作业 习题六、

2、Euler法式为 $y_{n+1} = y_n + h\lambda y_n$，即 $y_{n+1} - y_n - h\lambda y_n = 0$

有 $\rho(r) = r-1$，$\sigma(r) = 1$

第一特征多项式 $\rho(r)=0$ 仅有1个单根 $r=1$，故满足根条件

由于 $\rho(1)=0$，$\rho'(1) = 1 = \sigma(1)$． 故线性多步法是相容的

由定理6.6 线性多步法收敛的充分必要条件知，该初值问题Euler法是收敛的

5、证明：$T_n = C_0 y(x_n) + C_1 h y'(x_n) + \cdots + C_q h^q y^{(q)}(x_n) + \cdots$，$y_{n+1} = (\frac{1}{2}y_n + \frac{1}{2}y_{n-1}) + h(y'_{n+1} - \frac{1}{4}y'_n + \frac{3}{4}y'_{n-1})$

其中
$$\begin{cases} C_0 = 1 - (\frac{1}{2}+\frac{1}{2}) = 0 \\ C_1 = 1 - [(0\times\frac{1}{2} + 1\times\frac{1}{2}) + (1 - \frac{1}{4} + \frac{3}{4})] = 1 - [\frac{1}{2} + \frac{3}{2}] = 1-1 = 0 \\ C_2 = \frac{1}{2} - \frac{1}{2}\{0 + (-1)^2\cdot\frac{1}{2} + 2[1^2\cdot 1 + 0 + (-1)^1\cdot\frac{3}{4}]\} = \frac{1}{2} - \frac{1}{4} - (1-\frac{3}{4}) = 0 \\ C_3 = \frac{1}{3!} - \frac{1}{3!}\{0 + (-1)^3\cdot\frac{1}{2} + 3[1^2\cdot 1 + 0 + (-1)^2\cdot\frac{3}{4}]\} = \frac{1}{6} - \frac{1}{6}(-\frac{1}{2} + \frac{21}{4}) = -\frac{5}{8} \end{cases}$$

由 $C_0 = C_1 = C_2 = 0$，$C_3 \neq 0$ 知该方法是个2阶方法，局部截断误差主项为

$$C_{r+1} h^{r+1} y^{(r+1)}(x_n) = -\frac{5}{8} h^3 y^{(3)}(x_n)$$

11、具体计算过程见后附代码，以下仅讨论稳定性

由于 $y' = -20y$　故 $\bar{h} = h\lambda$，$\lambda = -20$．对于 $R_{44}(\bar{h}) = 1 + \bar{h} + \frac{1}{2!}\bar{h}^2 + \frac{1}{3!}\bar{h}^3 + \frac{1}{4!}\bar{h}^4$，绝对稳定区间为 $(-2.78, 0)$

故有 $\bar{h} \in (-2.78, 0) \Rightarrow h \in$ ~~(-0.139, 0)~~ $(0, 0.139)$ 故当 $h=0.1$ 是是稳定的，$h=0.2$ 时是不稳定的

11. 使用4阶Runge-Kutta方法求解如下所示：

函数定义如下：

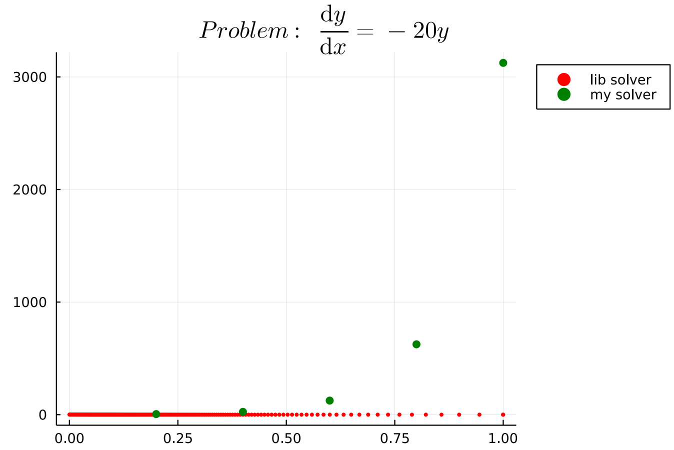
```julia
function rungekutta(f::Function, xspan, y0, num)
    a, b = xspan
    x0 = a
    h = (b - a) / num
    xs, ys = zeros(num), zeros(num)
    for n = 1:num
        K1 = h * f(x0, y0)
        K2 = h * f(x0 + h / 2, y0 + K1 / 2)
        K3 = h * f(x0 + h / 2, y0 + K2 / 2)
        K4 = h * f(x0 + h, y0 + K3)
        x1 = x0 + h
        y1 = y0 + 1 / 6 * (K1 + 2K2 + 2K3 + K4)
        xs[n], ys[n] = x0, y0 = x1, y1
    end
    xs, ys
end
```

