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PMATH 451 - Measure & Integration

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1 Some Introductory Ideas

REM **1.1.** Brief History of Ideas on Integration. Given a space X, $f : X \to \mathbb{R}$ function. For example, X a compact subset of $\mathbb{R}^n$ and f a bounded function. Given X, f want to find a number $\int_X f \in \mathbb{R}$ which "integrates together" the values of f on X. This idea goes back to Cauchy (circa 1800) who looked at when f is continuous on $[a, b] \subseteq \mathbb{R}$ (more generally, f continuous on $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$).

Riemann (circa 1850) introduced the concept of integrable functions on $[a, b]$. The Riemann Integral : $\int_a^b f(x)ds \approx \sum_{i=1}^m f(\xi_i)(t_i - t_{i-1})$. Turns out that which this approach, one cannot get too far from when f is continuous.

Stieltjes (circa 1870) extended the integral to $\int_a^b f(x)dG(x) \approx \sum_{i=1}^m f(\xi_i)(G(t_i) - G(t_{i-1}))$ where $G(x)$ is some increasing function. This leads to a richer theory of integration.

Lebesgue (circa 1900) introduced $\int_a^b f(x)d\mu(x)$ where μ is a finite positive measure on $[a, b]$. The idea is if we can measure sets, then we can integrate functions.

REM **1.2.** Review of Riemann Integral. Working in $\mathbb{R}^n$, denote $\mathcal{P}_n = \{P \subseteq \mathbb{R}^n \ : \ \exists a_1 < b_1, \ldots, a_n < b_n \in \mathbb{R}$ such that $P = (a_1, b_1] \times \cdots \times (a_n, b_n]$. Fix $P \in \mathcal{P}_n, f : P \to \mathbb{R}$ bounded. To go for "$\int_P f$", we use divisions of P, $\Delta = \{P_1, \ldots, P_r\}$ with $P_1, \ldots, P_r \in \mathcal{P}_n$ with $P_i \cap P_j = \emptyset$ for $i \neq j$ and $P_1 \cup \cdots \cup P_r = P$. For such Δ define Darboux Sums

$$U(f, \Delta) = \sum_{i=1}^r \sup_{P_i}(f) \cdot vol(P_i)$$
$$L(f, \Delta) = \sum_{i=1}^r \inf_{P_i}(f) \cdot vol(P)$$

Then have $L(f, \Delta') \leq U(f, \Delta'')$ (for every divisions Δ', Δ'' of P) hence

$$\int_P f := \sup\{L(f, \Delta') \ : \ \Delta' \text{ division of } P\} \leq \inf\{U(f, \Delta'') \ : \ \Delta'' \text{ division of P}\}$$

In the case that the lower sum and the upper sum are arbitrarily close we let the integral $\int_P f$ be equal to the common value. What key ingredients did we need for working with Darboux Sums?

- Every $P \in \mathcal{P}_n$ has $vol(P)$ and hence get a volume function $vol : \mathcal{P}_n \to [o, \infty)$.
- The Volume function is additive. If $\Delta = \{P_1, \ldots, P_r\}$ is a division of P, then $vol(P) = \sum_{i=1}^r vol(P_i)$.
- $\mathcal{P}_n$ has good properties with respect to Boolean operations. $P, Q \in \mathcal{P}_n \Rightarrow P \cap Q \in \mathcal{P}_n$ and $P, Q \in \mathcal{P}_n \Rightarrow P \backslash Q$ can be written as $P_1 \cup \cdots \cup P_r$ for some $P_1, \ldots, P_r \in \mathcal{P}_n$ pairwise disjoint. Let

$$\mathcal{A} = \{A \subseteq \mathbb{R}^n | \exists Q_1, \ldots, Q_s \in \mathcal{P}_n \text{ such that } Q_1 \cup \cdots \cup Q_s = A\}$$

Then every $A \in \mathcal{A}$ can be written as $A = P_1 \cup \cdots \cup P_r$ with $P_i \cap P_j = \emptyset$ for $i \neq j$ and for such $P_1, \ldots, P_r$, it is meaningful to define

$$vol(A) = \sum_{i=1}^{r} vol(P_i)$$

Hence, the volume function extends to $\mathcal{A}$, $vol : \mathcal{A} \to [o, \infty)$

Definition **1.3.** X a non-empty subset, a collection $\mathcal{A}$ of subsets of X is said to be an algebra of sets when it satisfies

1. $X \in \mathcal{A}$
2. If $A \in \mathcal{A}$, then $S \backslash A \in \mathcal{A}$
3. If $A, B \in A$, then $A \cup B \in A$

Definition **1.4.** X a non-empty subset, $\mathcal{A}$ an algebra of subsets of X. By additive set function on A, we understand a function $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in A$ are such that $A \cap B = \emptyset$.

2 An Idea of Lebesgue

EG **2.1.** Let X be a non-empty countable set, and suppose we are given a weight function $w : X \to [0, \infty)$, we can define

$$\mathcal{A} = \{A | A \subseteq X\}.$$

Now, we can define the additive set function $\mu : \mathcal{A} \to [0, \infty]$ defined by

$$\mu(A) = \sum_{x \in A} w(x).$$

In the case where A is infinite, we can define

$$\sum_{x \in A} w(x) = \sup \left\{ \sum_{x \in F} w(x) | F \subseteq A, F \text{ finite} \right\} \in [0, \infty].$$

We have that $\mathcal{A}$ is an algebra of sets because if $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$. In the special case when $w(x) = 1$ for every $x \in X$, μ is a counting measure (counts the number of elements).

We get another special case by fixing an element $x_0 \in X$ and considering the weight $w(x) = 1$ if $x = x_0$ and $w(x) = 0$ otherwise. In this case, μ is called the Dirac measure concentrated at x_0.

EG **2.2.** Let $\mathcal{J}$ be the following set of subsets of $\mathbb{R}$

$$\mathcal{J} = \{\emptyset\} \cup \{(a, b] | a < b \in \mathbb{R}\} \cup \{(-\infty, b] | b \in \mathbb{R}\} \cup \{(a, \infty) | a \in \mathbb{R}\} \cup \{\mathbb{R}\}$$

Next, we define

$$\mathcal{E} := \text{ the collection of finite unions of sets from } \mathcal{J}$$

Show that $\mathcal{E}$ is an algebra of sets.

REM **2.3.** If X is a set and $\mathcal{A}$ is an algebra of subsets of X. There are a few properties which follow from the definition of an algebra. We have

1. $\emptyset \in \mathcal{A}$
2. $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$ (using axiom 2, axiom 3, and then using DeMorgan's law)
3. $A, B \in \mathcal{A} \Rightarrow A \backslash B \in \mathcal{A}$ $(A \backslash B = A \cap (X \backslash B) \in A)$
4. Can take finite unions and intersections (induction)

REM **2.4.** Properties of an additive set function

1. μ is finitely additive (that is $\mu(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^n \mu(A_i)$ if $A_i \cap A_j = \emptyset$ for $i \neq j$).
2. $(A, B \in \mathcal{A}, A \subseteq B) \Rightarrow \mu(A) \leq \mu(B)$ (since $\mu(B) = \mu(A) + \mu(B \backslash A) \geq \mu(A)$)

3. The inclusion-exclusion formula :

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B).$$

Additivity implies

$$\begin{aligned} &\mu(A \cup B) = \mu(A \backslash B) + \mu(A \cap B) + \mu(B \backslash A), \text{ and} \\ &\mu(A \cap B) = \mu(A \cap B). \end{aligned}$$

Adding the above two equations together (which we can do even in the case of infinity), we get

$$\mu(A \cup B) + \mu(A \cap B) = \big(\mu(A \backslash B) + \mu(A \cap B)\big) + \big(\mu(B \backslash A) + \mu(A \cap B)\big) = \mu(A) + \mu(B).$$

4. Finite subadditivity. For every $A, B \in \mathcal{A}$ (Even if $A \cap B \neq \emptyset$) we have

$$\mu(A \cup B) \leq \mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B).$$

LEMMA **2.5.** *Let X be a non-empty set, and let $(\mathcal{A}_i)_{i \in I}$ be a family of algebras of subsets of X. Denote $\mathcal{A} := \bigcap_{i \in I} \mathcal{A}_i$. Then $\mathcal{A}$ also is an algebra of subsets of X.*

Proof. Checking the 3 axioms

1. $X \in A_i, \forall i \in I$ by axiom 1 for A_i, so $X \in \mathcal{A}$.
2. Fix $A \in \mathcal{A}$. For every $i \in I$ we have $A \in \mathcal{A}_i$, hence $X \backslash A \in \mathcal{A}_i$, for all $i \in I$ (by axiom 2 for $\mathcal{A}_i$), hence $X \backslash A \in A$.
3. Same trick as above.

∎

PROPOSITION **2.6.** *Let $\mathcal{U}$ a collection of subsets of X. Then there exists a smallest algebra $\mathcal{A}$ of subsets of X with the property that $\mathcal{U} \subseteq \mathcal{A}$.*

Proof. Let $(\mathcal{A}_i)_{i \in I}$ be the collection of all possible algebras of subsets of X which satisfy $\mathcal{A}_i \supseteq \mathcal{U}$ (Note: such algebras do exist, e.g. $\exists i_0 \in I$ such that $A_{i_0} = 2^X = \{A | A \subseteq X\}$). Put

$$\mathcal{A} := \bigcap_{i \in I} \mathcal{A}_i.$$

This is an algebra of sets by lemma 9. We also have $A \supseteq \mathcal{U}$ since $\mathcal{A}_i \supseteq \mathcal{U}$ for all $i \in I$. Finally, if B is an algebra of subsets of X such that $B \supseteq \mathcal{U}$, then $\exists i_1 \in I$ such that $B = A_{i_1}$, hence

$$B = A_{i_1} \supseteq \bigcap_{i \in I} \mathcal{A}_i = \mathcal{A}$$

∎

Definition **2.7.** Let X, Y be non-empty sets. Let $\mathcal{A}, \mathcal{B}$ be algebras of subsets of X and Y respectivly. A function $f : X \to Y$ is said to be $(\mathcal{A}, \mathcal{B})$**-measurable** to mean that $f^{-1}(B) \in \mathcal{A}, \forall B \in \mathcal{B}$.

Proposition **2.8.** *Let X, Y and $\mathcal{A}, \mathcal{B}$ as above and let $\mathcal{U} \subseteq \mathcal{B}$ be such that the algebra generated by $\mathcal{U}$ is equal to $\mathcal{B}$. Let $f : X \to Y$ be a function such that $f^{-1}(U) \in \mathcal{A}, \forall U \in \mathcal{U}$. Then f is $(\mathcal{A}, \mathcal{B})$-measurable.*

Proof. Consider the set of sets $\mathcal{S} = \{S \subseteq Y | f^{-1}(S) \in \mathcal{A}\}$ We claim that $\mathcal{S}$ is an algebra of subsets of Y. We check the various conditions

1. $Y \in \mathcal{S}$ because $f^{-1}(Y) = X \in \mathcal{A}$.
2. Suppose $S \in \mathcal{S}$. Then $f^{-1}(Y \backslash S) = X \backslash f^{-1}(S) \in \mathcal{A}$ by Axiom 2 for $\mathcal{A}$ since $f^{-1}(S) \in \mathcal{A}$.
3. Suppose $S_1, S_2 \in \mathcal{S}$. Then $f^{-1}(S_1 \cup S_2) = f^{-1}(S_1) \cup f^{-1}(S_2) \in \mathcal{A}$ by Axiom 3 for $\mathcal{A}$.

Next, we claim $\mathcal{S} \subseteq \mathcal{B}$. Our hypothesis that $f^{-1}(U) \in \mathcal{A}, \forall U \in \mathcal{U}$ says that $U \subseteq \mathcal{S}$. Since $\mathcal{S}$ is an algebra of sets, $\mathcal{S} \supseteq U$ implies that $\mathcal{S} \supseteq \mathcal{B}$ (since $\mathcal{B}$ is the smallest algebra of sets containing $\mathcal{U}$). ∎

eg **2.9.** On $\mathbb{R}$, consider the algebra ξ of half open intervals. Let $\mathcal{U} = \{(a, \infty) | a \in \mathbb{R}\}$. Then $\mathcal{U}$ generates ξ.

Definition **2.10.** Let (X, d) be a metric space, and let $\mathcal{A}$ be an algebra of subsets of X such that $D \in \mathcal{A}$ whenever $D \subseteq X$ is open. Then, every continuous function from $f : X \to \mathbb{R}$ will be $(\mathcal{A}, \xi)$ measurable.

Proposition **2.11.** *Let X be a non-empty set and let $\mathcal{A}$ be an algebra of subsets of X. Let $\mu : \mathcal{A} \to [0, \infty]$ be an additive set function with $\mu(X) < \infty$ (hence $\mu(A) < \infty$ for all $A \in \mathcal{A}$). Let $f : X \to \mathbb{R}$ be bounded and $(\mathcal{A}, \xi)$-measurable, where ξ is the algebra of half-open intervals in $\mathbb{R}$. Then,*

$$\overline{\int} f d\mu = \int_{-} f d\mu$$

Lemma **2.12.** *Let X, $\mathcal{A}$, μ, and f as in the above proposition.*

1. *Let $f : X \to \mathbb{R}$ be a bounded function. Let $\Delta = \{A_1, \ldots, A_r\}$ and $\Gamma = \{B_1, \ldots, B_s\}$ be measurable divisions of X such that Γ refines Δ (i.e. for every $1 \leq i \leq r$ there exist some $1 \leq j_n < \cdots < d_m \leq s$ such that $A_i = B_{d_1} \cup \cdots \cup B_{d_m}$). Then, $U(f, \Gamma) \leq U(f, \Delta)$ and $L(f, \Gamma) \geq L(f, \Delta)$.*

2. *For any measurable divisions Δ' and Δ'' of X, we have $L(f, \Delta') \leq U(f, \Delta'')$*

3. *$\int_{-} f d\mu \leq \int^{-} f d\mu$*

Proof. Details left as exercise (so much like calculus).

1. Direct calculation: For $A_i = B_{d_1} \cup \cdots \cup B_{d_m}$ use

$$\sup_{A_i}(f) \geq \sup_{B_{j_k}}(f)$$
$$\inf_{A_i}(f) \leq \inf_{B_{j_k}}(f)$$

   and the additivity of μ.

2. Use a common refinement ($\mathcal{A}$ closed under finite intersections) Γ of Δ' and Δ'', then

$$L(f, \Delta') \leq L(f, \Gamma) \leq U(f, \Gamma) \leq U(f, \Delta'').$$

3. We have

$$\begin{aligned} \int_{-} f d\mu &:= \sup\{L(f, \Delta) | \Delta \text{ a measurable division}\} \\ &\leq \inf\{U(f, \Delta) | \Delta \text{ a measurable division}\} \quad =: \quad \int^{-} f d\mu. \end{aligned}$$

∎

Claim. Let X, $\mathcal{A}$, μ, and f as in the above proposition. Given $\epsilon > 0$, one can find a measurable division Δ of X such that $U(f, \Delta) - L(f, \Delta) < \epsilon$.

Proof. Fix $\epsilon > 0$. Since f is bounded, can fix $\alpha < \beta \in \mathbb{R}$ such that $\alpha < f(x) < \beta, \forall x \in X$. Pick $k \in \mathbb{N}$ such that

$$k > \frac{\mu(X)(\beta - \alpha)}{\epsilon}$$

and write $(\alpha, \beta] = J_1 \cup J_2 \cup \cdots \cup J_k$ where

$$J_i = \left(\alpha + \frac{(i-1)(\beta-\alpha)}{k}, \alpha + \frac{i(\beta-\alpha)}{k}\right], \text{ for } 1 \leq i \leq k.$$

For every $1 \leq i \leq k$, put $A_i = f^{-1}(J_i) = \{x \in X | f(x) \in J_i\}$.

Observe that $A_i \in \mathcal{A}$ because f is $(\mathcal{A}, \xi)$-measurable.

Observe that for $1 \leq i \leq j \leq k$, we have $A_i \cap A_j = \{x \in X | f(x) \in J_i \text{ and } f(x) \in J_j\} = \emptyset$ and also $A_1 \cup \cdots \cup A_k = X$ (because for every $x \in X$, have $f(x) \in (\alpha, \beta]$). Hence, $\Delta = \{A_i | 1 \leq i \leq k, A_i \neq \emptyset\}$ is a measurable division of X. For this Δ, have

$$\begin{aligned} U(f, \Delta) &= \sum_{1 \leq i \leq k, A_i \neq \emptyset} \mu(A_i) \sup_{A_i}(f) \\ &\leq \sum_{1 \leq i \leq k, A_i \neq \emptyset} \mu(A_i) \left(\alpha + \frac{i(\beta-\alpha)}{k}\right) \end{aligned}$$

So we have

$$U(f, \Delta) \leq \sum_{i=1}^{k} \mu(A_i) \left(\alpha + \frac{i(\beta-\alpha)}{k}\right), \qquad (*)$$

and similarly we can get

$$L(f, \Delta) \geq \sum_{i=1}^{k} \mu(A_i) \left(\alpha + \frac{(i-1)(\beta-\alpha)}{k}\right). \qquad (**)$$

Subtract $(**)$ out of $(*)$ to get

$$\begin{aligned} U(f, \Delta) - L(f, \Delta) &\leq \sum_{i=1}^{n} \mu(A_i) \frac{\beta-\alpha}{k} \\ &= \frac{\beta-\alpha}{k} \sum_{i=1}^{k} \mu(A_i) \\ &= \frac{(\beta-\alpha)\mu(X)}{k} \\ &< \epsilon \end{aligned}$$

∎

CLAIM. $\int_{-} f d\mu = \int^{-} f d\mu$.

Proof. For every $\epsilon > 0$, pick Δ as in above claim and write

$$L(f, \Delta) \leq \int_{-} f d\mu \leq \int^{-} f d\mu \leq U(f, \Delta)$$

Hence,

$$0 \leq \int^{-} f d\mu - \int_{-} f d\mu \leq U(f, \Delta) - L(f, \Delta) < \epsilon$$

Since $\epsilon > 0$ was arbitrary, $\int^{-} f d\mu - \int_{-} f d\mu = 0$. ∎

EG **2.13.** Let $X = [0,1]$. Suppose we have an algebra $\mathcal{A}$ of subsets of X such that $D \in \mathcal{A}$ for all $D \subseteq X$ open and that we have an additive set function $\mu : \mathcal{A} \to [0,1]$ such that $\mu((a,b)) = b - a$ and $\mu(\{t\}) = 0$. Then,

1. Every continuous function $f : X \to \mathbb{R}$ is bounded and (A, ξ)-measurable. We can define $\int f d\mu$ for them.

2. A step function is still (A, ξ) - measurable so we can still do $\int f d\mu$.

3. What about the function g which is 0 on irrationals and 1 on rationals?

   The answer depends on the choice of $\mathcal{A}$.

   We can see immediately that g is $(\mathcal{A}, \xi)$-measurable if and only if $[0,1] \cap \mathbb{Q} \in \mathcal{A}$. For example, if $\mathcal{A}$ is the algebra of subsets of $[0,1]$ which is generated by the open sets, then $[0,1] \cap \mathbb{Q} \notin \mathcal{A}$. The second great idea of lebesgue; look at algebras $\mathcal{A}$ which are closed under countable unions (σ-algebras)! If $\mathcal{A}$ is a σ-algebra, then $\mathbb{Q} \cap [0,1] \in \mathcal{A}$ because it is countable.

3 σ-Algebras and Positive Measures

Definition **3.1.** Let X be a non-empty set. A set $\mathcal{A}$ of subsets of X is said to be a σ-algebra when it satisfies

1. $X \in \mathcal{A}$
2. $A \in \mathcal{A} \Rightarrow X \backslash A \in \mathcal{A}$
3. If $\{A_n\}_{n=1}^{\infty} \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} A_n \in \mathcal{A}$

rem **3.2.**

1. If $\mathcal{A}$ is a σ-algebra, then in particular, $\mathcal{A}$ is also an algebra of subset of X. Hence, all observations from Lecture 1 about algebras of sets hold for σ-algebras.

2. If $\mathcal{A}$ is a σ-algebra of X, we can safely do countable intersections of sets in $\mathcal{A}$ (Apply Demorgan's Law to the complement of the union).

Definition **3.3.** Let X be a non-empty set and let $\mathcal{A}$ be a σ-algebra of subsets of X. A function $\mu : \mathcal{A} \to [0, \infty]$ is said to be a **positive measure** when

1. $\mu(\emptyset) = 0$.
2. If $(A_n)_{n=1}^{\infty} \in \mathcal{A}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{n=1}^{\infty} \mu(A_n) = \sup\left\{\sum_{n=1}^{N} \mu(A_n) | N \in \mathbb{N}\right\} = \lim_{n\to\infty} \sum_{n=1}^{N} \mu(A_n).$$

If $\mu(X) < \infty$ then we say that μ is a finite positive measure. If $\mu(X) = 1$ then say that μ is a probability measure.

rem **3.4.** Let X, $\mathcal{A}$, and μ be as in the above definition, then μ is an additive set function in the sense of Lecture 1. Thus, results proved about additive set functions also hold for positive measures.

- μ is increasing ($A \subseteq B \Rightarrow \mu(A) \subseteq \mu(B)$)
- Inclusion-Exclusion
- Finite Sub-additivity:

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mu(A_i)$$

  $\forall n \in \mathbb{N}, \forall A_1, \ldots, A_n$.

A natural question is "shouldn't we have $\mu\left(\bigcup_{i=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$?". We will prove this, but first need a fact called "continuity of μ along chains".

Definition **3.5.** A **Measurable Space** is a pair $(X, \mathcal{A})$ where X is a non-empty set and $\mathcal{A}$ is a σ-algebra of subsets of X. A **Measure Space** is a triple $(X, \mathcal{A}, \mu)$ where $(X, \mathcal{A})$ is a measurable space and μ is a positive measure. If $\mu(X) = 1$, $(X, \mathcal{A}, \mu)$ is called a **Probability Space**.

rem **3.6.** The idea of using σ-algebras comes from Lebesgue around the year 1900. Why was this a significant idea? Cantor's proof that $\mathbb{R}$ is uncountable was in 1875. We will return the pros and cons for using σ-algebras and σ-additivity.

Proposition **3.7.** [“Continuity” Along Chains] *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

1. *If $(B_n)_{n=1}^{\infty}$ are sets in $\mathcal{A}$ such that $B_1 \subseteq B_2 \subseteq \cdots$, then $\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \mu(B_n)$*
2. *If $(C_n)_{n=1}^{\infty}$ are in $\mathcal{A}$ such that $C_1 \supseteq C_2 \supseteq \cdots$ and if $\exists n_0 \in \mathbb{N}$ such that $\mu(C_{n_0}) < \infty$ then $\mu\left(\bigcap_{i=1}^{\infty} C_n\right) = \lim_{n\to\infty} \mu(C_n)$.*

Proof. 1. Define $A_1 = B_1, A_2 = B_2 \backslash B_1, \cdots, A_n = B_n \backslash B_{n-1}, \cdots$. Then, by construction, the A_i are pairwise disjoint sets in $\mathcal{A}$ with $\bigcup_{i=1}^{n} A_i = B_n$ and $\bigcup_{i=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$ (immediate Boolean Algebra - Check!). So then

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} B_n\right) &= \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \\
&= \sum_{n=1}^{\infty} \mu(A_n) && (\sigma\text{-add}) \\
&= \lim_{n\to\infty} \left(\sum_{n=1}^{N} \mu(A_n)\right) \\
&= \lim_{n\to\infty} \mu\left(\bigcup_{n=1}^{N} A_n\right) = \lim_{n\to\infty} \mu(B_n).
\end{aligned}$$

2. We want $\mu(\bigcap_{n=1}^{\infty} C_n) = \lim_{n\to\infty} \mu(C_n)$ for $C_1 \supseteq C_2 \supseteq \cdots$ where $\exists n_0 \in \mathbb{N}$ with $\mu(C_{n_0}) < \infty$. What if we allow $\mu(C_n) = \infty, \forall n \in \mathbb{N}$, what goes wrong?

