Recovering Probability Moments from Inflation Option Prices

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Quantitative Methods in Finance

- **Report scope**
  - dataset: 1y inflation caps, floors, swaps; areas: EU, US
  - objective: option price time series → implied moments (mean, variance, skewness, kurtosis)
  - focus: identification, stability; no forecasting
- **Deliverables**
  - code: `code in github`
  - outputs: `moments_main.csv`, `moments_direct_bkm.csv`, `moments_parametric_lognormal.csv`, `moments_parametric_normal.csv`
- **LLM disclosure**
  - ChatGPT (GPT-5.2 Thinking)
  - usage: diagnostic narratives and LaTeX bullet structuring
  - no usage: data generation, result fabrication, code execution on hidden data
- **Literature review: method families & mappings**
  1. **Prices → pdf (risk-neutral density / state prices)**
     - Breeden–Litzenberger (1978) (nonparametric; differentiation of call price curve; ∼3315 cites)
     - Jackwerth–Rubinstein (1996) (nonparametric; constrained recovery / smoothing; ∼1614 cites)
     - Aït-Sahalia–Lo (1998/1995) (nonparametric; kernel / smoothing; ∼1591 cites)
     - spline/shape-constrained smoothing (convexity/monotonicity enforcement)
     - parametric densities (lognormal / mixtures) (pdf implied by calibrated parameters)
  2. **Pdf → moments (mean/var/skew/kurt)**
     - numerical quadrature on grid (trapz) on truncated support
     - analytic moments (parametric families; e.g., lognormal moments)
     - parametric fit to recovered pdf (diagnostic; e.g., Normal fit on grid)

3. **Prices → moments (direct)**
   - Bakshi–Kapadia–Madan style moment integrals (2003) (∼1858 cites)
   - static replication using OTM option payoffs; numerical integration over strikes

- **Citation-based ranking (high → low; by approximate cites)**
  1. **Prices → pdf**
     - Breeden–Litzenberger (1978) (∼3315)
     - Jackwerth–Rubinstein (1996) (∼1614)
     - Aït-Sahalia–Lo (1998/1995) (∼1591)
  2. **Prices → moments**
     - Bakshi–Kapadia–Madan (2003) (∼1858)
     - parametric calibration families (model-dependent; citations dispersed)
  3. **Pdf → moments**
     - numerical integration (standard; method-agnostic; citations dispersed)
     - analytic moments (family-specific; citations dispersed)

- **Implemented methods (cross-sectional per date/area)**
  1. **Most popular replication: Breeden–Litzenberger (two-sided)**
     - file: `method_main.py` → `moments_main.csv`
     - inputs: caps + floors + swaps
     - construction: call-equivalent curve (puts converted via parity)
     - smoothing: spline across strikes
     - density proxy: second derivative; clipping negatives; renormalization
     - moments: numerical integration on strike grid
     - gates: monotone decreasing call curve; convexity in strike; parity residual filter
  2. **Other replicated method: BKM direct moments (tail-stabilized)**
     - file: `method_moments_direct_bkm.py` → `moments_direct_bkm.csv`
     - inputs: caps + floors + swaps
     - preprocessing: deduplicate strikes; PV → undiscounted payoffs (divide by B)
     - OTM selection: puts ($K \leq K_\star$), calls ($K \geq K_\star$)
     - gates: parity residual filter; monotonicity of OTM payoffs; min strikes both wings
     - stabilization: synthetic tail points (zero payoff); optional winsorization
     - moments: raw moments (2–4) via strike integrals; central moments (var/skew/kurt)
  3. **Parametric benchmark: Lognormal calibration**
     - file: `method_parametric_lognormal.py` → `moments_parametric_lognormal.csv`
     - assumption: $X = I_T/I_0$ lognormal
     - calibration: fit (μ, σ) to OTM caps+floors; bounded σ; mean anchoring to $K_\star$
     - outputs: analytic mean/variance/skew/kurt; RMSE diagnostic
     - gates: parity residual filter; monotonicity of call/put PV curves; min strikes both wings
  4. **Diagnostic benchmark: Normal fit to BL-proxy pdf**