计算结果如下：

Runge-Kutta Solver:

x	h=0.2 Pred y	h=0.1 Pred y	h=0.05 Pred y	h=0.001 Pred y
0.10000000	NaN	0.33333333	0.14062500	0.13533528
0.20000000	5.00000000	0.11111111	0.01977539	0.01831564
0.30000000	NaN	0.03703704	0.00278091	0.00247875
0.40000000	25.00000000	0.01234568	0.00039107	0.00033546
0.50000000	NaN	0.00411523	0.00005499	0.00004540
0.60000000	125.00000000	0.00137174	0.00000773	0.00000614
0.70000000	NaN	0.00045725	0.00000109	0.00000083
0.80000000	625.00000000	0.00015242	0.00000015	0.00000011
0.90000000	NaN	0.00005081	0.00000002	0.00000002
1.00000000	3125.00000000	0.00001694	0.00000000	0.00000000

图1：左图为h=0.2时所求数值解，右图为h=0.1时所求数值解



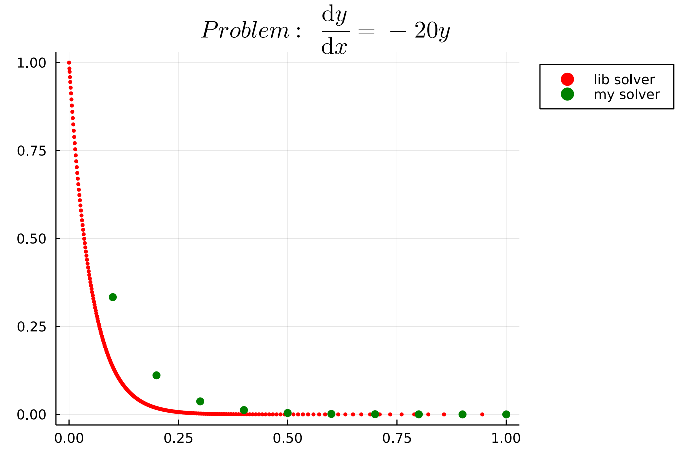
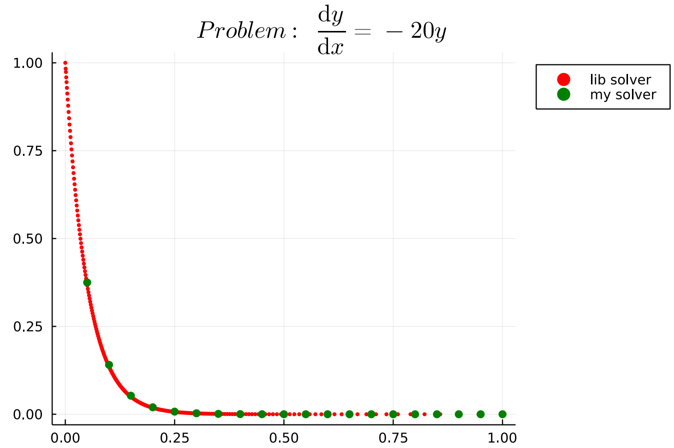
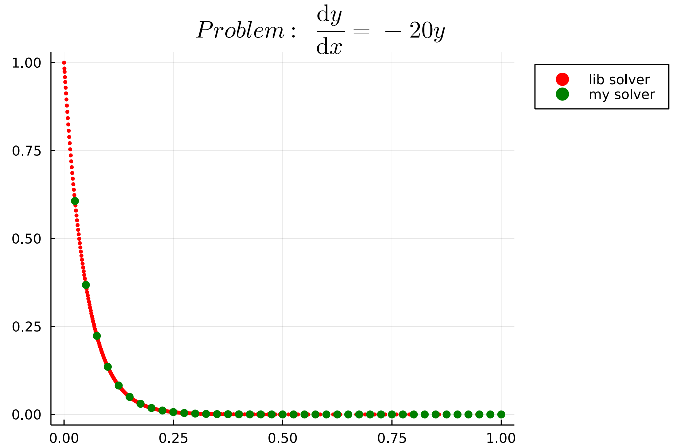


图2：左图为h=0.05时所求数值解，右图为h=0.025时所求数值解





具体的求解过程调用的`Julia`代码：

```
using DifferentialEquations
using Plots
using LaTeXStrings
using Statistics
using ImplicitEquations
using PrettyTables

for h in [0.2, 0.1, 0.05, 0.025]
    f(y, p, x) = -20y
    xspan = (0.0, 1.0)
    y0 = 1.0
    prob = ODEProblem(f, y0, xspan)
    alg = RK4()
    sol = solve(prob, alg, reltol=1e-8, abstol=1e-8)
    plot(title=L"~~~~~~~~~~~~ Problem:\ \frac{\mathrm{d} y}{\mathrm{d}
x}=-20y",legend=:outertopright)
    plot!(sol.t, sol.u, seriestype=:scatter, markersize=2, msw=0, color=:red, label="lib
solver")

    f(x, y) = -20y
    println("My Runge-Kutta Solver:")
    num = convert(Integer, 1.0 / h)
    xs, ys = rungekutta(f, xspan, y0, num)
    data = [xs ys]
    header = (["x", "Pred y"])
    pretty_table(
        data;
        alignment=[:c, :c],
        header=header,
        header_crayon=crayon"bold",
        # tf = tf_markdown,
        formatters=ft_printf("%14.8f"))
    p = plot!(xs, ys, seriestype=:scatter, markersize=4, msw=0, color=:green, label="my
solver")
    display(p)
end
```

