July 22, 2019

DEF: If X is a random variable with density $f_X(x)$ and Y is a random variable with density $f_Y(y)$, how would we describe the joint behavior of the tuple (X,Y) at the same time? The answer is: joint pdfs and joint cdfs.

A bivariate pdf is a function $f: \mathbb{R}^2 \to \mathbb{R}$ satisfying the following two properties:

1. $f(x,y) \geq 0 \quad \forall x \forall y$ in $\mathbb{R}^2$
2. $\int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y)\, dx\, dy = 1$

DEF: If $X_1, \ldots, X_n$ are r.v.'s with densities $f_{X_1}(x_1), f_{X_2}(x_2), \ldots, f_{X_n}(x_n)$, their joint density is given by some function $f: \mathbb{R}^n \to \mathbb{R}$ with

1. $f(x_1, \ldots, x_n) = f(\vec{x}) \geq 0$ for every $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$
2. $\underbrace{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}_{n\text{-fold}} f(x_1, \ldots, x_n)\, dx_n \cdots dx_1 = 1 = \int_{\mathbb{R}^n} f(\vec{x})\, d\vec{x}$

DEF: Suppose that $X \sim f_X$ and $Y \sim f_Y$, the bivariate cumulative distribution function (cdf) of (X,Y) is defined as:

$$F(s,t) = \mathbb{P}(X \leq s, Y \leq t) = \int_{-\infty}^{s} \int_{-\infty}^{t} f(x,y)\, dy\, dx$$

NOTE #1: $\lim_{\substack{t \to +\infty \\ s \to +\infty}} F(s,t) = 1$

NOTE #2: $\lim_{\substack{t \to -\infty \\ s \to -\infty}} F(s,t) = 0$

NOTE #3: F must be nondecreasing.

NOTE #4: F must be right-continuous.

NOTE #5: $\lim_{t\to\infty} F(s,t) = F_X(s)$ **NOTE #6**: $\lim_{s\to+\infty} F(s,t) = F_Y(t)$

DEF: Suppose that $X_i \sim f_i = f_{X_i}$ for $i=1,\dots,n$. The joint cdf of $(X_1,\dots,X_n)$ is a function $F:\mathbb{R}^n \to \mathbb{R}$ defined as

$$F(t_1,\dots,t_n) = \mathbb{P}(X_1 \le t_1, X_2 \le t_2, \dots, X_n \le t_n)$$
$$= \int_{-\infty}^{t_1}\int_{-\infty}^{t_2}\cdots\int_{-\infty}^{t_n} f(x_1,\dots,x_n)\,dx_n\dots dx_1$$

NOTE: The multivariate cdf inherits properties analogous to the bivariate cdf.

Qn: What does independence of r.v.'s look like in light of these definitions?

↳ **Recall**: If A and B are independent events, then $\mathbb{P}(A\cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Let $A = \{\omega\in\Omega \mid X(\omega)\le s\}$ and $B = \{\omega\in\Omega \mid Y(\omega)\le t\}$. Suppose that A and B are independent.

$$\Rightarrow \mathbb{P}(A\cap B) = \mathbb{P}(A)\mathbb{P}(B)$$
$$\Rightarrow \mathbb{P}(X\le s, Y\le t) = \mathbb{P}(X\le s)\,\mathbb{P}(Y\le t)$$

FORMAL DEFINITION: If, for every interval $[a,b]\subseteq\mathbb{R}$ and $[c,d]\subseteq\mathbb{R}$, $\mathbb{P}(X\in[a,b], Y\in[c,d]) = \mathbb{P}(X\in[a,b])\,\mathbb{P}(Y\in[c,d])$, then X and Y are independent random variables. An analogous definition holds in a multivariate context.

SAD FACE: Don't write $\mathbb{P}(X\cap Y) = \mathbb{P}(X)\mathbb{P}(Y)$.

<u>CONSEQUENCE:</u> Choose $[a,b] = (-\infty, s]$ and $[c,d] = (-\infty, t]$.

If X and Y are independent, then

$$\mathbb{P}(X \in (-\infty, s], Y \in (-\infty, t]) = \mathbb{P}(X \in (-\infty, s])\, \mathbb{P}(Y \in (-\infty, t])$$

$$\Rightarrow F(s,t) = F_X(s)\, F_Y(t).$$

This is called the <u>factorization property</u> of CDFs under independence.

