

实验题目3 四阶龙格-库塔(Runge-Kutta)方法

本实验完成了Runge-Kutta方法的编写，充分学习了微分方程求解、(隐)函数求导库等的用法，巩固了相关理论知识。

实验简介

本实验为Runge-Kutta方法实验，需要完成使用Runge-Kutta方法求解常微分方程初值问题数值解的任务，求解本次各个实验题目的问题。

本次实验过程中，主要为对Runge-Kutta方法代码完成编写，并充分体会Runge-Kutta方法的简洁性和相比于Euler方法的在准确性上的优点，同时从绘制的数值解图像注意到 n 的取值对于结果的重要影响。

实验的目的即为使用Runge-Kutta方法求解常微分方程初值问题的数值解。

该实验报告主要分为7个部分，大纲罗列如下：

- **实验简介**：即本部分的所有内容
- **数学原理**：即常微分方程初值问题的数学定义，和对Runge-Kutta方法的基本数学原理进行阐述
- **代码实现**：使用 Julia 语言，根据数学原理，编写实验代码
- **实验题目**：实验指导书中所要求的完成的实验题目，作有便于对照使用Runge-Kutta方法的 `lib solver` 和 `my solver` 与真实结果 `true result` 的曲线图，各题目均同时使用 `lib solver` 和 `my solver` 进行求解，熟悉了 Julia 库 `DifferentialEquations` 求解 ODE 问题的使用流程。

注：详细执行过程请参考附录：执行代码部分

- **问题1**：探究数值解法与解析解的关系，通过对于解为线性函数和非线性函数的常微分方程的数值求解，体会求出的数值解用于反推解析解的困难程度。
- **问题2&问题3**：探究 n 的大小对于求解精度的影响，首先是问题2变化的 n 对于求解精度的影响几乎可以不计，很容易求得精度较高的解，而在求解问题3时过小的 n 却根本无法对方程进行求解。这一定程度上说明了，求解的精度和 n 的选取很大程度上依赖于方程本身的性质。
- **思考题**：本部分为实验指导书中所要求的完成的思考题解答
- **参考资料**：本部分为完成实验过程中查阅的参考资料
- **附录**：本实验的非主体部分
 - **执行代码**：本部分是实验代码进行运行时封装的部分，将函数的调用细节隐藏在 `show_result()` 函数内部，便于直接从外部使用特定参数对函数进行调用。
 - **测试代码**：对程序的运行、输出进行测试的部分

Test 1 - Simple：使用教材上的例题对程序的正确性进行简单的测试，确保所写代码能完成实验任务。

数学原理

给定常微分方程初值问题

$$\begin{cases} \frac{dy}{dx} = f(x, y), a \leq x \leq b \\ y(a) = \alpha, h = \frac{b-a}{N} \end{cases}$$

记 $x_n = a + n \cdot h, n = 0, 1, \dots, N$ ，利用四阶Runge-Kutta方法，有

$$\begin{aligned} K_1 &= h \cdot f(x_n, y_n) \\ K_2 &= h \cdot f\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right) \\ K_3 &= h \cdot f\left(x_n + \frac{h}{2}, y_n + \frac{K_2}{2}\right) \\ K_4 &= h \cdot f(x_n + h, y_n + K_3) \end{aligned}$$

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4), n = 0, 1, \dots, N-1$$

可以逐次求出微分方程初值问题的数值解 $y_n, n = 1, 2, \dots, N$ 。

代码实现

首先导入需要的包。

`DifferentialEquations.jl` 是用于求解微分方程的标准库，本例中用于获取 `lib solver` 所需的数值解；

`ImplicitEquations.jl` 是用于支持隐函数的标准库，本例中仅在 `Test 1 - Simple` 部分用于支持绘制隐函数图像。

```
using DifferentialEquations
using Plots
using LaTeXStrings
using Statistics
using ImplicitEquations
using PrettyTables
```

根据数学原理和代码流程，可以很容易写出如下代码：

```
function rungekutta(f::Function, xspan, y0, num)
    a, b = xspan
    x0 = a
    h = (b - a) / num
    xs, ys = zeros(num), zeros(num)
    for n = 1:num
        K1 = h * f(x0, y0)
        K2 = h * f(x0 + h / 2, y0 + K1 / 2)
        K3 = h * f(x0 + h / 2, y0 + K2 / 2)
        K4 = h * f(x0 + h, y0 + K3)
        x1 = x0 + h
        y1 = y0 + 1 / 6 * (K1 + 2K2 + 2K3 + K4)
        xs[n], ys[n] = x0, y0 = x1, y1
    end
    xs, ys
end
```

实验题目

问题 1

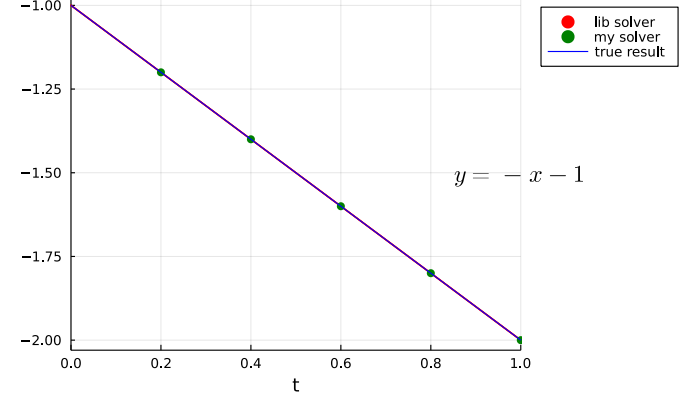
1.1

Problem 1.1 $\frac{dy}{dx} = x + y$

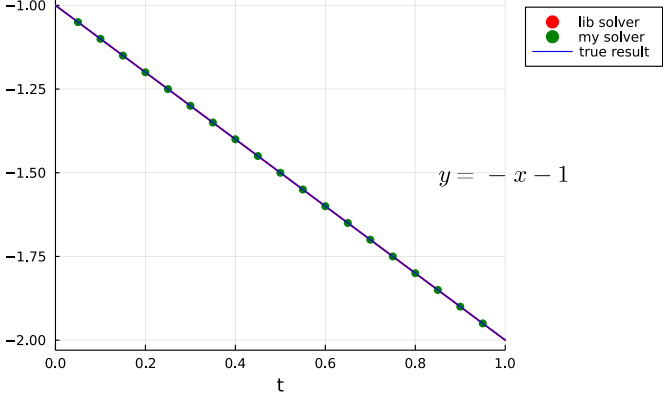
```
iternums = [5, 10, 20]

f1(y, p, x) = x + y      # lib RK4() solver
xspan = (0.0, 1.0)
y0 = -1.0
f2(x, y) = x + y          # my rungekutta() solver
f3(x) = -x - 1            # true result
title = L"Problem\ 1.1: \frac{\mathrm{d} y}{\mathrm{d} x} = x + y"
text = L"y = -x - 1"
show_result(f1, f2, f3, xspan, y0, iternums, true, true, title, text) # show, dense
```