Take $X = \mathbb{Z}$ and let $\mathcal{A}$ be the powerset of X. Let μ be the counting measure. Now take $C_n = \{m \in \mathbb{Z} | m \leq n\}$.

Proof starts here - Without loss of generality, assume $\mu(C_1) < \infty$. Let $B_n := C_1 \backslash C_n$ for all $n \in \mathbb{N}$. Then we have $\emptyset = B_1 \subseteq B_2 \subseteq \cdots$ and $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} (C_1 \backslash C_n) = C_1 \backslash \left(\bigcap_{n=1}^{\infty} C_n\right)$.

Using part (1) of the proposition for $(B_n)_{n=1}^{\infty}$. Get

$$\begin{aligned}\mu\left(\bigcup_{n=1}^{\infty} B_n\right) &= \lim_{n\to\infty} \mu(B_n) \Rightarrow \\ \mu\left(C_1 \backslash \bigcap_{n=1}^{\infty} C_n\right) &= \lim_{n\to\infty} \mu\left(C_1 \backslash C_n\right) \Rightarrow \\ \mu(C_1) - \mu\left(\bigcap_{n=1}^{\infty} C_n\right) &= \lim_{n\to\infty} \left(\mu(C_1) - \mu(C_n)\right).\end{aligned}$$

Here we use the fact that $\mu(C_1) < \infty$! We have that $\mu(C_1) = \mu(C_n) + \mu(C_1 \backslash C_n)$ and we use the arithmetic of $[0, \infty)$ not in $[0, \infty]$. Thus,

$$\lim_{n\to\infty} \left(\mu(C_1) - \mu(C_n)\right) = \mu(C_1) - \lim_{n\to\infty} \mu(C_n) \Rightarrow \mu\left(\bigcap_{n=1}^{\infty} C_n\right) = \lim_{n\to\infty} \mu(C_n).$$

∎

Corollary **3.8.** *Let $(X, \mathcal{A}, \mu)$ be a measure space. Then μ is countably subadditive. That is, for any $(A_n)_{n=1}^{\infty} \in \mathcal{A}$, we have*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu(A_n)$$

Proof. Let $B_n = A_1 \cup \cdots \cup A_n$ for all $n \in \mathbb{N}$. Then $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_n \subseteq \cdots$ in $\mathcal{A}$ with $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$. For every $n \in \mathbb{N}$ we have

$$\begin{aligned}\mu(B_n) &= \mu(A_1 \cup \cdots \cup A_n) \\ &\le \sum_{i=1}^{n} \mu(A_i) \\ &\le \sum_{i=1}^{\infty} \mu(A_i)\end{aligned}$$

because μ is in particular an additive set function as in lecture 1. Note that we have $\mu(B_n) \le \sum_{i=1}^{\infty} \mu(A_i)$ for all $n \in \mathbb{N}$. So we get

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \mu(B_n) \le \sum_{i=1}^{\infty} \mu(A_i).$$

∎

4 The Borel σ-Algebra

LEMMA **4.1.** *Let X be a non-empty set and let $(A_i)_{i\in I}$ be a family of σ-algebras on X. Let $\mathcal{A} = \bigcap_{i\in I} \mathcal{A}_i = \{A \subseteq X | A \in \mathcal{A}_i, \forall i \in I\}$. Then $\mathcal{A}$ is a σ-algebra.*

Proof. This is exactly as the proof from lecture 1 for algebras of sets. ∎

PROPOSITION **4.2.** *Let X be a non-empty set and let $\mathcal{U}$ be a collection of subsets of X. Then there exists a smallest σ-algebra, $\mathcal{A}$, on X such that $\mathcal{A} \supseteq \mathcal{U}$. This means that*

1. *$\mathcal{A}$ is a σ-algebra on X and $\mathcal{A} \supseteq \mathcal{U}$*
2. *Whenever $\mathcal{B}$ is some σ-algebra of subsets of X such that $B \supseteq \mathcal{U}$, it follows that $\mathcal{A} \subseteq \mathcal{B}$.*

Proof. This is exactly as the proof of proposition 10 in lecture 1: Let $(\mathcal{A}_i)_{i\in I}$ be the family of all σ-algebras which contain $\mathcal{U}$. Take $\mathcal{A} := \bigcap_{i\in I} \mathcal{A}_i$ and use lemma 26. ∎

DEFINITION **4.3.** This smallest σ-algebra $\mathcal{A}$ is called the **σ-algebra generated by $\mathcal{U}$**.

DEFINITION **4.4.** Let (X, d) be a metric space. Let $\mathcal{D} := \{D \subseteq X | D \text{ is open}\}$. The σ-algebra generated by $\mathcal{D}$ is called the **Borel σ-algebra** of (X, d) (Usually denoted by $\mathcal{B}_X$). The sets $B \in \mathcal{B}_X$ are called Borel sets.

REM **4.5.** We saw in lecture 2 that if (X, d) a metric space, we would like to look at bounded functions $f : X \to \mathbb{R}$ which are $(\mathcal{A}, \xi)$-measurable, where $\mathcal{A}$ is a σ-algebra which contains the open sets. The Borel σ-algebra $\mathcal{B}_X$ is the smallest possible such $\mathcal{A}$.

Why not allow a bigger $\mathcal{A}$? If $\mathcal{A}$ is bigger, it becomes harder to construct the positive measure on $\mathcal{A}$. Sometimes, it is not even possible to construct a good positive measure on large σ-algebras.

EG **4.6.** Suppose $X = \mathbb{Z}$. Let $\mathcal{A} = \{A | A \subseteq \mathbb{Z}\}$ and let μ be the counting measure. Then, μ is a translative invariant measure in the sense that $\mu(A + m) = \mu(A), \forall A \subseteq \mathbb{Z}, m \in \mathbb{Z}$, where $A + m := \{a + m | a \in A\}$.

EG **4.7.** Let $X = \mathbb{R}$, and take $L_{\mathbb{R}} = \{A | A \subseteq \mathbb{R}\}$. There exists no positive measure on $L_{\mathbb{R}}$ such that $(A + t) = \mu(A)$ for all $A \in L_{\mathbb{R}}$ and all $t \in \mathbb{R}$ and such that $\mu([0, 1]) = 1$. However, if we change $L_{\mathbb{R}}$ to $\mathcal{B}_{\mathbb{R}}$, then it does work.

REM **4.8.** From Yes $\neq$ No, it follows that $\mathcal{B}_{\mathbb{R}} \neq L_{\mathbb{R}}$. I.e. it follows that there are subsets $\mathcal{A} \subseteq \mathbb{R}$ which are not Borel. Specifically, by using the axiom of choice, one can find $E \subseteq [0, 1]$ such that

1. for every $t \in \mathbb{R}$, there exist $a \in E$ such that $t - a \in \mathbb{Q}$.

2. $a, b \in E$ and $a \neq b$, then $a - b \notin \mathbb{Q}$.

(Have an equivalence relation on $\mathbb{R}$ where $x \sim y$ if $x - y \in \mathbb{Q}$).

REM **4.9.** The Borel σ-algebra $\mathcal{B}_X$ is the framework for integration on (X, d). We still have the issue of how to construct positive measures $\mu : \mathcal{B}_X \to [0, \infty]$. The preferred method is one of Caratheodory. Find a nice algebra $\mathcal{A}$ of subsets of X, such that $\mathcal{A}$ generates $\mathcal{B}_X$ as a σ-algebra. Define $\mu_0 : \mathcal{A} \to [0, \infty]$, then extend μ_0 from $\mathcal{A}$ to $\mathcal{B}_X$.

5 The Caratheodory Extension Theorem

DEFINITION **5.1.** Let X be a non-empty set and let $\mathcal{A}$ be an algebra of subsets of X. Let μ_0 be an additive set function on $\mathcal{A}$. We say that μ_0 is a **pre-measure** when

- If $\{A_n\}_{n=1}^{\infty}$ from $\mathcal{A}$ are pairwise disjoint and if $\bigcup_{n=1}^{\infty} A_n \in A$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu_0(A_n).$$

REM **5.2.** Clearly, the above condition (we will call it pre-σ-additivity) is necessary if there is to be a chance that μ_0 extends to a positive measure on a σ-algebra $\mathcal{B} \supseteq \mathcal{A}$. We can rephrase the condition above as "If $\{A_n\}_{n=1}^{\infty} \in \mathcal{A}$ such that $A \subseteq \bigcup_{n=1}^{\infty} A_n$ for some $A \in \mathcal{A}$, then $\mu_0(A) \leq \sum_{n=1}^{\infty} \mu_0(A_n)$".

THEOREM **5.3.** [CARATHEODORY] *Let X be a non-empty set, let $\mathcal{A}$ be an algebra of subsets of X, and let μ_0 be a pre-measure on $(X, \mathcal{A})$. Then one can extend μ_0 to a positive measure μ on $(X, \mathcal{B})$ where $\mathcal{B}$ is the σ-algebra generated by $\mathcal{A}$.*

Proof. Use the trick of the outer measure. (A more detailed proof will follow) ∎

DEFINITION **5.4.** Let X be a non-empty set and let $\mathcal{L} = \{A | A \subseteq X\}$. A set function $\mu^* : \mathcal{L} \to [0, \infty]$ is said to be an an **outer measure** when it satisfies

1. $\mu^*(\emptyset) = 0$
2. $E \subseteq F \subseteq X \Rightarrow \mu^*(E) \leq \mu^*(F)$
3. $\mu^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$

LEMMA **5.5.** ["TRICK OF OUTER MEASURE"] *Let μ^* be an outer measure. We say that $G \subseteq X$ is "good for μ^*" when it has the following property: Whenever $E \subseteq G$ and $F \subseteq X \backslash G$ it follows that $\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$. Let $\mathcal{G}$ be subsets of X which are "good for μ^*". Then $\mathcal{G}$ is a σ-algebra on X and $\mu^*|_{\mathcal{G}}$ is a positive measure.*

Wednesday, September 26

Proof (of Caratheodory's Theorem). Recall that, by assumption, X is a non-empty set, $\mathcal{A}$ is an algebra of subsets of X, and μ_0 is a pre-measure on $(X, \mathcal{A})$. Let $L = \{E | E \subseteq X\}$ and define $\mu^* : L \to [0, \infty]$ to be such that for every $E \subseteq X$ we have

$$\mu^*(E) = \inf\left\{ \sum_{n=1}^{\infty} \mu_0(A_n) \,\middle|\, (A_n)_{n=1}^{\infty} \text{ Sets from } \mathcal{A} \text{ such that } \bigcup_{n=1}^{\infty} A_n \supseteq E \right\}$$

We will prove 3 claims about μ^*.

1. μ^* is an outer measure
2. $\mu^*(A) = \mu_0(A), \forall A \in \mathcal{A}$
3. Every $A \in \mathcal{A}$ is "good for μ^*".

Assume the claims are proven. Let $\mathcal{G} = \{G \subseteq X | G \text{ is good for } \mu^*\}$. Observe that $\mathcal{A} \subseteq \mathcal{G}$ (claim 3). $\mathcal{G}$ is a σ-algebra (by lemma). This implies that $\mathcal{B} \subseteq \mathcal{G}$ since $\mathcal{B}$ is the smallest σ-algebra containing $\mathcal{A}$.

Denote $\mu = \mu^*|_{\mathcal{B}}$, so $\mu : \mathcal{B} \to [0, \infty]$ is defined by $\mu(B) = \mu^*(B)$, for all $B \in \mathcal{B}$. Observe that μ extends μ_0 (for $A \in \mathcal{A}$ we have $\mu(A) = \mu^*(A) = \mu_0(A)$ by claim 2).

Finally observe that $\mu^*|_{\mathcal{G}}$ is a positive measure by lemma, this implies that $\mu^*|_{\mathcal{B}}$ is a positive measure on $\mathcal{B}$. This ends the proof modulo the verifications of claims 1,2,3.

∎

Claim (1). μ^* is an outer measure.

Proof. The first 2 properties of an outer measure are left as an exercise. Let's check property 3. Fix $E_1, \dots, E_n, \dots \subseteq X$ for which we will verify that $\mu^*\left(\bigcup_{n=1}^\infty E_n\right) \leq \sum_{n=1}^\infty \mu^*(E_n)$. If $\sum_{n=1}^\infty \mu^*(E_n) = \infty$, then it is clear. So assume $\sum_{n=1}^\infty \mu^*(E_n) < \infty$, it suffices to verify that

$$\mu^*\left(\bigcup_{n=1}^\infty E_n\right) \leq \epsilon + \sum_{n=1}^\infty \mu^*(E_n)$$

For every $\epsilon > 0$. So we fix an $\epsilon > 0$ as well. Look at

$$\mu^*(E_1) = \inf \left\{ \sum_{m=1}^\infty \mu_0(A_m) \,\middle|\, A_1, \dots, A_m, \dots \in \mathcal{A} \wedge \bigcup_{m=1}^\infty A_m \supseteq E_1 \right\}$$

By definition of infimum, we can pick $A_1^{(1)}, A_2^{(1)}, \dots, A_m^{(1)}, \dots \in \mathcal{A}$ with $\bigcup_{m=1}^\infty A_m^{(1)} \supseteq E_1$ and such that

$$\sum_{m=1}^\infty \mu_0(A_m^{(1)}) < \mu^*(E_1) + \frac{\epsilon}{2}.$$

In general for every $n \in \mathbb{N}$ pick $A_1^{(n)}, A_2^{(n)}, \dots, A_m^{(n)}, \dots \in \mathcal{A}$ such that $\bigcup_{m=1}^\infty A_m^{(n)} \supseteq E_n$ and such that

$$\sum_{m=1}^\infty \mu_0(A_m^{(n)}) < \mu^*(E_n) + \frac{\epsilon}{2^n}.$$

Note that $(A_m^{(n)})_{(m,n)\in\mathbb{N}^2} \in \mathcal{A}$ is a countable family, and that we also have

$$\bigcup_{(m,n)\in\mathbb{N}^2} A_m^{(n)} = \bigcup_{n=1}^\infty \left(\bigcup_{m=1}^\infty A_m^{(n)} \right) \supseteq \bigcup_{n=1}^\infty E_n.$$

From the definition of μ^* as inf, it follows that

$$\begin{aligned}\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) &\leq \sum_{(m,n)\in\mathbb{N}^2}^{\infty} \mu_0(A_m^{(n)}) \\ &= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} \mu_0(A_m^{(n)})\right) \\ &\leq \sum_{n=1}^{\infty}\left(\mu^*(E_n) + \frac{\epsilon}{2}\right) \\ &= \left(\sum_{n=1}^{\infty} \mu^*(E_n)\right) + \epsilon\end{aligned}$$

∎

Claim (2). $\mu^*(A) = \mu_0(A)$, for all $A \in \mathcal{A}$

Proof. Recall the remark about pre-σ-additivity which says if A and $(A_n)_{n=1}^{\infty}$ are in $\mathcal{A}$ and if $A \subseteq \bigcup_{n=1}^{\infty} A_n$, then $\mu_0(A) \leq \sum_{n=1}^{\infty} \mu_0(A_n)$. But then, for $A \in \mathcal{A}$, we have

$$\mu_0(A) \leq \inf\left\{\sum_{n=1}^{\infty} \mu_0(A_n) \middle| (A_n)_{n=1}^{\infty} \in \mathcal{A} \wedge \bigcup_{n=1}^{\infty} A_n \supseteq A\right\}.$$

"$\leq$" follows by definition of infimum. So we got $\mu^*(A) \geq \mu_0(A)$, for all $A \in \mathcal{A}$.

For the opposite inequality, look at the cover $A \subseteq \bar{A}_1 \cup \bar{A}_2 \cup \dots$ where $\bar{A}_1 = A$ ad $\bar{A}_n = \emptyset, \forall n \geq 2$. Thus, we have

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu_0(\bar{A}_n) = \mu_0(A) + 0 + 0 + \dots$$

Which implies that $\mu^*(A) \leq \mu_0(A)$, for all $A \in \mathcal{A}$. So $\mu^*(A) = \mu_0(A)$, for all $A \in \mathcal{A}$. ∎

Friday, September 28

Let $\mathcal{L} = \{E | E \subseteq X\}$ and let $\mu^* : \mathcal{L} \to [0, \infty]$, where for $E \subseteq X$, we put

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu_0(A_n) \middle| (A_n)_{n=1}^{\infty} \in \mathcal{A} \text{ with } \bigcup_{n=1}^{\infty} A_n \supseteq E\right\}.$$

We saw on Wednesday that μ^* is an outer measure which extends μ_0. Now to prove the last of our 3 claims.

Claim (3). Every $G \in \mathcal{A}$ is "good for μ^*" (Which means that for any sets $E \subseteq G$ and $F \subseteq X \backslash G$ we have that $\mu^*(E \cup F) = \mu^*(F) + \mu^*(F)$).

Proof. Pick a set $G \in \mathcal{A}$ and pick $E, F \subseteq X$ where $E \subseteq G$ and $F \subseteq X \backslash G$. We must show that $\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$. The inequality $\leq$ is obvious. We need to prove "$\geq$". If $\mu^*(E \cup F) = \infty$, then "$\geq$" is trivial, hence will assume that $\mu^*(E \cup F) < \infty$.

It suffices to prove that $\mu^*(E) + \mu^*(F) \leq \mu^*(E \cup F) + \epsilon, \forall \epsilon > 0$. Fix $\epsilon > 0$ for which we verify the above condition. By definition of μ^*, we can find $(A_n)_{n=1}^{\infty} \in \mathcal{A}$ such that $E \cup F \subseteq \bigcup_{n=1}^{\infty} A_n$ and such that $\sum_{n=1}^{\infty} \mu_0(A_n) < \mu^*(E \cup F) + \epsilon$. Observe that

$$E = (E \cup F) \cap G \subseteq \left(\bigcup_{n=1}^{\infty} A_n \right) \cap G = \bigcup_{n=1}^{\infty} (A_n \cap G)$$

with $A_n \cap G \in \mathcal{A}$, for all $n \geq 1$, because $A_n, G \in \mathcal{A}$ and $\mathcal{A}$ is an algebra of sets. The definition of μ^* implies that

$$\mu^*(E) \leq \sum_{n=1}^{\infty} \mu_0(A_n \cap G)$$

. Likewise, write $F = (E \cup F) \cap (X \backslash G)$, to get

$$\mu^*(F) \leq \sum_{n=1}^{\infty} \mu_0(A_n \cap (X \backslash G))$$

but then, adding the two inequalities together, we have

$$\mu^*(E) + \mu^*(F) \leq \sum_{n=1}^{\infty} \Big(\mu_0(A_n \cap G) + \mu_0\big(A_n \cap (X \backslash G)\big) \Big) = \sum_{n=1}^{\infty} \mu_0(A_n) < \mu^*(E \cup F) + \epsilon$$

as required. ∎

Thus, we have proven the three required claims and so the proof of Caratheodory's theorem holds.

6 An Application of Caratheodory / Lebesgue Stieljes Measures

Our goal is to try to describe all positive measures μ on $\mathcal{B}_{\mathbb{R}}$, the Borel σ-algebra of $\mathbb{R}$, such that μ is finite on compact sets. These measures are also called **Lebesgue-Stieltjes** measures.

Definition **6.1.** Let $\mu : \mathcal{B}_{\mathbb{R}} \to [0,1]$ be a probability measure. The probability distribution function of μ is $F : \mathbb{R} \to \mathbb{R}$ defined by $F(t) = \mu\big((-\infty, t]\big)$ where F has the following properties:

- F is increasing; $s \leq t \Rightarrow F(s) \leq F(t)$.
- F is continuous from the right; if we have a decreasing sequence $t_n \to a$ from above, then $F(t_n) \to F(a)$ from above.

Why? Because of the continuity of μ along decreasing chains, we have

$$\lim_{n\to\infty} F(t_n) = \lim_{n\to\infty} \mu\big((-\infty, t_n]\big) = \mu\left(\bigcap_{n=1}^{\infty} (-\infty, t_n]\right) = \mu\big((-\infty, a]\big) = F(a).$$

What about increasing convergent sequences $t_n \to b$. They have $\lim_{n\to\infty} F(t_n) \leq F(b)$ where the inequality may be strict. Why? Because

$$\lim_{n\to\infty} F(t_n) = \lim_{n\to\infty} \mu\big((-\infty, t_n]\big) = \mu\left(\bigcup_{n=1}^{\infty} (-\infty, t_n]\right) = \mu\big((-\infty, b)\big) = F(b) - \mu(\{b\}).$$

So equality holds if and only if b is not an atom for μ (where a singleton has positive measure).

Definition **6.2.** A function $F : \mathbb{R} \to \mathbb{R}$ is said to be a **Cadlag Function** when it is increasing (in the above sense) and it is continuous from the right at every $a \in \mathbb{R}$. Cadlag is an acronym for "continue à droite, limite à gauche". If μ is a probability measure, then the distribution function is Cadlag.

rem **6.3.** What if μ is the Lebesgue measure? Here $\mu\big((-\infty, t]\big) = \infty$ for all $t \in \mathbb{R}$.

Lemma **6.4.** *Let $\mu : \mathbb{R} \to [0, \infty]$ be a positive measure with $\mu(K) < \infty$ for all compact $K \subseteq \mathbb{R}$. There exists a function $G : \mathbb{R} \to \mathbb{R}$ that is uniquely determined such that $G(0) = 0$ and $\mu((a, b]) = G(b) - G(a), \forall a < b \in \mathbb{R}$. This G is a Cadlag function. If μ is a probability measure, then $G(t) = F(t) - F(0)$.*

Monday, October 1

rem **6.5.** In the special case when $\mu(\mathbb{R}) = 1$, we have $G(t) = F(t) - F(0)$ where F is the distribution function of μ, $F(t) = \mu((-\infty, t])$. In the special case when μ is the Lebesgue measure on $\mathbb{R}$, then $G(t) = t$.

Proof. Define G by putting

$$G = \begin{cases} \mu((0,t]) & \text{If } t > 0 \\ 0 & \text{If } t = 0 \\ -\mu((t,0]) & \text{If } t < 0 \end{cases}$$

Then G has all the required properties ("cadlag" checked exactly as we did for distribution functions on Friday - Exercise). To prove uniqueness of G, suppose $\tilde{G} : \mathbb{R} \to \mathbb{R}$ is such that $\tilde{G}(b) - \tilde{G}(a) = \mu((a,b])$, for all $a < b \in \mathbb{R}$. Then, for $t > 0$, we get

$$\tilde{G}(t) = \tilde{G}(t) - \tilde{G}(0) = \mu((0,t]) = G(t),$$

and for $t < 0$, we get

$$\tilde{G}(t) = -(\tilde{G}(0) - \tilde{G}(t)) = -\mu((t,0]) = G(t).$$

Hence, $\tilde{G} = G$. ∎

Definition **6.6.** Let $\mu : \mathcal{B}_{\mathbb{R}} \to [0,\infty]$. If $\mu(K) < \infty$ for all compact subsets of $\mathbb{R}$, we say that μ is a **Lebesgue-Stieltjes Measure**. If μ is a Lebesgue-Stieltjes measure, then the function G given by the previous lemma for this μ will be denoted as G_μ and we call it the the **Centered Stieltjes Function** of μ.