<u>CALCULUS ASIDE:</u> If X is a r.v., let its density be f_X and its cdf be F_X. By definition, if X is continuous, then

$$F_X(t) = \int_{-\infty}^{t} f(x)\,dx.$$

<u>QN:</u> What is $\frac{d}{dt} F_X(t)$? $F'(t) = \frac{d}{dt} F(t) = f(t)$, but why? <u>ANS:</u> FTC, Part 1.

<u>QN:</u> What happens in higher dimensions, say two? Assume that X and Y are independent.

$$\Rightarrow F(s,t) = F_X(s)\, F_Y(t)$$

$$\Rightarrow \frac{\partial}{\partial t} F(s,t) = \frac{\partial}{\partial t} \left\{ F_X(s)\, F_Y(t) \right\}$$

$$\Rightarrow \frac{\partial}{\partial t} F(s,t) = F_X(s)\, f_Y(t)$$

$$\Rightarrow \frac{\partial}{\partial s} \frac{\partial}{\partial t} F(s,t) = f_X(s)\, f_Y(t)$$

$$\Rightarrow \underbrace{f(s,t) = f_X(s)\, f_Y(t)}_{\text{we get this factorization property for densities too!}} \quad \text{(by FTC, Part 1 in 2-dimensional setting)}$$

<u>Qn</u>: How do we extend the concepts of conditional probability to a bivariate / multivariate context, i.e., to pmfs and pdfs in several dimensions.

↳ Continuous r.v.'s.
Let $f(x,y)$ be a joint pdf.

$\Rightarrow$ The conditional density of X given $Y=y$ is

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}.$$

NOTE: $$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

↳ Discrete r.v.'s.
Let $p(x,y)$ be a joint pmf.

$\Rightarrow$ The conditional pmf of X given $Y=y$ is

$$p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}.$$

$$p_Y(y) = \sum_{x \in A} p(x,y)$$

$$p_X(x) = \sum_{y \in B} p(x,y)$$

<u>Qn</u>: What would naturally be $\mathbb{E}[X|Y=y]$? (Assume that X and Y are continuous.)

$$\mathbb{E}[X|Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\,dx = \varphi(y)$$

↑ tracks the mean of X given $Y=y$

<u>Qn</u>: How might we define $Var(X|Y=y)$?

$$\int_{-\infty}^{\infty} (x - \mathbb{E}[X|Y=y])^2 f_{X|Y}(x|y)\,dx$$

$$\underline{\mathbb{E}[X^2|Y=y] = \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y)\,dx}$$

$$\Rightarrow \underline{\mathbb{E}[X^2|Y=y] - \mathbb{E}[X|Y=y]^2} = Var(X|Y=y)$$

NOTE: Let X and Y be independent random variables.

$$\Rightarrow \mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$$

PROOF: $\mathbb{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy$. But since X is independent of Y, $f(x,y) = f_X(x)\,f_Y(y)$.

So
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\,f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} y\, f_Y(y) \underbrace{\left\{\int_{-\infty}^{\infty} x\, f_X(x)\,dx\right\}}_{\mathbb{E}[X]} dy$$

$$= \mathbb{E}[X]\int_{-\infty}^{\infty} y\, f_Y(y)\,dy = \mathbb{E}[X]\,\mathbb{E}[Y]$$

WARNING: The converse statement is NOT true, i.e., if X and Y are r.v.'s such that $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$, it does not necessarily mean that $X \perp Y$.

"independent of"

EX: Let $X \sim N(0,1)$. Set $Y = X^2$. Clearly Y depends on X, i.e., they are not independent. But

$$\mathbb{E}[XY] = \mathbb{E}[X \cdot X^2] = \mathbb{E}[X^3] = \int_{-\infty}^{\infty} x^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = 0.$$

But $\mathbb{E}[X]\,\mathbb{E}[Y] = 0 \cdot 1 = 0$.

ANOTHER NOTE: Suppose that you want to verify that $X_1, \dots, X_n$ is a sequence of independent r.v.'s. Does it suffice to check that $X_i \perp X_j \;\forall i \neq j$?

BAD NEWS: No. You must in theory check all combinations of the X_i's for the property that probabilities of intersections equal products of probabilities (or, for r.v.'s, factorizations of joint densities).

QN: Suppose we have a random vector $(X_1, \dots, X_5)$? How would we define (a) the conditional density of (X_1, X_2, X_4, X_5) given X_3 and (b) the conditional density (X_1, X_4, X_5) given (X_2, X_3)?

