习题二

5. $A=\begin{bmatrix}5&7&9&10\\6&8&10&9\\7&10&8&7\\5&7&6&5\end{bmatrix}$ Doolittle 分解

$$A=LU=\underbrace{\begin{bmatrix}1&&&\\1.2&1&&\\1.4&-0.5&1&\\1&0&0.6&1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}5&7&9&10\\&-0.4&-0.8&-3\\&&-5&-8.5\\&&&0.1\end{bmatrix}}_{U}$$

由 $Ax=b \Longleftrightarrow \begin{cases}Ly=b\\Ux=y\end{cases}$

解得 $y=\begin{bmatrix}1\\-0.2\\-0.5\\0.3\end{bmatrix}$，代回入求得 $x=\begin{bmatrix}20\\-12\\-5\\3\end{bmatrix}$

对于 Crout 分解 $A=LU=LD\hat{U}=\hat{L}\hat{U}$

$$\hat{L}=\begin{bmatrix}5&&&\\6&-0.4&&\\7&0.2&-5&\\5&0&-3&0.1\end{bmatrix}\quad \hat{U}=\begin{bmatrix}1&1.4&1.8&2\\&1&2&7.5\\&&1&1.7\\&&&1\end{bmatrix}$$

同理，解得 $\hat{y}=\begin{bmatrix}0.2\\0.5\\0.1\\3\end{bmatrix}$ 代入求得 $\hat{x}=\begin{bmatrix}20\\-12\\-5\\3\end{bmatrix}$

7. $A=\begin{bmatrix}4&-1&1\\-1&4.75&2.75\\1&2.75&3.5\end{bmatrix}$ 平方根法 $A=\bar{L}\cdot\bar{L}^T=\begin{bmatrix}2&&\\-0.5&\sqrt{4.5}&\\0.5&3/\sqrt{4.5}&\sqrt{1.25}\end{bmatrix}\begin{bmatrix}2&-0.5&0.5\\&\sqrt{4.5}&3/\sqrt{4.5}\\&&\sqrt{1.25}\end{bmatrix}$

由 $Ax=b$
$\bar{L}\cdot\bar{L}^Tx=b$
$\Rightarrow\begin{cases}\bar{L}^Tx=y\\\bar{L}y=b\end{cases}$

解得 $y=\begin{bmatrix}2\\3.2998\\1.4161764\end{bmatrix}$

$x=\begin{bmatrix}0.8611111\\0.7111111\\1.266667\end{bmatrix}$

$b=[4\ \ 6\ \ 7.25]^T$

改进的平方根法
$A=LDL^T=\begin{bmatrix}1&&\\-0.25&1&\\0.25&2/3&1\end{bmatrix}\begin{bmatrix}4&&\\&4.5&\\&&1.25\end{bmatrix}\begin{bmatrix}1&-0.25&0.25\\&1&2/3\\&&1\end{bmatrix}$

由 $Ax=b$
$\begin{cases}L\bar{y}=b\\DL^Tx=y\end{cases}$

解得 $y=[4,7,19/12]^T$

~~$x=[0.861111$~~

$x=[31/36\ \ 32/45\ \ 19/15]$

8. 追赶法

(1) $A=\begin{bmatrix}2&1&&\\1&4&1&\\&1&4&1\\&&1&2\end{bmatrix}=\underbrace{\begin{bmatrix}2&&&\\1&7/2&&\\&1&26/7&\\&&1&45/26\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}1&1/2&&\\&1&2/7&\\&&1&7/26\\&&&1\end{bmatrix}}_{U}$

由 $\begin{cases}Ly=b\\Ux=y\end{cases}\Rightarrow y=\begin{bmatrix}1/2\\-5/7\\1\\-26/45\end{bmatrix}$ $x=\begin{bmatrix}46/45\\-47/45\\52/45\\-26/45\end{bmatrix}=\begin{bmatrix}1.0222\\-1.0444\\1.1556\\-0.5778\end{bmatrix}$

