 = |H|$ which implies $V' \cong V$.

$$\begin{aligned}\mathrm{Res}_H^G(\mathrm{Ind}_H^G W) &= W \oplus \bar{W} \\ \mathrm{Ind}_H^G(\mathrm{Res}_H^G V) &= V \otimes (U \oplus U')\end{aligned}$$

∎

Remark **164.** Consider the permutation representation of G on $\mathbb{P}^1(\mathbb{F})$. This has dim q+1. This contains the trivial representation, so remove it, and let this complementary representation be V.