Problem 1.1 : $\frac{dy}{dx} = x + y$



Problem 1.1 : $\frac{dy}{dx} = x + y$



Iternum: 5

x	True y	Pred y
0.20000000	-1.20000000	-1.20000000
0.40000000	-1.40000000	-1.40000000
0.60000000	-1.60000000	-1.60000000
0.80000000	-1.80000000	-1.80000000
1.00000000	-2.00000000	-2.00000000

Iternum: 10

x	True y	Pred y
0.10000000	-1.10000000	-1.10000000
0.20000000	-1.20000000	-1.20000000
0.30000000	-1.30000000	-1.30000000
0.40000000	-1.40000000	-1.40000000
0.50000000	-1.50000000	-1.50000000
0.60000000	-1.60000000	-1.60000000
0.70000000	-1.70000000	-1.70000000
0.80000000	-1.80000000	-1.80000000
0.90000000	-1.90000000	-1.90000000
1.00000000	-2.00000000	-2.00000000

Iternum: 20

x	True y	Pred y
0.05000000	-1.05000000	-1.05000000
0.10000000	-1.10000000	-1.10000000
0.15000000	-1.15000000	-1.15000000
0.20000000	-1.20000000	-1.20000000
0.25000000	-1.25000000	-1.25000000
0.30000000	-1.30000000	-1.30000000
0.35000000	-1.35000000	-1.35000000
0.40000000	-1.40000000	-1.40000000
0.45000000	-1.45000000	-1.45000000
0.50000000	-1.50000000	-1.50000000
0.55000000	-1.55000000	-1.55000000
0.60000000	-1.60000000	-1.60000000
0.65000000	-1.65000000	-1.65000000
0.70000000	-1.70000000	-1.70000000
0.75000000	-1.75000000	-1.75000000

	0.80000000		-1.80000000		-1.80000000	
	0.85000000		-1.85000000		-1.85000000	
	0.90000000		-1.90000000		-1.90000000	
	0.95000000		-1.95000000		-1.95000000	
	1.00000000		-2.00000000		-2.00000000	

1.2

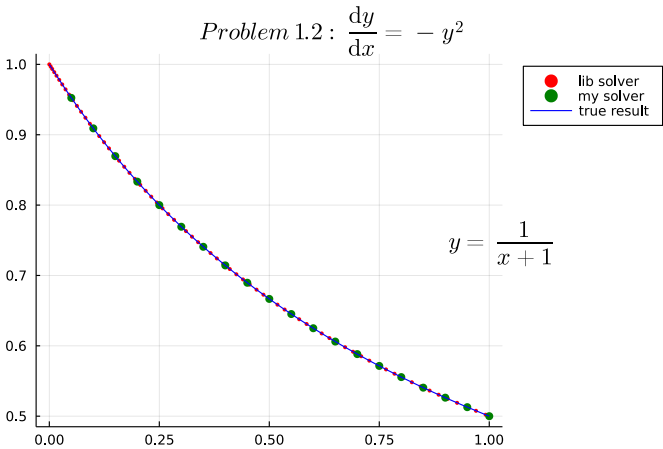
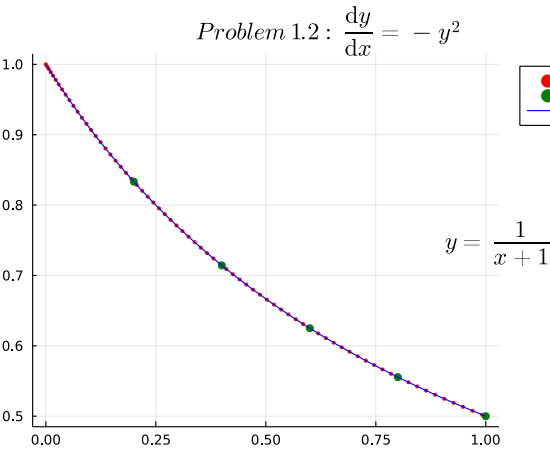
Problem 1.2 $\frac{dy}{dx} = -y^2$

```

iternums = [5, 10, 20]

f1(y, p, x) = -y^2
xspan = (0.0, 1.0)
y0 = 1.0
f2(x, y) = -y^2
f3(x) = 1 / (x + 1)
title = L"Problem\ 1.2: \frac{\mathrm{d} y}{\mathrm{d} x} = -y^2"
text = L"y = \frac{1}{x + 1}"
show_result(f1, f2, f3, xspan, y0, iternums, true, false, title, text) # show, dense

```



Iternum: 5

x	True y	Pred y
0.20000000	0.83333333	0.83333904
0.40000000	0.71428571	0.71429213
0.60000000	0.62500000	0.62500589
0.80000000	0.55555556	0.55556069
1.00000000	0.50000000	0.50000441

Iternum: 10

x	True y	Pred y
0.10000000	0.90909091	0.90909119
0.20000000	0.83333333	0.83333373
0.30000000	0.76923077	0.76923121
0.40000000	0.71428571	0.71428615
0.50000000	0.66666667	0.66666709
0.60000000	0.62500000	0.62500040
0.70000000	0.58823529	0.58823567
0.80000000	0.55555556	0.55555590

0.90000000	0.52631579	0.52631611
1.00000000	0.50000000	0.50000030

Iternum: 20

x	True y	Pred y
0.05000000	0.95238095	0.95238096
0.10000000	0.90909091	0.90909093
0.15000000	0.86956522	0.86956524
0.20000000	0.83333333	0.83333336
0.25000000	0.80000000	0.80000003
0.30000000	0.76923077	0.76923080
0.35000000	0.74074074	0.74074077
0.40000000	0.71428571	0.71428574
0.45000000	0.68965517	0.68965520
0.50000000	0.66666667	0.66666669
0.55000000	0.64516129	0.64516132
0.60000000	0.62500000	0.62500003
0.65000000	0.60606061	0.60606063
0.70000000	0.58823529	0.58823532
0.75000000	0.57142857	0.57142859
0.80000000	0.55555556	0.55555558
0.85000000	0.54054054	0.54054056
0.90000000	0.52631579	0.52631581
0.95000000	0.51282051	0.51282053
1.00000000	0.50000000	0.50000002

