July 11, 2019

Qn: How might we create new probability measures from old ones?

ANS: Conditional probability.

Recall: Suppose that $(S, \mathbb{P})$ is a probability space. Let A and B be events such that $\mathbb{P}(B) > 0$. Suppose that we propose a new probability measure $\mathbb{Q}$ and define

$$\mathbb{Q}_B(A) = \mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

"given"

ASIDE:
$(A_2 \cap B) \cap (A_5 \cap B)$
$= A_2 \cap A_5 \cap B \cap B$
$= \emptyset \cap B \cap B = \emptyset$

Qn: Is $\mathbb{Q}$ a probability measure in its own right? Check the axioms.

Ax. #1: $\mathbb{Q}_B(A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$. But, by ax. 1 for $\mathbb{P}$, $\mathbb{P}(A \cap B) \geq 0$ and by assumption $\mathbb{P}(B) > 0 \Rightarrow \mathbb{P}(B) \geq 0$. So,
$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \geq 0 \Longrightarrow \mathbb{Q}_B(A) \geq 0$. ☑

Ax. #2: $\mathbb{Q}_B(S) = \frac{\mathbb{P}(S \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$. ☑

Ax. #3: Let $A_1, A_2, \ldots$ be a sequence of mutually exclusive events.

$$\mathbb{Q}_B\left(\bigcup_{i=1}^{\infty} A_i\right) = \frac{\mathbb{P}\left(\left\{\bigcup_{i=1}^{\infty} A_i\right\} \cap B\right)}{\mathbb{P}(B)} = \frac{\mathbb{P}\left(\bigcup_{i=1}^{\infty} \{A_i \cap B\}\right)}{\mathbb{P}(B)}$$

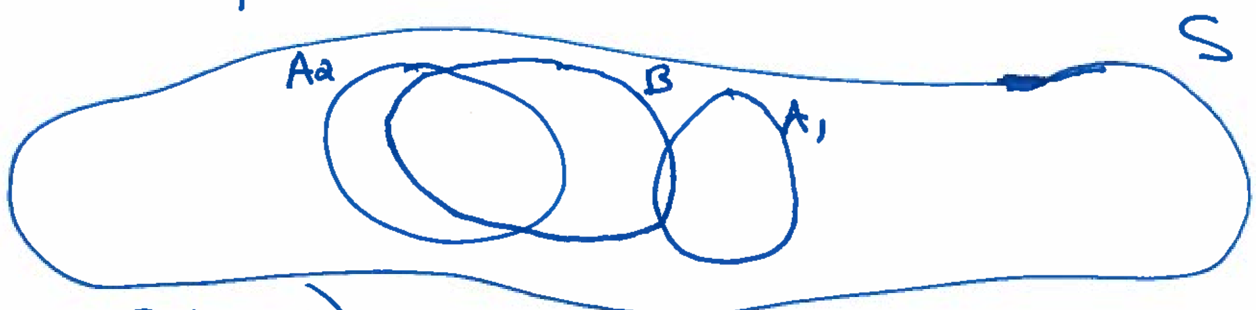
$$= \frac{\sum_{i=1}^{\infty} \mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} = \sum_{i=1}^{\infty} \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} = \sum_{i=1}^{\infty} \mathbb{Q}_B(A_i)$$ ☑

<u>NOTE</u>: This means that all theorems from last class hold for $\mathbb{Q}$.

<u>EX</u>: $\mathbb{Q}_B(A) = 1 - \mathbb{Q}_B(A^c) \iff \mathbb{P}(A|B) = 1 - \mathbb{P}(A^c|B)$

$$\mathbb{P}(A \cup B | C) = \mathbb{P}(A|C) + \mathbb{P}(B|C) - \mathbb{P}(A \cap B | C)$$

<u>INTUITION</u>: We want to "assume" that B occurs and then recompute the probabilities (e.g., of A) based on the assumption that it occurs.



$$\mathbb{Q}_B(A_1) = \frac{\mathbb{P}(A_1 \cap B)}{\mathbb{P}(B)} = \frac{\text{small}}{\text{relative larger}} \approx \text{small}$$

$$\mathbb{Q}_B(A_2) = \frac{\mathbb{P}(A_2 \cap B)}{\mathbb{P}(B)} = \frac{\text{relative large part of } B}{\text{relatively large}} \approx \text{relatively larger}$$