Proposition **6.7.** *Suppose we have 2 Lebesgue-Stiletjes measure μ, ν such that $G_\mu = G_\nu$, then $\mu = \nu$.*

Proof.

Claim (1). $\mu((a,b]) = \nu((a,b])$ for all $a < b \in \mathbb{R}$.
By definition, $\mu((a,b]) = G_\mu(b) - G_\mu(a) = G_\nu(b) - G_\nu(a) = \nu((a,b])$.

Claim (2). $\mu(I) = \nu(I)$ for every open interval $I \subseteq \mathbb{R}$.

If I is bounded, $I = (a,b)$ with $a < b$ in $\mathbb{R}$. Take $(t_n)_{n=1}^\infty \in (a,b)$ such that $t_n \to b$ from below. We get $(a,b) = \bigcup_{n=1}^\infty (a,t_n]$. Then, we apply the continuity of μ along an increasing chain to get

$$\mu((a,b)) = \lim_{n\to\infty} \mu((a,t_n]) = \lim_{n\to\infty} \nu((a,t_n]) = \nu((a,b)).$$

What if I was unbounded, say $I = (-\infty, b)$ for some $b \in \mathbb{R}$? Proceed in the same way, with $I = \bigcup_{n=1}^\infty (s_n, t_n]$ with $t_n \to b$ from below and $s_n \to -\infty$.

Claim (3). $\mu(D) = \nu(D)$, for all open $D \subseteq \mathbb{R}$.

We know from previous courses that an open set can be written as the disjoint union of open intervals: $D = \bigcup_{n=1}^\infty I_n$ (I_n pairwise disjoint) and every I_n is either an open interval or it is empty (the non-empty sets are connected components of D_n). Then,

$$\mu(D) = \sum_{n=1}^\infty \mu(I_n) = \sum_{n=1}^\infty \nu(I_n) = \nu(D).$$

We applied σ-additivity, claim 2, and then σ additivity again. We will now invoke the property of "Closed Regularity" as in Problem 4(b) of Homework 2.

Claim (4). $\mu(B) = \nu(B)$ for every bounded set $B \in \mathcal{B}_{\mathbb{R}}$.

Pick $n \in \mathbb{N}$ such that $B \subseteq (-n, n)$ and consider the metric space (X, d) where we have $X = (-n, n)$ and d is the usual distance on $\mathbb{R}$. Have $\mathcal{B}_X = \{B \in \mathcal{B}_{\mathbb{R}} | B \subseteq X\}$. Let $\tilde{\mu} = \mu|_{\mathcal{B}_X}$ and $\tilde{\nu} = \nu|_{\mathcal{B}_X}$. Then $\tilde{\mu}, \tilde{\nu}$ are finite positive measures on $\mathcal{B}_X$. We can write

$$\begin{aligned}
\mu(B) &= \tilde{\mu}(B) \\
&= \inf\{\tilde{\mu}(D) | D \subseteq X \text{ open, such that } D \supseteq B\} \\
&= \inf\{\mu(D) | D \subseteq \mathbb{R} \text{ open, such that } B \subseteq D \subseteq X\} \\
&= \inf\{\nu(D) | D \subseteq \mathbb{R} \text{ open, such that } B \subseteq D \subseteq X\} \\
&= \inf\{\tilde{\nu}(D) | D \text{ open in } X, D \supseteq X\} \\
&= \tilde{\nu}(B) = \nu(B).
\end{aligned}$$

Claim (5). $\mu(B) = \nu(B)$, for all $B \in \mathcal{B}_{\mathbb{R}}$

Write $B = \bigcup_{n=1}^{\infty} (B \cap (-n, n))$, an increase chain. We get

$$\mu(B) = \lim_{n\to\infty} \mu(B \cap (-n, n)) = \lim_{n\to\infty} \nu(B \cap (-n, n)) = \nu(B).$$

Above we used the continuity along increasing chains. ∎

rem **6.8.** We have a map from the Lebesgue Stieltjes measure into the cadlag functions, to every μ we associate G_μ. The proposition we proved above, says that this map is injective. Is this map also surjective? We will show that the answer is yes! This means that given a cadlag $G : \mathbb{R} \to \mathbb{R}$ with $G(0) = 0$ we are sure to have a μ such that $G = G_\mu$. This gives us a convenient way to construct measures on $\mathbb{R}$.

Wednesday, October 3

REM **6.9.** Recall an example from lecture 1;

$$\mathcal{J} := \{\emptyset\} \cup \{(a,b] | a < b \in \mathbb{R}\} \cup \{(-\infty, b] | b \in \mathbb{R}\} \cup \{(a,\infty) | a \in \mathbb{R}\} \cup \{\mathbb{R}\}.$$

Let Σ be the collection of finite unions of sets from $\mathcal{J}$. Then Σ is an algebra of subsets of $\mathbb{R}$ called the algebra of half-open intervals. We can get this by using problem 1 from homework 1. The conditions listed in that problem are satisfied by $\mathcal{J}$. This is a structure called a **semialgebra** of sets.

DEFINITION **6.10.** Given $G : \mathbb{R} \to \mathbb{R}$ is such that $s \leq t \Rightarrow G(s) \leq G(t)$. Define a set function $\mu_{00} : \mathcal{J} \to [0, \infty]$ by

- $\mu_{00}(\emptyset) = 0$
- $\mu_{00}((a,b]) = G(b) - G(a)$
- $\mu_{00}((-\infty, b]) = G(b) - L_-$
- $\mu_{00}((a,\infty)) = L_+ - G(a)$
- $\mu_{00}(\mathbb{R}) = L_+ - L_-$

Where $L_+ = \lim_{t\to\infty} G(t) \in \mathbb{R} \cup \{\infty\}$ and $L_- = \lim_{s\to-\infty} G(s) \in \mathbb{R} \cup \{-\infty\}$. Note that μ_{00} can be naturally extended to an additive set function $\mu_0 : \Sigma \to [0,\infty]$. Why? Every $E \neq \emptyset$ from Σ can be written uniquely as $E = J_1 \cup \cdots \cup J_k$ with $J_1, \ldots, J_k \in \mathcal{J}$, $J_i \neq \emptyset$ for $1 \leq i \leq k$ and where $\mathrm{cl}(J_i) \cap \mathrm{cl}(J_j) = \emptyset$ for $i \neq j$.

We are now facing the problem of how to extend $\mu_0 : \Sigma \to [0,\infty]$ to a positive measure $\mu : \mathcal{B}_{\mathbb{R}} \to [0,\infty]$.

LEMMA **6.11.** *Let Σ be as above. The σ-algebra generated by Σ is $\mathcal{B}_{\mathbb{R}}$.*

PROPOSITION **6.12.** *Let (X,d) be a metric space, let $\mathcal{A}$ be an algebra of subsets of X, and let $\mu_0 : \mathcal{A} \to [0,\infty]$ be an additive set function. Suppose that for every $A \in \mathcal{A}$, we have* **Pre-regularity***,*

$$\mu_0(A) = \inf\{\mu_0(U) | U \in \mathcal{A} \text{ and } \mathrm{int}(U) \supseteq A\}$$
$$\mu_0(A) = \sup\{\mu_0(H) | H \in \mathcal{A} \text{ where } \mathrm{cl}(H) \subseteq A \text{ and } \mathrm{cl}(H) \text{ is compact}\}$$

LEMMA **6.13.** *Let $G : \mathbb{R} \to \mathbb{R}$ be a cadlag function. Let Σ be the algebra of half-open intervals of $\mathbb{R}$ and let $\mu_0 : \Sigma \to [0,\infty]$ be as in the above definition. Then μ_0 satisfies Pre-regularity, hence it is a pre-measure.*

THEOREM **6.14.** *Let $G : \mathbb{R} \to \mathbb{R}$ a cadlag function and with $G(0) = 0$. Then there exists a Lebesgue-Stieltjes measure μ, uniquely determined such that $G_\mu = G$.*

Proof. Let $\mu_0 : \Sigma \to [0, \infty]$ be defined by starting from G as in the above remarks. Then, μ_0 is a pre-measure, by the above lemma. Hence Caratheodory applies and extends μ_0 to $\mu : \mathcal{B}_\mathbb{R} \to [0, \infty]$. It is immediate that μ is finite on compact sets (Lebesgue-Stieltjes) with $G_\mu = G$. ∎

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PROPOSITION **6.15.** *Let (X, d) be a metric space. Let $\mathcal{A}$ be an algebra of subsets of X and let μ_0 be an additive set function and suppose that μ_0 satisfies the following "pre regularity" conditions: For every $A \in \mathcal{A}$ we have that*

$$\mu_0(A) = \inf\{\mu_0(U) | U \in \mathcal{A} \text{ and } \mathrm{int}(U) \supseteq A\}.$$

Then μ_0 is a pre-measure.

Proof. In order to prove that μ_0 is a pre-measure, we verify the condition of pre-σ-additivity from problem 3 of homework 2. So let A and $(A_n)_{n=1}^\infty$ be in $\mathcal{A}$ such that $A \subseteq \bigcup_{n=1}^\infty A_n$. We want to show that

$$\mu_0(A) \leq \sum_{n=1}^\infty \mu_0(A_n).$$

If $\sum_{n=1}^\infty \mu_0(A_n) = \infty$, then we are done. So we will assume that $\sum_{n=1}^\infty \mu_0(A_n) < \infty$. From pre-regularity, we know that

$$\mu_0(A) = \sup\{\mu_0(H) | H \in \mathcal{A}, \mathrm{cl}(H) \subseteq A, \mathrm{cl}(H) \text{ compact}\}.$$

So the required equality will follow if we can prove that

$$\mu_0(H) \leq \sum_{n=1}^\infty \mu_0(A_n)$$

for every $H \in \mathcal{A}$ such that $\mathrm{cl}(H)$ is a compact subset of A. Lets fix such an H. We have our usual trick; it suffices to prove

$$\mu_0(H) \leq \epsilon + \sum_{n=1}^\infty \mu_0(A_n), \forall \epsilon > 0.$$

So besides H, let us also fix an $\epsilon > 0$. Apply pre-regularity to A_n, we can find $U_n \in \mathcal{A}$ with $U_n \supseteq \mathrm{int}(U_n) \supseteq A_n$ and such that $\mu_0(U_n) < \mu_0(A_n) + \frac{\epsilon}{2^n}$. Observe that

$$\mathrm{cl}(H) \subseteq A \subseteq \bigcup_{n=1}^\infty A_n \subseteq \bigcup_{n=1}^\infty \mathrm{int}(U_n).$$

By definition of compactness we can find $N \in \mathbb{N}$ such that $\mathrm{cl}(H) \subseteq \bigcup_{n=1}^{N} \mathrm{int}(U_n)$. So then $H \subseteq \bigcup_{n=1}^{N} U_n$ (because $H \subseteq \mathrm{cl}(H) \subseteq \bigcup_{n=1}^{N} \mathrm{int}(U_n) \leq \bigcup_{n=1}^{N} U_n$). Since μ_0 is known to be finitely sub-additive, we get

$$\mu_0(H) \leq \sum_{n=1}^{N} \mu_0(U_n) < \sum_{n=1}^{N} \left(\mu_0(A_n) + \frac{\epsilon}{2^n}\right) \leq \sum_{n=1}^{\infty} \left(\mu_0(A_n) + \frac{\epsilon}{2^n}\right) = \sum_{n=1}^{\infty} \mu_0(A_n) + \epsilon$$

Therefore we get the desired inequality and so QED. ∎

REM **6.16.** Where is the Cadlag for G used in order to verify Pre-Reg for μ_0? Take special case $A = (a, b] \in \Sigma$ for $a < b \in \mathbb{R}$. Lets check this, for this A that $\mu_0(A) = \inf\{\mu_0(U) | U \in \mathcal{A}$ such that $\mathrm{int}(U) \supseteq A\}$. Try $U_n = (a, b + \frac{1}{n}] \in \Sigma$. Have $\mathrm{int}(U) = (a, b + \frac{1}{n}) \supseteq A$, $\mu_0(U) = G(b + \frac{1}{n}) - G(a)$. Do we have $\mu_0(U_n) \to \mu_0(A)$? $G(b + \frac{1}{n}) - G(a) \to G(b) - G(a)$.

Yes, because, G is continous from the right. For same $A = (a, b]$, let us check that $\mu_0(A) = \sup\{\mu_0(H) | H \in \Sigma,$ such that $cl(H) \subseteq A$ and $cl(H)$ is compact $\}$. Try $H = (a_n, b] \in \Sigma$ where $a_n \to a$ from above. Then, $cl(H_n) = [a_n, b]$ compact, inside. Have $\mu_0(H_n) \to \mu_0(A)$ from the left. Thus, $G(b) - G(a_n) \to G(b) - G(b)$, i.e. $G(a_n) \to G(a)$ from the left and we are saved.

REM **6.17.** Aside : In class midterm - 2nd november.

7 The Space of Functions Bor$(X, \mathbb{R})$

DEFINITION **7.1.** A function $X \to Y$ is said to be $(\mathcal{A}, \mathcal{B})$-measurable when it has the property that $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$. We will now use this concept in the case when $\mathcal{A}, \mathcal{B}$ are σ-algebras (hence we will call $(X, \mathcal{A}), (Y, \mathcal{B})$ measurable spaces).

DEFINITION **7.2.** Let $(X, \mathcal{A})$ be a measurable space. A function $f : X \to \mathbb{R}$ which is $(\mathcal{A}, \mathcal{B}_{\mathbb{R}})$-measurable will be called a **Borel Function**. We denote

$$\mathrm{Bor}(X, \mathbb{R}) = \{f : X \to \mathbb{R} \mid f \text{ is a Borel function}\}.$$

THEOREM **7.3.** *Let* $(X, \mathcal{A})$ *be a measurable space. Then* $\mathrm{Bor}(X, \mathbb{R})$ *is a unital algebra of functions which is closed under pointwise convergence.*

REM **7.4.** "$\mathrm{Bor}(X, \mathbb{R})$ is an algebra of functions" means that

$$f, g \in \mathrm{Bor}(X, \mathbb{R}) \Longrightarrow \alpha f + \beta g \in \mathrm{Bor}(\mathrm{X}, \mathbb{R}) \text{ and } fg \in \mathrm{Bor}(X, \mathbb{R})$$

for any scalars $\alpha, \beta \in \mathbb{R}$. Unital means that $\mathbb{1} \in \mathrm{Bor}(X, \mathbb{R})$. Moreover, if f and $(f_n)_{n=1}^{\infty}$ are functions from $X \to \mathbb{R}$ such that $f_n(x) \to f(x), \forall x \in X$ and if $f_n \in \mathrm{Bor}(X, \mathbb{R}), \forall n \in \mathbb{N}$, then it follows that $f \in \mathrm{Bor}(X, \mathbb{R})$. We need 2 tools to prove theorem 59.

PROPOSITION **7.5.** [TOOL NO.1] *Consider the measurable spaces* $(X, \mathcal{M}), (Y, \mathcal{N})$, *and* $(Z, \mathcal{P})$. *Let* $f : X \to Y$ *be* $(\mathcal{M}, \mathcal{N})$ *measurable and let* $g : Y \to Z$ *be* $(\mathcal{N}, \mathcal{P})$*-measurable. Consider the composed function* $h = g \circ f : X \to Z$. *Then* h *is* $(\mathcal{M}, \mathcal{P})$*-measurable.*

Proof. For every $C \subseteq Z$ have $h^{-1}(C) = (g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$. So then $C \in \mathcal{P}$ implies $g^{-1}(C) \in \mathcal{N}$ (Since g is $(\mathcal{N}, \mathcal{P})$-measurable). Thus, $f^{-1}(g^{-1}(C)) \in \mathcal{M}$ (since f is $(\mathcal{M}, \mathcal{N})$ measurable). Thus, $h^{-1}(C) \in \mathcal{M}$. Hence, h is $(\mathcal{M}, \mathcal{P})$ measurable. ∎

PROPOSITION **7.6.** [TOOL NO.2] *Let* $(X, \mathcal{M})$ *and* $(Y, \mathcal{N})$ *be measurable spaces. Let* $\mathcal{U} \subseteq \mathcal{N}$ *be a collection of sets which generate* $\mathcal{N}$ *as a* σ*-algebra. Let* $f : X \to Y$ *be such that* $f^{-1}(U) \in \mathcal{M}$, *for all* $U \in \mathcal{U}$. *Then,* f *is* $(\mathcal{M}, \mathcal{N})$*-measurable.*

Proof. This very similar to the proof in lecture 2 (proposition 12), only that now we use σ-algebras. ∎

COROLLARY **7.7.** *Let* (X, d) *and* (Y, d') *be metric spaces. Let* $f : X \to Y$ *be continuous. Then* f *is* $(\mathcal{B}_X, \mathcal{B}_Y)$*-measurable.*

Proof. Let $\mathcal{D}_Y = \{\mathcal{D} \subseteq Y | D \text{ is open}\}$. Then, $\mathcal{D}_Y$ generates $\mathcal{B}_Y$ as a σ-algebra. By Tool #2, it suffices to check that $f^{-1}(D) \in \mathcal{B}_X$, for all $D \in \mathcal{D}_Y$. Indeed, $D \in \mathcal{D}_Y$ implies $f^{-1}(D)$ open in X since f is continuous, so $f^{-1}(D) \in \mathcal{B}_X$. ∎

PROPOSITION **7.8.** *Let $(X,\mathcal{M})$ be a measurable space and let $f : X \to \mathbb{R}^n$. We can write $f(x) = \big(f_1(x),\dots,f_n(x)\big) \in \mathbb{R}^n$. This defines functions $f_1,\dots,f_n : X \to \mathbb{R}$. We have that f is $(\mathcal{M},\mathcal{B}_{\mathbb{R}})$-measurable if and only if each of the $f_1,\dots,f_n$ is in* $\mathrm{Bor}(X,\mathbb{R})$.

Proof of "⇒". Fix $1 \le i \le n$ for which we will verify that $f_i \in \mathrm{Bor}(X,\mathbb{R})$. Let $P_i : \mathbb{R}^n \to \mathbb{R}$ be the projection on component i: $P(t_1,\dots,t_n) = t_i$. Note that P_i is continuous, hence it is $(\mathcal{B}_{\mathbb{R}^n},\mathcal{B}_{\mathbb{R}})$-measurable by corollary 63. Also note that f is $(\mathcal{M},\mathcal{B}_{\mathbb{R}^n})$-measurable (hypothesis). This implies $f_i = P_i \circ f$ is $(\mathcal{M},\mathcal{B}_{\mathbb{R}})$-measurable by Tool #1. ∎

REM **7.9.** [COURSE INFORMATION] Graduate Student T.A. Marking the Courses: Michael Ng. Office Hours: Mondays 2:30 - 3:30, office MC 5050. Email: ks2ng@uwaterloo.ca.

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Proof of "⇐". Let

$$\mathcal{U} = \{U \subseteq \mathbb{R}^n \mid U \text{ is of the form } U = (a_1,b_1)\times\cdots\times(a_n,b_n) \text{ for some } a_i < b_1 \in \mathbb{Q}\}.$$

Then, $\mathcal{U}$ is a basis of open sets for $\mathbb{R}^n$ (that is, for every $D \subseteq \mathbb{R}^n$ open and $a \in D$, one can find $U \in \mathcal{U}$ such that $a \in U \subseteq D$). Then, every non-empty open set $D \subseteq \mathbb{R}^n$ can be written as an (automatically countable!) union of sets from $\mathcal{U}$. This implies that $\mathcal{U}$ generates $\mathcal{B}_{\mathbb{R}^n}$ as a σ-algebra (exercise, check!). In this implication, " $\Leftarrow$ ", we know that $f_1,\dots,f_n \in \mathrm{Bor}(X,\mathbb{R})$ and want to prove that f is $(\mathcal{M},\mathcal{B}_{\mathbb{R}^n})$-measurable. We use Tool #2 (Prop. 62) which implies that it suffices to verify that $f^{-1}(U) \in \mathcal{M}$, for all $U \in \mathcal{U}$. Indeed, for $U = (a_1,b_1)\times\cdots\times(a_n,b_n) \in \mathcal{U}$, we get

$$\begin{aligned}
f^{-1}(U) &= \{x \in X \mid f(x) \in U\} \\
&= \{x \in X \mid \big(f_1(x),\dots,f_n(x)\big) \in (a_1,b_1)\times\cdots\times(a_n,b_n)\} \\
&= \{x \in X \mid f_i(x) \in (a_i,b_i), \text{for all } 1 \le i \le n\} \\
&= \bigcap_{i=1}^{n} f_i^{-1}\big((a_i,b_i)\big).
\end{aligned}$$

Each of the sets in the intersection is in $\mathcal{M}$ since f_i is $(\mathcal{M},\mathcal{B}_{\mathbb{R}})$-measurable. Hence, $f^{-1}(U)$ is a finite intersection of sets from $\mathcal{M}$ which implies that $f^{-1}(U) \in \mathcal{M}$. ∎

PROPOSITION **7.10.** *Let $(X,\mathcal{M})$ be a measurable space and let $f,g \in$ $\mathrm{Bor}(X,\mathbb{R})$, then we have $f+g, f\cdot g \in$* $\mathrm{Bor}(X,\mathbb{R})$.

Proof. Define $h : X \to \mathbb{R}^2$ by putting $h(x) := \big(f(x),g(x)\big)$. Then, h is $(\mathcal{M},\mathcal{B}_{\mathbb{R}^2})$-measurable by Prop. 64. Define, $S,P : \mathbb{R}^2 \to \mathbb{R}$ with $S\big((a,b)\big) = a+b$ and $P\big((a,b)\big) = ab$. Note that S and P are continuous functions, hence they are $(\mathcal{B}_{\mathbb{R}^2},\mathcal{B}_{\mathbb{R}})$-measurable by corollary 63. Using Tool #1 (Prop.61), we get that $S\circ h$ and $P\circ h$ are $(\mathcal{M},\mathcal{B}_{\mathbb{R}})$-measurable. Hence, $S\circ h, P\circ h \in \mathrm{Bor}(X,\mathbb{R})$ but $S\circ h = f+g$ and $P\circ h = f\cdot g$ so we are done. ∎

REM **7.11.** What about "unary" operations with Borel functions? Given a measurable space $(X, \mathcal{M})$ and $f \in \text{Bor}(X, \mathbb{R})$, one can form many new functions, such as αf for fixed $\alpha \in \mathbb{R}$. How about $|f|$ or $\exp(f)$? We claim that all of these functions are still in $\text{Bor}(X, \mathbb{R})$ because of Tool #1. Why? for example take $\exp(f)$, let $g(t) = \exp(t)$ which is continuous and so $(\mathcal{B}_{\mathbb{R}}, \mathbb{R})$-measurable, so composition maintains measurability.

