Physics as Quantized Measurement: The Farey Sequence as a Realization of the Arithmetic of Order

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Dedicated to the memory of John Horton Conway (1937–2020),
whose playful genius revealed that finite games, hidden symmetries,
and simple combinatorics can generate whole worlds—finite yet unbounded.

Abstract

This article formalizes the Farey sequence hierarchy ($\mathcal{F}_n$) as a concrete realization of the "Arithmetic of Order" (AoO) framework. This framework leverages the constructive machinery of ZF set theory, without assuming the Axiom of Infinity, to model physical reality as a procedural, finitistic unfolding. The core physical principle advanced is that the set-theoretic complement between successive powersets, $\mathcal{P}(\Omega_{n+1}) \setminus \mathcal{P}(\Omega_n)$, is the mathematical image of a resolution jump in physical measurement. This provides a rigorous engine for Cycle Clock Theory, demonstrating how a coherent, non-overlapping, and generative temporal process emerges from finite principles. By showing that the infinite is an emergent property, the AoO offers a unified, operational foundation for mathematics and physics.

Contents

1 Introduction: A Finitistic Recasting of the Continuum

The Farey sequence provides a rigorous mathematical prototype for the Arithmetic of Order (Arithmetic of Order) framework. This framework proposes a mathematics built from finite, observable, and constructive steps. It operates within the rigorous language of Zermelo-Fraenkel (ZF) set theory but adopts a procedural philosophy, using the power of ZF to define iterative functions rather than relying on abstract, non-constructive axioms.

The central thesis is that the Axiom of Infinity is not a necessary presupposition. Instead, the continuum emerges as the asymptotic limit of a simple, ordered progression, $1 \to n \to n+1$. The Farey sequence, generated algorithmically, provides a direct model of this principle [4].

Remark 1.1 (Operational Logic vs. Formal Logic)*.* While this work is formalized in the language of ZF set theory for rigor, the AoO does not assume the entire ZF foundation as primitive. Rather, the logical relations (unions, complements, cardinalities) emerge as natural descriptions of the iterative, ordered distinguishing process. Thus, AoO uses ZF language as a descriptive framework, not as an ontological axiom.

2 The "Arithmetic of Order": A Framework of Emergent Complexity

2.1 The Generative Progression $1 \to n \to n+1$

The Arithmetic of Order framework posits that mathematical structures emerge from the ordered progression $1 \to n \to n+1$. This progression is not added structure but the generating principle: order is not assigned to a pre-existing set, but brings the set into being by distinguishing elements in sequence. Each step adds a new, quantized degree of freedom, refining the system's resolution.

2.2 The Powerset Hierarchy as an Iterative Process

For a system with 'n' degrees of freedom, represented by the ordered set $\Omega_n = \{1, 2, \ldots, n\}$, the canonical state space is its powerset, $\mathcal{P}(\Omega_n)$. Within the AoO, this hierarchy of state spaces is not merely asserted by the Powerset Axiom but is understood as a sequence generated by an iterative function—a construction fully supported by the recursive principles of ZF set theory.

3 A Unification of Algebraic Structures

The correspondence between the Farey and powerset systems extends to the algebraic structures they embody, revealing a profound duality.

3.1 The Modular Group $\mathrm{SL}(2, \mathbb{Z})$: The Invariant Algebra of Farey Adjacency

The unimodular relation $bc - ad = 1$ for adjacent Farey fractions $\frac{a}{b}, \frac{c}{d}$ is the defining characteristic of the special linear group $\mathrm{SL}(2, \mathbb{Z})$ [6]. This infinite, non-abelian group governs the local, dynamic structure of the Farey sequence [3].

3.2 The Vector Space $\mathbb{F}_2^n$: The Canonical Algebra of System Configurations

The powerset $\mathcal{P}(\Omega_n)$, when equipped with the operation of symmetric difference (Δ), forms a finite abelian group isomorphic to the n-dimensional vector space over the finite field of two elements, $\mathbb{F}_2^n$ [5]. This is the canonical algebra for the global, static set of all possible configurations.

4 Realization of Cycle Clock Theories

Conceptual frameworks in cosmology and foundational physics, such as Roger Penrose's Cycle Clock Theory, posit that time does not flow along a pre-existing continuum but unfolds in discrete, generative "ticks" [7]. The AoO provides the precise mathematical engine for this vision.