<u>NOTE</u>: We often write

$$\mathbb{P}(A \cap B) = \underbrace{\mathbb{P}(A|B)\,\mathbb{P}(B)}_{\text{"multiplication rule for conditional probability"}}$$

<u>DEF</u>: If A and B are independent, $\mathbb{P}(A|B) = \mathbb{P}(A)$.

$$\Rightarrow \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \mathbb{P}(A) \Rightarrow \boxed{\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)}$$

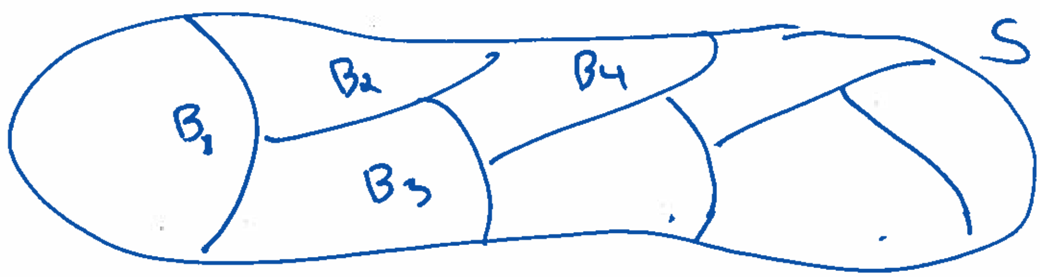
$$\Rightarrow \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \mathbb{P}(B) \Rightarrow \mathbb{P}(B|A) = \mathbb{P}(B)$$

IDEA: If A and B are positively correlated, $\mathbb{P}(A|B) > \mathbb{P}(A)$.

If A and B are negatively correlated, $\mathbb{P}(A|B) < \mathbb{P}(A)$.

DEF: Suppose that we have a sequence of events $B_1, \dots, B_n$ (finite) or $B_1, B_2, \dots$ (infinite) on a probability space $(S, \mathbb{P})$ and it satisfies:

1. $B_i \cap B_j = \emptyset \quad \forall i \neq j$ (mutually exclusive)
2. $\bigcup_{i=1}^{n \text{ or } \infty} B_i = S$ (covering property)
3. $\mathbb{P}(B_i) > 0$ (non-trivity)



We call such a sequence a <u>partition</u> or <u>decomposition</u> or <u>covering</u> of S.

<u>THE LAW OF TOTAL PROBABILITY:</u> If $B_1, \ldots, B_n$ is a partition of $\cancel{S}\ \Omega$ and A is any event, then
$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A|B_i)\mathbb{P}(B_i).$$

<u>PROOF:</u>
$$\sum_{i=1}^{n} \mathbb{P}(A|B_i)\mathbb{P}(B_i) = \sum_{i=1}^{n} \frac{\mathbb{P}(A\cap B_i)}{\cancel{\mathbb{P}(B_i)}}\cancel{\mathbb{P}(B_i)}$$
$$= \sum_{i=1}^{n} \mathbb{P}(A\cap B_i) = \mathbb{P}\left(\bigcup_{i=1}^{n}\{A\cap B_i\}\right)$$

