6. Complete the `Red_Black_Tree` class by coding the missing functions for removal. The functions erase and `find_largest_child` are adapted from the corresponding functions of the `Binary_Search_Tree` class. These adaptations are similar to those done for the AVL tree. A data field `fixup_required` performs the role analogous to the `decrease` data field in the AVL tree. It is set when a black node is removed. Upon return from a function that can remove a node, this variable is tested. If the removal is from the right, then a new function `fixup_right` is called. If the removal is from the left, then a new function `fixup_left` is called.

   The function `fixup_right` is called with a reference to the local root of the sub-tree whose right sub-tree's black height is one less than the left sub-tree. This local root is designated P in the figures that illustrate the various cases that must be considered. The right sub-tree is indicated by X in a dotted circle and with a dotted line. This node X represents a back leaf that has been deleted or it represents the root of the sub-tree whose black height has been reduced as shown in Figures 11.68, 11.69, 11.70, and 11.71.

   If the node X is red, then the black height can be easily be restored by setting it black. Otherwise, the `fixup_right` function must consider four cases as, as follows:

   - **Case 1**: The sibling of X (designated S in Figure 11.68(a)) is red. The parent (P) must be black, and if S has children, then they must be two black nodes (L, and R). We change the color of P to red, and S to black (Figure 11.68(b)) and then rotate right about P (Figure 11.68(c)). Now we have a case where X has a black sibling (R). Recall that null trees are considered black. This transforms the problem into one of the other cases where R is now the sibling of X.

FIGURE 11.68
Red-Black Removal Case 1

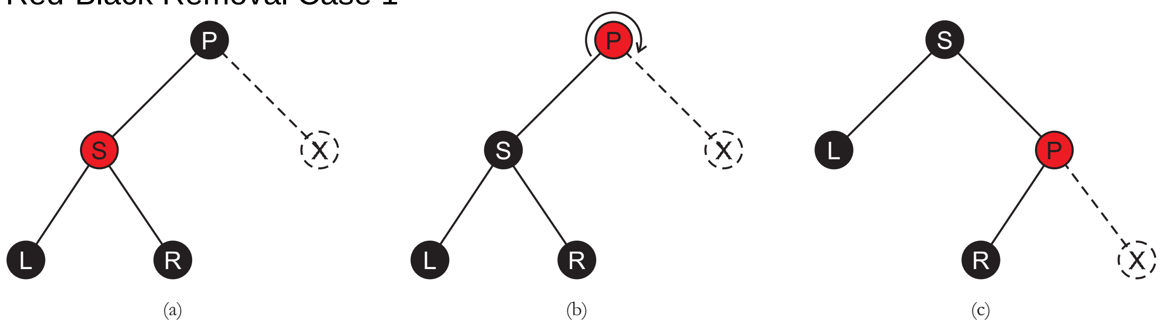
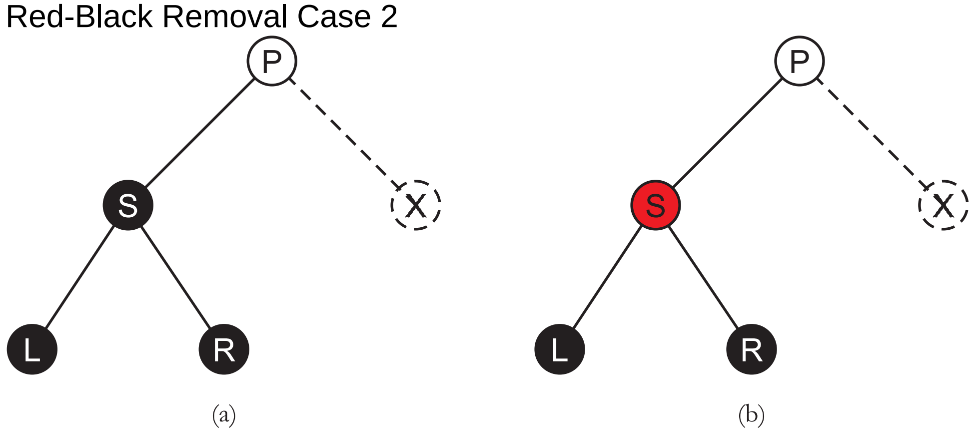


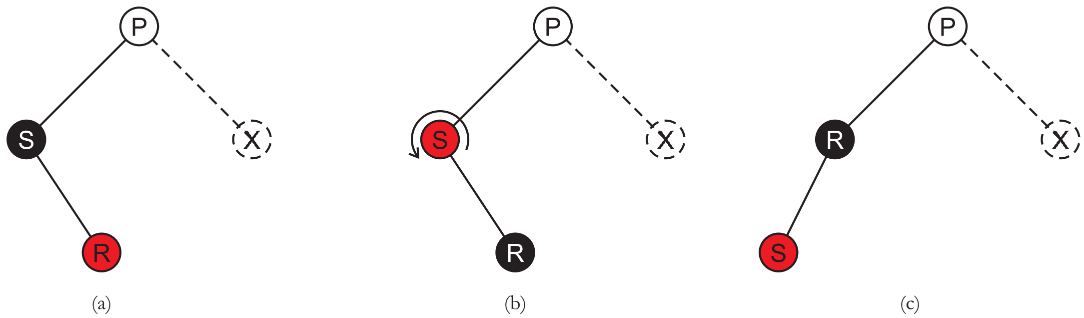
FIGURE 11.69
Red-Black Removal Case 2



- **Case 2**: The sibling of X (designated S in Figure 11.69(a)) is black, and it is either a leaf or it has two black children. Note that we do not care what color P has, so we show it in a white circle. We change the color of S to red (Figure 11.69(b)). This reduces the black height of the sub-tree whose root is S so it is now equal to the black height of the tree whose root is X. The overall black height of P has been reduced by one so we repeat the process at the next level (P's parent). Note that we may now have a red parent (P) with a red child (S), but this will be fixed at the next level.
- **Case 3**: The sibling of X (designated S in Figure 11.70(a)) is black and it has a red right child (R). S may also have a left child, but we do not care what its color is, so it is not shown. We change the color of S to red and the color of R to black. (Figure 11.70(b)). Then we rotate left about S (Figure 11.70(c)). This transforms the problem into Case 4.

FIGURE 11.70

Red-Black Removal Case 3



Note that before making this transformation, S was the root of a valid red-black tree. Therefore if S had a black left child, then R must have two back children. After performing the rotate, R is still the root of a valid red-black sub-tree. On the other hand, if S had a red left child, then after the rotate R's left child (S) is red and has a red left child. This will be fixed when we consider case 4. However the black heights of R's sub-trees remain balanced.

- **Case 4**: The sibling of X (designated S in Figure 11.71(a)) is black and it has a red left child (L). S is the root of a red-black sub-tree whose black height is balanced and is one greater than the black height of X. We change the color of L to black. (If we got here from case 3, the red-red problem is now fixed.) This increases the black height of L. We also change S to be the same color as P, and then change the color of P to black. (Figure 11.71(b)). By rotating right about P, we restore the black balance (Figure 11.77(c)).

FIGURE 11.71

Red-Black Removal Case 4