(a)
$$f_{(X_1,X_2,X_4,X_5)|X_3}(x_1, x_2, x_4, x_5 \mid x_3) = \frac{f(x_1, x_2, x_3, x_4, x_5)}{f_{X_3}(x_3)}$$

(b)
$$f_{(X_1,X_4,X_5)|(X_2,X_3)}(x_1, x_4, x_5 \mid x_2, x_3) = \frac{f(x_1, x_2, x_3, x_4, x_5)}{f_{(X_2,X_3)}(x_2, x_3)}$$

EX: Suppose that the joint pdf of X and Y is given by $f(x,y) = \frac{12}{5} x (2-x-y)$ for $0<x<1$ and $0<y<1$.

(a) what is $f_{X|Y}(x|y)$?

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\cancel{\frac{12}{5}} x(2-x-y)}{\cancel{\frac{12}{5}}\left(\frac{2}{3}-\frac{1}{2}y\right)} = \begin{cases} \dfrac{x(2-x-y)}{\frac{2}{3}-\frac{1}{2}y} & \text{if } \begin{array}{l} 0<x<1 \\ 0<y<1 \end{array} \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_0^1 \frac{12}{5} x(2-x-y)\,dx = \frac{12}{5}\int_0^1 2x - x^2 - xy \, dx$$

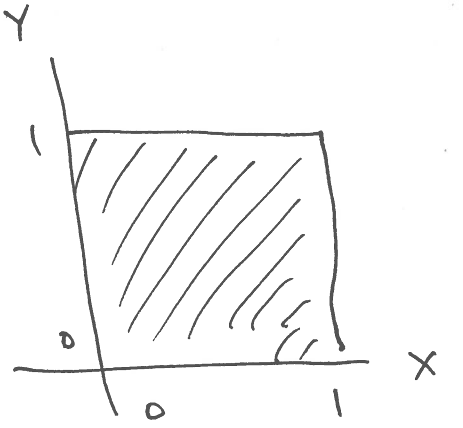
$$= \frac{12}{5}\left[\frac{2x^2}{2} - \frac{1}{3}x^3 - \frac{1}{2}x^2 y\right]_0^1$$

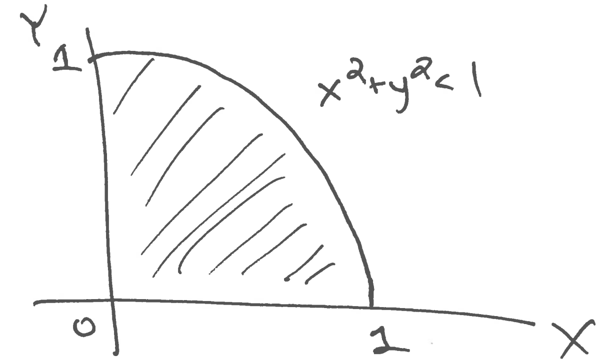
$$= \frac{12}{5}\left(1 - \frac{1}{3} - \frac{1}{2}y\right) = \frac{12}{5}\left(\frac{2}{3} - \frac{1}{2}y\right)$$

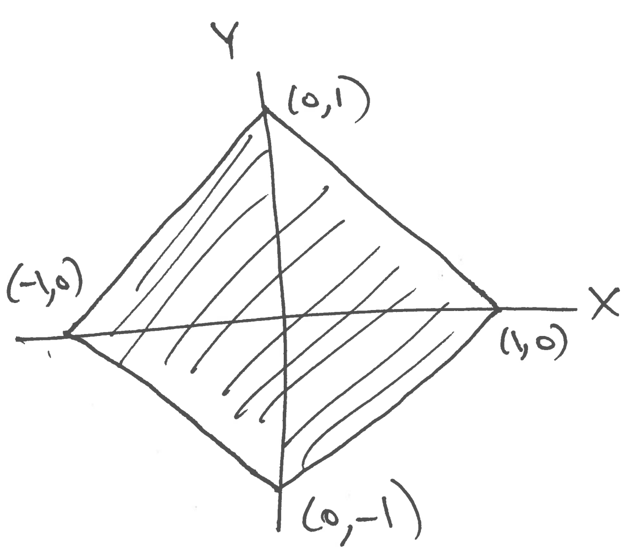
(b) what is the probability that $X \in \left(\frac{1}{2}, \frac{2}{3}\right)$ given $Y = \frac{1}{10}$?