PROPOSITION **7.12.** *Let $(X, \mathcal{M})$ be a measurable space.*

1. *Let f and $f_0, \ldots, f_n, \ldots$ be functions from $X \to \mathbb{R}$, such that $f(x) = \sup\{f_n(x) | n \in \mathbb{N}\}$. If, for all $n \in \mathbb{N}$, we have $f_n \in \text{Bor}(X, \mathbb{R})$, then it follows that $f \in \text{Bor}(X, \mathbb{R})$.*

2. *Similar statement for infimum.*

Proof. We must verify that $f^{-1}(B) \in \mathcal{M}$, for all $B \in \mathcal{B}_{\mathbb{R}}$. Tool #2 tells us that it suffices to check that $f^{-1}(U) \in \mathcal{M}$, for all $U \in \mathcal{U}$ where $\mathcal{U} = \{(-\infty, b] \mid b \in \mathbb{R}\}$ (Problem 1 form Homework 3 tells us that $\mathcal{U}$ generates $\mathcal{B}_{\mathbb{R}}$ as a σ-algebra). For $U = (-\infty, b]$, we have $f^{-1}(U) = \{x \in X | f(x) \leq b\}$. So it now suffices to verify that $\{x \in X | f(x) \leq b\} \in \mathcal{M}$ for every $b \in \mathbb{R}$. Let's fix $b \in \mathbb{R}$. Observe that for every $x \in X$ we have

$$f(x) \leq b \iff \sup_{n \geq 1} f_n(x) \leq b \iff f_n(x) \leq b, \forall n \in \mathbb{N}.$$

Therefore, we find that

$$\begin{aligned} \{x \in X | f(x) \leq b\} &= \bigcap_{n=1}^{\infty} \{x \in X | f_n(x) \leq b\} \\ &= \bigcap_{n=1}^{\infty} f_n^{-1}\big((-\infty, b]\big) \in \mathcal{M}. \end{aligned}$$

So we have proven 1. To prove 2, simply define $g := -f$ and $g_n := -f_n$ for all $n \in \mathbb{N}$. Then, $g_n \in \text{Bor}(X, \mathbb{R})$ for every $n \in \mathbb{N}$. We also have that $g(x) = \sup_{n \geq 1} g_n(x)$ directly from the definition of infimum and supremum. Apply part 1 of the proposition toget that g is a borel function. Hence, $f = -g$ is in $\text{Bor}(X, \mathbb{R})$ as well. ∎

PROPOSITION **7.13.** *Let $(X, \mathcal{M})$ be a measurable space.*

1. *If f and $(f_n)_{n=1}^{\infty}$ from $X \to \mathbb{R}$ are such that $f(x) = \limsup_{n \to \infty} f_n(x)$ and if we also have that $f_n \in \text{Bor}(X, \mathbb{R})$, then $f \in \text{Bor}(X, \mathbb{R})$.*

2. *If g and $(g_n)_{n=1}^{\infty}$ from $X \to \mathbb{R}$ are such that $g(x) = \liminf_{n \to \infty} g_n(x)$ and if we also have that $g_n \in \text{Bor}(X, \mathbb{R})$, then $g \in \text{Bor}(X, \mathbb{R})$.*

Proof. Recall that for a sequence $(t_n)_{n=1}^{\infty}$ in $\mathbb{R}$, we define

$$\limsup_{n \to \infty} t_n = \inf_{k \geq 1} \left(\sup_{n \geq k} t_n \right)$$

and

$$\liminf_{n\to\infty} t_n = \sup_{k\geq 1}\left(\inf_{n\geq k} t_n\right).$$

For every $x \in X$ we have $f(x) = \inf_{k\geq 1}(\sup_{n\geq k} f_n(x))$. So then, for every $k \in \mathbb{N}$, let us define

$$h_k(x) = \sup_{n\geq k} f_n(x).$$

By proposition 68, since each f_n is Borel for $n \geq k$, each h_k is measurable. So $h_k \in \mathrm{Bor}(X, \mathbb{R})$. But, the definition of limit supremum tells us that $f(x) = \inf_{k\geq 1} h_k(x)$. Using proposition 68 again, this time using part 2, yields that $f(x)$ is Borel. For part 2, this reduces to above proof by taking negatives as in proposition 68 (left as exercise). ∎

REM **7.14.** [Discussion of Limit Superior] Note that $\sup_{n\geq 1} t_n \geq \sup_{n\geq 2} t_n \geq \cdots \geq \sup_{n\geq k} t_n$ forms a decreasing sequence. Let $\sigma_k := \sup_{n\geq k} t_n$ and $\sigma := \lim_{k\to\infty} \sigma_k \in [-\infty, \infty]$. Note that the limit does exists since it is a non-increasing sequence. So it is $\lim_{n\to\infty}(\sup_{n\geq k} t_n)$. Another possibility is to look at all possible convergent subsequences of $\{t_n\}_{n=1}^{\infty}$ with limits $\lambda \in [-\infty, \infty]$. Among these possible limits λ, there exists a greatest and a smallest, λ_+, λ_-. Then, λ_+ is the limit superior and λ_- is limit inferior. As an exercise, show that the two definitions are equivalent. In particular, note that for a convergent sequence $(t_n)_{n=1}^{\infty}$ we have

$$\limsup_{n\to\infty} t_n = \liminf_{n\to\infty} t_n = \lim_{n\to\infty} t_n.$$

Corollary **7.15.** *Let $(X, \mathcal{M})$ be a measurable space, if $\{f_n\}_{n=1}^{\infty}$ a sequence of functions in* $\mathrm{Bor}(X, \mathbb{R})$ *such that $f(x) = \lim_{n\to\infty} f_n(x)$, then $f \in \mathrm{Bor}(X, \mathbb{R})$.*

Proof. Write $f(x) = \lim_{n\to\infty} \sup f_n(x)$ and then use proposition 69. ∎

REM **7.16.** [Conclusions] The theorem 59 is now proven. We showed that if $f, g \in \mathrm{Bor}(X, \mathbb{R})$, then $f + g, fg \in \mathrm{Bor}(X, \mathbb{R})$ and $\alpha f \in \mathrm{Bor}(X, \mathbb{R})$ for any $\alpha \in \mathbb{R}$. Note that $1_f \in \mathrm{Bor}(X, \mathbb{R})$ is trivial. So $\mathrm{Bor}(X, \mathbb{R})$ is a unital algebra of functions which is closed under pointwise limits (by corollary 71).

8 Integration of Non-Negative Borel Functions

REM **8.1.** We have a measurable space $(X, \mathcal{M})$ and an algebra of functions $\mathrm{Bor}(X, \mathbb{R})$. Denote

$$\mathrm{Bor}^+(X, \mathbb{R}) = \{f \in \mathrm{Bor}(X, \mathbb{R}) | f(X) \subseteq [0, \infty)\},$$

$$\mathrm{Bor}_s(X, \mathbb{R}) = \{f \in \mathrm{Bor}(X, \mathbb{R}) | f(X) \subseteq \mathbb{R} \text{ is finite }\},$$

and

$$\mathrm{Bor}_s^+(X, \mathbb{R}) = \mathrm{Bor}^+(X, \mathbb{R}) \cap \mathrm{Bor}_s(X, \mathbb{R}).$$

Given a measure space, we want to associate to every $f \in \mathrm{Bor}^+(X, \mathbb{R})$ a value in $[0, \infty]$ called its integral against μ.

LEMMA **8.2.** *We have a natural "functional" $L_s^+ : \mathrm{Bor}_s^+(X, \mathbb{R}) \to [0, \infty]$ uniquely determined with additivity, positive homogeneity and $L_s^+(I_A) = \mu(A)$ for measurable A.*

REM **8.3.** Operations in $[0, \infty]$ are the natural ones with the convention that $0 \cdot \infty = 0$. For the proof of the previous statement, how do we explicitly construct $L_s^+(u)$ for some $u \in \mathrm{Bor}_s^+(X, \mathbb{R})$. Observe that u has the canonical writing $u = \alpha_1 I_{A_1} + \cdots + \alpha_n I_{A_n}$ with $A_i \cap A_j = \emptyset$. Namely, we enumerate, $u(X) \backslash \{0\} = \{\alpha_1, \ldots, \alpha_n\}$ and put $A_i = u^{-1}\big(\{\alpha_i\}\big)$. Then, we define

$$L_s^+(u) := \sum_{i=1}^{n} \alpha_i \mu(A_i) \in [0, \infty].$$

It is trivial to see that $L_s^+(I_A) = \mu(A)$. Left to check that L_s^+ is additive and positive homogeneous. Exercise!

REM **8.4.** How do we define integrals for general functions $f \in \mathrm{Bor}(X, \mathbb{R})$? Lebesgue's answer: Take a supremum and that's it!

DEFINITION **8.5.** Let $f \in \mathrm{Bor}^+(X, \mathbb{R})$, then

$$\int f \, d\mu := \sup\{L_S^+(u) \ : \ u \in \mathrm{Bor}_s^+(X, \mathbb{R}), u \leq f\}.$$

THEOREM **8.6.** [PROPERTIES OF THE INTEGRAL]

1. *Monotonic*
2. *Additive*
3. *Positively Homogeneous*
4. *Simple functions have the integral you expect*
5. *Monotone Convergence Theorem*

Proof. Left as an exercise since it is so similar to PMATH 450. ∎

REM **8.7.** Part 5 is usually proved before part 2.

REM **8.8.** We will do a bit of a discussion around the proof of theorem 78, In particular #2. Proof of $\int f+g \geq \int f + \int g$ is easy, but proof of "$\leq$" is not that easy!

CLAIM. Fix $f, g \in \mathrm{Bor}^+(X, \mathbb{R})$. Then we have $\int f + g\, d\mu \geq \int f\, d\mu + \int g\, d\mu$.

Proof. If $\int f+g = \infty$, then it is clear. So let's assume that $\int f+g\, d\mu < \infty$. Then, we also have

$$\int f\, d\mu \leq \int f+g\, d\mu < \infty$$
$$\int g\, d\mu \leq \int f+g\, d\mu < \infty$$

(Because $f, g \leq f+g$ and by part 1 of the theorem seen on wednesday). It suffices to prove that

$$\epsilon + \int f+g\, d\mu \geq \int f\, d\mu + \int g\, d\mu,$$

for all $\epsilon > 0$. Let's fix some $\epsilon > 0$. From the definition of $\int f\, d\mu$, we can find $u \in \mathrm{Bor}_s^+(X, \mathbb{R})$ with $u \leq f$ and such that $L_S^+(u) > \int f\, d\mu - \frac{\epsilon}{2}$. Likewise, there exists a $v \in \mathrm{Bor}_s^+(X, \mathbb{R})$ such that $v \leq g$ and $L_s^+(v) > \int g\, d\mu - \frac{\epsilon}{2}$. Put $w := u + v \in \mathrm{Bor}_s^+(X, \mathbb{R})$. Then $w \leq f+g$, so

$$\int f+g\, d\mu \geq L_s^+(w) = L_s^+(u+v) = L_s^+(u) + L_s^+(v) > \int f\, d\mu + \int g\, d\mu - \epsilon.$$

∎

Claim 1 done. Can we do this with claim 2? Lets try and see what happens.

CLAIM. Fix $f, g \in \mathrm{Bor}^+(X, \mathbb{R})$. Then we have $\int f + g\, d\mu \leq \int f\, d\mu + \int g\, d\mu$.

Can we do this directly from the definition? We start with $w \in \mathrm{Bor}_s^+(X, \mathbb{R})$ such that $w \leq f+g$ and we need to prove that

$$L_s^+(w) \leq \int f\, d\mu + \int g\, d\mu.$$

To get this, would have to decompose $w = u + v$ with $u, v \in \mathrm{Bor}_s^+(X, \mathbb{R})$ such that $u \leq f$ and $v \leq g$. How? This is not easy! So how do people prove this claim? First prove Lebesgue Monotone Convergence Theorem without assuming additivity. Then prove that for every $f \in \mathrm{Bor}^+(X, \mathbb{R})$, one can make a sequence $(u_n)_{n=1}^{\infty}$ in $\mathrm{Bor}_s^+(X, \mathbb{R})$ such that $u_1 \leq \ldots \leq u_n \leq \ldots$ and $\lim_{n\to\infty} u_n(x) = f(x)$, for all $x \in X$. Let's write $u_n \to f$ to denote pointwise convergence

of functions.
Now go back to claim 2. Given $f, g \in \mathrm{Bor}^+(X, \mathbb{R})$, take sequences $u_n, v_n \in Bor_s^+(X, \mathbb{R})$ such that $u_n \to f, v_n \to g$ from below. Then, $u_n + v_n \to f + g$ and we get

$$\begin{aligned}
\int f + g \, d\mu &= \lim_{n\to\infty} \int (u_n + v_n) \, d\mu \\
&= \lim_{n\to\infty} L_s^+(u_n + v_n) \\
&= \lim_{n\to\infty} L_s^+(u_n) + L_s^+(v_n) \\
&\leq \int f \, d\mu + \int g \, d\mu.
\end{aligned}$$

REM **8.9.** How does the current lecture 8 relate to lecture 2? There it was a more restrictive case where we forced $\mu(X) < \infty$ and $f \in \mathrm{Bor}(X, \mathbb{R})$ bounded. There we had the hypothesis "f is $(\mathcal{M}, \Sigma)$-measurable with $\mathcal{M}$ a σ-algebra", this is the same as saying $f \in \mathrm{Bor}(X, \mathbb{R})$. We proved that $\int^+ f \, d\mu = \int_- f \, d\mu$ (make divisions, follow f) where $\int^+ f, \int_- f$ are Darboux sums. Suppose now that $f \geq 0$, hence $f \in \mathrm{Bor}^+(X, \mathbb{R})$ does the Darboux integral from above match the Lebesgue integral? Yes! The key-point is to relate upper and lower Darboux sums to simple functions. $\Delta = \{A_1, ..., A_r\} \Rightarrow u = \sum_{i=1}^r B_i I_{A_i}$ $U(f, \Delta) = L_s^+(u)$.

9 Integrable Functions

For the whole lecture, fix a measure space $(X, \mathcal{M}, \mu)$.

DEFINITION **9.1.** Let $\mathcal{L}^1(\mu) := \{f \in \mathrm{Bor}(X, \mathbb{R}) \ : \ \int |f| d\mu < \infty\}$. The functions in $\mathcal{L}^1(\mu)$ are said to be integrable with respect to μ.

REM **9.2.** $\mathcal{L}^1(\mu)$ is closed under linear combinations. For any $f, g \in \mathcal{L}^1(\mu)$ and any $\alpha, \beta \in \mathbb{R}$ we have $\alpha f + \beta g \in \mathcal{L}^1(\mu)$. Indeed,

$$\int |\alpha f + \beta g| \ d\mu \leq \int |\alpha| \, |f| + |\beta| \, |g| \ d\mu = |\alpha| \int |f| \ d\mu + |\beta| \int |g| \ d\mu < \infty.$$

But how do we define the integral on the space of $\mathcal{L}^1(\mu)$?

REM **9.3.** For $f : X \to \mathbb{R}$ define f_- and f_+ by $f_+(x) = \max(f(x), 0)$ and $f_-(x) = \max(-f(x), 0)$. It is immediate that $f = f_+ - f_-$, moreover $f_+ + f_- = |f|$. This shows that alternatively we can write

$$f_+ = \frac{|f| + f}{2} \text{ and } f_- = \frac{|f| - f}{2}.$$

It is now clear that

$$f \in \mathrm{Bor}(X, \mathbb{R}) \Longrightarrow f_+, f_- \in \mathrm{Bor}(X, \mathbb{R}).$$

Finally, observe that if $f \in \mathcal{L}^1(\mu)$, then

$$f_+ \leq |f| \Longrightarrow \int f_+ d\mu \leq \int |f| d\mu < \infty$$

and

$$f_- \leq |f| \Longrightarrow \int f_- d\mu \leq \int |f| d\mu < \infty.$$

In short, we also get $f_+, f_- \in \mathrm{Bor}^+(X, \mathbb{R}) \cap \mathcal{L}^1(\mu)$.

DEFINITION **9.4.** For $f \in \mathcal{L}^1(\mu)$ we define

$$\int f \, d\mu := \int f_+ \, d\mu - \int f_- \, d\mu \in \mathbb{R}$$

Note that the left integral is the new integral we have just defined and the right hand integral is the one from lecture 8 defined on positive measurable functions. Note that if $f \in \mathcal{L}^1(\mu) \bigcap \mathrm{Bor}^+(X, \mathbb{R})$, then the values "$\int f d\mu$" defined in Lecture 8 and in Lecture 9 coincide. Indeed, in this case we get $\int f_+ = f$ and $f_- = 0$. Hence, $\int f \, d\mu = \int f \, d\mu - 0$.

THEOREM **9.5.** [PROPERTIES OF THE INTEGRAL ON $\mathcal{L}^1(\mu)$]

1. *Additive:* $\int f + g\, d\mu = \int f\, d\mu + \int g\, d\mu$.
2. *Homogenous:* $\int \alpha f\, d\mu = \alpha \int f\, d\mu$.
3. *Increasing: If* $f, g \in \mathcal{L}^1(\mu)$ *and* $f \le g$*, then* $\int f\, d\mu \le \int g\, d\mu$.
4. $\left|\int f\, d\mu\right| \le \int |f|\, d\mu$ *for all* $f \in \mathcal{L}^1(\mu)$.

Proof. (1.) Let $h := f+g$ and consider the positive/negative parts $f_\pm, g_\pm, h_\pm \in \mathrm{Bor}^+(X, \mathbb{R}) \bigcap \mathcal{L}^1(\mu)$. We have,

$$h_+ - h_- = (f_+ + g_+) - (f_- + g_-)$$

so

$$h_+ + f_- + g_- = h_- + f_+ + g_+$$

so

$$\int h_+\, d\mu + \int f_-\, d\mu + \int g_-\, d\mu = \int h_-\, d\mu + \int f_+\, d\mu + \int g_+\, d\mu.$$

Note that all involved quantities are finite! Rearrange the terms to get

$$\begin{aligned}\int h\, d\mu &= \int h_+\, d\mu - \int h_-\, d\mu \\ &= \int f_+\, d\mu - \int f_-\, d\mu + \int g_+\, d\mu - \int g_-\, d\mu \\ &= \int f\, d\mu + \int g\, d\mu.\end{aligned}$$

(2.) Immediate directly form the definition of the integral on $\mathcal{L}^1(\mu)$. Verifications depend on the sign of α. E.g. for $\alpha < 0$, we get $(\alpha f)_+ = |\alpha| f_-$ and $(\alpha f)_- = |\alpha| f_+$. Hence,

$$\int \alpha f\, d\mu = \int |\alpha|\, f_-\, d\mu - \int |\alpha|\, f_+\, d\mu = -|\alpha| \int f\, d\mu = \alpha \int f\, d\mu.$$

(3.) Let $f, g \in \mathcal{L}^1(\mu)$ with $f \le g$. Then we have $g - f \in \mathrm{Bor}^+(X, \mathbb{R}) \bigcap \mathcal{L}^1(\mu)$. So we also have that $\int g - f\, d\mu \in [0, \infty)$. But $\int g - f\, d\mu = \int g\, d\mu - \int f\, d\mu$, and hence we get $\int g\, d\mu - \int f d\mu \ge 0$.

(4.) Let $f \in \mathcal{L}^1(\mu)$. We have $\pm|f| \in \mathcal{L}^1(\mu)$ with $-|f| \le f \le |f|$. Apply 3 of this theorem to get

$$-\int |f|\, d\mu = \int -|f|\, d\mu \le \int f\, d\mu \le \int |f|\, d\mu.$$

So we have that

$$-\left(\int |f| d\mu\right) \le \int f d\mu \le \int |f| d\mu$$

which gives $\left|\int f d\mu\right| \le \int |f| d\mu$. ∎

10 Fatou's Lemma and Lebesgue Dominated Convergence Theorem

EG **10.1.** Let $X = [0,1]$, let $\mathcal{M} = \mathcal{B}_X$ and let μ be the Lebesgue measure on $[0,1]$. For every $n \in \mathbb{N}$ let

$$f_n(x) = \begin{cases} (2n)^2 x + 1 & x \le \frac{1}{2n} \\ -(2n)^2 x + 4n + 1 & \frac{1}{2n} \le x \le \frac{1}{n} \\ 1 & \frac{1}{n} \le x \end{cases}$$

Note that $f_n(x) \to 1$ but $\int f_n(x)\,dx = 2$. We get different integrals. Could we construct an example such that the limit is less than 1? Fatou says no, we cannot do that.

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REM **10.2.** Fatou's lemma for f_n in $\mathrm{Bor}^+(X, \mathbb{R})$ says that we always have

$$\lim_{n\to\infty} \left(\int f_n \, d\mu \right) \ge \int \lim_{n\to\infty} f_n \, d\mu.$$

Question: Could we adjust the example to make the f_n uniformly bounded? Dominated convergence theorem says no. If c dominates all the f_n and since $\int c\,d\mu = c < \infty$, then by LDCT it must follow that

$$\lim_{n\to\infty} \int f_n \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu$$

Note: The dominating constant function c is specific to case when $\mu(X) < \infty$ but LDCT applies in the case when $\mu(X) = \infty$, but there we use integrable dominating functions instead.

REM **10.3.** There is an analogy between LMCT and "continuity along chains" from lecture 3. Recall general principle sets are functions with values in $\{0,1\}$.

LEMMA **10.4.** *Let $u \in \mathrm{Bor}^+(X, \mathbb{R})$ and let $(u_n)_{n=1}^{\infty}$ be a sequence of functions from $\mathrm{Bor}^+(X, \mathbb{R})$ with $u_n \to u$ such that*

$$u_1 \ge u_2 \ge \cdots \ge u_n \ge \cdots$$

and such that $\int u_1 \, d\mu < \infty$, then $\int u \, d\mu = \lim_{n\to\infty} \int u_n \, d\mu$.

Proof. Proof left as an exercise. The idea for the proof: Denote $V_n := u_1 - u_n \in \mathrm{Bor}^+(X, \mathbb{R})$. Then we have $0 = v_1 \le v_2 \le \cdots$. Apply LMCT to $\{v_n\}$ (proof is truly analaogous to what we did in Lecture 3). ∎

PROPOSITION **10.5.** [FATOU'S LEMMA]

1. *Suppose we have f and $(f_n)_{n=1}^\infty$ in $\mathrm{Bor}^+(X,\mathbb{R})$ such that $f(x) = \liminf_{n\to\infty} f_n(x)$, for all $x \in X$. Then,*
$$\int \liminf f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu.$$
2. *Suppose we have g and $(g_n)_{n=1}^\infty$ in $\mathrm{Bor}^+(X,\mathbb{R})$ such that $g(x) = \limsup_{n\to\infty} g_n(x)$ for all $x \in X$. Suppose also that there exists a dominating function $h \in \mathrm{Bor}^+(X,\mathbb{R}) \cap \mathcal{L}^1(\mu)$ such that $g_n \leq h$ for all $n \in \mathbb{N}$, then*
$$\int \limsup g_n \, d\mu \geq \limsup_{n\to\infty} \int g_n \, d\mu.$$

REM **10.6.** Fatou's Lemma, both 1 and 2, is proved by the "Trick of Fatou"; take the inf/sup for the tail of your sequence.