问题 2

2.1

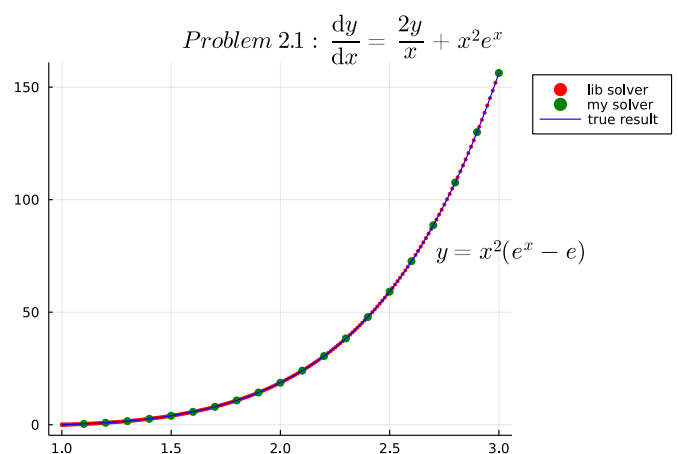
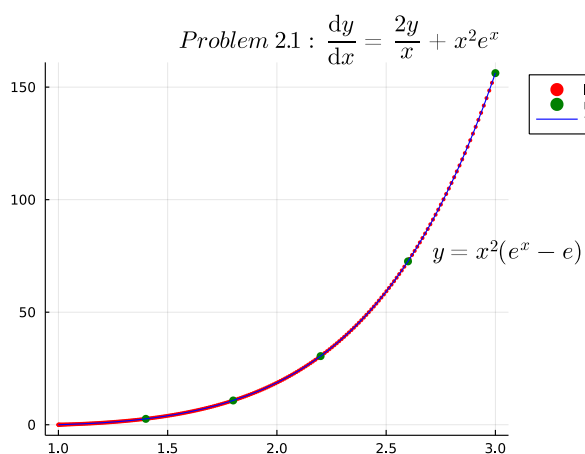
Problem 2.1 $\frac{dy}{dx} = \frac{2y}{x} + x^2 e^x$

```

iternums = [5, 10, 20]

f1(y, p, x) = 2 * y / x + x^2 * exp(x)
xspan = (1.0, 3.0)
y0 = 0.0
f2(x, y) = 2 * y / x + x^2 * exp(x)
f3(x) = x^2 * (exp(x) - exp(1))
title = L"Problem\ 2.1:\frac{\mathrm{d} y}{\mathrm{d} x}=\frac{2y}{x}+x^2 e^x"
text = L"y=x^2(e^x - e)"
show_result(f1, f2, f3, xspan, y0, iternums, true, false, title, text) # show, dense

```



Iternum: 5

x	True y	Pred y
1.40000000	2.62035955	2.61394279
1.80000000	10.79362466	10.77631317
2.20000000	30.52458129	30.49165420
2.60000000	72.63928396	72.58559861
3.00000000	156.30529585	156.22519828

Iternum: 10

x	True y	Pred y
1.20000000	0.86664254	0.86637911
1.40000000	2.62035955	2.61974052
1.60000000	5.72096153	5.71989528
1.80000000	10.79362466	10.79201760
2.00000000	18.68309708	18.68085236
2.20000000	30.52458129	30.52159814
2.40000000	47.83619262	47.83236583
2.60000000	72.63928396	72.63450354
2.80000000	107.61470115	107.60885199
3.00000000	156.30529585	156.29825744

Iternum: 20

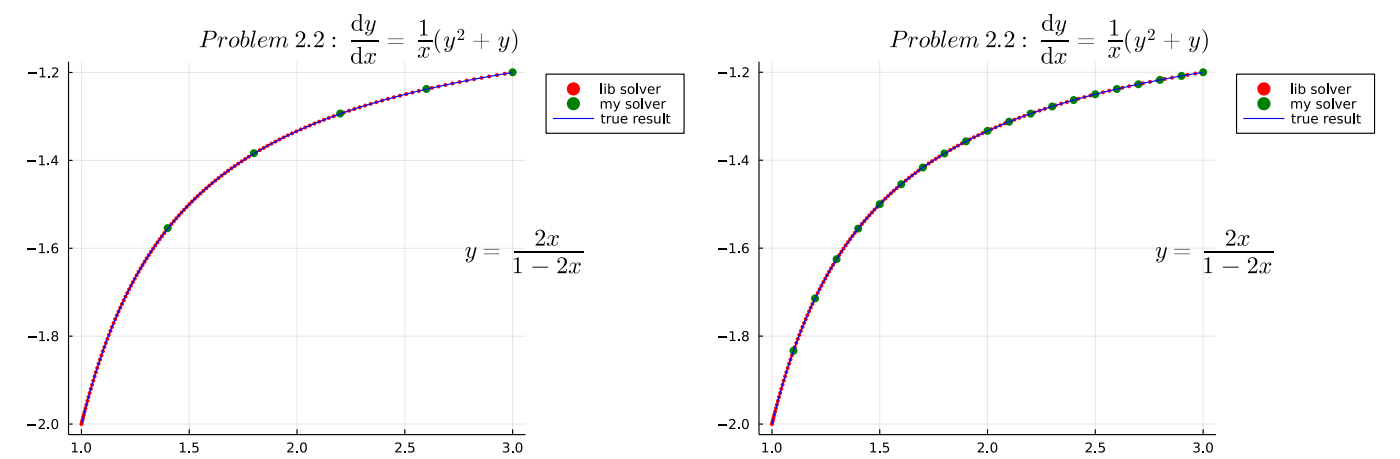
x	True y	Pred y
1.10000000	0.34591988	0.34591029
1.20000000	0.86664254	0.86662169
1.30000000	1.60721508	1.60718135
1.40000000	2.62035955	2.62031131
1.50000000	3.96766629	3.96760190
1.60000000	5.72096153	5.72087932
1.70000000	7.96387348	7.96377179
1.80000000	10.79362466	10.79350178
1.90000000	14.32308154	14.32293573
2.00000000	18.68309708	18.68292657
2.10000000	24.02518645	24.02498942
2.20000000	30.52458129	30.52435589
2.30000000	38.38371431	38.38345866
2.40000000	47.83619262	47.83590478
2.50000000	59.15132583	59.15100383
2.60000000	72.63928396	72.63892578
2.70000000	88.65696974	88.65657333
2.80000000	107.61470115	107.61426439
2.90000000	129.98381238	129.98333312
3.00000000	156.30529585	156.30477188

2.2

Problem 2.2 $\frac{dy}{dx} = \frac{1}{x}(y^2 + y)$

```
iternums = [5, 10, 20]

f1(y, p, x) = (y^2 + y) / x
xspan = (1.0, 3.0)
y0 = -2.0
f2(x, y) = (y^2 + y) / x
f3(x) = 2x / (1 - 2x)
title = L"Problem\ 2.2:\frac{\mathrm{d} y}{\mathrm{d} x}=\frac{1}{x}(y^2+y)"
text = L"y=\frac{2x}{1-2x}"
show_result(f1, f2, f3, xspan, y0, iternums, true, false, title, text) # show, dense
```



Iternum: 5		
x	True y	Pred y
1.40000000	-1.55555556	-1.55398900
1.80000000	-1.38461538	-1.38361729
2.20000000	-1.29411765	-1.29340153
2.60000000	-1.23809524	-1.23754016
3.00000000	-1.20000000	-1.19954796

