7/9/2019

DEFINITION: A random experiment is a process or action whose outcome is not determined, i.e., it is stochastic.

EX: $f(x) = x^2$

↑ input   ↖ output

If you know x, and you can do calculations, x^2 is known.

NOT RANDOM

EX: Roll a die.
Deal out a deck of cards.
The weather at 12 noon on 7/9/2019 at 80.

DEFINITION: The set of all possible outcomes of a random experiment is called the sample space.
We typically denote the sample space by S or Ω.

DEFINITIONS: We call the elements of S (or Ω) outcomes (or atoms or singletons) and they are denoted by s or ω.

NOTE: We will make statements like $s \in S$ or $\omega \in \Omega$.

EX: Flip a coin three times. (Order matters.)

S

HHH HTT
HHT THT
HTH TTH
THH TTT

$A \subseteq S$

DEF: An event (typically denoted with capital letters like A and B) is (kind of) any subset of S.

↳ We can write things like $A \subseteq S$ or $A \cap B \subseteq S$.

↳ All set operations (e.g., $\cup$, $\cap$, ^{c}, $\sim$, etc) can be used with events.

QN: You have seen notation like $\mathbb{P}(A)$ or $\mathbb{P}(B) = \mathbb{P}(X \geq 5)$.

↳ $B = \{ s \in S \mid X(s) \geq 5 \}$

What makes $\mathbb{P}$ a probability measure on S?

ANS: The axioms of Kolmogorov.

the power set of S, i.e., the set of all subsets

FRAMEWORK: Let S be a sample space and let $\mathbb{P}$ be a function $\mathbb{P} : \mathcal{P}(S) \to \mathbb{R}$. We say that $\mathbb{P}$ is a probability measure and $(S, \mathbb{P})$ a probability space if:

1. $\mathbb{P}(A) \geq 0$ for any event $A \subseteq S$. (nonnegativity)
2. $\mathbb{P}(S) = 1$ (unity)
3. If $A_1, A_2, \ldots$ is a sequence of mutually exclusive events, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

(countable additivity)

DEFINITION: Events A and B from S are mutually exclusive if $A \cap B = \emptyset$. So, if $A_1, A_2, \ldots$ is a sequence of mutually exclusive events, then $\forall i, \forall j$, $A_i \cap A_j = \emptyset$.

`\mathbb{P}`

THEOREM #1: $\mathbb{P}(\emptyset) = 0$

PROOF: Assume that I have a probability space $(S, \mathbb{P})$. Let $A_1 = \emptyset, A_2 = \emptyset, A_3 = \emptyset, \dots$.

Because $\emptyset \cap \emptyset = \emptyset$, $\{A_i\}_{i=1}^{\infty}$ is mutually exclusive.

$\Rightarrow$ can apply Axiom 3

$$\Rightarrow \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

Note that $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} \emptyset\right) = \mathbb{P}(\emptyset)$.

But also, $\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \sum_{i=1}^{\infty} \mathbb{P}(\emptyset)$.

By Axiom 1, $\mathbb{P}(\emptyset) \geq 0$.

CASE #1: $\mathbb{P}(\emptyset) > 0$.

$\mathbb{P}(\emptyset) = c > 0$.

But then

$$c = \sum_{i=1}^{\infty} c.$$

Since $0 < c < 1 < \infty$,

this is a contradiction.

$\bigotimes$

CASE #2: $\mathbb{P}(\emptyset) = 0$.

THEOREM #2: (finite additivity) Let $A_1, A_2, A_3, \ldots, A_n$ be a finite sequence of mutually exclusive events. Then

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i)$$

PROOF: Construct an infinite sequence of events as follows:

$$B_1 = A_1$$
$$B_2 = A_2$$
$$B_3 = A_3$$
$$\vdots$$
$$B_n = A_n$$
$$B_{n+1} = \emptyset$$
$$B_{n+2} = \emptyset$$
$$\vdots$$

CHECK that

1. $B_i \cap B_j$ when $i, j \leq n$, $i \neq j$,

   $B_i \cap B_j = A_i \cap A_j = \emptyset$ by assumption

2. $B_i \cap B_j$ when $i, j \geq n+1$, $i \neq j$, means $B_i \cap B_j = \emptyset \cap \emptyset = \emptyset$

3. If $i \leq n$ and $j > n$, then $B_i \cap B_j = A_i \cap \emptyset = \emptyset$.

$\Rightarrow$ the (infinite) sequence $\{B_i\}_{i=1}^{n}$ is mutually exclusive

$\Rightarrow$ I can use Ax. 3

$$\Rightarrow \mathbb{P}\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(B_i) = \sum_{i=1}^{n} \mathbb{P}(B_i) + \sum_{i=n+1}^{\infty} \mathbb{P}(B_i)$$

$$= \sum_{i=1}^{n} \mathbb{P}(B_i) \quad \text{by THEOREM \#1}$$

$$= \sum_{i=1}^{n} \mathbb{P}(A_i)$$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} B_i\right) = \mathbb{P}\left(\left\{\bigcup_{i=1}^{n} B_i\right\} \cup \left\{\bigcup_{i=n+1}^{\infty} B_i\right\}\right) = \mathbb{P}\left(\left\{\bigcup_{i=1}^{n} B_i\right\} \cup \emptyset\right) = \mathbb{P}\left(\bigcup_{i=1}^{n} B_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right)$$

THEOREM #3: If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

(NOTE: First use of disjunctification.)

write a set as the union of mutually exclusive sets.