Proof. We will do part 2. Denote $\int g_n \, d\mu := \gamma_n$. We want to prove that $\int g d\mu \geq \limsup_{n\to\infty} \gamma_n$. For every $n \in \mathbb{N}$ define $u_n : X \to \mathbb{R}$ by
$$u_n(x) = \sup(g_n(x), g_{n+1}(x), \ldots, g_m(x), \ldots) \in X.$$
Note that $0 \leq u_n(x) \leq h(x)$ for all $n \in \mathbb{N}$ and $x \in X$. We have $u_n \in \mathrm{Bor}^+(X,\mathbb{R})$ by some proposition in lecture 7. We also have
$$u_1 \geq u_2 \geq \cdots \geq u_n \geq \cdots$$
and
$$\lim_{n\to\infty} u_n(x) := \limsup_{n\to\infty} g_n(x) = g(x)$$
for all $x \in X$. Note that $\int u_1 \, d\mu \leq \int h \, d\mu < \infty$. Hence, we can apply lemma 90 to $(u_n)_{n=1}^\infty$. We now get
$$\lim_{n\to\infty} \int u_n \, d\mu = \int g \, d\mu.$$
Now fix $n \in \mathbb{N}$, observe that $\forall m \geq n$ we have
$$\int u_n \, d\mu \geq \int g_m \, d\mu = \gamma_m$$
so $\int u_n \, d\mu \geq \sup(\gamma_n, \gamma_{n+1}, \ldots, \ldots)$. Last point, unfix n and make $n \to \infty$ in the above point. So
$$\begin{aligned}\int g \, d\mu &= \lim_{n\to\infty} \int u_n \, d\mu \\ &\geq \lim_{n\to\infty} (\sup(\gamma_n, \gamma_{n+1} m, \ldots,)) \\ &= \limsup_{n\to\infty} \gamma_n\end{aligned}$$
∎

THEOREM **10.7.** [LEBESGUE DOMINATED CONVERGENCE THEOREM] *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let f and $(f_n)_{n=1}^{\infty}$ be functions from Bor$(X, \mathbb{R})$ such that $\lim_{n\to\infty} f_n(x) = f(x)$, $\forall x \in X$. Suppose moreover that there exists a function $h \in$ Bor$^+(X, \mathbb{R}) \cap \mathcal{L}^1(\mu)$ which dominates all the f_n, in the sense that ve have $|f_n| \leq h$, $\forall n \in \mathbb{N}$. Then f and $f_1, f_2, \ldots, f_n, \ldots$ are all in $\mathcal{L}^1(\mu)$, and*

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu \tag{1}$$

In fact, even stronger that (1), we have

$$\lim_{n\to\infty} \int |f_n - f| \, d\mu = 0 \tag{2}$$

Proof. Do first (2). For every $g_n = |f_n - f| \in \mathrm{Bor}^+(X, \mathbb{R})$. Then, $g_n(x) = |f_n(x) - f(x)| \to 0$ for all $x \in X$. Will apply corollary 10.5 to the g_ns. Is this a dominating function for the g_ns?

We know that $|f_n(x)| \leq h(x)$. Note that for every $x \in X$, we also have that

$$|f(x)| = \lim_{n\to\infty} |f_n(x)| \leq h(x)$$

Hence, $g_n(x) = |f_n(x) - f(x)| \leq |f_n(x)| + |f(x)| \leq 2h(x)$. So the previous lemma applies, and gives $\int g_n \to 0$ hence $\int |f_n - f| \to 0$. We have

$$|\int f_n - \int f| = |\int f - f| \leq \int |f_n - f| = \int g_n \to 0$$

∎

11 Densities and Signed Measures

REM **11.1.** For this lecture, let $(X, \mathcal{M}, \mu)$ be a measure space, $f \in \mathrm{Bor}^+(X, \mathbb{R})$, and $A \in \mathcal{M}$. By $\int_A f d\mu$ we mean $\int f \cdot I_A \, d\mu$, in the sense of lecture 8, where $I_A \in \mathrm{Bor}^+(X, \mathbb{R})$ is the indicator function of A.

PROPOSITION **11.2.** *For every $A \in \mathcal{M}$, let us put*

$$\nu(A) := \int_A f \, d\mu$$

Then, $\nu : \mathcal{M} \to [0, \infty]$ is a positive measure.

Proof. Firstly, $\nu(\emptyset) = \int f \cdot I_\emptyset \, d\mu = \int 0 \, d\mu = 0$. So the first condition is satisfied. Next, let $(A_n)_{n=1}^\infty$ be in $\mathcal{M}$ such that each A_i is pairwise disjoint with the other elements in the collection. Consider the union $U := \bigcup_{n=1}^\infty A_n$. We want to check that $\nu(U) = \sum_{n=1}^\infty \nu(A_n)$. Indeed we have

$$\begin{aligned} \sum_{n=1}^{\infty} \nu(A_n) &= \lim_{N\to\infty} \left(\sum_{n=1}^{N} \nu(A_n) \right) \\ &= \lim_{N\to\infty} \left(\sum_{n=1}^{N} \int f \cdot I_{A_n} \, d\mu \right) \\ &= \lim_{N\to\infty} \left(\int f (I_{A_1} + \cdots + I_{A_n}) \, d\mu \right) \\ &= \lim_{N\to\infty} \left(\int f \cdot I_{A_1 \cup \cdots \cup A_n} \, d\mu \right). \end{aligned}$$

It is immediate that we have $f \cdot I_{A_1 \cup \cdots \cup A_n}$ converges from below to $f \cdot I_U$. So therefore, by LMCT, we have

$$\lim_{N\to\infty} \int f \cdot I_{A_1 \cup \cdots \cup A_n} \, d\mu = \int f \cdot I_U \, d\mu = \nu(U).$$

So altogether, we get $\sum_{n=1}^\infty \nu(A_n) = \nu(U)$ as we wanted. ∎

LEMMA **11.3.** *Let $f \in \mathrm{Bor}^+(X, \mathbb{R})$, and let $\nu : \mathcal{M} \to [0, \infty]$ be as in the above proposition. Then, for every $g \in \mathrm{Bor}^+(X, \mathbb{R})$, we have*

$$\int g \, d\nu = \int g f \, d\mu. \tag{$\diamond$}$$

Proof. Let $G = \{g \in \mathrm{Bor}^+(X, \mathbb{R}) \ : \ (\diamond) \text{ holds for } g\}$. We want to prove that $G = \mathrm{Bor}^+(X, \mathbb{R})$.

CLAIM (1)**.** $I_A \in G$ for all $A \in \mathcal{M}$.

Verification of Claim 1. For $g = I_A$, we get

$$\int g\,d\nu = \int I_A\,d\nu = \nu(A) = \int f \cdot I_A\,d\mu = \int gf\,d\mu.$$

$\blacksquare$

CLAIM (2). If $g_1, g_2 \in G$ and $\alpha_1, \alpha_2 \in [0, \infty)$, then $g := \alpha_1 g_1 + \alpha_2 g_2$ is in G as well.

Verification of Claim 2. We know that $\int g_1\,d\nu = \int g_1 f\,d\mu$ and $\int g_2\,d\nu = \int g_2 f\,d\mu$, so if we do the linear combination of these two equalities with coefficients α_1, α_2 we get $\int g\,d\nu = \int gf\,d\mu$ due the linearity of the integral with respect to μ and to ν. Hence, $g \in G$. $\blacksquare$

CLAIM (3). Suppose g and $(g_n)_{n=1}^{\infty}$ in $\mathrm{Bor}^+(X, \mathbb{R})$ are such that $g_n \to g$ from below. If $g_n \in G$ for each $n \in \mathbb{N}$, then $g \in G$.

Verification of Claim 3. It is immediate that $g_n \cdot f \to gf$. We know that $\int g_n\,d\nu = \int g_n\,f d\mu$ since $g_n \in G$. Use LMCT on both sides of the the equality to get $\int g\,d\nu = \int gf\,d\mu$ and hence $g \in G$. $\blacksquare$

CLAIM (4). $G \supseteq \mathrm{Bor}_s^+(X, \mathbb{R})$.

Verification of Claim 4. This follows from claims 1 and 2 and the fact that every $g \in \mathrm{Bor}_s^+(X, \mathbb{R})$ can be written as $\sum_{i=1}^{n} \alpha_i I_{A_i}$. $\blacksquare$

CLAIM (5). $G = \mathrm{Bor}^+(X, \mathbb{R})$.

Verification of Claim 5. This follows from claims 3 and 4 and the fact that for every g in $\mathrm{Bor}^+(X, \mathbb{R})$, one can find a sequence $(g_n)_{n=1}^{\infty}$ in $\mathrm{Bor}_s^+(X, \mathbb{R})$ with $g_n \to g$ (This last fact is similar to problem 4 in homework 4). $\blacksquare$

Claim 5 ends the proof. $\blacksquare$

REM **11.4.** One uses the notation $d\nu = f\,d\mu$. This only really makes sense in the context of integrals. Look at the formula $(\diamond)$ and "cancel" the integral for convenience (or laziness).

REM **11.5.** Suppose $f \in \mathcal{L}^1(\mu)$. For every $A \in \mathcal{M}$ have that $f \cdot I_A \in \mathcal{L}^1(\mu)$ as well (Why? Beacause $f \cdot I_A \in \mathrm{Bor}(X, \mathbb{R})$ since f and I_A are Borel and we have $|f \cdot I_A| \le |f|$. Hence $\int |f \cdot I_A|\,d\mu \le \int |f|\,d\mu < \infty$). So we can define

$$\int_A f\,d\mu := \int f \cdot I_A\,d\mu \in \mathbb{R}$$

Hence, we can define $\nu : \mathcal{M} \to \mathbb{R}$ as $\nu(A) = \int_A f\,d\mu$, for all $A \in \mathcal{M}$.

If $f \ge 0$ (that is, $f \in \mathcal{L}^1(\mu) \cap \mathrm{Bor}^+(X, \mathbb{R})$), then ν is a finite positive measure. What if f is not in $\mathrm{Bor}^+(X, \mathbb{R})$. Then, we get for ν what is called a finite signed measure.

DEFINITION **11.6.** Let $(X, \mathcal{M})$ be a measurable space. A **finite signed measure** is a set function $\nu : \mathcal{M} \to \mathbb{R}$ such that whenever $(A_n)_{n=1}^{\infty}$ are in $\mathcal{M}$, pairwise disjoint, it follows that

$$\sum_{n=1}^{\infty} |\nu(A_n)| < \infty \qquad (*)$$

and

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n) \qquad (**)$$

We denote

$$\mathrm{Meas}^{\pm}(X, \mathcal{M}) = \{\nu : \mathcal{M} \to \mathbb{R} \ : \ \nu \text{ is a finite signed measure}\}$$

and we also denote

$$\begin{aligned} \mathrm{Meas}^{+}(X, \mathcal{M}) &= \{\nu \in \mathrm{Meas}^{\pm}(X, \mathcal{M}) \ : \ \nu(A) \geq 0, \forall A \in \mathcal{M}\} \\ &= \{\nu : \mathcal{M} \to [0, \infty) \ : \ \nu \text{ is a finite positive measure}\}. \end{aligned}$$

REM **11.7.** In definition 99, we did not ask (as we usually do) that $\nu(\emptyset) = 0$. This actually follows from (*). Take $A_n = \emptyset$, for all $n \in \mathbb{N}$. Then (*) gives us $\sum_{n=1}^{\infty} |\nu(\emptyset)| < \infty$, which implies $\nu(\emptyset) = 0$.

Note that $\nu \in \mathrm{Meas}^{\pm}(X, \mathcal{M})$ implies that ν is finitelly additive since $\nu(\bigcup_{i=1}^{m} A_i) = \sum_{i=1}^{m} \nu(A_i)$ for pairwise disjoint sets (use (**)).

Warning: A property that $\nu \in \mathrm{Meas}^{\pm}(X, \mathcal{M})$ generally does not have is monotonicity. It is generally not true that $A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$ because $\nu(B) - \nu(A) = \nu(B \backslash A)$ could be less than 0.

REM **11.8.** $\mathrm{Meas}^{\pm}(X, \mu)$ is a vector space over $\mathbb{R}$ with natural operations for additions and scalar multiplication, $\nu := \alpha_1 \nu_1 + \alpha_2 \nu_2$ by $\nu(A) = \alpha_1 \nu_1(A) + \alpha_2 \nu_2(A)$. It is immediate that ν is a vector space. One does a linear combination of absolutely convergent series.

PROPOSITION **11.9.** *Suppose $(X, \mathcal{M}, \mu)$ a measure space. Then,*

1. *Let $f \in \mathcal{L}^1(\mu)$ and define $\nu : \mathcal{M} \to \mathbb{R}$ by $\nu(A) = \int_A f \, d\mu = \int f \cdot I_A \, d\mu$, for all $A \in \mathcal{M}$. Then $\nu \in \mathrm{Meas}^{\pm}(X, \mathcal{M})$.*

2. *The map $\Lambda : f \to \nu$ (as in 1) is a linear map from $\mathcal{L}^1(\mu) \to \mathrm{Meas}^{\pm}(X, \mathcal{M})$.*

Proof.

1. Write $f = f_+ - f_-$, with $f_{\pm} = \frac{1}{2}(|f| \pm f) \in \mathrm{Bor}^{+}(X, \mathbb{R}) \cap \mathcal{L}^1(\mu)$ as in lecture 9. Define $\nu_+, \nu_- : \mathcal{M} \to \mathbb{R}$ by $\nu_+(A) = \int_A f_+ \, d\mu$ and $\nu_-(A) = \int_A f_- \, d\mu$, for all $A \in \mathcal{M}$.

Then, ν_+, ν_- are positive measures by proposition 95. In fact, ν_+, ν_- are *finite* positive measures since

$$\nu_\pm(X) = \int f_\pm \, d\mu \leq \int |f| \, d\mu < \infty$$

Hence, $\nu_+ - \nu_- \in \mathrm{Meas}^\pm(X, \mathcal{M})$ (because $\mathrm{Meas}^+(X, \mathcal{M}) \subseteq \mathrm{Meas}^\pm(X, \mathcal{M})$ and because $\mathrm{Meas}^\pm(X, \mathcal{M})$ is closed under linear combinations), but $\nu_+ - \nu_- = \nu$. Hence we have $\nu \in \mathrm{Meas}^\pm(X, \mathcal{M})$ as required.

2. Let $f_1, f_2 \in \mathcal{L}^1(\mu)$ and let $\alpha_1, \alpha_2 \in \mathbb{R}$. Put $f = \alpha_1 f_1 + \alpha_2 f_2 \in \mathcal{L}^1(\mu)$ and consider

$$\begin{aligned} f_1 &\to \nu_1 \\ f_2 &\to \nu_2 \\ f &\to \nu. \end{aligned}$$

We want to check that $\nu = \alpha_1 \nu_1 + \alpha_2 \nu_2$. This is immediate; for every $A \in \mathcal{M}$ we have

$$\begin{aligned} \nu(A) &= \int f I_A \, d\mu \\ &= \int (\alpha_1 f_1 + \alpha_2 f_2) I_A \, d\mu \\ &= \alpha_1 \int f_1 I_A \, d\mu + \alpha_2 \int f_2 I_A \, d\mu \\ &= \alpha_1 \nu_1(A) + \alpha_2 \nu_2(A). \end{aligned}$$

∎

REM **11.10.** Some natural questions about the linear map $\Lambda : f \to \nu \in \mathrm{Meas}^\pm(X, \mathcal{M})$ from prop 11.9.2 where $f \in \mathcal{L}^1(\mu)$. What is $\ker(\Lambda)$? Homework problem; Show that $\Lambda(f) = \Lambda(g) \iff f = g$ a.e.

What is $\mathrm{Ran}(\Lambda)$? This is the Radon-Nikodyn Theorem: Assume μ is σ-finite. The we get that $\mathrm{Ran}(\Lambda) = \{\nu \in \mathrm{Meas}^\pm(X, \mathcal{M}) \ : \ \nu \text{ is absolutely continuous w.r.t } \mu\}$. Some natural questions about $\mathrm{Meas}^\pm(X, \mathcal{M})$.

1. Is it true that $\mathrm{Span}(\mathrm{Meas}^+(X, \mathcal{M})) = \mathrm{Meas}^\pm(X, \mathcal{M})$?
2. Do we have a natural norm on $\mathrm{Meas}^\pm(X, \mathcal{M})$?

The answers to both these questions are yes and shown in the following lecture.

12 Positive, Negative, and Total Variation for Finite Signed Measures

Definition **12.1.** Let $(X, \mathcal{M})$ be a measurable space and let $\nu \in \text{Meas}^{\pm}(X, \mathcal{M})$. For every $A \in \mathcal{M}$, we define

$$V^+(A) = \sup\{\nu(B) | B \in \mathcal{M}, B \subseteq A\}$$

and

$$V^-(A) = \sup\{-\nu(B) | B \in \mathcal{M}, B \subseteq A\} = -\inf\{\nu(B) | B \in \mathcal{M}, B \subseteq A\}.$$

We call $V^+(A)$ and $V^-(A)$ the **Positive and Negative Variations** of ν on A. The sum $\nu^+(A) + \nu^-(A)$ is called the total variation ν on A.

rem **12.2.** When we replace ν by $-\nu$ (which still belongs to $\text{Meas}^{\pm}(X, \mathcal{M})$), the roles of V^+ and V^- are swapped.

rem **12.3.** If $(X, \mathcal{M})$ and $\nu \in \text{Meas}^{\pm}(X, \mathcal{M})$, then $A \to V^+(A)$ is a set function. Why is it that $V^+(A) = \sup\{\nu(B) | B \in \mathcal{M}, B \subseteq A\} \geq 0$? because the supremum includes $\emptyset$ and $V^+(\emptyset) = 0$. A priori, it looks possible that $V(A) = \infty$ (we will rule this out).

Lemma **12.4.** *Let $V^+(A)$ be the positive variation of ν (a signed measure), then V^+ is a positive measure.*

Proof. First we have that $V^+(\emptyset) = \sup\{\nu(B) | B \in \mathcal{M}, B \subseteq \emptyset\} = \nu(\emptyset) = 0$. Now fix $(A_n)_{n=1}^\infty$ in $\mathcal{M}$ such that $A_n \cap A_m = \emptyset$ for $n \neq m$ and denote $A := \bigcup_{n=1}^\infty A_n$. We want to prove that

$$V^+(A) = \sum_{n=1}^\infty V^+(A_n).$$

Let's first prove "$\leq$". From the definition of $V^+(A)$ as a supremum, it suffices to check that we have $\nu(B) \leq \sum_{n=1}^\infty \nu^+(A_n)$ for every $B \in \mathcal{M}$ with $B \subseteq A$. Fix $B \in \mathcal{M}$ with $B \subseteq A$. We have

$$B = B \cap A = B \cap \left(\bigcup_{n=1}^\infty A_n \right) = \bigcup_{n=1}^\infty (B \cap A_n).$$

Note that this is still a union over disjoint sets, therefore,

$$\nu(B) = \sum_{n=1}^\infty \nu(B \cap A_n) \leq \sum_{n=1}^\infty V^+(A_n).$$

The first equality comes from absolute convergence and second inequality comes form the fact that $\nu(B \cap A_n) \leq V^+(A_n)$ because $B \cap A_n \in \mathcal{M}$ and $B \cap A_n \subseteq A_n$.
Now let's prove "$\geq$". That is, $V^+(A) \geq \sum_{n=1}^\infty V^+(A_n)$. It suffices to check that

$$V^+(A) \geq \sum_{n=1}^N V^+(A_n)$$

for all $N \geq 1$. Once this is done, we only have to make $N \to \infty$. So fix an $N \in \mathbb{N}$. By playing with supremums, it is easy to see that

$$\sum_{n=1}^{N} V^+(A_n) = \sup\{\nu(B_1) + \cdots + \nu(B_n) \ : \ B_1, \ldots, B_n \in \mathcal{M}, B_i \subseteq A_i\}.$$

So it suffices to check that $\nu(B_1)+\cdots+\nu(B_n) \leq V^+(A)$ whenever $B_1, \ldots, B_n \in \mathcal{M}$ are such that $B_i \subseteq A_i$. And indeed, for any such $B_1, \ldots, B_N$, we have $B_i \cap B_j \subseteq A_i \cap A_j = \emptyset$ for $i \neq j$. Hence,

$$\nu(B_1) + \cdots + \nu(B_n) = \nu(B_1 \cup \cdots \cup B_n) \leq V^+(A)$$

The first equality by finite additivity and the second inequality because $B_1 \cup \cdots \cup B_N \subseteq A$. ∎

Theorem **12.5.** *Let $(X, \mathcal{M})$ be a measurable space and and let $\nu \in \mathrm{Meas}^{\pm}(X, \mathcal{M})$. Let $V^+, V^- : \mathcal{M} \to [0, \infty]$ be the positive/negative variations of ν. Recall that*

$$\mathrm{Meas}^+(X, \mathcal{M}) = \{\mu \in \mathrm{Meas}^{\pm}(X, \mathcal{M}) \ : \ \mu(A) \geq 0, \forall A \in \mathcal{M}\} = \{\mu \textit{ a finite signed measure}\}.$$

Then,

1. $V^+ \in \mathrm{Meas}^+(X, \mathcal{M})$
2. $V^- \in \mathrm{Meas}^+(X, \mathcal{M})$
3. $V^+ - V^- = \nu$ *(This is called the Jordan Decomposition of V)*

Definition **12.6.** Let $\nu \in \mathrm{Meas}^{\pm}(X, \mathcal{M})$. The measure $V^+ + V^- \in \mathrm{Meas}^+(X, \mathcal{M})$ is called the **Total Variation Measure** of ν, denoted by $|\nu|$. The number $|\nu|(X) = V^+(X) + V^-(X)$ is called the **Total Variation** of V denoted by $\|\nu\|$.

Lemma **12.7.** *Let $\nu \in \mathrm{Meas}^{\pm}(X, \mathcal{M})$ and let V^+ be the positive variation of ν. Then, $V^+(X) < \infty$ and hence $V^+ \in \mathrm{Meas}^+(X, \mathcal{M})$.*

Proof.

Claim (1)**.** Let $E \in \mathcal{M}$ and suppose that $V^+(E) = \infty$. Then, can write $E = A \cup B$ with $A \cap B = \emptyset$ sucht that $|\nu(A)| \geq 1$ and $|\nu(B)| \geq 1$.

Verification of Claim 1. By definition, we can find $B \in \mathcal{M}$ such that $B \subseteq E$ with the property that $\nu(B) > 1 + |\nu(E)|$. Put $A = E \backslash B$, then $\nu(A) = \nu(E) - \nu(B)$. Hence,

$$|\nu(A)| = |\nu(B) - \nu(E)| \geq |\nu(B)| - |\nu(E)| = \nu(B) - \nu(E) \geq 1$$

∎

Claim (2)**.** Let $E \in \mathcal{M}$ and suppose $V^+(E) = \infty$. Then, A, B from Claim 1 can be picked such that (in addition to the properties form Claim 1) we have $V^+(A) = \infty$.

Verification of Claim 2. V^+ is a positive measure and hence

$$V^+(A) + V^+(B) = V^+(A \cup B) = V^+(E) = \infty.$$

Therefore, at least one of $V^+(A), V^+(B)$ is infinite. By swapping A and B if necessary, we may assume that $V^+(A) = \infty$. ∎

Claim (3). $V^+(X) < \infty$.