Iternum: 10		
x	True y	Pred y
1.20000000	-1.71428571	-1.71424518
1.40000000	-1.55555556	-1.55552288
1.60000000	-1.45454545	-1.45451975
1.80000000	-1.38461538	-1.38459451
2.00000000	-1.33333333	-1.33331586
2.20000000	-1.29411765	-1.29410266
2.40000000	-1.26315789	-1.26314480
2.60000000	-1.23809524	-1.23808362
2.80000000	-1.21739130	-1.21738087
3.00000000	-1.20000000	-1.19999054

Itemnum: 20

x	True y	Pred y
1.10000000	-1.83333333	-1.83333283
1.20000000	-1.71428571	-1.71428517
1.30000000	-1.62500000	-1.62499950
1.40000000	-1.55555556	-1.55555511
1.50000000	-1.50000000	-1.49999961
1.60000000	-1.45454545	-1.45454510
1.70000000	-1.41666667	-1.41666635
1.80000000	-1.38461538	-1.38461510
1.90000000	-1.35714286	-1.35714260
2.00000000	-1.33333333	-1.33333309
2.10000000	-1.31250000	-1.31249978
2.20000000	-1.29411765	-1.29411744
2.30000000	-1.27777778	-1.27777759
2.40000000	-1.26315789	-1.26315771
2.50000000	-1.25000000	-1.24999983
2.60000000	-1.23809524	-1.23809508
2.70000000	-1.22727273	-1.22727258
2.80000000	-1.21739130	-1.21739116
2.90000000	-1.20833333	-1.20833320
3.00000000	-1.20000000	-1.19999987

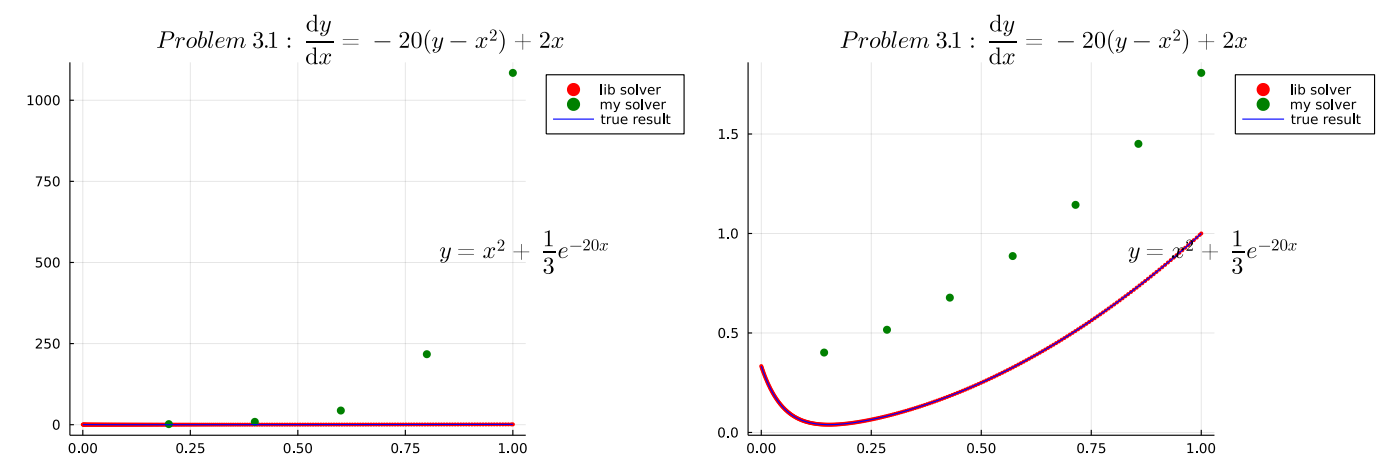
问题 3

3.1

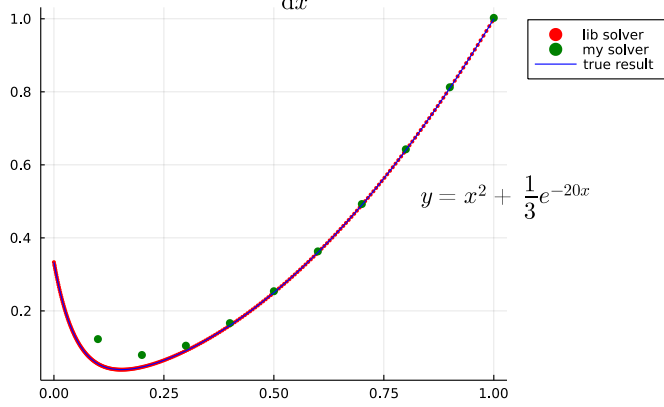
Problem 3.1 $\frac{dy}{dx} = -20(y - x^2) + 2x$

```
iternums = [5, 7, 10, 20] # 为观察方便，添加了n=7的作图，表格数据仍为所求[5, 10, 20]

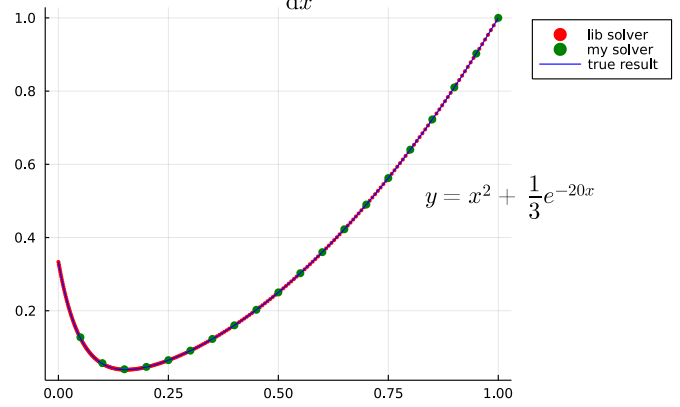
f1(y, p, x) = -20(y - x^2) + 2x
xspan = (0.0, 1.0)
y0 = 1 / 3
f2(x, y) = -20(y - x^2) + 2x
f3(x) = x^2 + 1 / 3 * exp(-20x)
title = L"Problem\ 3.1: \frac{\mathrm{d} y}{\mathrm{d} x}=-20(y-x^2)+2x"
text = L"y=x^2+\frac{1}{3}e^{-20x}"
show_result(f1, f2, f3, xspan, y0, iternums, true, false, title, text) # show, dense
```



Problem 3.1 : $\frac{dy}{dx} = -20(y - x^2) + 2x$



Problem 3.1 : $\frac{dy}{dx} = -20(y - x^2) + 2x$



Iternum: 5

x	True y	Pred y
0.20000000	0.04610521	1.76000000
0.40000000	0.16011182	8.81333333
0.60000000	0.36000205	43.68000000
0.80000000	0.64000004	217.29333333
1.00000000	1.00000000	1084.32000000

Iternum: 10

x	True y	Pred y
0.10000000	0.05511176	0.12277778
0.20000000	0.04610521	0.07925926
0.30000000	0.09082625	0.10475309
0.40000000	0.16011182	0.16658436
0.50000000	0.25001513	0.25386145
0.60000000	0.36000205	0.36295382
0.70000000	0.49000028	0.49265127
0.80000000	0.64000004	0.64255042
0.90000000	0.81000001	0.81251681
1.00000000	1.00000000	1.00250560