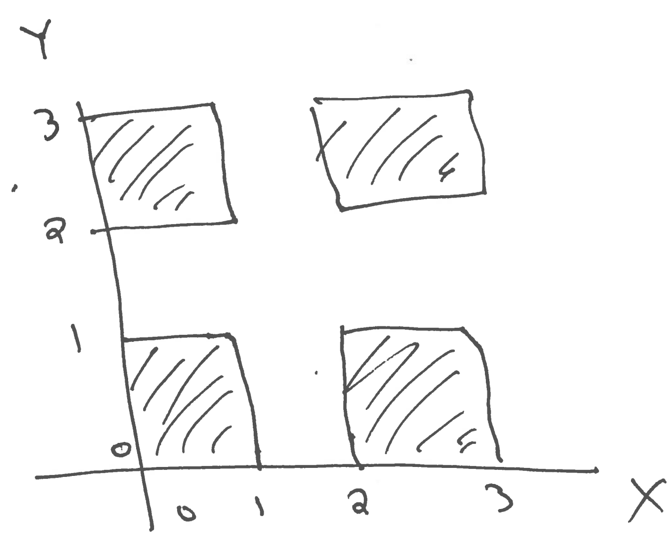
$$\mathbb{P}\left(\frac{1}{2} \le X \le \frac{2}{3} \,\middle|\, Y = \frac{1}{10}\right) = \int_{\frac{1}{2}}^{\frac{2}{3}} \frac{x\left(2-x-\frac{1}{10}\right)}{\frac{2}{3} - \frac{1}{2}\cdot\frac{1}{10}}\,dx$$

Qn: What does the shape of the support set of the tuple (X,Y) (for example) tell us about the possibility that X and Y are independent?





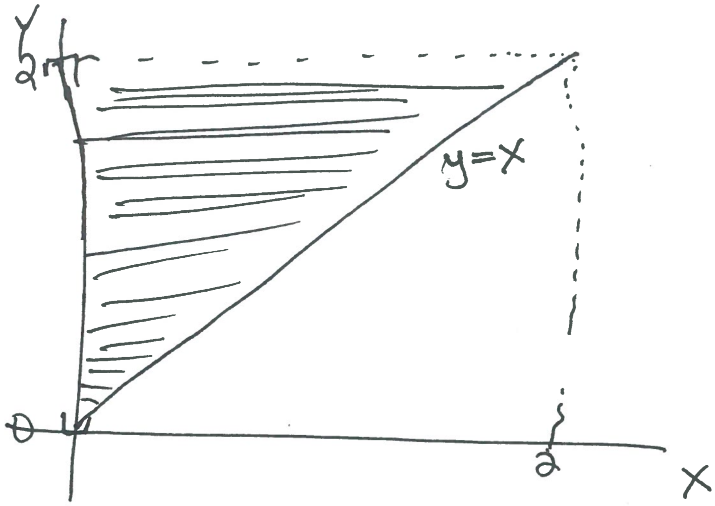




KEY REQUIREMENT: A necessary condition for independence is that the support set of their joint of r.v.'s is that the support set of their joint density must be defined on a multi-dimensional rectangle, i.e., any number of intervals that are unioned, intersected, complemented, etc, and then crossed (in a Cartesian sense) with another such set.

↳ Then, you have to check factorization condition.

EX: Suppose that $f(x,y) = \begin{cases} \frac{x+y}{4} & \text{if } 0<x<y<2 \\ 0 & \text{otherwise} \end{cases}$



Qn: What is $f_{X|Y}(x|y)$? Or, $f_{Y|X}(y|x)$?

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} \qquad f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

$$\boxed{f_Y(y) = \int_0^y \frac{1}{4}(x+y)\,dx = \begin{cases} \frac{3}{8}y^2 & 0<y<2 \\ 0 & \text{otherwise} \end{cases}}$$

$$\boxed{f_X(x) = \int_x^2 \frac{1}{4}(x+y)\,dy = \begin{cases} \frac{1}{2}+\frac{1}{2}x-\frac{3}{8}x^2 & 0<x<2 \\ 0 & \text{otherwise} \end{cases}}$$

$$= \begin{cases} \dfrac{\frac{1}{4}(x+y)}{\frac{3}{8}y^2} & 0<x<y<2 \\ 0 & \text{otherwise} \end{cases} \qquad = \begin{cases} \dfrac{\frac{1}{4}(x+y)}{\frac{1}{2}+\frac{1}{2}x-\frac{3}{8}x^2} & 0<x<y<2 \\ 0 & \text{otherwise} \end{cases}$$