(2) $A=\begin{bmatrix}4&-1&&&\\-1&4&-1&&\\&-1&4&-1&\\&&-1&4&-1\\&&&-1&4\end{bmatrix}=\begin{bmatrix}4&&&&\\-1&15/4&&&\\&-1&56/15&&\\&&-1&209/56&\\&&&-1&780/209\end{bmatrix}\begin{bmatrix}1&-1/4&&&\\&1&-4/15&&\\&&1&-15/56&\\&&&1&-56/209\\&&&&1\end{bmatrix}$

$\cancel{\Rightarrow y=\begin{bmatrix}25\\20/3\\25/14\\22/409\\3761/10\end{bmatrix}}$ $\cancel{x=\begin{bmatrix}866/105\\83/14\\22/35\\3761/70\end{bmatrix}}$

$b=[100\ \ 0\ \ 0\ \ 0\ \ 200]^T$

$y=\begin{bmatrix}25\\20/3\\25/14\\100/209\\2095/39\end{bmatrix}$ $x=\begin{bmatrix}1055/39\\320/39\\75/13\\580/39\\2095/39\end{bmatrix}=\begin{bmatrix}27.0528\\8.2051\\5.7692\\14.8718\\53.7179\end{bmatrix}$

10. $A=\begin{bmatrix}1&1&-1\\2&1&0\\1&-1&0\end{bmatrix}$

$$C=\left[\begin{array}{ccc:ccc}1&1&-1&1&0&0\\2&1&0&0&1&0\\1&-1&0&0&0&1\end{array}\right]\xrightarrow{r_2\leftrightarrow r_1}\left[\begin{array}{ccc:ccc}\boxed{2}&1&0&0&1&0\\1&1&-1&1&0&0\\1&-1&0&0&0&1\end{array}\right]\xrightarrow{\text{消元1}}\begin{bmatrix}1&1/2&0&0&1/2&0\\&1/2&-1&1&-1/2&0\\&\boxed{-3/2}&0&0&-1/2&1\end{bmatrix}$$

$$\xrightarrow{r_2\leftrightarrow r_3}\begin{bmatrix}1&1/2&0&0&1/2&0\\&\boxed{-3/2}&0&0&-1/2&1\\&1/2&-1&1&-1/2&0\end{bmatrix}\xrightarrow{\text{消元2}}\begin{bmatrix}1&0&0&0&\frac{1}{3}&1/3\\&1&0&0&1/3&-2/3\\&0&\boxed{-1}&1&-4/6&1/2\end{bmatrix}=\left[\begin{array}{ccc:ccc}1&&&0&1/3&1/3\\&1&&0&1/3&-2/3\\&&1&-1&2/3&-1/3\end{array}\right]=(I:A^{-1})$$

即 $A^{-1}=\begin{bmatrix}0&1/3&1/3\\0&1/3&-2/3\\-1&2/3&-1/3\end{bmatrix}$

15. $A=\begin{bmatrix}10&4&4\\4&10&8\\4&8&10\end{bmatrix}, b=\begin{bmatrix}13\\11\\25\end{bmatrix}$

$A=L+D+U$

Jacobi: $x_{k+1}=-D^{-1}(L+U)x_k+D^{-1}b$

$$x_{k+1}=\begin{pmatrix}0&-0.4&-0.4\\-0.4&0&-0.8\\-0.4&-0.8&0\end{pmatrix}x_k+\begin{pmatrix}1.3\\1.1\\2.5\end{pmatrix}$$

Gauss-Seidel:

$x_{k+1}=-(D+L)^{-1}U\cdot x_k+(D+L)^{-1}b$

$$x_{k+1}=-\begin{bmatrix}10&&\\4&10&\\4&8&10\end{bmatrix}^{-1}\begin{bmatrix}0&4&4\\&0&8\\&&0\end{bmatrix}\cdot x_k+\begin{bmatrix}10&&\\4&10&\\4&8&10\end{bmatrix}^{-1}\begin{bmatrix}13\\11\\25\end{bmatrix}=\begin{bmatrix}0&-0.4&-0.4\\0&0.16&-0.64\\0&0.032&0.672\end{bmatrix}x_k+\begin{bmatrix}1.3\\0.58\\1.516\end{bmatrix}$$