Iternum: 20

x	True y	Pred y
0.05000000	0.12512648	0.12755208
0.10000000	0.05511176	0.05694661
0.15000000	0.03909569	0.04015706
0.20000000	0.04610521	0.04667348
0.25000000	0.06474598	0.06505464
0.30000000	0.09082625	0.09101007
0.35000000	0.12280396	0.12293086
0.40000000	0.16011182	0.16021366
0.45000000	0.20254114	0.20263220
0.50000000	0.25001513	0.25010166
0.55000000	0.30250557	0.30259021

0.60000000 0.36000205 0.36008591
0.65000000 0.42250075 0.42258430
0.70000000 0.49000028 0.49008370
0.75000000 0.56250010 0.56258347
0.80000000 0.64000004 0.64008338
0.85000000 0.72250001 0.72258335
0.90000000 0.81000001 0.81008334
0.95000000 0.90250000 0.90258334
1.00000000 1.00000000 1.00008333

3.2

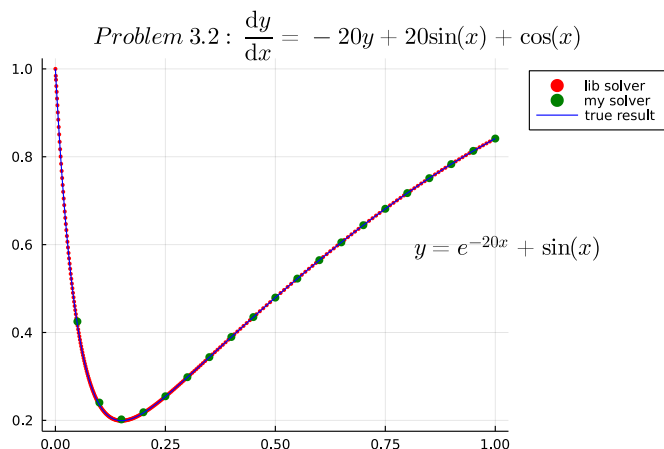
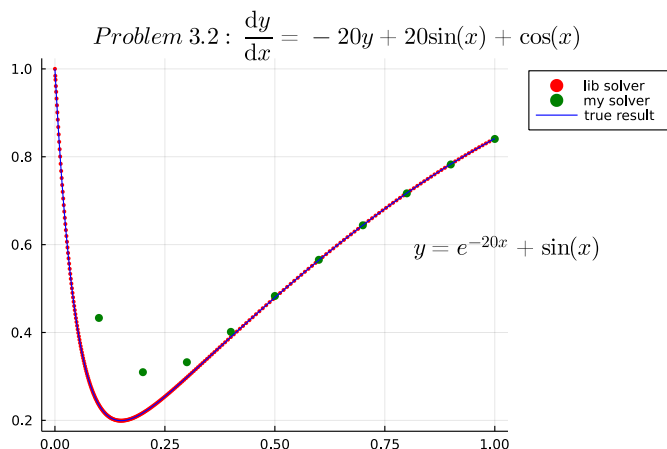
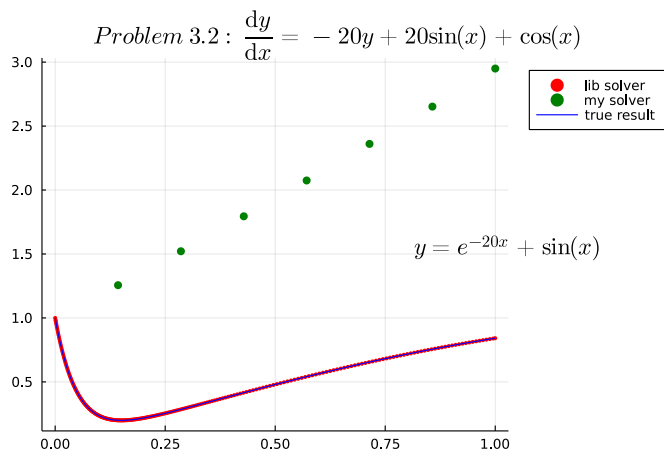
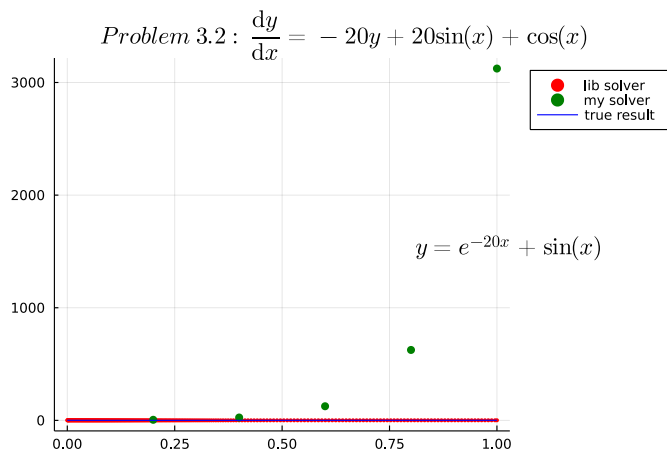
Problem 3.2 $\frac{dy}{dx} = -20y + 20\sin(x) + \cos(x)$

```

iternums = [5, 7, 10, 20] # 为观察方便, 添加了n=7的作图, 表格数据仍为所求[5, 10, 20]

f1(y, p, x) = -20y + 20sin(x) + cos(x)
xspan = (0.0, 1.0)
y0 = 1.0
f2(x, y) = -20y + 20sin(x) + cos(x)
f3(x) = exp(-20x) + sin(x)
title = L"Problem\ 3.2: \frac{\mathrm{d} y}{\mathrm{d} x}=-20y+20\sin(x)+\cos(x)"
text = L"y=e^{-20x}+\sin(x)"
show_result(f1, f2, f3, xspan, y0, iternums, true, false, title, text) # show, dense

```



Iternum: 5

x	True y	Pred y
0.20000000	0.21698497	5.19733811
0.40000000	0.38975380	25.37617070
0.60000000	0.56464862	125.48681526
0.80000000	0.71735620	625.31209552
1.00000000	0.84147099	3123.79515095

Iternum: 10

x	True y	Pred y
0.10000000	0.23516870	0.43313900
0.20000000	0.21698497	0.30966047
0.30000000	0.29799896	0.33232467
0.40000000	0.38975380	0.40141397
0.50000000	0.47947094	0.48307434
0.60000000	0.56464862	0.56543528
0.70000000	0.64421852	0.64398900
0.80000000	0.71735620	0.71672235
0.90000000	0.78332692	0.78249915
1.00000000	0.84147099	0.84052572