  - file: `method_parametric_normal.py` $\rightarrow$ `moments_parametric_normal.csv`
  - steps: single-instrument price curve $\rightarrow$ spline $\rightarrow$ BL-proxy pdf $\rightarrow$ fit Normal on grid
  - filters: convexity/monotonicity; boundary-mass threshold; variance sanity caps
  - role: diagnostic only (positivity mismatch; truncation sensitivity)

- **Comparison logic (identical inputs)**
  - common keys: (date, area)
  - intersection sample: retain rows where all compared methods return valid moments
  - plots produced: time series of mean; time series of variance (log scale)
  - diagnostics stored: strike span proxies; counts of OTM puts/calls; tail parameters; parity residual pass/fail

- **Empirical interpretation: identification & stability**
  1. **Mean (first moment)**
     - strong cross-method agreement: BL $\approx$ Lognormal $\approx$ BKM
     - driver: near-ATM information; swap-anchored center ($K_\star$); parity enforcement
     - Normal benchmark: flat/spiky behavior; diagnostic only
  2. **Variance (second moment)**
     - systematic gaps: BKM variance lower than BL/Lognormal in sparse-strike periods
     - driver: tail dependence; missing far-OTM strikes; truncation sensitivity
     - convergence episodes: improved strike coverage / liquidity (stress periods)
  3. **Skewness (third central moment)**
     - dependence: left-right tail balance (puts vs calls)
     - instability sources: wing asymmetry; sparse tail quotes; regularization sensitivity
     - interpretation: caution; conditional on balanced OTM coverage
  4. **Kurtosis (fourth central moment)**
     - dependence: extreme tails; highest sensitivity to missing strikes
     - instability sources: truncation; tail-point assumptions; outlier quotes
     - interpretation: very cautious; mainly qualitative tail-risk indicator

- **Disagreement diagnosis**
  - trigger: material divergence in mean/variance across methods
  - checks: OTM strike counts; observed strike span; tail-point usage; parity residual filter; monotonicity failures
  - primary cause: variance divergence aligned with limited far-OTM coverage; BKM downward bias

- **Why option-implied inflation can fall below the swap forward (notably 2021–2022)**
  - **Insufficient right-wing strike coverage**
    * limited availability of far-OTM caps ($K > K_\star$) during stressed or illiquid periods
    * weak identification of the upper tail of the risk-neutral distribution

    * mean estimation becomes dominated by near-ATM information
  - **Right-tail truncation bias**
    * when the swap-anchored center $K_\star$ approaches or exceeds the maximum observed strike $K_{\max}$
    * the recovered density is effectively truncated on the right
    * missing mass beyond $K_{\max}$ mechanically pulls the implied mean below the forward
  - **Smoothing and regularization effects**
    * spline interpolation across sparse strikes
    * clipping of negative density estimates and renormalization
    * regularization stabilizes estimation but compresses tails, biasing higher moments downward
  - **Cross-market inconsistency and microstructure noise**
    * caps/floors and swaps may reflect different liquidity, bid–ask spreads, or quote timestamps
    * imperfect put–call parity and forward anchoring in practice
    * residual misalignment can shift option-implied means away from the swap forward

- **Main conclusion: best way to work on moments**

  1. **Mean**
     - preferred: any method using both wings + parity + swap anchoring
     - robustness: high; low tail dependence
  2. **Variance**
     - preferred: tail-regularized methods (parametric lognormal; smoothed two-sided BL)
     - secondary: BKM with explicit tail stabilization; interpret as lower-bound under sparse tails
  3. **Skewness**
     - preferred: parametric/regularized methods; require balanced wing coverage
     - caution: high sensitivity to asymmetric strike availability
  4. **Kurtosis**
     - preferred: parametric with strong tail structure; report with sensitivity flags
     - caution: weakest identification; treat as qualitative