Verification of Claim 3. Assume by contradiction that $V^+(X) = \infty$. By Claim 2, we can find $A_1, B_1 \in \mathcal{M}$ with $A \cup B = X$, $A_1 \cap B_1 = \emptyset$, $V^+(A_1) = \infty$, and $|\nu(B_1)| \geq 1$. Now apply claim 2 to A. This gives us a partition $A_1 = A_2 \cup B_2$ with $A_2, B_2 \in \mathcal{M}$ such that $A_2 \cap B_2 = \emptyset$, $V^+(A_2) = \infty$, and $|\nu(B_2)| \geq 1$. Continue recursively, get sequences of sets $(A_n), (B_n)$ in $\mathcal{M}$ where

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \ldots$$

and

$$B_1 = X \backslash A_1,\ B_2 = A_1 \backslash A_2,\ \ldots,\ B_n = A_{n-1} \backslash A_n,\ \ldots$$

and where $|\nu(B_n)| \geq 1$, for all $n \in \mathbb{N}$. Note that our construction gives $B_n \cap B_m = \emptyset, \forall n \neq m$. The definition of a finite signed measure implies that

$$\sum_{n=1}^{\infty} |\nu(B_n)| < \infty.$$

So the assumption $V^+(X) = \infty$ leads to a contradiction. Hence, $V^+(X) < \infty$. ∎

Claim 3 ends the proof. ∎

Proof of Theorem 108.

1. Follows from Lemma 110
2. Follows from 1, applied to the measure $-\nu \in \text{Meas}^{\pm}(X, \mathcal{M})$
3. Will prove the required equality $V^+(A) - V^-(A) = \nu(A)$ by double inequality.

Claim (“$\geq$”). If $\nu \in \text{Meas}^{\pm}(X, \mathcal{M})$ and $A \in \mathcal{M}$, then $\nu(A) \geq V^+(A) - V^-(A)$.

Verification of “$\geq$”. By the definition of $V^+(A)$, we can find a sequence $(B_n) \in \mathcal{M}$ with $B_n \subseteq A$ for all $n \in \mathbb{N}$ such that $\nu(B_n) \to V^+(A)$ (from below). For every $n \in \mathbb{N}$ we put $C_n = A \backslash B_n$. We have, $\nu(C_n) = \nu(A) - \nu(B_n)$. Note that

$$C_n \in \mathcal{M}, C_n \subseteq A \implies \nu(C_n) \geq \inf\{\nu(C) \ :\ C \in \mathcal{M}, C \subseteq A\} = -V^-(A).$$

So we get $\nu(A) - \nu(B_n) = \nu(C_n) \geq -V^-(A)$, for all $n \in \mathbb{N}$. Let $n \to \infty$ and we get

$$\nu(A) - V^+(A) \geq -V^-(A).$$

Hence, $\nu(A) \geq V^+(A) - V^-(A)$ as claimed. ∎

Claim ("$\leq$"). If $\nu \in \mathrm{Meas}^{\pm}(X, \mathcal{M})$ and $A \in \mathcal{M}$, then we have $\nu(A) \leq V^+(A) - V^-(A)$.

Verification of "$\leq$". Use claim "$\geq$" for the measure $-\nu$. Let us denote $-\nu =: \sigma$ and let W^+, W^- denote the positive/negative variations for σ. By claim "$\geq$" for σ we have

$$\sigma(A) \geq W^+(A) - W^-(A).$$

But we know that $W^+ = V^-$ and $W^- = V^+$ so we get

$$-\nu(A) \geq V^-(A) - V^+(A)$$

which implies

$$\nu(A) \leq V^+(A) - V^-(A).$$

∎

This ends the proof of the Jordan Decomposition Theorem. ∎

13 Hahn Decomposition for Finite Signed Measures

DEFINITION **13.1.** Let $(X, \mathcal{M})$ be a measurable space.

1. Let $\mu \in \text{Meas}^+(X, \mathcal{M})$ and let $A \in \mathcal{M}$. We say that μ is concentrated on A if and only if $\mu(X \setminus A) = 0$.

2. Let $\mu_1, \mu_2 \in \text{Meas}^+(X, \mathcal{M})$. We say that μ_1, μ_2 are mutually singular (denoted as $\mu_1 \perp \mu_2$) when there exists $A_1, A_2 \in \mathcal{M}$ such that $A_1 \cap A_2 = \emptyset$ and such that μ_1 is concentrated on A_1 and μ_2 is concentrated on A_2.

THEOREM **13.2.** *Let $(X, \mathcal{M})$ be a measurable space. Let $\nu \in \text{Meas}^\pm(X, \mathcal{M})$. Let $V^+, V^- \in \text{Meas}^+(X, \mathcal{M})$ be the positive and negative variations of ν. Then $V^+ \perp V^-$.*

REM **13.3.** Let $(X, \mathcal{M})$, ν, V^+, and V^- be as above. Then $V^+ \perp V^-$ implies that there exists $Y, Z \in \mathcal{M}$ with $Y \cap Z = \emptyset$ such that V^+ is concentrated on Y and that V^- is concentrated on Z. Put $Y^+ := Y$ and $Y^- := X \setminus Y \supseteq Z$. Then we have

$$\text{(H-Dec.)} \begin{cases} Y^+, Y^- \in \mathcal{M} \\ Y^+ \cup Y^- = X \\ Y^+ \cap Y^- = \emptyset \\ V^+ \text{ concentrated on } Y^+ \\ V^- \text{ concentrated on } Y^- \end{cases} .$$

A pair (Y^+, Y^-) satisfying (H-Dec.) is called a Hahn decomposition of ν. The previous theorem says that some Hahn decompositions do exist!

REM **13.4.** [WHY IS (H-DEC.) GOOD?] Let Y^+, Y^- be a Hahn decomposition of ν. Then

$$\left.\begin{array}{c} A \in \mathcal{M} \\ A \subseteq Y^+ \end{array}\right\} \Longrightarrow \nu(A) = V^+(A),$$

and

$$\left.\begin{array}{c} A \in \mathcal{M} \\ A \subseteq Y^- \end{array}\right\} \Longrightarrow \nu(A) = -V^-(A).$$

Why is this true? We have that

$$\begin{aligned} V^+ \text{ concentrated on } Y^+ &\Longrightarrow V^+(X \setminus Y^+) = 0 \\ &\Longrightarrow V^+(Y^-) = 0 \\ &\Longrightarrow V^+(A) = 0, \forall A \in \mathcal{M} \text{ such that } A \subseteq Y^-. \end{aligned}$$

Likewise, we also have

$$V^- \text{ concentrated on } Y^- \Longrightarrow V^-(A) = 0, \forall A \in \mathcal{M} \text{ such that } A \subseteq Y^+.$$

Now we recall that $\nu = V^+ - V^-$, so

$$\left.\begin{array}{r} A \in \mathcal{M} \\ A \subseteq Y^+ \end{array}\right\} \Longrightarrow \nu(A) = V^+(A) - V^-(A) = V^+(A),$$

and

$$\left.\begin{array}{r} A \in \mathcal{M} \\ A \subseteq Y^- \end{array}\right\} \Longrightarrow \nu(A) = V^+(A) - V^-(A) = -V^-(A).$$

So for a general set $E \in \mathcal{M}$ we have

$$\nu(E) = \nu(E \cap Y^+) + \nu(E \cap Y^-) = V^+(E \cap Y^+) - V^-(E \cap Y^-).$$

REM **13.5.** [AN IDEA OF THE PROOF] We have that $(X, \mathcal{M})$ is a measurable space, we have $\nu \in \text{Meas}^{\pm}(X, \mathcal{M})$, and we have the variations V^+ and V^- for ν. We need $Y, Z \in \mathcal{M}$, such that $Y \cap Z = \emptyset$, that V^+ is concentrated on Y, and that V^- is concentrated on Z. Suppose that we found such Y, Z. Then, since V^+ is concentrated on Y, we observe that

$$\nu(Y) = V^+(Y) - V^-(Y) = V^+(Y) = V^+(Y) + V^+(X \setminus Y) = V^+(X)$$

and since V^- is concentrated on Z, we observe that

$$\nu(Z) = V^+(Z) - V^-(Z) = -V^-(Z) = -V^-(Z) - V^-(X \setminus Z) = -V^-(X).$$

Hence, the Y, Z that we're looking for must have $\nu(Y) = V^+(X) = \sup\{\nu(A) : A \in \mathcal{M}\}$ and $\nu(Z) = -V^-(X) = \inf\{\nu(A) : A \in \mathcal{M}\}$.

Proof of Theorem 113. We have by assusmption that $(X, \mathcal{M})$ is a measurable space, that $\nu \in \text{Meas}^{\pm}(X, \mathcal{M})$, and that V^+, V^- are the variations of ν. Since

$$V^+(X) = \sup\{\nu(A) : A \in \mathcal{M}\},$$

for every $n \in \mathbb{N}$ we can find $A_n \in \mathcal{M}$ such that $\nu(A_n) > V^+(X) - \frac{1}{2^n}$.

(What happens if we try $Y = \bigcup_{n=1}^{\infty} A_n$? This is not good, can't conclude that $\nu(Y) \geq \nu(A_n)$.)

We will have 3 claims:

1. $V^+(X \setminus A_n) \leq \frac{1}{2^n}$, for all $n \geq 1$.
2. $V^-(A_n) \leq \frac{1}{2^n}$, for all $n \geq 1$.
3. Let $T = \bigcap_{k=1}^{\infty} \left(\bigcup_{n=k}^{\infty} A_n\right)$. Then $V^-(T) = 0$ and $V^+(X \setminus T) = 0$, hence V^- is concentrated on $X \setminus T$ and V^+ is concentrated on T.

This will finish the proof, since we can simply take $Y = X \setminus T$ and $Z = T$.
(TO BE CONTINUED NEXT CLASS) ∎

Monday, November 12

Now we verify the claims. Have $(\chi, \mathcal{M}), \nu \in \text{Meas}^{\pm}(X, \mathcal{M})$ and $V^+, V^- \in \text{Meas}^+(X, \mathcal{M})$ are the positive/negative variation of ν. For every $n \in \mathbb{N}$, we pick $A_n \in \mathcal{M}$ with $\nu(A_n) > V^+(X) - \frac{1}{2^n}$. Recall that $V^+(X) = \sup\{\nu(A) | A \in \mathcal{M}\} < \infty$.

Proof of Claim 1. Have $V^+(X \backslash A_n) = \sup\{\nu(B) | \mathcal{B} \in \mathcal{M}, B \subseteq X \backslash A_n\}$. We must show that $\nu(B) \leq \frac{1}{2^n}, \forall B \in \mathcal{M}$ such that $B \subseteq X \backslash A_n$. Assume by contradiction that $\exists B \in \mathcal{M}, B \subseteq X \backslash A_n$ such that $\nu(B) > \frac{1}{2^n}$. Then,

$$\nu(B \cup A_n) = \nu(B) + \nu(A_n) > \frac{1}{2^n} + \left(V(X) - \frac{1}{2^n}\right) = V^+(X)$$

which contradicts the definition of $V^+(X)$. ∎

Proof of Claim 2. From Jordan Decomposition we have $\nu(A_n) = V^+(A_n) - V^-(A_n) \Rightarrow$

$$\begin{aligned} V^+(A_n) - V^-(A_n) = \nu(A_n) > \nu^+(X) - \frac{1}{2^n} \geq \nu^+(A_n) - \frac{1}{2^n} \Rightarrow \\ V^+(A_n) - V^-(A_n) \geq V^+(A_n) - \frac{1}{2^n} - \frac{1}{2^n} \Rightarrow \\ V^-(A_n) \leq \frac{1}{2^n} \end{aligned}$$

∎

Verification of Claim 3. The fact that $V^-(T) = 0$ follows from the trick of the tail set (Homework 2, Problem 1) since $V^-(A_n) < \frac{1}{2^n}$ for all $n \in \mathbb{N}$ (by claim 2) with $\sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$. For $V^+(X \backslash T)$, let us write

$$\begin{aligned} X \backslash T &= X \backslash \bigcap_{k=1}^{\infty} \left(\bigcup_{n=k}^{\infty} A_n \right) \\ &= \bigcup_{k=1}^{\infty} \left(X \backslash \left(\bigcup_{n=k}^{\infty} A_n \right) \right) \\ &= \bigcup_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} (X \backslash A_n) \right) \end{aligned}$$

Call the inner term (the intersection) B_k. So have $X \backslash T = \bigcup_{k=1}^{\infty} B_k$ where $B_k = \bigcap_{n=k}^{\infty} (X \backslash A_n), \forall k \in \mathbb{N}$. For every $k, n \in \mathbb{N}$ such that $k \geq n$, we have

$$B_k \subseteq X \backslash A_n \Rightarrow V^+(B_k) \leq V^+(X \backslash A_n) \leq \frac{1}{2^n}$$

Last inequality follows by claim 1. Fix $k \in \mathbb{N}$, make $n \to \infty$, get $V^+(B_k) = 0$. Now unfix $k \in \mathbb{N}$ and use subadditivity for V^+ to get

$$V^+(X \backslash T) \leq \sum_{k=1}^{\infty} V^+(B_k) = 0$$

hence $V^+(X \backslash T) = 0$. ∎

14 Absolute Continuity, Radon-Nikodyn, and Lebesgue Decomposition Theorem

DEFINITION **14.1.** Let $(X, \mathcal{M})$ a measurable space, $\mu, \nu : \mathcal{M} \to [0, \infty]$ a positive measures. Say that ν is **absolutely continuous** with respect to μ and write $\nu \ll \mu$ to mean that $A \in \mathcal{M}$ and $\mu(A) = 0$ implies $\nu(A) = 0$.

DEFINITION **14.2.** $(X, \mathcal{M})$ a measurable space and say that a positive measure $\mu : \mathcal{M} \to [0, \infty]$ is **σ-finite** to mean there exists a sequence $(A_n)_{n=1}^{\infty}$ in $\mathcal{M}$ such that $\bigcup_{n=1}^{\infty} A_n = X$ and such that $\nu(A_n) < \infty, \forall n \in \mathbb{N}$.

THEOREM **14.3.** *$(X, \mathcal{M})$ a measurable space and let $\mu, \nu : \mathcal{M} \to [0, \infty]$ be positive measures on $\mathcal{M}$ where ν is finite ($\nu(X) < \infty$) and μ is σ-finite. Then, the following are equivalent*

1. *$\nu \ll \mu$.*
2. *$\forall \epsilon > 0, \exists \delta > 0$ such that $A \in \mathcal{M}, \mu(A) < \delta \Rightarrow \nu(A) < \epsilon$. (Called the Absolute continuity $\epsilon - \delta$)*
3. *$\exists h \in \mathrm{Bor}^+(X, \mathbb{R}) \cap \mathcal{L}^1(\mu)$ such that $d\nu = h d\mu$. That is, we have, $\forall A \in \mathcal{M}$,*

$$\nu(A) = \int_A h d\mu = \int h I_A d\mu$$

REM **14.4.**
1. Some of the implications are immediate, and do not equire the hypothesis of finite/σ-finite for ν and μ, (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1). For (2) $\Rightarrow$ (1) proof for every $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that $\mu(B) < \delta_n \Rightarrow \nu(B) < \frac{1}{n}$. Make $B = A$ and let $n \to \infty$ and done.
2. For (3) implies (1) assume $\nu = h d\mu$. But then for $A \in \mathcal{M}$ we must have $\mu(A) = 0 \Rightarrow h I_A = 0$ almost everywhere μ implies that $\int h I_A d\mu = \int 0 d\mu = 0$.
3. For the implication (1) $\Rightarrow$ (2) needs the hypothesis $\nu(X) < \infty$ (but does not require μ to be σ-finite). If we allow $\nu(X) = \infty$, it is easy to find examples when (1) holds but (2) does not (Homework Problem).
4. (1) $\Rightarrow$ (3) is the Radon-Nikodyn Theorem. In this implication, we may allow both μ and ν to be σ-finite. The essence of the proof is the case when $\mu(X), \nu(X) < \infty$, after that one bootstraps to the σ-finite case.

PROPOSITION **14.5.** *$(X, \mathcal{M})$ a measurable space. $\mu, \nu : \mathcal{M} \to [0, \infty]$ positive measures, where $nu(X) < \infty$. If $\nu ll \mu$, then Absolute Continuity w.r.t $\epsilon - \delta$ holds. ((1) $\Rightarrow$ (2)) in the theorem)*

Proof. Assume by contradiction that there exists $\epsilon > 0$ for which no $\delta > 0$ works in (Absolute Continuity $\epsilon - \delta$. For every $k \in \mathbb{N}$, let us record how $\delta = \frac{1}{2^k}$ fails to work. ∎

PROPOSITION **14.6.** *Let $(X, \mathcal{M})$ be a measurable space and suppose $\mu, \nu : \mathcal{M} \to [0, \infty)$ are finite positive measures such that $\nu \leq \mu$ (in the sense that $\nu(A) \leq \mu(A)$ for every $A \in \mathcal{M}$). Then there exists $g \in \mathrm{Bor}(X, \mathbb{R})$ with $0 \leq g(x) \leq 1, \forall x \in X$ and such that $d\nu = g d\mu$.*

PROPOSITION **14.7.** *Let $(X, \mathcal{M})$ be a measurable space. Let $\mu, \nu : \mathcal{M} \to [0, \infty]$ be positive measures, where $\nu(X) < \infty$. If $\nu << \mu$, then absolute continuity ϵ-δ holds. In other words, (1)$\Longrightarrow$ (2).*

Proof. Assume by way of contradiction that there exists $\epsilon > 0$ for which no δ works in absolute continuity ϵ-δ. For every $k \in \mathbb{N}$, let us record how $\delta = \frac{1}{2^k}$ fails to work. Not true that

$$\left.\begin{array}{c} A \in \mathcal{M} \\ \mu(A) < \frac{1}{2^k} \end{array}\right) \Longrightarrow \nu(A) < \epsilon.$$

hence there exists $A_k \in \mathcal{M}$ such that $\mu(A_k) < \frac{1}{2^k}$ but $\nu(A_k) \geq \epsilon$. Look at the tail-swt

$$T = \bigcap_{k=1}^{\infty} \left(\bigcup_{n=k}^{\infty} A_n \right) \in \mathcal{M}$$

Since $\sum_{k-1}^{\infty} \mu(A_k) \leq \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty$, we know that $\mu(T) = 0$ (from problem 1 in homework 2, Borel-Cantelli lemma). We now have

$$T = \bigcap_{k=1}^{\infty} T_k$$

with

$$T_1 \supseteq T_2 \supseteq \cdots \supseteq T_k \supseteq \cdots$$

where $T_k = \bigcap_{n=k}^{\infty} A_n$. For every $k \in \mathbb{N}$ we have

$$T_k \supseteq A_k \Longrightarrow \nu(T_k) \geq \nu(A_k) \geq \epsilon.$$

Hence, we find

$$\nu(T) = \lim_{k \to \infty} \nu(T_k) \geq \epsilon > 0$$

by continuity along decreasing chains (we use here that ν is finite). So we found $\mu(T) = 0$ but $\nu(T) \neq 0$. This is a contradiction with $\nu << \mu$. ∎

REM **14.8.** Bootstrapping in RN can in fact be started from the special case of the "bounded Radon-Nikodyn Derivative"

PROPOSITION **14.9.** *Let $(X, \mathcal{M})$ be a measurable space. Let $\mu, \nu \in \mathrm{Meas}^+(X, \mathcal{M})$. Suppose that $\nu \leq \mu$, in the sense that $\nu(A) \leq \mu(A)$ for all $A \in \mathcal{M}$. Then there exists a $g \in \mathrm{Bor}(X, \mathbb{R})$ with $0 \leq g(x) \leq 1, \forall x \in X$, such that $d\nu = g\, d\mu$.*

REM **14.10.** Note that $\nu \leq \mu$ implies $\nu << \mu$. So the previous proposition has a stronger hypothesis, but also a stronger conclusion than (1)$\Longrightarrow$(3) of theorem 14.3.

REM **14.11.** How to find the idea of the proof from the proposition? Fix $t \in [0,1]$ and look at the sets

$$P = \{x \in X \;:\; g(x) \geq t\}$$
$$Q = \{x \in X \;:\; g(x) \leq t\}$$

For every $A \in \mathcal{M}$ with $A \subseteq P$, we have

$$\nu(A) = \int gI_A d\mu \geq \int t \cdot I_A d\mu = t\mu(A).$$

Note that $gI_A \geq tI_A$ because $g(x) \geq t, \forall x \in A$. Likewise for $B \in \mathcal{M}, B \subseteq Q$ we have,

$$\begin{aligned}\nu(B) &= \int gI_B d\mu \\ &\leq \int tI_B d\mu = t\mu(B)\end{aligned}$$

$gI_B \leq tI_B$ because $g(x) \leq t, \forall x \in B$. Write these facts in terms of the signed measure $\nu - t$ $mu \in \text{Meas}^{\pm}(X, \mathcal{M})$, we get

$$(A \in \mathcal{M}, A \subseteq P) \Rightarrow (\nu 0 t\mu)A \geq 0$$
$$(B \in \mathcal{M}, B \subseteq Q) \Rightarrow (\nu - t\mu)(B) \leq 0$$

It follows that $(P, X \backslash P)$ is a Hahn decomposition for $V - t\mu$

Proof. Given $\mu \, \nu \in \text{Meas}^{+}(X, \mathcal{M})$ such that $\nu \leq \mu$. For every $t \in [0,1]$, consider a Hahn decomposition (P_t, Q_t) for the signed measure $\nu - t\mu \in \text{Meas}^{\pm}(X, \mathcal{M})$. ∎

Friday, November 16

Today we prove "Proposition 14.6" (Radon-Nikodyn with bounded derivative).

Proof started on Wednesday. Given $\mu, \nu \in \text{Meas}^{+}(X, \mathcal{M})$ such that $\nu \leq \mu$. For every $t \in [0,1]$, consider the Hahn decomposition (P_t, Q_t) for the signed measure $\nu - t\mu \in \text{Meas}^{\pm}(X, \mathcal{M})$. For $t = 0$, we have $\nu - t\mu = \nu - 0\mu = \nu \in \text{Meas}^{+}(X, \mathcal{M})$ so we can pick $P_0 = X$ and $Q_0 = \emptyset$. For $t = 1$, we have $\nu - t\mu = \nu - \mu = -(\mu - \nu) \in -\text{Meas}^{+}(X, \mathcal{M})$ so we can pick $P_1 = \emptyset$ and $Q_1 = X$. For general $t \in [0,1]$ we have

$$P_t, Q_t \in \mathcal{M} \text{ and } P_t \cup Q_t = X \text{ and } P_t \cap Q_t = \emptyset$$

we also have

$$(\nu - t\mu)(A) \geq 0, \forall A \in \mathcal{M}, A \subseteq P_t$$

and

$$(\nu - t\mu)(B) \leq 0, \forall B \in \mathcal{M}, B \subseteq Q_t.$$

We will focus on the sets $(Q_t)_{t \in [0,1]}$. (We would be happy if $Q_s \subseteq Q_t$ for $s \leq t$, but we're not that lucky...)

Claim (1). For $0 \leq s < t \leq 1$ we have $\mu(Q_s \setminus Q_t) = 0$.