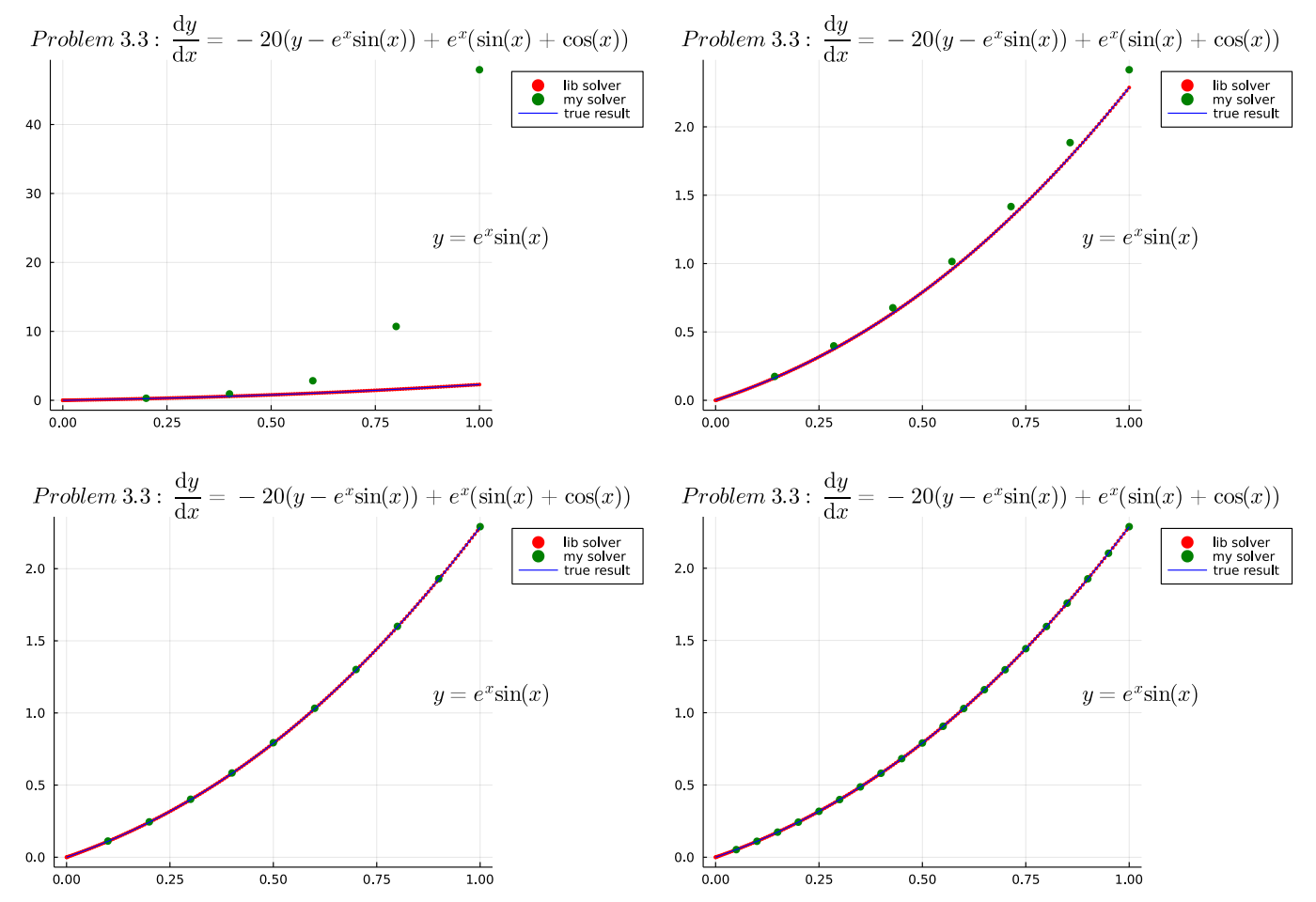
Iternum: 20

x	True y	Pred y
0.05000000	0.41785861	0.42497852
0.10000000	0.23516870	0.24045622
0.15000000	0.19922520	0.20216844
0.20000000	0.21698497	0.21843866
0.25000000	0.25414191	0.25481165
0.30000000	0.29799896	0.29829102
0.35000000	0.34380969	0.34392855
0.40000000	0.38975380	0.38979534
0.45000000	0.43508894	0.43509617
0.50000000	0.47947094	0.47946262
0.55000000	0.52270393	0.52268809
0.60000000	0.56464862	0.56462864
0.65000000	0.60518867	0.60516599
0.70000000	0.64421852	0.64419376
0.75000000	0.68163907	0.68161253
0.80000000	0.71735620	0.71732804
0.85000000	0.75128045	0.75125076
0.90000000	0.78332692	0.78329581
0.95000000	0.81341551	0.81338305
1.00000000	0.84147099	0.84143727

Problem 3.3 $\frac{dy}{dx} = -20(y - e^x \sin(x)) + e^x(\sin(x) + \cos(x))$

```
iternums = [5, 7, 10, 20] # 为观察方便, 添加了n=7的作图, 表格数据仍为所求[5, 10, 20]

f1(y, p, x) = -20(y - exp(x)sin(x)) + exp(x) * (sin(x) + cos(x))
xspan = (0.0, 1.0)
y0 = 0.0
f2(x, y) = -20(y - exp(x)sin(x)) + exp(x) * (sin(x) + cos(x))
f3(x) = exp(x) * sin(x)
title = L"Problem\ 3.3: \frac{\mathrm{d} y}{\mathrm{d} x}=-20(y-e^{\wedge}x \sin(x))+e^{\wedge}x (\sin(x) + \cos(x))"
text = L"y=e^{\wedge}x \sin(x)"
show_result(f1, f2, f3, xspan, y0, iternums, true, false, title, text) # show, dense
```



Iternum: 5

x	True y	Pred y
0.20000000	0.24265527	0.29864621
0.40000000	0.58094390	0.92721987
0.60000000	1.02884567	2.83547734
0.80000000	1.59650534	10.71088533
1.00000000	2.28735529	47.94144638

Iternum: 10

x	True y	Pred y
0.10000000	0.11033299	0.11205511

0.20000000	0.24265527	0.24511651
0.30000000	0.39891055	0.40177810
0.40000000	0.58094390	0.58409696
0.50000000	0.79043908	0.79382205
0.60000000	1.02884567	1.03241831
0.70000000	1.29729511	1.30101499
0.80000000	1.59650534	1.60032101
0.90000000	1.92667330	1.93052103
1.00000000	2.28735529	2.29115692

Iternum: 20

x	True y	Pred y
0.05000000	0.05254166	0.05259504
0.10000000	0.11033299	0.11040899
0.15000000	0.17362234	0.17370939
0.20000000	0.24265527	0.24274900
0.25000000	0.31767297	0.31777169
0.30000000	0.39891055	0.39901355
0.35000000	0.48659515	0.48670207
0.40000000	0.58094390	0.58105449
0.45000000	0.68216175	0.68227577
0.50000000	0.79043908	0.79055629
0.55000000	0.90594922	0.90606933
0.60000000	1.02884567	1.02896834
0.65000000	1.15925927	1.15938414
0.70000000	1.29729511	1.29742175
0.75000000	1.44302927	1.44315720
0.80000000	1.59650534	1.59663402
0.85000000	1.75773083	1.75785967
0.90000000	1.92667330	1.92680163
0.95000000	2.10325633	2.10338342
1.00000000	2.28735529	2.28748035