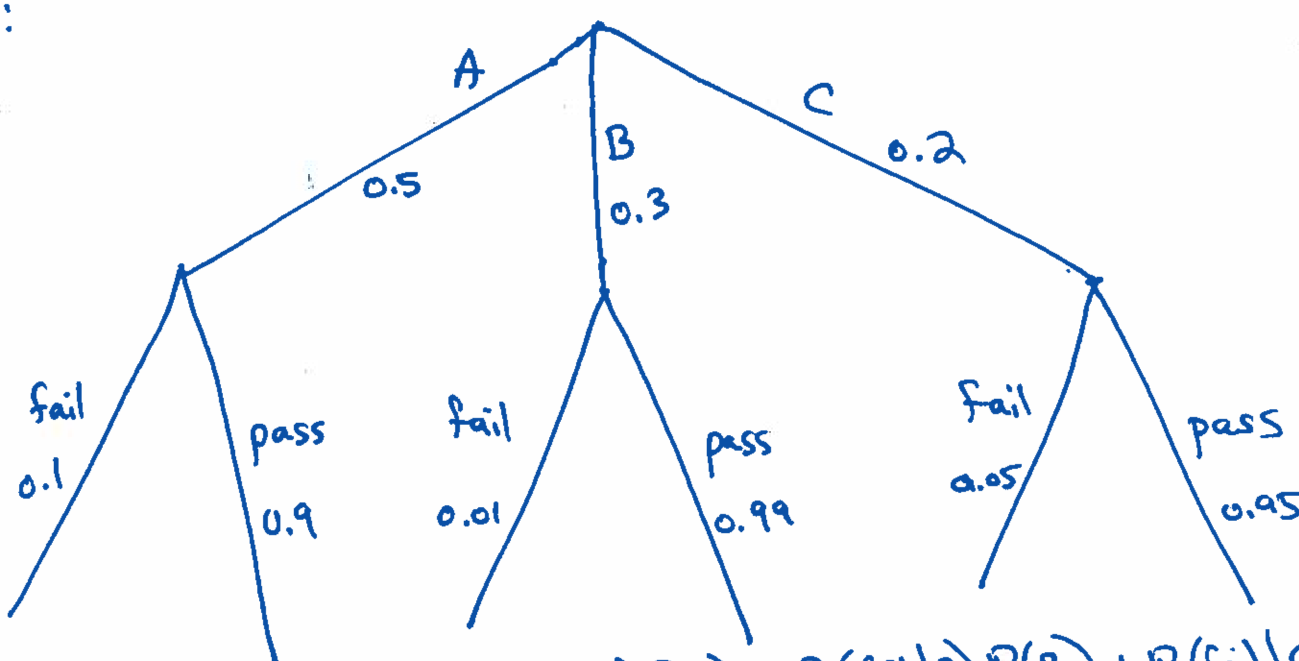
finite additivity

<u>NOTE:</u> $A\cap B_i$'s mutually exclusive

$$= \mathbb{P}(A)$$

element-wise argument

<u>EX</u>:



$$\mathbb{P}(\text{widget fails QA}) = \mathbb{P}(\text{fail}|A)\mathbb{P}(A) + \mathbb{P}(\text{fail}|B)\mathbb{P}(B) + \mathbb{P}(\text{fail}|C)\mathbb{P}(C)$$
$$= (0.1)(0.5) + (0.01)(0.3) + (0.05)(0.2)$$
$$= 0.05 + 0.003 + 0.01$$
$$= 0.063$$

BAYES' THEOREM: Suppose that $(S, \mathbb{P})$ is a probability space and $B_1, \dots, B_n, \dots$ is a partition of S.

this is easier to compute

$$\underbrace{\mathbb{P}(B_k|A)}_{\text{this is hard to compute}} = \frac{\mathbb{P}(A|B_k)\,\mathbb{P}(B_k)}{\underbrace{\mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2) + \dots}_{\text{finite sum if partition is finite}}}$$

$$= \frac{\mathbb{P}(A|B_k)\,\mathbb{P}(B_k)}{\sum_{i=1}^{\infty} \mathbb{P}(A|B_i)\,\mathbb{P}(B_i)}$$

PROOF :

$$\frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\sum_{i=1}^{\infty}\mathbb{P}(A|B_i)\mathbb{P}(B_i)} = \frac{\frac{\mathbb{P}(A\cap B_k)}{\cancel{\mathbb{P}(B_k)}}\cancel{\mathbb{P}(B_k)}}{\sum_{i=1}^{\infty}\frac{\mathbb{P}(A\cap B_i)}{\cancel{\mathbb{P}(B_i)}}\cancel{\mathbb{P}(B_i)}}$$

Some textbooks call this Bayes' Theorem

$$= \boxed{\frac{\mathbb{P}(A\cap B_k)}{\sum_{i=1}^{\infty}\mathbb{P}(A\cap B_i)}}$$

$$= \frac{\mathbb{P}(A\cap B_k)}{\mathbb{P}\left(\bigcup_{i=1}^{\infty}\{A\cap B_i\}\right)}$$