6. (1) $T_n = c_0 y(x_n) + c_1 h y'(x_n) + c_2 h^2 y''(x_n) + \cdots$

$c_0 = 1 - a_0 - a_1$

$c_1 = 1 - [a\cdot 0 + (-1)\cdot a_1 + b_0 + b_1] = 1 + a_1 - b_0 - b_1$

$c_2 = \frac{1}{2} - \frac{1}{2}[(-0)^2 \cdot a_0 + (-1)^2 \cdot a_1 + 2(-0)\cdot b_0 + 2(-1) b_1] = \frac{1}{2} - \frac{1}{2}a_1 + b_1$

$c_3 = \frac{1}{6} - \frac{1}{6}\{(-0)^3 a_0 + (-1)^3 a_1 + 3[(-0)^2 \cdot b_0 + (-1)^2 \cdot b_1]\} = \frac{1}{6} + \frac{1}{6}a_1 - \frac{1}{2}b_1$

$c_4 = \frac{1}{4!} - \frac{1}{4!}\{(-0)^4 a_0 + (-1)^4 a_1 + 4[(-0)^3 \cdot b_0 + (-1)^3 \cdot b_1]\} = \frac{1}{24} - \frac{1}{24}a_1 + \frac{1}{6}b_1$

令 $c_0 = 0 \Rightarrow a_0 + a_1 = 1$，即 $a_1 = 1 - a_0$

令 $c_1 = 0 \Rightarrow 1 + a_1 - b_0 - b_1 = 0$ 即 $b_0 + b_1 = 2 - a_0$

令 $c_2 = 0 \Rightarrow$ ~~$\frac{1}{6} + \frac{1}{6}a_1 - \frac{1}{2}b_1 = 0$ 即 $b_1 = \frac{2}{3} - \frac{1}{3}a_0$，$b_0 = \frac{4}{3} - \frac{2}{3}a_0$~~

$\frac{1}{2} - \frac{1}{2}a_1 + b_1 = 0$. 即 $b_1 = -\frac{1}{2}a_0$，$b_0 = 2 - \frac{1}{2}a_0$

此时 $c_3 = \frac{1}{6} + \frac{1}{6}a_1 - \frac{1}{2}b_1 = \frac{1}{6} + \frac{1}{6} - \frac{1}{6}a_0 + \frac{1}{4}a_0 = \frac{1}{3} + \frac{1}{12}a_0$ 不恒为0.

故由 $c_0 = c_1 = c_2 = 0$ 且 c_3 知 知该法至少是二阶的 此时有 $a_1 = 1 - a_0$，$b_0 = 2 - \frac{1}{2}a_0$，$b_1 = -\frac{1}{2}a_0$

(2) 若满足根条件，则 $\rho(r) = r^2 - a_0 r - a_1$ 两根均满足模不大于1且为1的是单根.

由 $a_1 + a_0 = 1$ 得 $r^2 - a_0 r - (1 - a_0) = (r-1)[r + (1 - a_0)] \Rightarrow r_1 = 1$，$r_2 = a_0 - 1$

由于 $r_1 = 1$ 则 $|a_0 - 1| \neq 1$，$|a_0 - 1| < 1 \Rightarrow 0 < a_0 < 2$

即当 $0 < a_0 < 2$ 时 该法满足根条件（假定第一题条件成立）

(3) $a_0 = 0$ 时 $a_1 = 1$ 有 $y_{n+1} = y_{n-1} + h(b_0 y_n' + b_1 y_{n-1}')$（假设第(1)题条件成立）

由 $b_0 = 2$，$b_1 = \frac{1}{2}a_0 = 0$ 得 $y_{n+1} = y_{n-1} + 2hy_n'$（步长为2的Euler法）

$a_0 = 1$ 时，$a_1 = 0$，$b_0 = \frac{3}{2}$，$b_1 = -\frac{1}{2}$ 得 $y_{n+1} = y_{n-1} + h(\frac{3}{2}y_n' - \frac{1}{2}y_{n-1}')$（步长为2的显式Adams）

(4) 令 $c_3 = 0$，得 $a_0 = -4$，有 $|r_2| = |-5| > 1$ 不满足根条件