Note: if $A \subseteq B$, then $B = A \cup (B \sim A)$.

(Aside: $B \sim A = B \cap A^c$.)

Qn: Are A and $B \sim A$ clearly mutually exclusive? $\boxed{\checkmark}$

$$\Rightarrow \mathbb{P}(B) = \mathbb{P}(A \cup (B \sim A)) \overset{\text{by Theorem \#2}}{=} \mathbb{P}(A) + \mathbb{P}(B \sim A)$$

$$\Rightarrow \mathbb{P}(B \sim A) = \mathbb{P}(B) - \mathbb{P}(A)$$

$\Rightarrow$ Axiom 1 requires that

$$0 \leq \mathbb{P}(B \sim A) = \mathbb{P}(B) - \mathbb{P}(A)$$

$$\Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$$

ASIDE: If $\mathbb{P}(A) = 0$, can you conclude that $A = \emptyset$?

If $U \sim \text{Unif}(0,1)$, then $\mathbb{P}(U = \frac{1}{\pi}) = 0$ but $\{\frac{1}{\pi}\} \neq \emptyset$.

THEOREM #4: $\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$ (complementarity)

PROOF: For any event A, $S = A \cup A^c$.

$$\Rightarrow 1 = \mathbb{P}(S) = \mathbb{P}(A) + \mathbb{P}(A^c)$$

$$\Rightarrow \mathbb{P}(A) = \cancel{\mathbb{P}(A^c)}\ 1 - \mathbb{P}(A^c)$$

#5

THEOREM: For any event $A \subseteq S$, $0 \le \mathbb{P}(A) \le 1$.

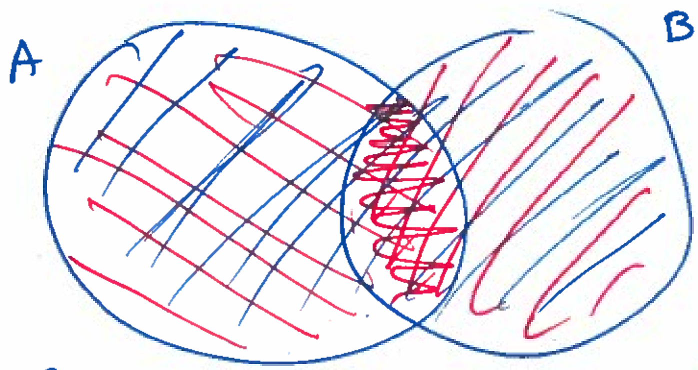
PROOF: Recall $\emptyset \subseteq A \subseteq S$. By Theorem #3,

$$\mathbb{P}(\emptyset) \le \mathbb{P}(A) \quad \text{and} \quad \mathbb{P}(A) \le \mathbb{P}(S).$$

But then $\underbrace{0 = \mathbb{P}(\emptyset)}_{\text{Theorem \#1}} \le \mathbb{P}(A) \le \underbrace{\mathbb{P}(S) = 1}_{\text{Ax. 2}}$

THEOREM #6: (HIGH SCHOOL THEOREM)

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$



Note that $A \cup B = (A \sim B) \cup (B \sim A) \cup (A \cap B)$.

True but not helpful. Instead,

$$A \cup B = A \cup (B \sim (A \cap B)).$$

(Check with an elementwise proof.)

Recall: we already showed that if $W \subseteq V$, then $\mathbb{P}(V \sim W) = \mathbb{P}(V) - \mathbb{P}(W)$.

$$\Rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A \cup (B \sim A \cap B)) = \mathbb{P}(A) + \mathbb{P}(B \sim (A \cap B))$$
$$= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

if you're convinced that

$$A \cap (B \sim (A \cap B)) = \emptyset.$$

NOTE: This theorem has a generalization called the inclusion-exclusion principle that tells you how to compute probabilities of unions of more than two sets

EX: $$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(B \cap C) - \mathbb{P}(A \cap C) + \mathbb{P}(A \cap B \cap C)$$

DEF: Two events A and B are equally likely if $\mathbb{P}(A) = \mathbb{P}(B)$.

THEOREM #7: Let S be a discrete and finite sample space, i.e., $S = \bigcup_{i=1}^{n} \{s_i\}$. If the members of S are equally likely, then $\mathbb{P}(\{s_i\}) = \frac{1}{n}$, when $n = |S|$ (i.e., the number of elements in S).

$$1 = \mathbb{P}(S) = \mathbb{P}\left(\bigcup_{i=1}^{n} \{s_i\}\right) = \sum_{i=1}^{n} \mathbb{P}(\{s_i\})$$

by Ax. 2

$$= \sum_{i=1}^{n} \mathbb{P}(\{s_i\})$$

$$= \mathbb{P}(\{s_i\}) \sum_{i=1}^{n} 1$$

$$= n\,\mathbb{P}(\{s_i\}) \Rightarrow \mathbb{P}(\{s_i\}) = \frac{1}{n}$$