思考题

1. 对实验 1，数值解和解析解相同吗？为什么？试加以说明。

对于问题1.1，数值解和解析解是相同的，因为本题的解是线性函数，能够通过所得数值解的两个点确定直线的方程，即等价于得到了解析解。

本例中，待求解微分方程为 $\frac{dy}{dx} = x + y$ ，解为 $y = -x - 1$ ，而 `rungekutta()` 函数求解的任意两点（如 $(0.2, -1.2)$ ， $(1.0, -2.0)$ ）所决定的直线方程即为 $y = -x - 1$ 。

而对于问题1.2，虽然数值解和解析解之间差异已经极小（绝对误差在 $1e-7 \sim 1e-5$ 数量级，仅仅对比相同 x 所在的 y 取值，如下表所示），但对于非线性函数 $y = \frac{1}{1+x}$ ，在未知函数解析式类型的情况下，是几乎不可能仅仅通过数值解所求得的点来推断准确的函数解析式的，此时不能认为所求得的数值解就是解析解。

Test x	True y	5-Iter Pred y	10-Iter Pred y	20-Iter Pred y
0.20000000	0.83333333	0.83333904	0.83333373	0.83333336
0.40000000	0.71428571	0.71429213	0.71428615	0.71428574
0.60000000	0.62500000	0.62500589	0.62500040	0.62500003
0.80000000	0.55555556	0.55556069	0.55555590	0.55555558
1.00000000	0.50000000	0.50000441	0.50000030	0.50000002

2. 对实验 2, N 越大越精确吗? 试加以说明。

虽然确实N越大越精确, 但从本例实验的结果来看, 因为当n=5的时候已经获得足够精确的数值解了, 再增大n的值只是增加了计算量, 却不能再明显提高结果的精度, 此时我们不能一味的增大N, 而要根据所需要达到的精度要求及时终止计算。

本例中, $y = x^2(e^x - e)$, 在迭代次数从5增加到20的时候, 数值上的精度只增加了2位, 继续增大n对于所求数值解精度改变很小, 很难继续使用Runge-Kutta方法继续进行求解, 并且这样的计算资源成本是不可忽略的。

Test x	True y	5-Iter Pred y	10-Iter Pred y	20-Iter Pred y
1.40000000	2.62035955	2.61394279	2.61974052	2.62031131
1.80000000	10.79362466	10.77631317	10.79201760	10.79350178
2.20000000	30.52458129	30.49165420	30.52159814	30.52435589
2.60000000	72.63928396	72.58559861	72.63450354	72.63892578
3.00000000	156.30529585	156.22519828	156.29825744	156.30477188

3. 对实验 3, N 较小会出现什么现象? 试加以说明

当n较小的时候所得数值解和正确结果相差较大, 结果失真, 说明在一定条件下确实需要更大的n来更好的获得数值解。而具体这个n的大小如何选取则取决于待求解微分方程性质, 这里应该涉及到更深入的课程或者研究。

对本例而言, 从下表以及所绘制的图像都很容易能看到, 当n较小的时候会导致求得数值解偏差极大, 甚至于几乎就完全是错误的(大约与正确结果相差1e3的量级), 所以选择充分大的n, 并设置结果收敛的措施, 才能确保最终可以得到精度合适的数值解的同时不会造成太大的计算资源浪费。

下表为了便于对齐, 略去了多余的x数据, 方程的解析解为 $y = e^{-20x} + \sin(x)$, 数值解如下所示:

x	True y	5-Iter Pred y	10-Iter Pred y	20-Iter Pred y
0.20000000	0.04610521	1.76000000	0.07925926	0.04667348
0.40000000	0.16011182	8.81333333	0.16658436	0.16021366
0.60000000	0.36000205	43.68000000	0.36295382	0.36008591
0.80000000	0.64000004	217.29333333	0.64255042	0.64008338
1.00000000	1.00000000	1084.32000000	1.00250560	1.00008333

以下为方程 $\frac{dy}{dx} = -20(y - e^x \sin(x)) + e^x(\sin(x) + \cos(x))$ 的部分数值解表格, 为便于集中观察而总结如下, 解析解为 $y = e^x \sin(x)$,

x	True y	5-Iter Pred y	10-Iter Pred y	20-Iter Pred y
0.20000000	0.24265527	0.29864621	0.24511651	0.24274900
0.40000000	0.58094390	0.92721987	0.58409696	0.58105449
0.60000000	1.02884567	2.83547734	1.03241831	1.02896834
0.80000000	1.59650534	10.71088533	1.60032101	1.59663402
1.00000000	2.28735529	47.94144638	2.29115692	2.28748035

参考资料

1. julia ordinary differential equations tutorial https://diffeq.sciml.ai/stable/tutorials/ode_example/
2. intro to solving differential equations in julia <https://www.youtube.com/watch?v=KPEqYtEd-zY>
3. julia ode solver type: Runge-Kutta https://diffeq.sciml.ai/stable/solvers/ode_solve/#Explicit-Runge-Kutta-Methods
4. julia ode problem type https://diffeq.sciml.ai/stable/types/ode_types/#ode_prob
5. julia ode speed up perf https://diffeq.sciml.ai/stable/features/performance_overloads/#performance_overload
6. julia ode common solver option https://diffeq.sciml.ai/stable/basics/common_solver_opts/#solver_options
7. 《计算方法实验指导》实验题目 3 四阶龙格—库塔(Runge—Kutta)方法

附录

执行代码

本部分代码用于将需要呈现的结果封装在一个 `show_result()` 函数中，作图时调用重载的三个作图函数 `show_plot()`，分别绘制出 `lib solver`，`my solver` 和 `true result` 的图像，用于观察结果。在运行的循环中，打印出每次执行时的数据，以表格方式呈现。