Verification of Claim 1. Denote $A := Q_s \setminus Q_t = Q_s \cap (X \setminus Q_t) = Q_s \cap P_t$. Then we have

$$A \subseteq Q_s \Longrightarrow (\nu - s\mu)(A) \leq 0 \Longrightarrow \nu(A) \leq s\mu(A)$$

and

$$A \subseteq P_t \Longrightarrow (\nu - t\mu)(A) \geq 0 \Longrightarrow \nu(A) \geq t\mu(A).$$

Hence we have

$$t\mu(A) \leq \nu(A) \leq s\mu(A)$$

which implies

$$(s-t)\mu(A) \geq 0$$

So $\mu(A)$ must be 0 since $(s-t) < 0$. ∎

Claim (2). Denote

$$N := \bigcup_{s,t \in [0,1] \cap \mathbb{Q}} (Q_S \setminus Q_t)$$

Then $N \in \mathcal{M}$ with

$$\mu(N) = \nu(N) = 0$$

Verification of Claim 2. We have

$$0 \leq \nu(N) \leq \mu(N) \leq \sum_{s,t \in [0,1] \cap \mathbb{Q}} (Q_S \setminus Q_t) = 0$$

∎

Claim (3). For every $t \in \mathbb{Q} \cap [0,1]$, put

$$\tilde{Q}_t = Q_t \cup N \text{ and } \tilde{P}_t = X \setminus \tilde{Q}_t = P_t \setminus N.$$

Then $(\tilde{P}_t, \tilde{Q}_t)$ still is a Hahn decomposition for $\nu - t\mu$. Moreover we have that $s < t$ in $\mathbb{Q} \cap [0,1]$ implies that $\tilde{Q}_s \subseteq \tilde{Q}_t$. We also have that $\tilde{Q}_1 = X$ and $\tilde{Q}_0 = N$

Verification of Claim 3. ∎

Fix $t \in [0,1] \cap \mathbb{Q}$.

$$A \in \mathcal{M}, A \subseteq \tilde{P}_t \Rightarrow A \in \mathcal{M}, A \subseteq P_t \Rightarrow (\nu - t\mu)(A) \geq 0$$

For $B \in \mathcal{M}, B \subseteq \tilde{Q}_t$, we write $B = (B \cap Q_t) \cup (B \cap (N \backslash Q_t)) \Rightarrow$

$$(\nu - t\mu)(B) = (\nu - t\mu)(B \cap Q_t) + (\nu - t\mu)(B \cap (N \backslash Q_t))$$

Let B' be the right part of the sum. We have $B' \in, B' \subseteq N \Rightarrow (\nu - t\mu)(B') = 0$. The left part of the sum is less than or equal to 0 because $B \cap Q_t \subseteq Q_t$, use that (P_t, Q_t) is a Hahn Decomposition. Hence, $(\nu - t\mu)(B) \leq 0$. Moreover, for $s < t \in \mathbb{Q} \cap [0,1]$ we have

$$\tilde{Q}_s = Q_s \cup N = (Q_s \cap Q_t) \cup (Q_s \backslash Q_t) \cup N = (Q_s \cap Q_t) \cup N \subseteq Q_t \cup N = \tilde{Q}_t$$

Note that $(Q_s \backslash Q_t) \cup N$, this is N, by the definition of N. Finally,

$$\tilde{Q}_1 = Q_1 \cup N = X \cup N = X$$
$$\tilde{Q}_0 = Q \cup = \emptyset \cup N = N$$

∎

Claim (4). There exists $g \in \mathrm{Bor}(X, \mathbb{R})$ with $0 \leq g(x) \leq 1$ such that for every $t \in \mathbb{Q} \cap [0,1]$ we have

$$\begin{cases} x \in \tilde{P}_t & \Longrightarrow & g(x) \geq t \\ x \in \tilde{Q}_t & \Longrightarrow & g(x) \leq t \end{cases}$$

rem **14.12.** For Claim 4 we use Lemma 14.8 (which will be on homework 7).

$$B_t \text{ from Lemma 14.8} \longleftrightarrow \tilde{Q}_t \text{ here}$$

The proof of the lemma is elementary. Hint:

$$g(x) := \inf\{t \in \mathbb{Q} \cap [0,1] \ : \ x \in B_t\}$$

Claim (5). THe function g found in Claim 4 satisfies

$$\nu(A) = \int_A g d\mu, \forall A \in \mathcal{M} \tag{$*$}$$

So indeed, is such that $d\nu = g d\mu$

Verification of Claim 5. Fix $A \in \mathcal{M}$ for which we prove $(*)$. In fact we will prove

$$\left|\nu(A) - \int_A g d\mu\right| < \frac{\mu(x)}{n}, \forall n \in \mathbb{N} \tag{$**$}$$

(Of course, $(**) \Longrightarrow (*)$ when $n \to \infty$). So let us also fix $n \in \mathbb{N}$, for which we verify $(**)$. The Trick will be to find a partition $A = A_0 \cup A_1 \cup \cdots \cup A_n$ such that

$$\sum_{i=1}^{n} \frac{i-1}{n} \mu(A_i) \leq \nu(A), \int_A g d\mu \leq \sum_{i=1}^{n} \frac{i}{n} \mu(A_i)$$

∎

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Continuing proof of Claim 5. Look a the sets

$$N = \tilde{Q}_0 \subseteq \tilde{Q}_{\frac{1}{n}} \subseteq \cdots \subseteq \tilde{Q}_{\frac{n-1}{n}} \subseteq \tilde{Q}_1 = X$$

We get a partition of X, namely, $X = \tilde{Q}_0 \cup (\tilde{Q}_{\frac{1}{n}} \setminus \tilde{Q}_0) \cup \cdots \cup (\tilde{Q}_1 \setminus \tilde{Q}_{\frac{n-1}{n}})$. This gives us a partition of $A = A_0 \cup A_1 \cup \cdots \cup A_n$ where $A_0 = A \cap \tilde{Q}_0 = A \cap N$ and where for $1 \leq i \leq n$ we put

$$\begin{aligned} A_i &= A \cap \left(\tilde{Q}_{\frac{i}{n}} \setminus \tilde{Q}_{\frac{n-1}{n}}\right) \\ &= A \cap \tilde{Q}_{\frac{i}{n}} \cap \tilde{P}_{\frac{i-1}{n}} \end{aligned}$$

Then,

$$\nu(A) = \sum_{i=0}^{n} \nu(A_i) = \sum_{i=1}^{n} \nu(A_i).$$

$\nu(A_0) = 0$ because $A_0 \subseteq N$ by claim 2. For $1 \leq i \leq n$ we observe that

$$\begin{aligned} A_i \subseteq \tilde{Q}_{\frac{i}{n}} &\implies (\nu - \frac{i}{n}\mu)(A_i) \leq 0 \\ &\implies \nu(A_i) \subseteq \frac{i}{n}\mu(A_i) \\ A_i \subseteq \tilde{P}_{\frac{i-1}{n}} &\implies (\nu - \frac{i-1}{n}\mu)(A_i) \geq 0 \\ &\implies \nu(A_i) \geq \frac{i-1}{n}\mu(A_i) \end{aligned}$$

So we get

$$\frac{i-1}{\mu}(A_i) \leq \nu(A_i) \leq \frac{i}{n}\mu(A_i), 1 \leq i \leq n$$

Summing over i yields

$$\sum_{i=1}^{n} \frac{i-1}{n}\mu(A_i) \leq \nu(A) \leq \sum_{i=1}^{n} \frac{i}{n}\mu(A_i) \tag{$\diamond$}$$

Now look at $\int_A g\, d\mu$. Partition again, $A = A_0 \cup A_1 \cup \cdots \cup A_n$ we get

$$\int_A g\, d\mu = \sum_{i=0}^{n} \int_{A_i} g\, d\mu = \sum_{i=1}^{n} \int_{A_i} g\, d\mu$$

For the last equality on the right, $\int_{A_0} g d\mu = 0$ since $A_0 \subseteq N$, hence $\mu(A_0) = 0$. For $1 \leq i \leq n$ (since $A_i = A \cap \tilde{Q}_{\frac{i}{n}} \cap \tilde{P}_{\frac{i-1}{n}}$) we have

$$\begin{aligned} A_i \subseteq \tilde{Q}_{\frac{i}{n}} &\Rightarrow g(x) \leq \frac{i}{n} \text{for all x in } A_i \\ A_i \subseteq \tilde{P}_{\frac{i-1}{n}} &\Rightarrow g(x) \geq \frac{i-1}{n} \text{ for all } x \in A_i \end{aligned}$$

So we get that

$$\frac{i-1}{n}\mu(A_i) \le \int_{A_i} g \le \frac{i}{n}\mu(A_i), 1 \le i \le n$$

Sum over i to get

$$\sum_{i=1}^{n} \frac{i-1}{n}\mu(A_i) \le \int_A g d\mu \le \sum_{i=1}^{n} \frac{i}{n}\mu(A_i) \tag{$\diamond\diamond$}$$

Putting ($\diamond$) and ($\diamond\diamond$) together gives us

$$\sum_{i=1}^{n} \frac{i-1}{n}\mu(A_i) \le \begin{matrix} \nu(A) \\ \int_A g\, d\mu \end{matrix} \le \sum_{i=1}^{n} \frac{i}{n}\mu(A_i)$$

It follows that

$$\begin{aligned} \left|\nu(A) - \int_A g\, d\mu\right| &\le \left(\sum_{i=1}^{n} \frac{i}{n}\mu(A_i)\right) - \left(\sum_{i=1}^{n} \frac{i-1}{n}\mu(A_i)\right) \\ &= \sum_{i=1}^{n} \frac{1}{n}\mu(A_i) = \frac{1}{n}\sum_{i=1}^{n}\mu(A_i) \le \frac{1}{n}\mu(X). \end{aligned}$$

This finishes the proof, but I didn't get the last thing he wrote... ∎

REM **14.13.** Given $(X, \mathcal{M})$ and $\mu, \nu \in \mathrm{Meas}^+(X, \mathcal{M})$ (no relation assumed between them!). Idea: We can consider $\sigma := \mu + v \in \mathrm{Meas}^+(X, \mathcal{M})$ abd we gave $\mu \le \sigma, \nu \le \sigma$. Hence, by the Radon-Nikodym Theorem, can be applied to ν andσ. Easy fact about σ we have

$$\int f d\sigma = \int f d\mu + \int f d\nu \tag{$\circ$}$$

for all $f \in \mathrm{Bor}^+(X, \mathbb{R})$. Why? If $f = I_A$ then ($\circ$) becomes $\sigma(A) = \mu(A) + \nu(A)$. Then do linear combinations, use LMCT.

Wednesday, November 21

PROPOSITION **14.14.** *14.10 Let $(X, \mathcal{M})$ be a measurable space and let $\nu, \mu : \mathcal{M} \to [0, \infty)$ be two finite positive measures. There exists a function $g \in \mathrm{Bor}(X, \mathbb{R})$ such that $0 \le g(x) \le 1$ for all $x \in X$ and such that*

$$\int fg d\mu = \int f(1-g) d\nu, \forall f \in \mathrm{Bor}_b^+(X, \mathbb{R})$$

Proof. Let $\sigma \in \mu + \nu \in \mathrm{Meas}^+(X, \mathcal{M})$. Then, $\nu \le \sigma$, hence proposition 14.6 (Radon Nikodym) gives us $g \in \mathrm{Bor}(X, \mathbb{R})$ with $0 \le g(x) \le 1$ and such that $d\nu = g d\sigma$. For every $f \in \mathrm{Bor}^+(X, \mathbb{R})$. We then get that

$$\int f d\nu = \int fg d\sigma = \int fg d\mu + \int fg d\nu$$

If we also assume that f is bounded, then it follows that $\int fgd\nu < \infty$. Hence, we can do algebra to get

$$\int ff\nu - \int fgd\nu = \int fgd\nu \Rightarrow$$
$$\int f(1-g)d\nu = \int fgd\mu$$

■

LEMMA **14.15.** $(X, \mathcal{M}), \nu, \mu \in \mathrm{Meas}^+(X, \mathcal{M})$. *Let* $g \in \mathrm{Bor}(X, \mathbb{R})$ *connecting function as in Proposition 14.10. Let* $N = \{x \in X | g(x) = 1\}$, $(N \in \mathcal{M})$. *Then,*

1. $\mu(N) = 0$.
2. $A \in \mathcal{M}, \mu(A) = 0 \Rightarrow \nu(A \cap (X \backslash N)) = 0$.

Proof. We know that $0 \leq g(x) \leq 1, \forall x \in X$ and

$$f(1-g)d\nu = \int fgd\mu \qquad ((3))$$

for all $f \in \mathrm{Bor}_b^+(X, \mathbb{R})$.

1. Put $f = I_N$ in (3). Then, $f(1-g) = I_N(1-g) = 0$, $fg = I_N g = I_N$. $x \in X \backslash N \Rightarrow I_N(x) = 0$ and $x \in N \Rightarrow (1-g)(x) = 1-1 = 0$. (3) gives $\int 0d\nu = \int I_n d\mu \Rightarrow 0 = \mu(N)$.
2. In (3) we put $f = I_A$ to get

$$\int I_A(1-g)d\nu = \int I_A g d\mu = \int 0 d\mu = 0$$
$$I_A f = 0 \text{ a.e. } \mu$$

(It is 0 on $X \backslash A$ where $\mu(A) = 0$. So we get that

$$\int I_A(1-g)d\nu = 0 \Rightarrow$$
$$I)A(1-g) = 0 \text{a.e. } \nu$$

This implies that

$$\mu\left(\{x \in X | I_A(x)(1-g(x)) \neq 0\}\right) = 0 \qquad (\diamond)$$

But $I_A(x)(1-g(x)) \neq 0$ if and only if $I_A(x) \neq 0$ and $g(x) \neq 1$ if and only if $x \in A \cap (X \backslash N)$. Then, $\diamond$ is $A \cap (X \backslash N)$ and so the result follows immediately.

■

THEOREM **14.16.** [RADON - NIKODYM] $\nu, \mu \in \mathrm{Meas}^+(X, \mathcal{M})$ *such that* $\nu l \mu$ *(We have* $A \in \mathcal{M}, \mu(A) \Rightarrow \nu(A) = 0$*). Then there exists* $h \in \mathrm{Bor}^+(X, \mathbb{R}) \cap \mathcal{L}^1(\mu)$ *such that* $d\nu = hd\mu$.

Proof. Let $g \in \mathrm{Bor}(X, \mathbb{R})$ be a connecting function between μ and ν (as in proposition 14.10). Let $N := \{x \in X | g(x) = 1\}$ (as in Lemma 14.11). Then, $\mu(N) = 0$ by above lemma part 1, hence $\nu(N) = 0$ as well (since $\nu \ll \mu$). Let $\tilde{g} : X \to \mathbb{R}$ defined by

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \in X \backslash N \\ 0 & \text{if} x \in N \end{cases}$$

Then, $\tilde{g} \in \mathrm{Bor}(X, \mathbb{R})$ (patching), $0 \leq \tilde{g}(x) < 1, \forall x \in X$. We claim that $\tilde{g}$ is still a connecting function between μ and ν. That is, we have

$$\int f(1 - \tilde{g}) d\nu = \int f \tilde{g} d\mu, \forall f \in \mathrm{Bor}_b^+(X, \mathbb{R}) \tag{$\tilde{3}$}$$

Indeed,

$$\int f(1 - \tilde{g}) d\nu = \int f(1 - g) d\nu = \int f g d\mu = \int f \tilde{g} d\mu$$

first equallity because $f(1 - g) = f(1 - \tilde{g})$ almost everywhere μ. Second equality by use of (3) and third equality because $fg = f\tilde{g}$ a.e. μ. Define

$$h \in \frac{\tilde{g}}{1 - \tilde{g}} \in \mathrm{Bor}^+(X, \mathbb{R}), x \in X)$$

Use here that $0 \leq \tilde{g}(x) \leq 1, \forall x \in X$. Given $f\ in \mathrm{Bor}_b^+(X, \mathbb{R})$, use ($\tilde{3}$) for the function $u = \frac{f}{1 - \tilde{g}}$. We claim that $\tilde{g}$ still a connecting function between μ and ν. That is we have by ($\tilde{3}$

$$\int u(1 - \tilde{g}) d\nu = \int u \tilde{g} d\mu \Rightarrow \int \frac{f}{1 - f} \tilde{g} d\nu = \int \frac{f}{1 - \tilde{g}} \tilde{g} d\mu$$

Where last equality because $\int f d\nu = \int f h d\mu$. We have a little problem to fix. ($\tilde{3}$) can be used for bounded functions, and we donky know it $u \frac{f}{1 - \tilde{g}}$ is bounded. Way to go: Instead of u, we use in ($\tilde{3}$) the function

$$u_n = f(1 + \tilde{g} + \cdots + \tilde{g}^n)$$

then make $n \to \infty$ and use LMCT on both sides. ∎

Friday, November 23

REM **14.17.** Some remarks about theorem 14.16.

1. h is uniquely determined a.e.-μ.
2. σ-finite case, do $h_n = \frac{d\nu_n}{d\mu_n}$.

THEOREM **14.18.** [LEBESGUE DECOMPOSITION THEOREM] *Let $(X, \mathcal{M})$ be a measurable space and let $\mu, \nu : \mathcal{M} \to [0, \infty)$ be two finite positive measures (where no relation is assumed between μ and ν). Then, one can write $\nu = \nu_1 + \nu_2$ where $\nu_1, \nu_2 \in \mathrm{Meas}^+(X, \mathcal{M})$ are such that $\nu \ll \mu$ and $\nu_2 \perp \mu$.*

Proof. Apply prop 14.14, find connecting function $g \in \text{Bor}(X, \mathbb{R})$ between μ and ν. So $0 \leq g(x) \leq 1, \forall x \in X$ and (3) holds. Put $N = \{x \in X | g(x) = 1\}$. Then, lemma 14.15 says that $\mu(N) = 0$ and $A \in \mathcal{M}, \mu(A) = 0 \Rightarrow \nu(A \cap (X \backslash N)) = 0$. Define $nu_1, \nu_2 : \mathcal{M} \to [0, \infty)$ by

$$\nu_1(A) = \nu(A \cap (X \backslash N))$$
$$\nu_2(A) = \nu(A \cap N)$$

CLAIM (1). $\nu_1, \nu_2 \in \text{Meas}^+(X, \mathcal{M})$ and $\nu_1 + \nu_2 = \nu$.

Verification of Claim 1. Immediate from formulas defining ν_1 and ν_2. ∎

CLAIM (2). $\nu_1 \ll \mu$.

Verification of Claim 2. By Lemma 14.15.2, $A \in \mathcal{M}, \mu(A) = 0 \Rightarrow \nu(A \cap (X \backslash N)) = 0$ implies that $\nu_1(A) = 0$. ∎

CLAIM (3). $\nu_2 \perp \mu$

Proof. We have $\nu_2(X \backslash N) = \nu(X \backslash N) \cap N) = \nu(\emptyset) = 0$. Where first equality follows by definition of ν_2. Hence, ν is concentrated on N. On the other hand, $\mu(N) = 0$ (by Lemma 14.15.1), hence μ is concentrated on $X \backslash N$. So then $\nu_2 \perp \mu$, as they are concentrated on N and $X \backslash N$. ∎

This concludes the proof.

∎

REM **14.19.** The following remarks conclude our discussion of lecture 14.

1. Uniqueness of the Lebesgue Decomposition

2. We can extend to the case when μ is σ-finite.

15 Product of Two (σ)-finite measures and the Fubini-Tonelli Theorem

DEFINITION **15.1.** Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. For $M \in \mathcal{M}, N \in \mathcal{N}$, can form

$$M \times N := \{(x, y) | x \in M, y \in M\} \subseteq X \times Y$$

Let

$$\mathcal{P} = \{M \times N | M \in \mathcal{M}, N \in \mathcal{N}\}$$

The σ-algebra of subsets of $X \times Y$ that is generated by $\mathcal{P}$ is called the direct product of $\mathcal{M}$ and $\mathcal{N}$, denoted as $\mathcal{M} \times \mathcal{N}$.

NOTE. $\mathcal{M} \times \mathcal{N}$ is usually strictly larger than $\mathcal{P}$, but it still good to be able to write that $(X \times Y, \mathcal{M} \times \mathcal{N})$. This is the definition of the **Direct Product** of Measurable Space.

REM **15.2.** $(X, \mathcal{M}), (Y, \mathcal{N})$ and $\mathcal{P} = \{M \times N | M \in \mathcal{M}, N \in \mathcal{N}\}$ as above then,

1. $\mathcal{P}$ is a **semialgebra** of subsets of X. It is a semialgebra in the sense that

   (a) $\emptyset \in \mathcal{P}$

   (b) $U_1, \ldots, U_k \in \mathcal{P} \Rightarrow U_1 \cap \cdots \cap U_k \in \mathcal{P}$

   (c) $U \in \mathcal{P} \Rightarrow (X \times Y) \setminus U$ can be written as a union of sets in $\mathcal{P}$.

   Verifications. We verify the three required conditions,

   (a) $\emptyset = \emptyset \times \emptyset \in \mathcal{P}$.

   (b) Write $U_i = M_i \times N_i, 1 \leq i \leq k$, where $M_i \in \mathcal{M}, N_i \in \mathcal{N}$, then

   $$U_1 \cap \cdots \cap U_k = M \times N$$

   Where $M = \bigcap_{i=1}^{k} M_i \in \mathcal{M}$ and $N = \bigcap_{i=1}^{k} N_i \in \mathcal{N}$.

   (c) Write $U = M \times N$ with $M \in \mathcal{M}, N \in \mathcal{N}$. Then

   $$(X \times Y) \setminus U = (M \times (Y \setminus N)) \cup (N \times (X \setminus M)) \cup ((X \setminus M) \times (Y \setminus N))$$

   ∎

2. $A = \{U \subseteq X \times Y | U$ can be written as a finite union of sets form $\mathcal{P}\}$. Then $\mathcal{A}$ is an algebra of subsets of $X \times Y$ and the σ-algebra generated by A is equal to the σ-algebra generated by $\mathcal{P}$ which is defined to be $\mathcal{M} \times \mathcal{N}$.