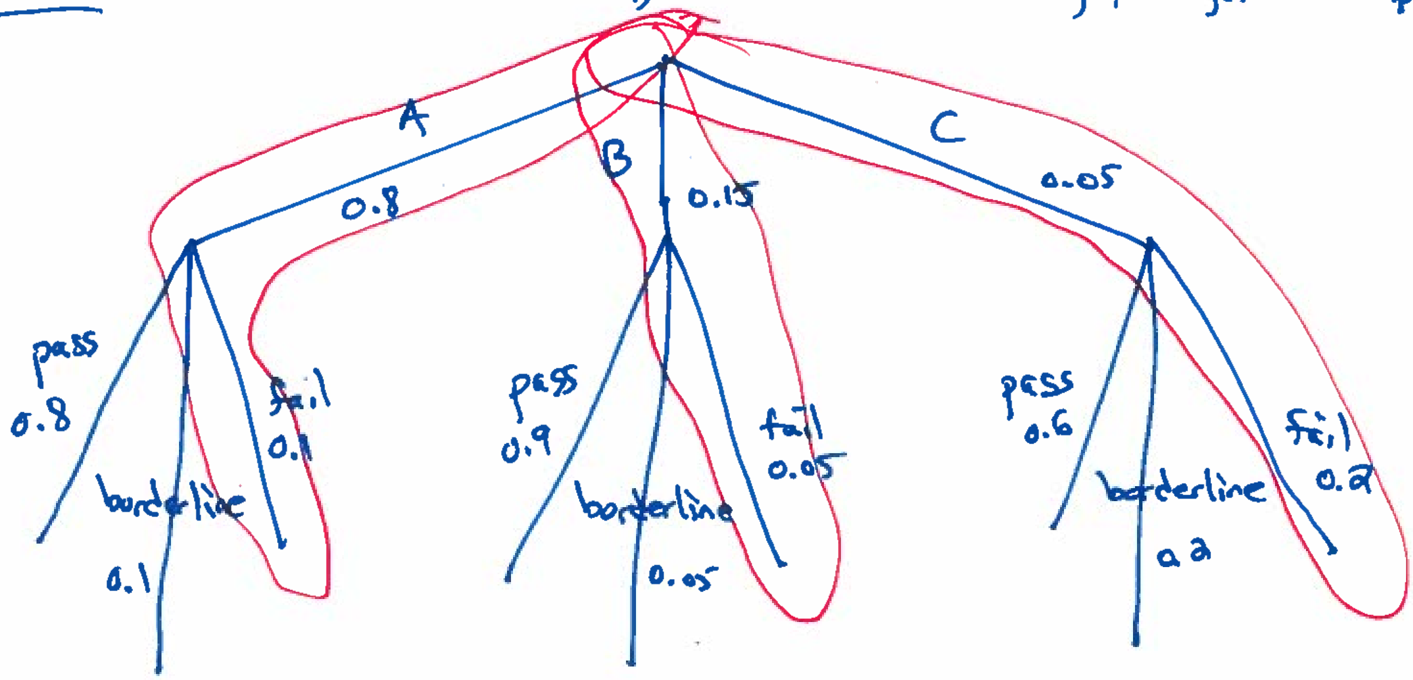
b/c B_i's are a cover

$$= \frac{\mathbb{P}(A\cap B_k)}{\mathbb{P}(A)}$$

$$= \frac{\mathbb{P}(B_k\cap A)}{\mathbb{P}(A)}$$

$$= \mathbb{P}(B_k|A)$$

RECALL: A (modified version) of the factory/widget example.