Remark 4.1 (AoO as the Mathematical Engine for Cycle Clock Theory). Cycle Clock Theory conceptualizes physical time as a sequence of discrete, generative "ticks"—each cycle marking not just passage, but the creation of new states. The *Arithmetic of Order* (AoO) provides the rigorous mathematical engine for this vision:

- **Ordered Progression:** The step $1 \to n \to n + 1$ in AoO is the formal "tick" of the universal clock, each step generating a new, quantized layer of reality.
- **Generative Mechanism:** The powerset complement $\mathcal{P}(\Omega_{n+1}) \setminus \mathcal{P}(\Omega_n)$ precisely defines the new states actualized at each tick.
- **Farey Sequence as Refinement:** The mediant operation in the Farey hierarchy models the arithmetic rule for how each cycle refines measurable structure.
- **Unimodular Coherence:** The condition $bc - ad = 1$ ensures that each generative step is globally consistent and non-overlapping.
- **Algebraic Duality:** The local, dynamic law of $\mathrm{SL}(2, \mathbb{Z})$ (cycles/refinements) and the global, static structure of $\mathbb{F}_2^n$ (state space) together capture the full symmetry of the process.

Synthesis: *Cycle Clock Theory describes time as fundamentally cyclical and discrete; AoO provides the explicit arithmetic and combinatorial rules that generate, organize, and unify these cycles into a coherent, finitistic engine for physical reality.*

5 Measurement as a Generative Process

The Arithmetic of Order framework necessitates a re-evaluation of the concept of measurement itself. Traditional views treat measurement as a passive act of reading a pre-existing value. In contrast, AoO defines measurement as the very act of generation that unfolds structure.

Definition 5.1 (Measurement in the Arithmetic of Order)**.** Let n denote the current degree of freedom (resolution) of a system. A **measurement** is the operational act that increases the system's degree of freedom from n to $n + 1$, thereby generating the new set of distinguishable states

$$\mathcal{M}_{n+1} := \mathcal{P}(\Omega_{n+1}) \setminus \mathcal{P}(\Omega_n)$$

or, in the Farey sequence context, the set of new mediants inserted between neighbors in $\mathcal{F}_n$ to form $\mathcal{F}_{n+1}$. Measurement thus unfolds reality in discrete, complementary layers.

6 Conclusion and Outlook

The Farey hierarchy provides a direct, operational instantiation of the Arithmetic of Order. This framework demonstrates that a rich mathematical universe, capable of describing physical reality, can be generated by a simple, iterative procedure grounded in the ordered progression of natural numbers. It provides a rigorous, finitistic engine for physical models based on discrete, generative cycles.

This procedural approach, while formalized within the language of ZF set theory as a descriptive tool, shows that the Axiom of Infinity is not a necessary prerequisite. Instead, the infinite arises as the asymptotic horizon of a finitistic, physically motivated construction.

By clarifying how the same local symmetry group ($\mathrm{SL}(2, \mathbb{Z})$), the nested global powerset, and the constraint-driven Farey refinement produce coherent, non-overlapping layers, the Arithmetic of Order framework unifies perspectives across scales. It naturally supports cosmological models like Penrose's Conformal Cyclic Cosmology and Planck-scale lattice models like Irwin's Quasicrystal Clock — showing that both can be understood as scale-specific realizations of the same universal arithmetic process.

Future work will develop explicit models of space-time, quantum fields, and measurement built purely from these finitistic complement operations, bypassing assumed continua. The Arithmetic of Order thus invites a fundamental rethinking of how measurement, logic, and finitude shape the physical and mathematical worlds.

A Appendix: Key Properties

A.1 Proof Sketch of the Unimodular Relation

Sketch. Let $\frac{a}{b}$ and $\frac{c}{d}$ be adjacent fractions in a Farey sequence. The unimodular relation $bc - ad = 1$ can be proven geometrically. Consider the triangle in the integer lattice with vertices at $(0, 0)$, (b, a), and (d, c). The area of this triangle is given by $\frac{1}{2}|ad - bc|$. According to Pick's Theorem, the area of a simple lattice polygon is $A = I + \frac{B}{2} - 1$, where I is the number of interior lattice points and B is the number of lattice points on the boundary. Because $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent in a Farey sequence, there are no lattice points in the interior of the triangle ($I = 0$) or on the edges between the vertices (other than the vertices themselves, so $B = 3$). Thus, the area is $0 + \frac{3}{2} - 1 = \frac{1}{2}$. Equating the two expressions for the area gives $\frac{1}{2}|ad - bc| = \frac{1}{2}$, which implies $|ad - bc| = 1$. Since the fractions are ordered, $bc - ad = 1$. $\square$

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