Monday, November 26

THEOREM **15.3.** *Let $(X, \mathcal{M}, \mu)$ and $(Y\mathcal{N}, \nu)$ be measure spaces where there measure μ, ν are σ-finite. Consider the measurable space $(X \times Y, \mathcal{M} \times \mathcal{N})$. There exists a positive measure $\pi : \mathcal{M} \times \mathcal{N} \to [0, \infty]$ uniquely determined such that*

$$\pi(M \times N) = \mu(M) \cdot \nu(N), \forall M \in \mathcal{M}, N \in \mathcal{N}$$

Proof of Theorem 15.3 by assuming proposition 15.4. Have $(X, \mathcal{M}, \mu), (Ym\mathcal{N}, \nu)$ with $\mu\nu$ are σ-finite. Pick increasing chains $(X_k)_{k=1}^{\infty} \in \mathcal{M}$ and $(Y_k)_{k=1}^{\infty} \in \mathcal{N}$ such that $\bigcup_{k=1}^{\infty} X_k = X$ and $\mu(X_k) < \infty$, while $\bigcup_{k=1}^{\infty} Y_k = Y$ and $\nu(Y_k) < \infty$ for each $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ put

$$\mathcal{M}_k = \{M \in \mathcal{M} | M \subseteq X_k\}, \, \mu_k = \mu|_{\mathcal{M}_k}$$
$$\mathcal{N}_k = \{N \in \mathcal{N} | N \subseteq Y_k\|, \, \nu_k = \nu|_{\mathcal{N}_n}$$

Proposition 15.4 applies to $(X_k, \mathcal{M}_k, \mu_k)$ and $(Y_k, \mathcal{N}_k, \nu_k)$ and gives a finit e positive $\pi_k : \mathcal{M}_k \times \mathcal{N}_k \to [0, \infty)$. We introduce some notation

$$\begin{aligned}
&\mathcal{P} = \{M \times N | M \in \mathcal{M}, N \in \mathcal{N}\} \\
&\mathcal{A} = \text{Collection of finite unions of sets from } \mathcal{P} \\
&\mathcal{U} = \mathcal{M} \times \mathcal{N} = \sigma - \text{algebra generated by either } P \text{ or } \mathcal{A}
\end{aligned}$$

Likewise for any $k \in \mathbb{N}$ we have

$$\begin{aligned}
\mathcal{P}_k &= \{M \times N | M \in \mathcal{M}_k, N \in \mathcal{N}_k\} \\
&= \{M \times N | M \in \mathcal{M}, M \subseteq C_k, N \in \mathcal{N}, N \subseteq Y_k\} \\
A_k &= \text{finite unions from } \mathcal{P}_k \\
\mathcal{U}_k &= \mathcal{M}_l \times \mathcal{N}_k = \sigma\text{-algebra of } \mathcal{P}_k \text{ or } A_k
\end{aligned}$$

CLAIM (1)**.** $U \in \mathcal{U} \Rightarrow U \cap (X_k \times Y_k) \in U_k, \forall k \in \mathbb{N}$

Verification of Claim 1. We have $\mathcal{A}$ =algebra of subsets of $X \times Y$ and A_k = algebra of subsets of subsets of $X_k \times Y_k$. Observe that

$$A_k = \{A \in \mathcal{A} | A \subseteq X_k \times Y_k\}$$

Indeed both sides of this equality are equal to

$$\{A | A = (M_1 \times N_1) \cup \cdots \cup (M_r \times N_r) \text{where } M_1, \ldots, M_r \in \mathcal{M}, M_1 \ldots, M_r \subseteq X_u, \\ N_1, \ldots, N_r \in \mathcal{N}, N_1 \ldots N_r \subseteq Y_k\}$$

Then the σ algebra generated by A is $\mathcal{U}$. Put

$$V_k = \{V \in \mathcal{U} | U \subseteq X_k \times Y_k\}$$

Problem 7 in homework 1 says that the σ-algebra generated by A_k is V_k. On the other hand we know that the σ-algebra generated by A_k is U_k $(= \mathcal{M}_k \times \mathcal{N}_k)$. Hence,

$$U_k = V_k\{v \in U | V \subseteq X_k \times Y_k\}$$

So then for every $U \in \mathcal{U}$ we have $U \cap (X_k \times Y_k) \in V_k \Rightarrow U \cap (X_k \times Y_k) \in U_k$. ∎

Claim (2). For every $k \in \mathbb{N}$ we have $\mathcal{U}_k \subseteq \mathcal{U}_{k+1}$ and $\pi_{k+1}|_{\mathcal{U}_k} = \pi_k$

Verification of Claim 2. It is immediate that $\{U \in \mathcal{U}_{k+1} | U \subseteq X_k \times Y_k\}$ is a σ-algebra which contains $\mathcal{P}_k = \{M \times N | M \subseteq X_k, N \subseteq Y_k\}$. Hence, it contains the σ-algebra generated by $\mathcal{P}_k$ which is U_k. Hence,

$$U_k \subseteq \{U \in U_{k+1} | U \subseteq X_k \times Y_k\} \subseteq U_{k+1}$$

Then, observe that $\tilde{\pi}_k = \pi_{k+1}|_{\mathcal{U}_k}$. (1 line missing) From the uniqueness property of π_k it follows that $\pi_k = \tilde{\pi}_k = \pi_{k+1}|_{\mathcal{U}_k}$. ∎

Claim (3). For every $U \in \mathcal{U}(= \mathcal{M} \times \mathcal{N})$, it makes sense to define

$$\pi(U) = \lim_{n\to\infty} \pi_k(U \cap (X_k \cap Y_k)) \in [0, \infty]$$

Verification of Claim 3. We have $U \cap (X_k \times Y_k) \in \mathcal{U}_k$ (by claim 1). Hence, $\pi_k(U \cap (X_k \times Y_k)$ makes sense. Moreover, for every $k \in \mathbb{N}$ we have

$$\begin{aligned} \pi_k(U \cap X_k \times Y_k)) &= \pi_{k+1}(U \cap (X_k \times Y_k)) && \text{(By Claim 2)} \\ &\leq \pi_{k+1}(U \cap (X_{k+1} \times Y_{k+1}) \end{aligned}$$

Last inequality follows because $U \cap (X_k \times Y_k) \subseteq U \cap (X_{k+1} \times Y_{k+1})$. Hence, the limit as $k \to \infty$ must exist.

∎

Claim (4). The set function $\pi : \mathcal{U} \to [0, \infty]$ is a s positive measure such that $\pi(M \times N) = \mu(M)\nu(N)$ for all $M \in \mathcal{M}, N \in \mathcal{N}$. (left as exercise)

Claim (5). Let $\tilde{\pi} : \mathcal{U} \to [0, \infty)$ be a positive measure such that $\tilde{\pi}(M \times N) = \mu(M) \cdot \nu(N)$, for all $M \in \mathcal{M}, N \in \mathcal{N}$. Then, $\tilde{\pi} = \pi$ with π defined as in claim 3.

Verification of claim 5. For every $k \in \mathbb{N}$ we have $\tilde{\pi}|_{\mathcal{U}_k} = \pi_k$ due to uniqueness of π_k, so then for every $U \in \mathcal{U}$ we write

$$U = \bigcup_{k=1}^{\infty} (U \cap (X_k \times Y_k))$$

So then

$$\begin{aligned} \tilde{\pi}(U) &= \lim_{n\to\infty} \tilde{\pi}(U \cap (X_k \times Y_k)) && \text{(CAIC)} \\ &= \lim_{k\to\infty} \pi_k(U \cap (X_k \times Y_k)) \\ &= \pi(U) \end{aligned}$$

∎

∎

PROPOSITION **15.4.** *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces, where the measures μ and ν are finite $\mu(X), \nu(Y) < \infty$. Consider the measurable space $(X \times Y, \mathcal{M} \times \mathcal{N})$. There exists a finite positive measure $\pi : \mathcal{M} \times \mathcal{N} \to [0, \infty)$, uniquely determined such that*

$$\pi(M \times N) = \mu(M) \cdot \nu(N), \forall M \in \mathcal{M}, N \in \mathcal{N}$$

This was not proven in class.

REM **15.5.** Let $(X, \mathcal{M}, \mu)$ $(Y, \mathcal{N}, \nu)$ be measure spaces where $\mu(X), \nu(Y) < \infty$. Let

$$\mathcal{P} = \{M \times N \mid M \in \mathcal{M}, N \in \mathcal{N}\}$$

We denote

$\mathcal{A} =$ (Algebra of subsets of $X \times Y$ which is generated by $\mathcal{P}$)
$\mathcal{U} = \mathcal{M} \times \mathcal{N} =$ (σ-algebra of subsets of $X \times Y$ which is generated by $\mathcal{P}$ (or by $\mathcal{A}$))

We define $\pi_0 : \mathcal{P} \to [0, \infty)$ by

$$\pi_0(M \times N) = \mu(M)\nu(N)$$

We want to extend π_0 to an additive set function $\tilde{\pi}_0 : \mathcal{A} \to [0, \infty)$ then to a finite positive measure $\pi : \mathcal{U} \to [0, \infty)$. The key point is we will see that π_0 behaves well under divisions of a set $P = M \times N$ in $\mathcal{P}$.

REM **15.6.** Comments on theorem on side A of handout from Nov 28. Idea for proof of part 1. of theorem. Given $A \in \mathcal{A}$, we write $A = S_1 \cup \cdots \cup S_r$ with $S_1, \ldots, S_r \in \mathcal{S}$ and $S_i \cap S_j = \emptyset$ for $i \neq j$. Then, define $\tilde{\mu}_0(A) = \sum_{i=1}^r \mu_0(S_i)$ (this quantity depends only on A, not on how it is written as $\bigcup_{i=1}^r S_i$, use idea of common refinement).

We now look at the idea of the proof for part 2. Get $\tilde{\mu}_0 : \mathcal{A} \to [0, \infty)$ as in part 1. The hypothesis that μ_0 respects countable divisions in $\mathcal{S}$ implies that $\tilde{\mu}_0$ is a pre-measure on A. Use the Caratheodory Extension Theorem to extend from A to U.

Uniqueness of Statements? In part 1, it is clear that if $\tilde{\mu}_0 : \mathcal{A} \to [0, \infty)$ is additive and extends μ_0, then for any $A \in \mathcal{A}$ written as $A = S_1 \cup \cdots \cup S_r$ we have

$$\tilde{\mu}_0(A) = \sum_{i=1}^r \tilde{\mu}_0(S_i) = \sum \mu_0(S_i)$$

Second equality follows since $\tilde{\mu}_0$ extends μ_0.

For uniqueness in part 2, we do a bit of a digression on why uniqueness in Caratheodory extension theorem.

REM **15.7.** Comment on side B of today's handout. Let $X, \mathcal{A}, \mathcal{U}$ be as in proposition on side B of handout. Suppose we have two finite positive measure $\mu, \nu : U \to [0, \infty)$ such that

$$\mu(A) = \nu(A), \forall A \in \mathcal{A} \tag{*}$$

Look at

$$\mathcal{C} = \{c \in \mathcal{U} \mid \mu(C) = \nu(C)\}$$

This is a monotone class! ($\mu(\emptyset) = \nu(\emptyset) = 0$ and $\mu(X) = \nu(X) < \infty$ by putting $A = X$ in (*)). (MC2), (MC3) follows from continuity along increasing/decreasing chains of μ and of ν. Hence, we have $\mathcal{C}$ a monotone class, $\mathcal{C} \supseteq \mathcal{A}$ by (*) and $\mathcal{C} \subseteq \mathcal{U}$ by definition implies that $\mathcal{C} = \mathcal{U}$. Thus, this implies that $\mu(c) = \nu(c)$ for all $c \in \mathcal{U}$.

We return to the framework of $(C, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ a semialgebra

$$\mathcal{P} = \{M \times N \mid M \in \mathcal{M}, N \in \mathcal{N}\}$$

and $\pi_0 : \mathcal{P} \to [0, \infty)$, $\pi_0(M \times N) = \mu(M)\nu(N)$. We want to extend π_0 to a finite positive measure $\pi : \mathcal{U} \to [0, \infty)$ where $\mathcal{U} = \mathcal{M} \times \mathcal{N}$. Due to side A of the handout, we only need a lemma.

LEMMA **15.8.** $\pi_0 : \mathcal{P} \to [0, \infty)$ *respects countable divisions. Let* $P = M \times N$ *in* $\mathcal{P}$*. Suppose that* $P = \bigcup_{i=1}^{\infty} P_i$ *where* $P_i = M_i \times N_i \in \mathcal{P}, \forall i \in \mathbb{N}$ *with* $P_i \cap P_j = \emptyset$ *for* $i \neq j$*. Then,*

$$\pi_0(P) = \sum_{i=1}^{\infty} \pi_0(P_i)$$

Proof. Write $P = M \times N$ and $P_i = M_i \times N_i, \forall i \in \mathbb{N}$.

CLAIM (1)**.** For every $x \in X$, we have $\sum_{i=1}^{\infty} I_{M_i}(x)\nu(N_i) = I_M(x)\nu(N)$.

Verification of claim 1. Fix $x \in X$ ad write

$$\begin{aligned}
\sum_{i=1}^{\infty} I_{M_i}(x)\nu(N_i) &= \lim_{k\to\infty} \left(\sum_{i=1}^{k} I_{M_i}(x)\nu(N_i) \right) \\
&= \lim_{n\to\infty} \left(\sum_{1\le i\le k} \nu(N_i) \right) \\
&= \lim_{k\to\infty} \left(\nu \left(\bigcup_{1\le i\le k} N_i \text{ such that } x \in M_i \right) \right)
\end{aligned}$$

In the last step we use here if $i \neq j$ where $1 \le i, j \le k$ such that $x \in M_i$, $x \in M_j$, the $N_i \cap N_j = \emptyset$. We want to check this claim. Suppose there exists $y \in N_i \cap N_j$. Then,

$(x,y) \in (M_i \times N_i) \cap M_j \times N_j) = P_i \cap P_j = \emptyset$. Hence,

$$\begin{aligned}
\sum_{i=1}^{n} I_{M_i}(x)\nu(N_i) &= \lim_{k\to\infty}\left(\nu\left(\bigcup_{1\le i\le k \text{ such that } x\in M_i} N_i\right)\right)\\
&= \nu\left(\bigcup_{i\in\mathbb{N}} N_i \text{ such that } x\in M_i\right)\\
&= \begin{cases}\nu(N) & \text{If} x\in M\\ 0 & \text{if } x\in X\setminus M\end{cases}
\end{aligned}$$

Second last union is equal to N if $x \in M$ and $\emptyset$ if $x \in X \setminus M$. Indeed if $x \in X \setminus M$ then $x \notin M_i$, hence $\cup_{i,x\in X_i} N_i = \emptyset$. If $x \in M$, for every $y \in N$ we write $(x,y) \in M \times N = \bigcup_{i=1}^{\infty} M_i \times N_i$ hence there exists $i \in \mathbb{N}$ such that $(x,y) \in M_i \times N_i$ implies there exists i such that $x \in M_i, y \in M \Rightarrow y \in \bigcup_{i\in\mathbb{N}, x\in M_i} N_i$. Hence,

$$\sum_{i=1}^{\infty} I_{M_i}(x)\nu(N_i) = \begin{cases}\nu(N) & x\in M\\ 0 & x\in X\setminus M\end{cases} = I_m(x)\nu(N)$$

∎

Claim (2). We have $\sum_{i=1}^{\infty}\mu(M_i)\nu(N_i) = \mu(M)\nu(N)$

Proof. For every $k \in \mathbb{N}$ define $g_k : X \to \mathbb{R}$

$$g_k(x) = \sum_{i=1}^{k} I_{M_i}(x)\nu(N_i)$$

Then $g_k \in \text{Bor}^+(X,\mathbb{R})$ with $\int g_k d\mu = \sum_{i=1}^{k}\mu(M_i)\nu(N_i)$. Claim 1 says that $g_k \to g$ from below where $g : X \to \mathbb{R}$. $g(x) = I_M(x)\nu(N), x \in X$. Apply LMCT to get $\int g_k d\mu \to \int g d\mu$. Hence, $\sum_{i=1}^{k}\mu(M_i)\nu(N_i) \to \mu(M)\nu(N)$. Done with claim 2. ∎

∎

16 The Fubini - Tonelli Theorem

Definition **16.1.** X, Y are non-empty sets,

1. Let E be a subset of X. For every $x \in X$ we denote
$$E_{(x)} = \{y \in Y | (x, y) \in\}$$
For every $y \in Y$ we write
$$E^{(y)} = \{x \in X | (x, y) \in E\}$$
2. Let $f : X \times Y \to \mathbb{R}$ be a function. For every $x \in X$ we define $f_{(x)} : Y \to \mathbb{R}$ by $f_{(x)} = f(x, y), \forall y \in Y$. For every $y \in Y$ we define $f^{(y)} : X \to \mathbb{R}$ by $f^{(y)}(x) = f(x, y), \forall x \in X$.

Proposition **16.2.** $(X, \mathcal{M}), (Y, \mathcal{N})$ *measurable spaces. Consider direct product* $(X \times Y, \mathcal{M} \times \mathcal{N}$.

1. *If* $E \in \mathcal{M} \times \mathcal{N}$, *then* $E_{(x)} \in \mathcal{N}, \forall x \in X$, $E^{(y)} \in \mathcal{M}, \forall y \in Y$.
2. *If* $f \in \mathrm{Bor}(X \times Y, \mathbb{R})$ *then* $f_{(x)} \in \mathrm{Bor}(Y, \mathbb{R}), \forall x \in X$ *and* $f^{(y)} \in \mathrm{Bor}(X, \mathbb{R}), \forall y \in Y$.

Proof. 1. Will do proof for $E_{(x)}$. Fix $x_0 \in X$. We want $E_{(x_0)} \in \mathcal{N}$. Define $\varphi : Y \to X \times Y$ by $\varphi(y) = (x_0, y)$ for al $y \in Y$. Observe that
$$\begin{aligned} E_{(x_0)} &= \{y \in Y \mid (x_0, y) \in E\} \\ &= \{y \in Y \mid \varphi(y) \in E\} \\ &= \varphi^{-1}(E) \end{aligned}$$
So if we prove that φ is $(\mathcal{N}, \mathcal{M} \times \mathcal{N}$ - measurable then we are done (φ^{-1} set $\mathcal{M} \times \mathcal{N}$ must be in $\mathcal{N}$). Remember that $\mathcal{M} \times \mathcal{N}$ is the σ-algebra generated by
$$\mathcal{P} = \{M \times N | M \in \mathcal{M}, N \in \mathcal{N}\}$$
So in order to prove measurability of φ sufices to check that $\varphi^{-1}(P) \in \mathcal{N}$ for all $P \in \mathcal{P}$ (by Tool no 2 form lecture 7). And indeed for $P = M \times N \in \mathcal{P}$ we have
$$\begin{aligned} \varphi^{-1}(P) &= \{y \in N \mid \varphi(y) \in P\} \\ &= \{y \in N \mid (x_0, y) \in M \times N\} \end{aligned}$$
where the above equals $\emptyset$ if $x_0 \in X \setminus M$ and N if $x_0 \in \mathcal{M}$. So in any case, $\varphi^{-1}(P) \in \mathcal{N}$

2. We do verifications for $f_{(x)}$. Fix $x \in X$, we must show that
$$f^{-1}_{(x)}(B) \in \mathcal{N}, \forall B \in \mathcal{B}_{\mathbb{R}}$$
. Have $f : X \times Y \to \mathbb{R}$ with $f_{(x)} : Y \to \mathbb{R}$ by $f_{(x)}(y) = f(x, y)$. Denote
$$E := f^{-1}(B) \in \mathcal{M} \times \mathcal{N}$$
Then, $f^{-1}_{(x)}(B) = E_{(x)} \in \mathcal{N}$ by part 1. Check this equality.

∎

We form a product $(X \times Y, \mathcal{M} \times \mathcal{N}, \mu \times \nu$ Pick a bounded non negative borel function f. For every $x \in X$ consider a partial function $f_{(x)} : Y \to \mathbb{R}$ and $f_{(x)}(y) = f(x, y)$ for all $y \in Y$. $f_{(x)}$ is a Borel function by prop 16.2. In fact here $f_{(x)} \in \text{Bor}_b^+(Y, \mathbb{R})$ for each $x \in X$. $f_{(x)}$ is bounded and non-negative because

$$f_{(x)}(Y) \subseteq f(X \times Y) \subseteq [0, c]$$

Where c is someuppser bound for f. Put $F(x) = \int_Y f_{(x)} d\nu, \forall x \in X$. Have $0 \leq F(x) \leq c \cdot \nu(Y)$ for all $x \in X$. In this way we get a bounded non-negative function $F : X \to \mathbb{R}$.

Proposition **16.3.** [Special Case of Tonelli] *In the above notations have that* $f \in \text{Bor}_b^+(X, \mathbb{R})$ *with* $\int_X F d\mu = \int f d(\mu \times \nu)$.

Ideas of proof. Degnote

$$\mathcal{G} = \{f \in \text{Bor}_b^+(X \times Y, \mathbb{R}) | \text{prop 16.4 holds true}\}$$

We want $\mathcal{G} = \text{Bor}_b^+(X \times Y, \mathbb{R})$. Our plan is

$$I_E \in \mathcal{G}, \forall E \in \mathcal{M} \times \mathcal{N} \Rightarrow \text{Bor}_s^+(X \times Y, \mathbb{R}) \subseteq \mathcal{G} \Rightarrow \text{Bor}_b^+(X \times Y, \mathbb{R}) \subseteq \mathcal{G}$$

which will force equality. The first implication follows because G is closed under linear combinations with coefficients in $[0, \infty)$. Do linear combinations of I_E. The second implication because approximation with simple functions use LMCT. Why do we have the first part of implication chain?

To prove that $I_E \in \mathcal{G}$ for all $E \in \mathcal{M} \times \mathcal{N}$, we denote

$$\Sigma = \{E \in \mathcal{M} \times \mathcal{N} | I_E \in \mathcal{G}\}$$

We want $\Sigma = \mathcal{M} \times \mathcal{N}$. Find the properties of Σ. LEt

$$P = M \times N, M \in \mathcal{M}, N \in \mathcal{N} \Rightarrow P \in \Sigma \tag{P1}$$

Why?

$$f = I_p = f_{(x)} = \begin{cases} I_N & \text{If } x \in M \\ 0 & \textit{else} \end{cases}$$

This implies that $F(x) = 0$ if $\nu(N)$ if $x \in \mathcal{M}$ and 0 otherwise. $F = \nu(N) \cdot I_m$. We get that F is Borel on X with

$$\begin{aligned} \int F d\mu &= \nu(N)\mu(M) \\ &= (\mu \times \nu)(P) \\ &= \int f d(\mu \times \nu) \end{aligned}$$

Let

$$E, F \in \Sigma, E \cap F = \emptyset \Rightarrow E \cup F \in \Sigma \tag{P2}$$

$$\Sigma \text{ is a monotone class (Use LMCT, LDCT)} \tag{P3}$$

Use P(1) + P(2) + P(3) and the trick of the monotone class to get $\Sigma = \mathcal{M} \times \mathcal{N}$. ∎

REM **16.4.** Nice way to write the statement of Prop 16.4 with

$$\int_X F(x)d\mu(x) = \int_X \left(\int_Y f(x,y)d\nu(y) \right) d\mu(x)$$

PROPOSITION **16.5.** [TONELLI'S THEOREM] *See handout.*

Proof. In general frame need to deal with the set $A = \{x \in X | \int f_{(x)}d\nu = \infty\}$ ∎

THEOREM **16.6.** [FUBINI] *is for function in $\mathcal{L}^1(\mu \times \nu)$ we apply tonelli's theorem for positive functions and subtract what tonelli gives for f^+ and f^-.*

EG **16.7.** $(X, \mathcal{M}, \mu)$ a probability space, f a positive borel function. Define

$$u(t) = \mu(\{x \in X | f(x) \geq t\})$$

We claim that $\int f d\mu = \int_0^\infty u(t)dt$. Why? Pit $Y = [0, \infty)$, B_+ is the borel σ algebra of $[0, \infty)$. Let b_+ be the positive lebesgue measure. Form the product $(X \times Y, \mathcal{M} \times \mathcal{B}_+, \mu \times b_+)$. Look at

$$E = \{(x,t) \in X \times Y | f(x) \geq t\}$$

Then $F \in \mathcal{M} \times \mathcal{B}_+$. Why¿ Let $g : X \times Y \to \mathbb{R}$, $g(x,t) = f(x) - t$. Then, $g \in \mathrm{Bor}(X, \times Y, \mathbb{R})$ and $E = g^{-1}([0, \infty))$. Apply tonelli to $I_E \in \mathrm{Bor}^+(X \times Y, \mathbb{R})$. Do it in both ways!

$$\begin{aligned} &\int_X \int_0^\infty I_E(x,t)dt d\mu(x) \\ &\qquad = \int_X f(x)d\mu(x) \\ &\qquad = \int f d\mu \end{aligned}$$

The other order of iteration

$$\int_0^\infty \int_X I_E(x,t)d\mu(x)dt = \int_0^\infty = \int_0^\infty \mu(\{x \in X | f(x) \geq t\})dt = \int_0^\infty u(t)dt$$

where I_E is an indicator function 1 if $f(x) \geq t$ otherwise 0.