SOR:

$x_{k+1}=(D+\omega L)^{-1}((1-\omega)D-\omega U)x_k+\omega(D+\omega L)^{-1}b$

$$x_{k+1}=\begin{bmatrix}-0.35&-0.54&-0.54\\0.189&-0.0584&-0.7884\\-0.0152&0.354672&0.793072\end{bmatrix}x_k+\begin{bmatrix}1.755\\0.5373\\1.847016\end{bmatrix}$$

收敛性判断

Jacobi $B_J=\begin{bmatrix}0&-0.4&-0.4\\-0.4&0&-0.8\\-0.4&-0.8&0\end{bmatrix}$, $|\lambda E-B_J|=\begin{vmatrix}\lambda&0.4&0.4\\0.4&\lambda&0.8\\0.4&0.8&\lambda\end{vmatrix}=\lambda(\lambda^2-0.64)-0.4(0.4\lambda-0.32)+0.4(0.32-0.4\lambda)=0$

$\Rightarrow f(\lambda)=\lambda^3-0.96\lambda+0.256=0.$

$$\begin{cases}\lambda_1=\dfrac{-2-2\sqrt3}{5}\approx-1.09282\\ \lambda_2=4/5=0.8\\ \lambda_3=\dfrac{-2+2\sqrt3}{5}\approx0.29282\end{cases}$$

$\Rightarrow \rho(B_J)=\max\{|\lambda_1|,|\lambda_2|,|\lambda_3|\}=|\lambda_1|>1$

所以、Jacobi 迭代不收敛

Gauss-Seidel:

$B_G=\begin{bmatrix}0&-0.4&-0.4\\0&0.16&-0.64\\0&0.032&0.672\end{bmatrix}$, $|\lambda E-B_G|=\begin{vmatrix}\lambda&0.4&0.4\\0&\lambda-0.16&0.64\\0&-0.032&\lambda-0.672\end{vmatrix}=\lambda[(\lambda-0.16)(\lambda-0.672)-0.64\times(-0.032)]=0$

$\Rightarrow f(\lambda)=\lambda(\lambda^2-0.832\lambda+0.128)=0$

$\lambda_1=0$, $\lambda_2\approx0.6283$, $\lambda_3\approx0.2037$

$\Rightarrow \rho(B_G)=\max\{|\lambda_1|,|\lambda_2|,|\lambda_3|\}=|\lambda_2|<1$，所以 Gauss-Seidel 迭代收敛

SOR

$$B_s = \begin{bmatrix} -0.35 & -0.54 & -0.54 \\ 0.189 & -0.0584 & -0.7884 \\ -0.0152 & 0.354672 & 0.793072 \end{bmatrix}, \quad |\lambda E - B_s| = \begin{vmatrix} \lambda+0.35 & 0.54 & 0.54 \\ -0.189 & \lambda+0.0584 & 0.7884 \\ 0.0152 & -0.354672 & \lambda-0.793072 \end{vmatrix} = 0.$$

$$\Rightarrow f(\lambda) = (\lambda+0.35)\big((\lambda+0.0584)(\lambda-0.793072)+0.7884\times0.354672\big)$$
$$+0.189\big(0.54\times(\lambda-0.793072)+0.54\times0.354672\big)$$
$$+0.0152\big(0.54\times0.7884-0.54(\lambda+0.0584)\big)=0$$

$$\Rightarrow f(\lambda) = \lambda^3 - 0.384672\lambda^2 + \overset{0.0700248}{\cancel{0.\ldots}}\,\lambda + 0.04290636 = 0$$

$\lambda_1 = \cancel{-0.1901516}\ -0.21535$

$\lambda_{2,3} = \cancel{0.2874118004 \pm 0.3782042318i}\ \ 0.30 \pm 0.33051i$

$\rho(B_s) = \max\{|\lambda_1|, |\lambda_{2,3}|\} = |\lambda_{2,3}| < 1$. 所以 SOR 方法收敛.