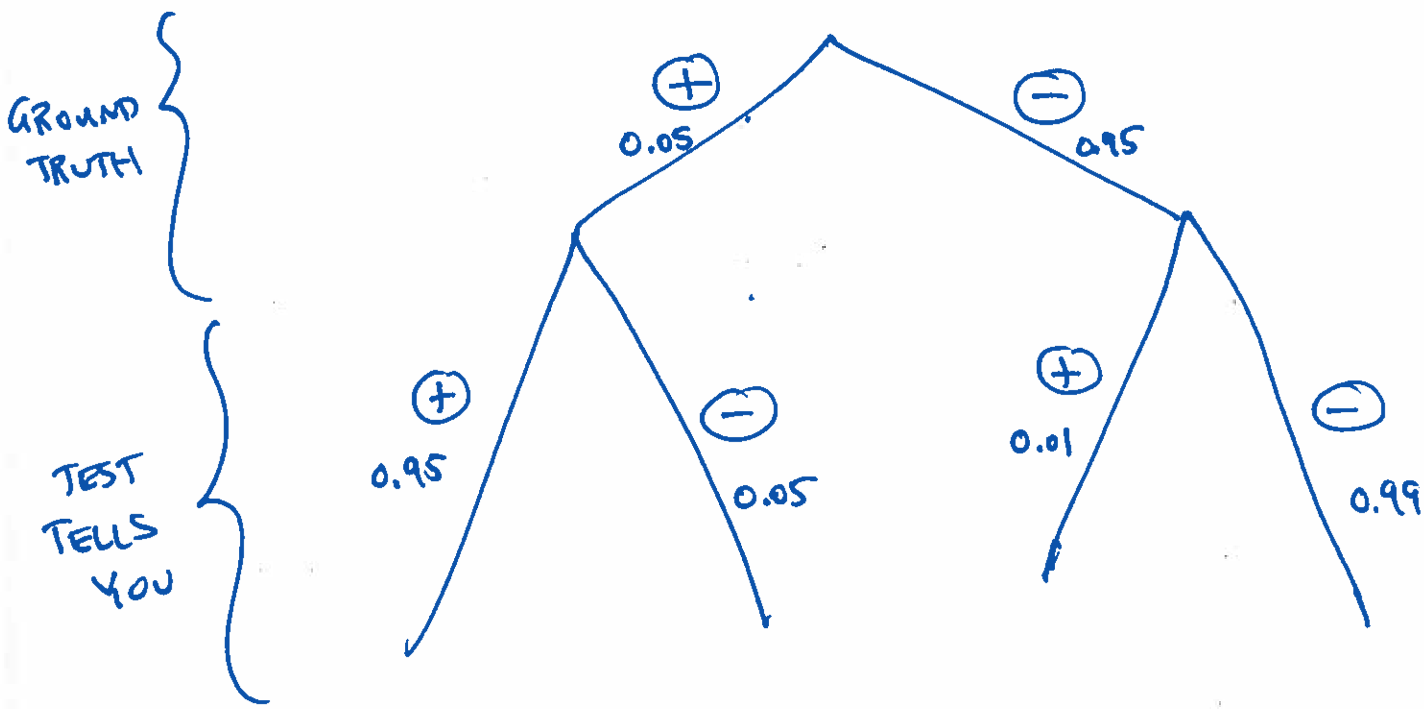
Qn: Given that a pair of underwear has failed, what is the probability that it came from factory C?

Suppose that B_1 = factory A, B_2 = factory B, B_3 = factory C, D = underwear failure.

$$\mathbb{P}(B_3|D) = \frac{\mathbb{P}(D|B_3)\mathbb{P}(B_3)}{\mathbb{P}(D|B_1)\mathbb{P}(B_1) + \mathbb{P}(D|B_2)\mathbb{P}(B_2) + \mathbb{P}(D|B_3)\mathbb{P}(B_3)}$$

$$= \frac{(0.2)(0.05)}{(0.1)(0.8) + (0.05)(0.15) + (0.2)(0.05)}$$

$$= \frac{0.01}{0.08 + 0.0075 + 0.01} \approx 0.1025$$



$$\mathbb{P}(\text{HIV}\oplus \mid \text{test says } \oplus) = \frac{\mathbb{P}(\text{test says } \oplus \mid \oplus)\,\mathbb{P}(\oplus)}{\mathbb{P}(\text{test says } \oplus \mid \oplus)\,\mathbb{P}(\oplus) + \mathbb{P}(\text{test } \oplus \mid \ominus)\,\mathbb{P}(\ominus)}$$

$$= \frac{(0.95)(0.05)}{(0.95)(0.05) + (0.01)(0.95)}$$

$$= \frac{5}{6} \approx 0.83$$