在本部分之后，是各个问题的逐一求解过程，因题目本身不带更多条件，为标准的常微分方程初值问题求解，故仅按部就班完成了代码的编写和求解，以及结果展示。

为便于区分题目，所绘制的图像中给出了题目的微分方程和标准解的解析式，可供参考。考虑到图片整洁性的原因，略去对于x范围和初值的呈现，前者可直接从x轴范围看出，后者可从标准解的y坐标大致读出。

```
function show_plot(p, f::Function, tspan, u0::Float64, reltol, abstol, dense::Bool)
    prob = ODEProblem(f, u0, tspan)
    alg = RK4()
    sol = solve(prob, alg, reltol=1e-8, abstol=1e-8)
    if dense
        p = plot!(sol, seriestype=:scatter, markersize=1, msw=0, color=:red, label="lib
solver")
    else
        p = plot!(sol.t, sol.u, seriestype=:scatter, markersize=2, msw=0, color=:red,
label="lib solver")
    end
    p, sol
end
function show_plot(p, f::Function, xspan, y0::Float64, iternum::Integer)
    xs, ys = rungekutta(f, xspan, y0, iternum)
    p = plot!(xs, ys, seriestype=:scatter, markersize=4, msw=0, color=:green, label="my
solver")
    p, xs, ys
end
function show_plot(p, f::Function, xs, show::Bool, text)
    x = xlims(p)[2]
    y = mean(ylims(p))
    annotate!(x, y, text, :black)
    if show
        p = plot!(f, color=:blue, label="true result")
    else
        p = plot!(f, color=:blue, label="true result")
    end
    p, xs, f.(xs)
end
function show_result(f1::Function, f2::Function, f3::Function, xspan, y0, iternums, show::Bool,
dense::Bool, title, text)
    println("\n\n" * title)
```

```

for itemnum in itemnums
    print("\nItemnum: $itemnum\n")
    p = plot(legend=:outertopright, title=L"~~~~~" * title)
    p, sol = show_plot(p, f1, xspan, y0, 1e-8, 1e-8, dense)
    p, xs, ys = show_plot(p, f2, xspan, y0, itemnum)
    p, xt, yt = show_plot(p, f3, xs, show, text)
    data = [xt yt ys]
    header = (["x", "True y", "Pred y"])
    pretty_table(
        data;
        alignment=[:c, :c, :c],
        header=header,
        header_crayon=crayon"bold",
        # tf = tf_markdown,
        formatters=ft_printf("%14.8f"))
    display(p)
end
end

```

测试代码

这是一段从教材上选取的测试代码。

待求微分方程为 $\frac{dy}{dx} = y - \frac{2x}{y}$ ，解析解为抛物线 $y^2 = 2x + 1$ ，编写的 `rungekutta()` 函数进行数值求解时只求解了 $y > 0$ 的情形。

除此以外，调用 `DifferentialEquations.jl` 库中经 `ODEProblem()` 返回类型重载了的 `solve()` 方法获得了更精确的数值解。

因本部分仅做测试用，运行过程未经过封装，略显零乱，但考虑到与本实验问题求解并无直接关联，故未作更多修改。

Test 1 - Simple

```

f(y, p, x) = y - 2x / y
xspan = (0.0, 1.0)
y0 = 1.0
prob = ODEProblem(f, y0, xspan)
alg = RK4()
sol = solve(prob, alg, reltol=1e-8, abstol=1e-8)
plot(title=L"~~~~~ Problem:\ \frac{\mathrm{d}}{y}\ x=y-\frac{2x}{y}")
plot!(sol.t, sol.u, seriestype=:scatter, markersize=3, msw=0, color=:red, label="lib solver")

f(x, y) = y - 2x / y
println("My Runge-Kutta Solver:")
num = convert(Integer, 1.0 / 0.2)
xs, ys = rungekutta(f, xspan, y0, 5)
yt = .√(2 .* xs .+1)
data = [xs yt ys]
header = (["x", "True y", "Pred y"])
pretty_table(
    data;
    alignment=[:c, :c, :c],
    header=header,
    header_crayon=crayon"bold",
    formatters=ft_printf("%14.8f"))
p = plot!(xs, ys, seriestype=:scatter, markersize=5, msw=0, color=:green, label="my solver")

f(x, y) = y^2 - 2x - 1
p = plot!(f == 0.0, color=:green, linewidth=0.1, label="true result") # \Equal[Tab]
p = plot!(legend=:outertopright, xlim=(-0.51, 1.53), ylim=(-1.9, 1.9))

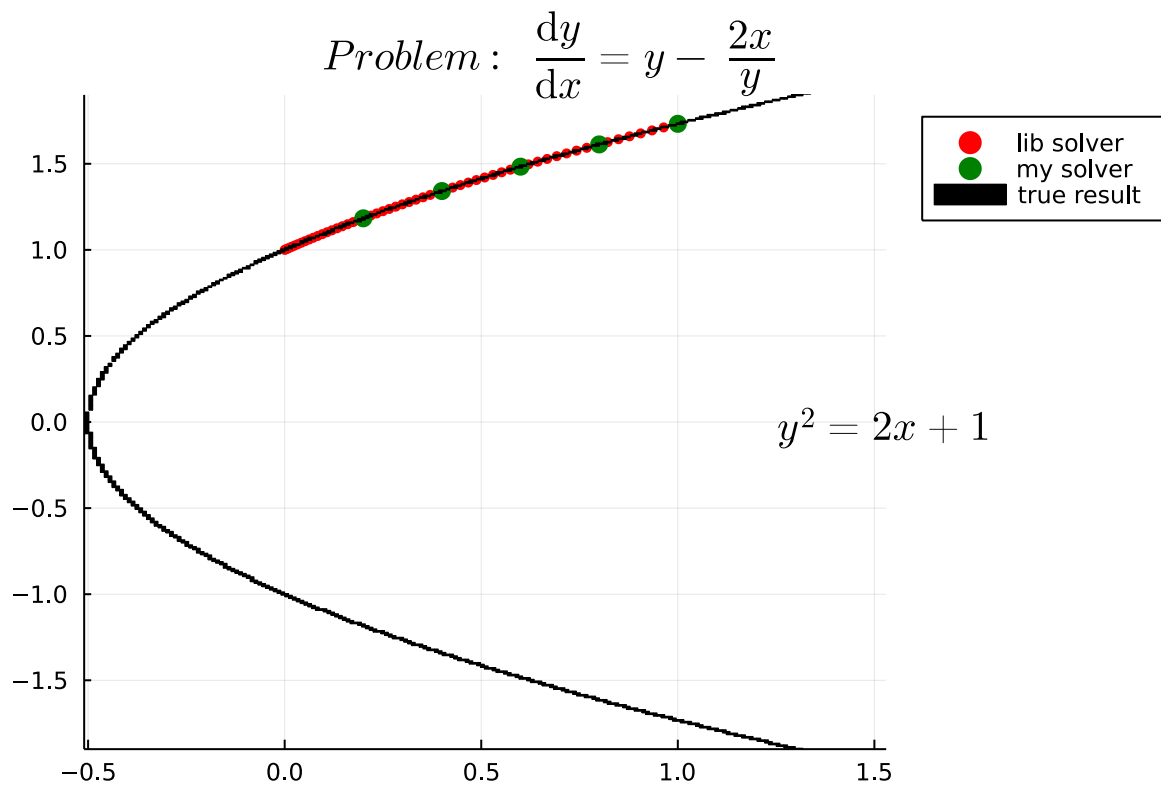
```



```

x = xlims(p)[2]
y = mean(ylims(p))
ymax = ylims(p)[2]
annotate!(x, y, L"$y^2=2x+1$", :black)
display(p)

```



My Runge-Kutta Solver:

x	True y	Pred y
0.20000000	1.18321596	1.18322929
0.40000000	1.34164079	1.34166693
0.60000000	1.48323970	1.48328146
0.80000000	1.61245155	1.61251404
1.00000000	1.73205081	1.73214188